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Yamamoto's Theorem for Generalized Singular Values

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ABSTRACT

We extend Yamamoto's theorem, an asymptotic result that relates the ith largest absolute eigenvalue to the ith singular value for complex matrices, to various generalized singular values associated with a class of functions weaker than norms.

1. INTRODUCTION

Let $\mathbf{C}^n$ and $\mathbf{M}_n$ denote, respectively, the vector spaces of n-component column vectors and n by n matrices over the complex numbers. For $A \in \mathbf{M}_n$, we arrange the eigenvalues $\lambda_1, \ldots, \lambda_n$ in order of nonincreasing absolute value. We number the singular values of A (nonnegative square roots of the eigenvalues of A^*A) $\sigma_1, \ldots, \sigma_n$, similarly.

It is well known when $\|\cdot\|$ is a matrix norm, and remains true when $\|\cdot\|$ is only a (generalized matrix) norm on $\mathbf{M}_n$ [1, p. 322], that

$$\rho(A) \equiv |\lambda_1| = \lim_{k \to \infty} \|A^k\|^{1/k}.$$

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This is sometimes called the *spectral radius formula*. When $\|\cdot\|$ is the spectral norm, $\|\cdot\| = \sigma_1(\cdot)$, the spectral radius formula becomes

$$|\lambda_1| = \lim_{k\to\infty} \sigma_1(A^k)^{1/k}.$$

In [7] this fact was attractively extended by noting that similar formulae hold for the absolute values of the remaining eigenvalues in terms of the remaining singular values, i.e.,

$$|\lambda_i| = \lim_{k\to\infty} \sigma_i(A^k)^{1/k}, \qquad i = 1,\dots,n.$$

A variety of generalizations of singular values are currently being studied by focusing upon known properties of the usual singular values and upon norms other than the spectral norm on $\mathbf{M}_n$ and Euclidean norm on $\mathbf{C}^n$ (which are naturally tied to the usual singular values via known properties). See [5], for example. Our purpose here is to note that Yamamoto's observation may be extended to these generalized singular values. This, of course, makes such quantities seem more natural as generalizations of singular values. Yamamoto's original proof [7] used compounds to show that the spectral radius formula could be algebraically extended to further eigenvalues and singular values. We give two proofs. One relies upon Yamamoto's observation and exploits the comparability among functions on a vector space even more general than a norm, but the other deals with the remaining eigenvalues directly (in a parallel way) and may be viewed as an alternative to Yamamoto's original proof.

2. NORMS AND GENERALIZATIONS

Let $\mathbf{X}$ denote either the vector space $\mathbf{C}^n$ or $\mathbf{M}_n$. A norm on $\mathbf{X}$ is a function $\|\cdot\|: \mathbf{X} \to \mathbf{R}$ satisfying the following three axioms:

(1a) $\|x\| \geqslant 0$ for all $x \in \mathbf{X}$;
(1b) $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\alpha x\| = |\alpha|\,\|x\|$ for all complex numbers α and vectors x.
(3) $\|x + y\| \leqslant \|x\| + \|y\|$ for all vectors x and y.

Common examples of norms on $\mathbf{C}^n$ are the Euclidean norm $\|\cdot\|_2$, defined by $\|x\|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$, and the sup norm $\|\cdot\|_\infty$, defined by $\|x\|_\infty = \max\{|x_1|,\dots,|x_n|\}$.

Any norm $\|\cdot\|$ on $\mathbf{C}^n$ naturally induces a norm $|||\cdot|||$ on $\mathbf{M}_n$ via the following maximization:

$$|||A||| = \max\{ \|Ax\| : x \in \mathbf{C}^n,\ \|x\| = 1\}.$$

The norm induced on $\mathbf{M}_n$ by the Euclidean norm $\|\cdot\|_2$ is the *spectral norm*, which we denote by $|||\cdot|||_2$ [and which, of course, is the same as $\sigma_1(\cdot)$].

For any norm $\|\cdot\|$ on $\mathbf{C}^n$ and any nonsingular $S \in \mathbf{M}_n$ we may define a norm $\|\cdot\|_S$ on $\mathbf{C}^n$ by $\|x\|_S = \|Sx\|$. With this construction, the norm $|||\cdot|||_S$ (on $\mathbf{M}_n$) induced by $\|\cdot\|_S$ is related to the norm $|||\cdot|||$ induced by $\|\cdot\|$ by the identity

$$|||A|||_S = |||SAS^{-1}||| \qquad \text{for all} \quad A \in \mathbf{M}_n.$$

A prenorm is a function $f(\cdot): X \to R$ satisfying norm axioms (1) (a and b) and (2), but with (3) replaced by

(3′) f is continuous on $\mathbf{X}$.

All norms are prenorms, but not all prenorms are norms. The following comparability, well known for norms on finite dimensional spaces, may be found in [1, p. 272].

PROPOSITION 2.1. *Let $f(\cdot)$ and $g(\cdot)$ be any prenorms on a finite dimensional vector space $\mathbf{X}$ (e.g. $\mathbf{C}^n$ or $\mathbf{M}_n$). Then there exist positive constants c_m and c_M such that for all $x \in \mathbf{X}$,*

$$c_m f(x) \leqslant g(x) \leqslant c_M f(x).$$

There is an equivalence relation on functions suggested by Proposition 2.1. Given functions f and g from $\mathbf{X}$ into the nonnegative reals, we will say f and g are equivalent if there exist positive constants c_m and c_M such that the conclusion of Proposition 2.1 holds. Our main results are valid to at least the level of generality of functions on $\mathbf{X}$ that are equivalent to a norm on $\mathbf{X}$. We segregate this special class of functions in the following definition.

DEFINITION 2.2. Let $\|\cdot\|$ be a norm on the finite dimensional vector space $\mathbf{X}$. A function $f(\cdot): \mathbf{X} \to \mathbf{R}$ is called *norm-equivalent* if there exist fixed positive constants c_m and c_M such that $c_m\|x\| \leqslant f(x) \leqslant c_M\|x\|$ for all $x \in \mathbf{X}$.

It is clear that all norm-equivalent functions satisfy norm axiom 1 (a and b); in view of Proposition 2.1, the set of norm-equivalent functions is independent of which norm $\|\cdot\|$ is chosen. Also, all prenorms are norm-equivalent functions. The function $f(x) \equiv 2\|x\|+\sin\|x\|$ is norm-equivalent but is not a prenorm.

3. GENERALIZED SINGULAR VALUES AND YAMAMOTO'S THEOREM

DEFINITION 3.1. Let $\|\cdot\|$ be a norm on $\mathbf{C}^n$, $|||\cdot|||$ the norm on $\mathbf{M}_n$ induced by $\|\cdot\|$, and $A \in \mathbf{M}_n$. Let $i \in \{1,\dots,n\}$.

The ith *min-max number* of A is denoted by $\alpha_i(A, \|\cdot\|)$ and defined by $\alpha_i(A, \|\cdot\|) = \inf\{\sup\{\|Ax\|: \|x\|=1,\ x \in \mathbf{L}\}: \dim \mathbf{L} = n-i+1\}$.

The ith *max-min number* of A is denoted by $\omega_i(A, \|\cdot\|)$ and defined by $\omega_i(A, \|\cdot\|) = \sup\{\inf\{\|Ax\|: \|x\|=1,\ x \in \mathbf{L}\}: \dim \mathbf{L} = i\}$.

The ith *approximation number* of A is denoted by $\delta_i(A, \|\cdot\|)$ and defined by $\delta_i(A, \|\cdot\|) = \inf\{|||A-X|||: X \in \mathbf{M}_n, \operatorname{rank} X < i\}$.

Such numbers have been studied by several authors. See, for example, [4], [2], [3], and [5]. In the following proposition we state the known facts about these numbers that we need.

PROPOSITION 3.2. *Let $A \in \mathbf{M}_n$. Let $\|\cdot\|$ be a norm on $\mathbf{C}^n$, and $|||\cdot|||$ the corresponding induced norm on $\mathbf{M}_n$. Let the ω, α, and δ numbers be as in Definition 3.1. Suppressing the arguments of ω, α, and δ numbers, we then have:*

(a) $\delta_1 = \alpha_1 = \omega_1 = |||A|||$.

(b) *For $i \in \{2,\dots,n\}$, $\delta_i \geq \alpha_i \geq \omega_i$.*

(c) *If the norm used to define the numbers is the Euclidean norm, then for all $i \in \{1,\dots,n\}$, $\sigma_i(A) = \omega_i = \alpha_i = \delta_i$.*

(d) *If* rank $A < i$, *then $0 = \omega_i = \alpha_i = \delta_i$.*

(e) *For each $i \in \{1,\dots,n\}$ and any $\beta \in \mathbf{C}$, $\delta_i(\beta A) = |\beta|\delta_i(A)$, and similarly for the min-max and max-min numbers.*

(f) *If A is nonsingular, then $\delta_n = \alpha_n = \omega_n = |||A^{-1}|||^{-1}$.*

We may now state the first of our two primary observations, which says that the asymptotic result of Yamamoto holds also for these generalized singular values (with respect to any norm). Note that the proof does not rely upon Yamamoto's result.

THEOREM 3.3. *Let $A \in \mathbf{M}_n$ with eigenvalues numbered in order of nonincreasing absolute value. Let $\|\cdot\|$ be a norm on $\mathbf{C}^n$. For each $i \in \{1,\dots,n\}$ we have*

(a) $|\lambda_i| = \lim_{k\to\infty} \omega_i(A^k, \|\cdot\|)^{1/k}$,
(b) $|\lambda_i| = \lim_{k\to\infty} \alpha_i(A^k, \|\cdot\|)^{1/k}$, *and*
(c) $|\lambda_i| = \lim_{k\to\infty} \delta_i(A^k, \|\cdot\|)^{1/k}$.

Proof. Assume that $\lambda_i \neq 0$, since otherwise for $m \geqslant n$ we have rank $A^m < i$, from which the result is a consequence of (d) of Proposition 3.2. Assume that $i \geqslant 2$, since $i = 1$ is just the spectral radius formula.

Assume that A is upper triangular with diagonal $\lambda_1,\dots,\lambda_n$, since by Schur's triangularization theorem there exists a unitary matrix U such that U^*AU has this form, and if we define the norm $\|\cdot\|_U$ on $\mathbf{C}^n$ by $\|x\|_U = \|Ux\|$, then by consideration of Definition 3.1,

$$\omega_i(A^k, \|\cdot\|)^{1/k} = \omega_i\big((U^*AU)^k, \|\cdot\|_U\big)^{1/k},$$

$$\alpha_i(A^k, \|\cdot\|)^{1/k} = \alpha_i\big((U^*AU)^k, \|\cdot\|_U\big)^{1/k},$$

$$\delta_i(A^k, \|\cdot\|)^{1/k} = \delta_i\big((U^*AU)^k, \|\cdot\|_U\big)^{1/k}$$

for each $k = 1,2,\dots$. Thus the limiting behavior of the sequences on the left is the same as that of those on the right.

Assume that $|\lambda_i| = 1$, since

$$\omega_i\big((A/|\lambda_i|)^k, \|\cdot\|\big)^{1/k} = \frac{1}{|\lambda_i|}\omega_i(A^k, \|\cdot\|)^{1/k},$$

$$\alpha_i\big((A/|\lambda_i|)^k, \|\cdot\|\big)^{1/k} = \frac{1}{|\lambda_i|}\alpha_i(A^k, \|\cdot\|)^{1/k},$$

$$\delta_i\big((A/|\lambda_i|)^k, \|\cdot\|\big)^{1/k} = \frac{1}{|\lambda_i|}\delta_i(A^k, \|\cdot\|)^{1/k}.$$

Let B denote the i by i submatrix of A in the first i rows and columns, $\|\cdot\|^*$ the norm on $\mathbf{C}^i$ obtained by appending $n-i$ zeros onto a vector in $\mathbf{C}^i$

and evaluating $\|\cdot\|$ on the resulting member of $\mathbf{C}^n$, and $|||\cdot|||^*$ the norm on $\mathbf{M}_i$ induced by $\|\cdot\|^*$. Then

$$\omega_i(A^k, \|\cdot\|) \geqslant \inf\{\|A^k x\| : x \in \operatorname{span}(e_1,\dots,e_i),\ \|x\| = 1\}$$

$$= \inf\{\|B^k y\|^* : y \in \mathbf{C}^i,\ \|y\|^* = 1\}$$

$$= \frac{1}{|||(B^{-1})^k|||^*}.$$

Since the spectral radius of B^{-1} is 1,

$$\lim_{k\to\infty} |||(B^{-1})^k|||^{*1/k} = 1.$$

Now, let C_k be the matrix obtained from A^k by setting the last $n-i+1$ rows equal to zero, and let $D = A - C_1$. Since A is upper triangular,

$$A^m - C_m = D^m.$$

Then, since $\operatorname{rank} C_k < i$,

$$\delta_i(A^k, \|\cdot\|)^{1/k} \leqslant |||A^k - C_k|||^{1/k} = |||D^k|||^{1/k}.$$

Since the spectral radius of D is 1,

$$\lim_{k\to\infty} |||D^k|||^{1/k} = 1.$$

The conclusion of the theorem now follows from inequality (b) of Proposition 3.2. ■

DEFINITION 3.4. Let f and g be norm-equivalent functions on $\mathbf{C}^n$, and h a norm-equivalent function on $\mathbf{M}_n$. Let $A \in \mathbf{M}_n$ and $i \in \{1,\dots,n\}$.

The ith *norm-equivalent min-max number* of A is denoted by $\alpha_i(A, f, g)$ and defined by

$$\alpha_i(A, f, g) = \inf\left\{ \sup\left\{ \frac{f(Ax)}{g(x)} : x \in \mathbf{L},\ x \neq 0 \right\} : \dim \mathbf{L} = n - i + 1 \right\}.$$

The ith *norm-equivalent max-min number* of A is denoted by $\omega_i(A, f, g)$ and defined by

$$\omega_i(A, f, g) = \sup\left\{ \inf\left\{ \frac{f(Ax)}{g(x)} : x \in \mathbf{L},\ x \neq 0 \right\} : \dim \mathbf{L} = i \right\}.$$

The ith *norm-equivalent approximation number* of A is denoted by $\delta_i(A, h)$ and defined by $\delta_i(A, h) = \inf\{h(A - X) : X \in \mathbf{M}_n,\ \operatorname{rank} X < j\}$.

THEOREM 3.5. *Let $A \in \mathbf{M}_n$ with eigenvalues numbered in order of nonincreasing absolute value. Let f and g be norm-equivalent functions on $\mathbf{C}^n$, and h a norm-equivalent function on $\mathbf{M}_n$. For each $i \in \{1, \dots, n\}$ we have*

(a) $|\lambda_i| = \lim_{k \to \infty} \omega_i(A^k, f, g)^{1/k}$,
(b) $|\lambda_i| = \lim_{k \to \infty} \alpha_i(A^k, f, g)^{1/k}$, *and*
(c) $|\lambda_i| = \lim_{k \to \infty} \delta_i(A^k, h)^{1/k}$.

Proof. The approach is to bound the norm-equivalent ω, α, and δ numbers with constant multiples of the usual singular values, apply the fact that the limit of the kth root of a positive constant as k increases is one, and apply Theorem 3.3 or Yamamoto's theorem. This, of course, gives an alternate proof of Theorem 3.4 via Yamamoto's theorem and norm comparability.

Let c_m, c_M, d_m, and d_M be positive constants such that for all $x \in \mathbf{C}^n$,

$$c_m \|x\|_2 \leqslant f(x) \leqslant c_M \|x\|_2,$$

$$d_m \|x\|_2 \leqslant g(x) \leqslant d_M \|x\|_2.$$

Let $i \in \{1, \ldots, n\}$ and $A \in \mathbf{M}_n$ be given. Then

$$\begin{aligned}
\omega_i(A, f, g) &= \sup\left\{ \inf\left\{ \frac{f(Ax)}{g(x)} : x \in \mathbf{L},\ x \neq 0 \right\} : \dim \mathbf{L} = i \right\} \\
&\geqslant \sup\left\{ \inf\left\{ \frac{c_m \|Ax\|_2}{d_M \|x\|_2} : x \in \mathbf{L},\ x \neq 0 \right\} : \dim \mathbf{L} = i \right\} \\
&= \frac{c_m}{d_M} \sup\left\{ \inf\left\{ \frac{\|Ax\|_2}{\|x\|_2} : x \in \mathbf{L},\ x \neq 0 \right\} : \dim \mathbf{L} = i \right\} \\
&= \frac{c_m}{d_M} \sup\{ \inf\{ \|Ax\|_2 : x \in \mathbf{L},\ \|x\|_2 = 1 \} : \dim \mathbf{L} = i \} \\
&= \frac{c_m}{d_M} \omega_i(A, \|\cdot\|_2) = \frac{c_m}{d_M} \sigma_i(A).
\end{aligned}$$

For a lower bound,

$$\begin{aligned}
\omega_i(A, f, g) &= \sup\left\{ \inf\left\{ \frac{f(Ax)}{g(x)} : x \in \mathbf{L},\ x \neq 0 \right\} : \dim \mathbf{L} = i \right\} \\
&\leqslant \sup\left\{ \inf\left\{ \frac{c_M \|Ax\|_2}{d_m \|x\|_2} : x \in \mathbf{L},\ x \neq 0 \right\} : \dim \mathbf{L} = i \right\} \\
&= \frac{c_M}{d_m} \sup\left\{ \inf\left\{ \frac{\|Ax\|_2}{\|x\|_2} : x \in \mathbf{L},\ x \neq 0 \right\} : \dim \mathbf{L} = i \right\} \\
&= \frac{c_M}{d_m} \sup\{ \inf\{ \|Ax\|_2 : x \in \mathbf{L},\ \|x\|_2 = 1 \} : \dim \mathbf{L} = i \} \\
&= \frac{c_M}{d_m} \omega_i(A, \|\cdot\|_2) = \frac{c_M}{d_m} \sigma_i(A).
\end{aligned}$$

Thus, we have shown that

$$\frac{c_m}{d_M} \sigma_i(A) \leqslant \omega_i(A, f, g) \leqslant \frac{c_M}{d_m} \sigma_i(A).$$

It follows that

$$\limsup \omega_i(A^k, f, g)^{1/k} \leqslant \left[\lim (c_M/d_m)^{1/k}\right]\left[\lim \sigma_i(A^k)^{1/k}\right]$$

$$= \lim \sigma_i(A^k)^{1/k}$$

and

$$\liminf \omega_i(A^k, f, g)^{1/k} \geqslant \left[\lim (c_m/d_M)^{1/k}\right]\left[\lim \sigma_i(A^k)^{1/k}\right]$$

$$= \lim \sigma_i(A^k)^{1/k}.$$

The corresponding fact for the α numbers is analogously proven.

For the δ numbers, let e_m and e_M be positive constants such that for all A in $\mathbf{M}_n$,

$$e_m |||A|||_2 \leqslant h(A) \leqslant e_M |||A|||_2.$$

Then by applying the spectral norm approximation number characterization of the singular values, we have

$$e_m \sigma_i(A) \leqslant \delta_i(A, h) \leqslant e_M \sigma_i(A),$$

from which the result for the δ numbers follows. ∎

4. OTHER GENERALIZED SINGULAR VALUES

The primary intent of this paper has been to see to what extent we could replace the Euclidean norm (in the well-known properties of the usual singular values suggested by Definition 3.1) with other norms or, more generally, norm-equivalent functions. A helpful suggestion of the referee and a conversation with Harald Wimmer lead to Yamamoto type theorems for other generalized singular values.

In the work [4], an axiomatic definition of singular values for operators on Banach spaces is given. The α, ω, and δ numbers of Definition 3.1 are examples of these numbers.

DEFINITION 4.1. An *s-number sequence* is a function s which for each $n,\ p\in \mathbf{N}$ associates with every $A\in \mathbf{M}_{n,p}$, norm $\|\cdot\|_x$ on $\mathbf{C}^n$, and norm $\|\cdot\|_y$ on $\mathbf{C}^p$ a sequence of nonnegative real numbers

$$s_i(A,\|\cdot\|_x,\|\cdot\|_y),\qquad i=1,2,\dots,$$

that satisfies the following axioms:

(1) $\max\{\|Ax\|_x : \|x\|_y=1\}=s_1(A)\geqslant s_2(A)\geqslant\cdots\geqslant s_n(A)\geqslant 0$,
(2) $s_i(A+B)\leqslant s_i(A)+s_1(B)$,
(3) $s_i(ABC,\|\cdot\|_w,\|\cdot\|_z)\leqslant s_1(A,\|\cdot\|_w,\|\cdot\|_x)s_i(B,\|\cdot\|_x,\|\cdot\|_y)s_1(C,\|\cdot\|_y,\|\cdot\|_z)$,
(4) $\operatorname{rank} A<i$ implies $s_i(A)=0$, and
(5) $i\leqslant n$ implies $s_i(I_n,\|\cdot\|_x,\|\cdot\|_x)=1$.

From [4] we have the following lemma, which will establish Yamamoto's theorem for these numbers:

LEMMA 4.2. *There exists a constant c_i dependent only on i such that for every $A\in\mathbf{M}_n$ every s-number satisfies*

$$\delta_i(A,\|\cdot\|_x,\|\cdot\|_y)\geqslant s_i(A,\|\cdot\|_x,\|\cdot\|_y)\geqslant c_i\delta_i(A,\|\cdot\|_x,\|\cdot\|_y).$$

THEOREM 4.3. *Let $A\in\mathbf{M}_n$ with eigenvalues numbered in order of nonincreasing absolute value. Let $\|\cdot\|_x$ and $\|\cdot\|_y$ be norms on $\mathbf{C}^n$. For each $i\in\{1,\dots,n\}$ we have*

$$|\lambda_i|=\lim_{k\to\infty} s_i(A^k,\|\cdot\|_x,\|\cdot\|_y)^{1/k}.$$

Proof. This result follows from Theorem 3.5 by the method of proof used for that theorem, since we have the s-number bounded above and below by a constant multiple of the approximation number. ■

Another generalization of the usual singular values arose in conversations between Harald Wimmer and the first author. It is based on the following formula for the usual singular values that follows from the multiplicavity of the compound:

$$\sigma_i(A) = \frac{|||C_i(A)|||_2}{|||C_{i-1}(A)|||_2},$$

where $C_i(A)$ is the ith compound of the matrix A. The matrix $C_0(A)$ is defined to be the one by one matrix [1]. See Reference [1] for the definition of compound matrices. The generalization is obtained by replacing the two occurrences of the spectral norm with some other norm, or more generally, some other norm-equivalent function:

$$\chi_i(A) = \frac{f_i(C_i(A))}{g_{i-1}(C_{i-1}(A))}.$$

Note that since the size of $C_i(A)$ depends on i, we must allow the norm-equivalent functions to depend on i also.

THEOREM 4.4. *Let $A \in \mathbf{M}_n$. For each $i = 0, \ldots, n$ let f_i and g_i be norm-equivalent functions defined on $\mathbf{M}_{c(i,n)}$, where*

$$c(k,n) = \frac{n!}{k!(n-k)!}.$$

Define $\chi_i(A)$ as above. For each $i \in \{1, \ldots, n\}$ we have

$$|\lambda_i| = \lim_{k \to \infty} \chi_i(A^k)^{1/k}.$$

Proof. This result is proven using the norm equivalence argument of Theorem 3.5. Let $i \in \{1, \ldots, n\}$ be given. By the definition of norm-equivalent functions, there exist positive constants c_m, c_M, d_m, and d_M such that for all matrices B of the appropriate dimensions,

$$c_m |||B|||_2 \leqslant f_i(B) \leqslant c_M |||B|||_2$$

and

$$d_m |||B|||_2 \leqslant g_{i-1}(B) \leqslant d_M |||B|||_2.$$

Combining these inequalities, we have

$$\frac{c_m|||C_i(A)|||_2}{d_M|||C_{i-1}(A)|||_2} \leqslant \frac{f_i(C_i(A))}{g_{i-1}(C_{i-1}(A))}$$

$$\leqslant \frac{c_M|||C_i(A)|||_2}{d_m|||C_{i-1}(A)|||_2},$$

which is the same as

$$\frac{c_m}{d_M}\sigma_i(A) \leqslant \chi_i(A) \leqslant \frac{c_M}{d_m}\sigma_i(A).$$

The result now follows by the same argument used in the proof of Theorem 3.5. ∎

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