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State morphism MV-algebras ☆

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ABSTRACT

We present a complete characterization of subdirectly irreducible MV-algebras with internal states (SMV-algebras). This allows us to classify subdirectly irreducible state morphism MV-algebras (SMMV-algebras) and describe single generators of the variety of SMMV-algebras, and show that we have a continuum of varieties of SMMV-algebras.

1. Introduction

States on MV-algebras have been introduced by Mundici in [18]. A *state* on an MV-algebra **A** is a map s from A into $[0, 1]$ such that:

(a) $s(1) = 1$, and
(b) if $x \odot y = 0$, then $s(x \oplus y) = s(x) + s(y)$.

Special states are the so called $[0, 1]$-*valuations* on **A**, that is, the homomorphisms from **A** into the standard MV-algebra $[0, 1]_{MV}$ on $[0, 1]$.

States are related to $[0, 1]$-valuations by two important results. First of all, $[0, 1]$-valuations are precisely the *extremal states*, that is, those states that cannot be expressed as non-trivial convex combinations of other states. Moreover, by the Krein–Milman Theorem, every state belongs to the convex closure of the set of all $[0, 1]$-valuations with respect to the topology of weak convergence. Finally, every state coincides locally with a convex combination of $[0, 1]$-valuations (see [19,16]). More precisely, given a state s on an MV-algebra **A** and given elements $a_1, \ldots, a_n$ of A, there are $n+1$ extremal states $s_1, \ldots, s_{n+1}$ and $n+1$ elements $\lambda_1, \ldots, \lambda_{n+1}$ of $[0, 1]$ such that $\sum_{h=1}^{n+1} \lambda_h = 1$ and for $j = 1, \ldots, n$, $\sum_{i=1}^{n+1} \lambda_i s_i(a_j) = s(a_j)$.

Another important relation between states and $[0, 1]$-valuations is the following: let X_A be the set of $[0, 1]$-valuations on **A**. Then X_A becomes a compact Hausdorff subspace of $[0, 1]^A$ equipped with the Tychonoff topology. To every element a of A we can associate its Gelfand transform $\widehat{a}$ from X_A into $[0, 1]$, defined for all $v \in X_A$, by $\widehat{a}(v) = v(a)$. Now Panti [20] and Kroupa [14] independently showed that to any state s on **A** it is possible to associate a (uniquely determined) Borel regular probability measure μ on X_A such that for all $a \in A$ one has $s(a) = \int \widehat{a}\, d\mu$. Hence, every state has an integral representation.

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Yet another important result motivating the use of states, related to de Finetti's interpretation of probability in terms of bets, is Mundici's characterization of coherence [19]. That is, given an MV-algebra **A**, given $a_1, \ldots, a_n \in A$ and $\alpha_1, \ldots, \alpha_n \in [0, 1]$, the following are equivalent:

(1) There is a state s on **A** such that, for $i = 1, \ldots, n$, $s(a_i) = \alpha_i$.

(2) For every choice of real numbers $\lambda_1, \ldots, \lambda_n$ there is a [0, 1]-valuation v such that $\sum_{i=1}^{n} \lambda_i(\alpha_i - v(a_i)) \geq 0$.

These results show that the notion of state on an MV-algebra is a very important notion and the first one shows an important connection between states and [0, 1]-valuations. However, MV-algebras with a state are not universal algebras, and hence they do not provide for an algebraizable logic in the sense of [1] for reasoning on probability over many-valued events.

In [11] the authors find an algebraizable logic for this purpose, whose equivalent algebraic semantics is the variety of SMV-algebras. An SMV-algebra (see the next section for a precise definition) is an MV-algebra **A** equipped with an operator τ whose properties resemble the properties of a state, but, unlike a state, is an internal unary operation (called also an *internal state*) on **A** and not a map from A into [0, 1]. The analogue for SMV-algebras of an extremal state (or equivalently of a [0, 1]-valuation) is the concept of *state morphism*. By this terminology we mean an idempotent endomorphism from **A** into **A**. MV-algebras equipped with a state morphism form a variety, namely, the variety of SMMV-algebras, which is a subvariety of the variety of SMV-algebras. The following are some motivations for the study of SMMV-algebras:

(1) Let $(\mathbf{A}, \tau)$ be an SMV-algebra, and assume that $\tau(\mathbf{A})$, the image of **A** under τ, is simple. Then $\tau(\mathbf{A})$ is isomorphic to a subalgebra of $[0, 1]_{MV}$, and τ may be regarded as a state on **A**. Moreover, by Di Nola's theorem [6], **A** is isomorphic to a subalgebra of $[0, 1]^{*^I}$ for some ultrapower $[0, 1]^*$ of $[0, 1]_{MV}$ and for some index set I. Finally, using a result by Kroupa [15] stating that any state on a subalgebra **A** of an MV-algebra **B** can be extended to a state on **B**, we obtain that τ can be extended to a state τ^* on $[0, 1]^{*^I}$. Note that, after identifying a real number $\alpha \in [0, 1]$ with the function on I which is constantly equal to α, τ^* is also an internal state, and it makes $[0, 1]^{*^I}$ into an SMV-algebra. Moreover, by the Krein–Milman theorem, for every real number $\varepsilon > 0$ there is a convex combination $\sum_{i=1}^{n} \lambda_i v_i$ of [0, 1]-valuations $v_1, \ldots, v_n$ such that for every $a \in A$, $|\tau(a) - \sum_{i=1}^{n} \lambda_i v_i(a)| < \varepsilon$. After identifying $v_i(a)$ with the function from I into $[0, 1]^*$ which is constantly equal to $v_i(a)$, these valuations can be regarded as idempotent endomorphisms on $[0, 1]^{*^I}$, and hence each of them makes $[0, 1]^{*^I}$ into an SMMV-algebra. Summing up, if $(\mathbf{A}, \tau)$ is an SMV-algebra and $\tau(\mathbf{A})$ is simple, then τ can be approximated by convex combinations of state morphisms on (an extension of) **A**.

(2) All subdirectly irreducible SMMV-algebras were described in [7,9], but the description of all subdirectly irreducible SMV-algebras remains open, [11].

(3) As shown in [8], if $(\mathbf{A}, \tau)$ is an SMV-algebra and $\tau(\mathbf{A})$ belongs to a finitely generated variety of MV-algebras, then $(\mathbf{A}, \tau)$ is an SMMV-algebra. In particular, MV-algebras from a finitely generated variety only admit internal states which are state morphisms.

(4) A linearly ordered SMV-algebra is an SMMV-algebra, [8]. Moreover, we will see that representable SMV-algebras form a variety which is a subvariety of the variety of SMMV-algebras.

The goal of the present paper is to continue in the algebraic investigations on SMMV-algebras which begun in [8] and in [7,9].

The paper is organized as follows. After preliminaries in Section 2, we give in Section 3 a complete characterization of subdirectly irreducible SMV-algebras. This solves an open problem posed in [11]. In Section 4 we present a classification of subdirectly irreducible SMMV-algebras introducing four types of subdirectly irreducible SMMV-algebras. In Section 5, we describe some prominent varieties of SMMV-algebras and their generators. In particular, we answer in the positive to an open question from [7] that the diagonalization of the real interval [0, 1] generates the variety of SMMV-algebras. Section 6 shows that every subdirectly irreducible SMMV-algebra is subdiagonal. Finally, Section 7 describes an axiomatization of some varieties of SMMV-algebras, including a full characterization of representable SMMV-algebras. We show that in contrast to MV-algebras, there is a continuum of varieties of SMMV-algebras. In addition, some open problems are formulated.

2. Preliminaries

For all concepts of Universal Algebra we refer to [2]. For concepts of many-valued logic, we refer to [12], for MV-algebras in particular, we will also refer to [5], and for reasoning about uncertainty, we refer to [13].

Definition 2.1. An *MV-algebra* is an algebra $\mathbf{A} = (A, \oplus, \neg, 0)$, where $(A, \oplus, 0)$ is a commutative monoid, $\neg$ is an involutive unary operation on A, $1 = \neg 0$ is an absorbing element, that is, $x \oplus 1 = 1$, and letting $x \to y = (\neg x) \oplus y$, the identity $(x \to y) \to y = (y \to x) \to x$ holds.

In any MV-algebra **A**, we further define $x \odot y = \neg(\neg x \oplus \neg y)$, $x \ominus y = \neg(\neg x \oplus y)$, $x \vee y = (x \to y) \to y$, $x \wedge y = x \odot (x \to y)$, and $x \leftrightarrow y = (x \to y) \odot (y \to x)$. With respect to $\vee$ and $\wedge$, **A** becomes a distributive lattice with top element 1 and bottom element 0.

We also define nx for $x \in \mathbf{A}$ and natural number n by induction as follows: $0x = 0$; $(n + 1)x = nx \oplus x$.

MV-algebras constitute the equivalent algebraic semantics of *Łukasiewicz logic* Ł, cf. [12] for an axiomatization.

The *standard MV-algebra* is the MV-algebra $[0, 1]_{MV} = ([0, 1], \oplus, \neg, 0)$, where $r \oplus s = \min\{r + s, 1\}$ $\neg r = 1 - r$. For the derived operations one has:

$$r \ominus s = \max\{r - s, 0\}, \quad r \odot s = \max\{r + s - 1, 0\}, \quad r \to s = \min\{1 - r + s, 1\},$$
$$r \vee s = \max\{r, s\}, \quad r \wedge s = \min\{r, s\}.$$

The variety of all MV-algebras is generated as a quasi variety by $[0, 1]_{MV}$. It follows that in order to check the validity of an equation or a quasi equation in all MV-algebras, it is sufficient to check it in $[0, 1]_{MV}$. We will tacitly use this fact in the sequel.

Definition 2.2. A *filter* of an MV-algebra $\mathbf{A}$ is a subset F of A such that $1 \in F$ and if a and $a \to b$ are in F, then $b \in F$.

Dually, an *ideal* of $\mathbf{A}$ is a subset J of A such that $0 \in J$ and if a and $b \ominus a$ are in J, then $b \in J$. A filter F (an ideal J respectively) of $\mathbf{A}$ is called *proper* if $0 \notin F$ ($1 \notin J$ respectively) and *maximal* if it is proper and it is not properly contained in any proper filter (ideal respectively). The radical, $Rad(\mathbf{A})$, of $\mathbf{A}$, is the intersection of all its maximal ideals, and the co-radical, $Rad_1(\mathbf{A})$, of $\mathbf{A}$ is the intersection of all its maximal filters. An MV-algebra $\mathbf{A}$ is called *semisimple* if $Rad(\mathbf{A}) = \{0\}$, and is called *local* if it has exactly one maximal filter.

It is well-known (and easy to prove) that an MV-algebra $\mathbf{A}$ is semisimple iff $Rad_1(\mathbf{A}) = \{1\}$, and it is local iff it has exactly one maximal filter.

Both the lattice of ideals and the lattice of filters of an MV-algebra $\mathbf{A}$ are isomorphic to its congruence lattice via the isomorphisms $\theta \mapsto \{a \in A : (a, 0) \in \theta\}$ and $\theta \mapsto \{a \in A : (a, 1) \in \theta\}$, respectively. The inverses of these isomorphisms are:
$J \mapsto \left\{(a, b) \in A^2 : \neg(a \leftrightarrow b) \in J\right\}$ and $F \mapsto \left\{(a, b) \in A^2 : a \leftrightarrow b \in F\right\}$, respectively.

It follows that an MV-algebra is semisimple iff it has a subdirect embedding into a product of simple MV-algebras.

Definition 2.3. *A Wajsberg hoop is a subreduct (subalgebra of a reduct) of an MV-algebra in the language* $\{1, \odot, \to\}$.

Definition 2.4. A *lattice ordered abelian group* is an algebra $\mathbf{G} = (G, +, -, 0, \vee, \wedge)$ such that $(G, +, -, 0)$ is an abelian group, $(G, \vee, \wedge)$ is a lattice, and for all $x, y, z \in G$, one has $x + (y \vee z) = (x + y) \vee (x + z)$.

A *strong unit* of a lattice ordered abelian group $\mathbf{G}$ is an element $u \in G$ such that for all $g \in G$ there is $n \in \mathbf{N}$ such that $g \leq \underbrace{u + \cdots + u}_{n \text{ times}}$.

If $\mathbf{G}$ is a lattice-ordered abelian group and u is a strong unit of $\mathbf{G}$, then $\Gamma(\mathbf{G}, u)$ denotes the algebra $\mathbf{A}$ whose universe is $\{x \in G : 0 \leq x \leq u\}$, equipped with the constant 0 and with the operations $\oplus$ and $\neg$ defined by $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$. It is well-known [17] that $\Gamma(\mathbf{G}, u)$ is an MV-algebra, and every MV-algebra can be represented as $\Gamma(\mathbf{G}, u)$ for some lattice ordered abelian group $\mathbf{G}$ with strong unit u.

In the sequel, $\mathbf{Z} \times_{\text{lex}} \mathbf{Z}$ denotes the direct product of two copies of the group $\mathbf{Z}$ of integers, ordered lexicographically, i.e., $(a, b) \leq (c, d)$ if either $a < c$ or $a = c$ and $b \leq d$. For every positive natural number n, $\mathbf{S}_n$ and $\mathbf{C}_n$ denote $\Gamma(\mathbf{Z}, n)$ and $\Gamma(\mathbf{Z} \times_{\text{lex}} \mathbf{Z}, (n, 0))$ respectively. The algebra $\mathbf{C}_1$, that is $\Gamma(\mathbf{Z} \times_{\text{lex}} \mathbf{Z}, (1, 0))$, is also referred to as Chang's algebra (cf. [3]).

Definition 2.5. A *state* on an MV-algebra $\mathbf{A}$ (cf. [18]) is a map s from A into $[0, 1]$ satisfying:

(1) $s(1) = 1$.
(2) $s(x \oplus y) = s(x) + s(y)$ for all $x, y \in A$ such that $x \odot y = 0$.

Definition 2.6. An *MV-algebra with an internal state* (*SMV-algebra* in the sequel) is an algebra $(\mathbf{A}, \tau)$ such that:

(a) $\mathbf{A}$ is an MV-algebra.
(b) τ is a unary operation on $\mathbf{A}$ satisfying the following equations:
(b_1) $\tau(1) = 1$.
(b_2) $\tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \odot y))$.
(b_3) $\tau(\neg x) = \neg \tau(x)$.
(b_4) $\tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y)$.

An operator τ is said to be also an *internal state*. An operator τ is *faithful* if $\tau(a) = 1$ implies $a = 1$.

A *state morphism MV-algebra* (*SMMV-algebra* for short) is an SMV-algebra further satisfying:

(c) $\tau(x \oplus y) = \tau(x) \oplus \tau(y)$.

The following facts are easily provable:

Lemma 2.7 (see [11,8]). (1) *In an SMV-algebra* $(\boldsymbol{A}, \tau)$*, the following conditions hold:*

(1a) $\tau(0) = 0$.
(1b) *If* $x \odot y = 0$*, then* $\tau(x) \odot \tau(y) = 0$ *and* $\tau(x \oplus y) = \tau(x) \oplus \tau(y)$.
(1c) $\tau(\tau(x)) = \tau(x)$.
(1d) *Let* $\tau(A) := \{\tau(a) : a \in A\}$*. Then* $\tau(\boldsymbol{A}) = (\tau(A), \oplus, \neg, 0)$ *is an MV-subalgebra of* $\boldsymbol{A}$*, and* τ *is the identity on it.*
(1e) *If* $x \le y$*, then* $\tau(x) \le \tau(y)$.
(1f) $\tau(x) \odot \tau(y) \le \tau(x \odot y)$.
(1g) $\tau(x \to y) = \tau(x) \to \tau(x \wedge y)$.
(1h) *If* $(\boldsymbol{A}, \tau)$ *is subdirectly irreducible, then* $\tau(A)$ *is linearly ordered.*

(2) *The following conditions on SMMV-algebras hold:*

(2a) *In an SMMV-algebra* $(\boldsymbol{A}, \tau)$*,* $\tau(\boldsymbol{A})$ *is a retract of* $\boldsymbol{A}$*, that is,* τ *is a homomorphism from* $\boldsymbol{A}$ *onto* $\tau(\boldsymbol{A})$*, the identity map is an embedding from* $\tau(\boldsymbol{A})$ *into* $\boldsymbol{A}$*, and the composition* $\tau \circ \mathrm{Id}_{\tau(A)}$*, that is, the restriction of* τ *to* $\tau(\boldsymbol{A})$ *is the identity on* $\tau(\boldsymbol{A})$.
(2b) *An algebra* $(\boldsymbol{A}, \tau)$ *is an SMMV-algebra iff* $\boldsymbol{A}$ *is an MV-algebra and* τ *is an idempotent endomorphism on* $\boldsymbol{A}$.
(2c) *An SMV-algebra* $(\boldsymbol{A}, \tau)$ *is an SMMV-algebra iff it satisfies* $\tau(x \vee y) = \tau(x) \vee \tau(y)$ *iff it satisfies* $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$.
(2d) *Any linearly ordered SMV-algebra is an SMMV-algebra.*

3. Subdirectly irreducible SMV-algebras

In this section we characterize and classify subdirectly irreducible SMV-algebras which answers to an open problem posed in [11]. Our result also characterizes subdirectly irreducible SMMV-algebras.

Definition 3.1. *Let* $(\boldsymbol{A}, \tau)$ *be any SMV-algebra. Any filter* F *of* $\boldsymbol{A}$ *such that* $\tau(F) \subseteq F$ *is said to be a* τ*-filter.*

Clearly, $F_\tau(A)$ *is a* τ*-filter of* $\boldsymbol{A}$*, and hence* $\boldsymbol{F}_\tau(\boldsymbol{A}) = (F_\tau(A), \to, 0, 1)$ *is a Wajsberg subhoop of* $\boldsymbol{A}$*. Say that two Wajsberg subhoops,* $\boldsymbol{B}$ *and* $\boldsymbol{C}$*, of an MV-algebra* $\boldsymbol{A}$ *have the disjunction property if for all* $x \in B$ *and* $y \in C$*, if* $x \vee y = 1$*, then either* $x = 1$ *or* $y = 1$*.*

We recall that τ-filters are in a bijection with SMV-congruences, and hence an SMV-algebra is subdirectly irreducible iff it has a minimum τ-filter.

Lemma 3.2. *Suppose that* $(\boldsymbol{A}, \tau)$ *is a subdirectly irreducible SMV-algebra. Then:*

(1) *If* $F_\tau(A) = \{1\}$*, then* $\tau(\boldsymbol{A})$ *is subdirectly irreducible.*
(2) $\boldsymbol{F}_\tau(\boldsymbol{A})$ *is (either trivial or) a subdirectly irreducible hoop.*
(3) $\boldsymbol{F}_\tau(\boldsymbol{A})$ *and* $\tau(\boldsymbol{A})$ *have the disjunction property.*

Proof. Let F denote the minimum τ-filter of $(\mathbf{A}, \tau)$.

(1) Suppose $F_\tau(A) = \{1\}$. If $\tau(A) \cap F \neq \{1\}$, then $\tau(A) \cap F$ is the minimum non-trivial filter of $\tau(\mathbf{A})$ and $\tau(\mathbf{A})$ is subdirectly irreducible. If $\tau(A) \cap F = \{1\}$, then for all $x \in F$, $\tau(x) = 1$ (because $\tau(x) \in \tau(A) \cap F$) and $F \subseteq F_\tau(A) = \{1\}$ is the trivial filter, a contradiction.

(2) Suppose that $\mathbf{F}_\tau(\mathbf{A})$ is non-trivial. Then $F_\tau(A)$ is a non-trivial τ-filter. If $(\mathbf{A}, \tau)$ is subdirectly irreducible, it has a minimum non-trivial τ-filter, F say. So, $F \subseteq F_\tau(A)$, and hence F is the minimum non-trivial filter of $\mathbf{F}_\tau(\mathbf{A})$. Hence, $\mathbf{F}_\tau(\mathbf{A})$ is subdirectly irreducible.

(3) Suppose, by way of contradiction, that for some $x \in F_\tau(A)$ and $y = \tau(y) \in \tau(A)$ one has $x < 1$, $y < 1$ and $x \vee y = 1$. Then since the MV-filters generated by x and by y, respectively, are τ-filters (easy to verify), they both contain F. Hence, the intersection of these filters contains F. Now let $c < 1$ be in F. Then there is a natural number n such that $x^n \le c$ and $y^n \le c$. It follows that $1 = (x \vee y)^n = x^n \vee y^n \le c$, a contradiction. □

Corollary 3.3. *If* $(\boldsymbol{A}, \tau)$ *is subdirectly irreducible, then* $\tau(\boldsymbol{A})$ *and* $\boldsymbol{F}_\tau(\boldsymbol{A})$ *are linearly ordered.*

Proof. That $\tau(\mathbf{A})$ is linearly ordered follows from [11]. As regards to $\mathbf{F}_\tau(\mathbf{A})$, by Lemma 3.2, $\mathbf{F}_\tau(\mathbf{A})$ is a (possibly trivial) subdirectly irreducible Wajsberg hoop, and hence it is linearly ordered. □

Theorem 3.4. *Suppose that* $(\boldsymbol{A}, \tau)$ *is an SMV-algebra satisfying conditions* (1)–(3) *in Lemma* 3.2*. Then* $(\boldsymbol{A}, \tau)$ *is subdirectly irreducible, and hence, the above conditions constitute a characterization of subdirectly irreducible SMV-algebras.*

Proof. *Claim. Let* F *be the MV-filter of* $\boldsymbol{A}$ *generated by a filter* F_0 *of* $\tau(\boldsymbol{A})$*. Then* F *is a* τ*-filter.* Indeed, if $x \in F$, then there are $\tau(a) \in F_0$ and a natural number n such that $\tau(a)^n \le x$. It follows that $\tau(x) \ge \tau(\tau(a)^n) = \tau(a)^n$, and $\tau(x) \in F$.

Now suppose first that $F_\tau(A) = \{1\}$ and that $\tau(\mathbf{A})$ is subdirectly irreducible. Let F_0 be the minimum non-trivial filter of $\tau(\mathbf{A})$ and let F be the MV-filter of $\mathbf{A}$ generated by F_0. By Claim 1, F is a τ-filter. We claim that F is the minimum non-trivial τ-filter of $(\mathbf{A}, \tau)$. Let G be a non-trivial τ-filter of $(\mathbf{A}, \tau)$, and let $G_0 = \tau(G) = G \cap \tau(\mathbf{A})$. Then G_0 is a filter of $\tau(\mathbf{A})$, and it is non-trivial. Indeed, since $F_\tau(A) = \{1\}$ we have that if $c \in G$ and $c < 1$, then $\tau(c) \in G_0$ and $\tau(c) < 1$. Since F_0 is minimal, $F_0 \subseteq G_0$. Finally, since F is the MV-filter generated by F_0 and $F_0 \subseteq G_0 \subseteq G$, we have that F is the minimum non-trivial τ-filter of $(\mathbf{A}, \tau)$, as desired.

Now suppose that $\mathbf{F}_\tau(\mathbf{A})$ is non-trivial. By condition (2), $\mathbf{F}_\tau(\mathbf{A})$ is subdirectly irreducible. Thus, let F be the minimum filter of $\mathbf{F}_\tau(\mathbf{A})$. Then F is a non-trivial τ-filter, and it is left to prove that F is the minimum non-trivial τ-filter of $(\mathbf{A}, \tau)$. Let G be any non-trivial τ-filter of $(\mathbf{A}, \tau)$. If $G \subseteq F_\tau(A)$, then it contains the minimal filter, F, of $\mathbf{F}_\tau(\mathbf{A})$, and $F \subseteq G$. Otherwise, G contains some $x \notin F_\tau(A)$, and hence it contains $\tau(x) < 1$. Now by the disjunction property, for all $y < 1$ in $F_\tau(A)$, $\tau(x) \vee y < 1$ and $\tau(x) \vee y \in F_\tau(A) \cap G$. Thus, G contains the filter generated by $\tau(x) \vee y$, which is a non-trivial filter of $\mathbf{F}_\tau(\mathbf{A})$, and hence it contains F, the minimum non-trivial filter of $\mathbf{F}_\tau(\mathbf{A})$. This settles the claim. □

Theorem 3.5. (1), (2) *and* (3) *are independent conditions, and hence none of them is redundant in Theorem 3.4.*

Proof. (1) Let $\mathbf{C}_1$ be Chang's MV-algebra, let τ_1 be the identity on $\mathbf{C}_1$ and τ_2 be the function defined by $\tau_2(x) = 0$ if x is an infinitesimal and $\tau_2(x) = 1$ otherwise. Clearly, both $(\mathbf{C}_1, \tau_1)$ and $(\mathbf{C}_1, \tau_2)$ are SMV-algebras, and so is their direct product $(\mathbf{B}, \tau) = (\mathbf{C}_1, \tau_1) \times (\mathbf{C}_1, \tau_2)$. Let $(\mathbf{D}, \tau)$ be the subalgebra of $(\mathbf{B}, \tau)$ generating by all pairs (x, y) such that x is infinitesimal iff y is infinitesimal. Clearly, $(\mathbf{D}, \tau)$ is not subdirectly irreducible. However, $\tau(\mathbf{D})$ consists of all pairs $(x, 0)$ such that x is infinitesimal and all pairs $(y, 1)$ such that y is not infinitesimal, and hence it is subdirectly irreducible (the minimum filter is the set of all $(y, 1)$ such that y is not infinitesimal. Moreover, $F_\tau(D)$ consists of all elements of the form $(1, y)$ such that y is not infinitesimal, and hence it is subdirectly irreducible, by the same argument. Clearly (3) does not hold (e.g., if x is not infinitesimal and $x < 1$, then $(1, x) \in F_\tau(D)$, $(x, 1) \in \tau(D)$, and $(1, x) \vee (x, 1) = (1, 1)$, but $(x, 1) < (1, 1)$ and $(1, x) < (1, 1)$).

(2) Let $\mathbf{A}$ be an ultrapower of $[0, 1]_{MV}$, and let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by all the infinitesimals. Let τ be defined by $\tau(x) = 0$ if x is an infinitesimal and $\tau(x) = 1$ otherwise. Then $\tau(\mathbf{B})$ is subdirectly irreducible, being the MV-algebra with two elements, and the disjunction property holds because $\mathbf{B}$ is linearly ordered, but $\mathbf{F}_\tau(\mathbf{B})$ consists of all infinitesimals and hence it is not subdirectly irreducible. (If F is any non-trivial τ-filter and $1 - \epsilon \in F$, with ϵ a positive infinitesimal, then the filter generated by $1 - \epsilon^2$ is a non-trivial τ-filter strictly contained in F).

(3) Let $\mathbf{B}$ be as in (2) and let τ be the identity on $\mathbf{B}$. Then $\mathbf{F}_\tau(\mathbf{B})$ is subdirectly irreducible, being a trivial algebra, and the disjunction property holds because B is linearly ordered, but $\tau(\mathbf{B}) = \mathbf{B}$ is not subdirectly irreducible. □

Lemma 3.6. *If $(\mathbf{A}, \tau)$ is a subdirectly irreducible SMMV-algebra, then for all $a \in A$, either $a \le \tau(a)$ or $\tau(a) \le a$.*

Proof. Since $(\mathbf{A}, \tau)$ is subdirectly irreducible, $\mathbf{F}_\tau(\mathbf{A})$ is subdirectly irreducible and hence it is linearly ordered. Hence, 1 is join irreducible in $\mathbf{F}_\tau(\mathbf{A})$. Now $(a \to \tau(a)) \vee (\tau(a) \to a) = 1$, and hence either $a \to \tau(a) = 1$ and $a \le \tau(a)$, or $\tau(a) \to a = 1$ and $\tau(a) \le a$. □

Subdirectly irreducible SMMV-algebras also enjoy another interesting property, namely:

Theorem 3.7. *Let $(\mathbf{A}, \tau)$ be a subdirectly irreducible SSMV-algebra and let $a \in A$. Then there are uniquely determined $b \in \tau(A)$ and $c \in F_\tau(A)$ such that exactly one of the following two conditions holds:*

(a) *$a = b \odot c$, and c is the greatest element with this property, when $a \le \tau(a)$, or*
(b) *$a = c \to b$ and $b < c < 1$ when $\tau(a) < a$.*

Proof. First of all, note that $\tau(a \to \tau(a)) = \tau(\tau(a) \to a) = \tau(a) \to \tau(a) = 1$, and hence, for every $a \in A$, $a \to \tau(a)$ and $\tau(a) \to a$ belong to $F_\tau(A)$.

Let $b = \tau(a)$ and let $c = b \to a$ if $a \le b$, and $c = a \to b$ otherwise.

Suppose $a \le b$. Then $a = a \wedge b = b \odot (b \to a) = b \odot c$. Finally, c is the greatest element such that $b \odot c = a$, by the definition of residuum, and $\tau(c) = 1$.

Now suppose $b < a$. Then $c \to b = (a \to b) \to b = a \vee b = a$. Moreover, $c < 1$, as $b < a$. Finally, $b < c$. Indeed, $b \le a \to b = c$, and it cannot be $c = b$, as $\tau(c) = 1$ and $\tau(b) = b < a$.

Now we discuss uniqueness. (i) Let $a \le \tau(a)$. If $a = b' \odot c'$, with $b' \in \tau(A)$ and $c' \in F_\tau(A)$, then $\tau(a) = \tau(b') \odot \tau(c') = b' \odot 1 = b' = \tau(b')$. Thus $b' = \tau(a)$ is uniquely determined; we denote it by b. Moreover, $a \le b$, $b \odot c' = a$ and c' is the greatest element with this property. Hence, $c' = a \to b$.

(ii) Let $\tau(a) < a$. Then $a < 1$. If $a = c' \to b'$ with $b' < c' \in F_\tau(A) \setminus \{1\}$ and $b' \in \tau(A)$, then by Lemma 2.7(1g), $\tau(a) = \tau(c') \to \tau(c' \wedge b') = \tau(c') \to \tau(b') = 1 \to b' = b'$, and b' is uniquely determined; we denote it by b. Then $b < a$. Finally, in any MV-algebra, if $z \le x$, $z \le y$ and $x \to z = y \to z$, then $x = y$ (this property is expressed as a quasi equation and holds in $[0, 1]_{MV}$, and hence it holds in any MV-algebra). Now $b < c' < 1$, $b \le (a \to b) \to b$, and $c' \to b = (a \to b) \to b$. It follows that $c' = a \to b$, and uniqueness of c' is proved. □

Let $(\mathbf{A}, \tau)$ be a subdirectly irreducible SMMV-algebra. For all $b \in \tau(A)$, the define $M(b) = \{x \in A : \tau(x) = b\}$. Then A is a disjoint union of the sets $M(b)$ for $b \in \tau(A)$.

We assert that every $M(b)$ is linearly ordered. Indeed, let $x, y \in M(b)$. Due to Lemma 3.6, there are three cases: (i) $x \le b$, $y > b$ or $x > b$, $y \le b$, (ii) $x \le b$, $y \le b$, and (iii) $x > b$ and $y > b$. In the case (i), x and y are comparable. In the case (ii), by Lemma 2.7(1b), $\tau(x \oplus \neg b) = \tau(x) \oplus \tau(\neg b) = 1$ and $\tau(y \oplus \neg b) = \tau(y) \oplus \tau(\neg b) = 1$ which by Corollary 3.3 entails $x \oplus \neg b$ and $y \oplus \neg b$ are comparable. Because $x \odot \neg b = 0 = y \odot \neg b$, we have x and y are also comparable. In the case (iii), $\neg x < \neg b$ and $\neg y < \neg b$, and in the same way as in (ii) we can prove $\neg x$ and $\neg y$ are comparable, consequently, x and y are comparable.

Thus, although $\mathbf{A}$ need not be linearly ordered, it is close to be such. More precisely, let $M = \{\pm c : c \in F_\tau(A),\ c < 1\} \cup \{1\}$. We define a poset $\mathbf{M}$ on M letting $-c < -d$ iff $d < c$, and $c < 1 < -d$ for all $c, d \in F_\tau(A) \setminus \{1\}$. Now given $x \in M(b)$, by Lemma 3.6, it follows $x \le b$ or $b < \tau(x)$. By Theorem 3.7, in the first case we can associate x with $(b, b \to x)$ and in the second case with $(b, -(x \to b))$ to obtain an order isomorphism from A into $\tau(A) \times M$. That is, $\mathbf{A}$ as a poset is isomorphic to a quotient of a subposet of the product of two chains. This suggests that either $\mathbf{A}$ is a chain or a subalgebra of a product of two chains. This conjecture will be proved in Section 6. More precisely:

Definition 3.8. An SMMV-algebra $(\mathbf{A}, \tau)$ is said to be *diagonal* if there are MV-chains $\mathbf{B}$ and $\mathbf{C}$ such that $\mathbf{B} \subseteq \mathbf{C}$, $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ and τ is defined, for all $b \in B$ and $c \in C$, by $\tau(b, c) = (b, b)$.

An SMMV-algebra is said to be *subdiagonal* if it is a subalgebra of a diagonal SMMV-algebra.

In Section 6 we will prove:

Theorem 3.9. *Every subdirectly irreducible SMMV-algebra is subdiagonal.*

4. A classification of subdirectly irreducible SMMV-algebras

We present a classification of SMMV-algebras introducing four types of subdirectly irreducible SMMV-algebras, type $\mathcal{I}$, identity, type $\mathcal{L}$, local, type $\mathcal{D}$, diagonalization, and type $\mathcal{K}$, killing infinitesimals.

The following theorem was proved in [7,9,10].

Theorem 4.1. *Let $(\mathbf{A}, \tau)$ be a subdirectly irreducible SMMV-algebra. Then $(\mathbf{A}, \tau)$ belongs to exactly one of the following classes:*

(i) *$\mathbf{A}$ is linearly ordered, τ is the identity on A and the MV-reduct of $\mathbf{A}$ is a subdirectly irreducible MV-algebra.*
(ii) *The state morphism operator τ is not faithful, $\mathbf{A}$ has no non-trivial Boolean elements and is a local MV-algebra. Moreover, $\mathbf{A}$ is linearly ordered if and only if $Rad_1(\mathbf{A})$ is linearly ordered, and in such a case, $\mathbf{A}$ is a subdirectly irreducible MV-algebra such that the smallest non-trivial τ-filter of $(\mathbf{A}, \tau)$, and the smallest non-trivial MV-filter for $\mathbf{A}$ coincide.*
(iii) *The state morphism operator τ is not faithful, $\mathbf{A}$ has a non-trivial Boolean element. There are a linearly ordered MV-algebra $\mathbf{B}$, a subdirectly irreducible MV-algebra $\mathbf{C}$, and an injective MV-homomorphism $h : \mathbf{B} \to \mathbf{C}$ such that $(\mathbf{A}, \tau)$ is isomorphic to $(\mathbf{B} \times \mathbf{C}, \tau_h)$, where $\tau_h(x, y) = (x, h(x))$ for any $(x, y) \in B \times C$.*

Note that while every SMMV-algebra satisfying (i) or (iii) is subdirectly irreducible, the same is not true of SMMV-algebras satisfying (ii). A full classification of subdirectly irreducible SMMV-algebras is obtained by combining Theorem 4.1, Theorem 3.9, and Theorem 3.4.

Let us consider the following classes of SMMV-algebras:

Definition 4.2. *Type $\mathcal{I}$ (identity).* The MV-reduct, $\mathbf{A}$, of $(\mathbf{A}, \tau)$ is a subdirectly irreducible MV-algebra and τ is the identity function on A.

Type $\mathcal{L}$ (local). $(\mathbf{A}, \tau)$ is subdiagonal, the MV-reduct, $\mathbf{A}$, of $(\mathbf{A}, \tau)$ is a local MV-algebra (hence it has no Boolean non-trivial elements), $\mathbf{F}_\tau(\mathbf{A})$ is a non-trivial subdirectly irreducible hoop, $\mathbf{F}_\tau(\mathbf{A})$ and $\tau(\mathbf{A})$ have the disjunction property.

Type $\mathcal{D}$ (diagonalization). The MV-reduct, $\mathbf{A}$, of $(\mathbf{A}, \tau)$ is of the form $\mathbf{B} \times \mathbf{C}$, where $\mathbf{C}$ is a subdirectly irreducible MV-algebra and $\mathbf{B}$ is a subalgebra of $\mathbf{C}$. Moreover, τ is defined by $\tau(b, c) = (b, b)$.

Theorem 4.3. *An SMMV-algebra is subdirectly irreducible if and only if it is of one of the types $\mathcal{I}$, $\mathcal{L}$ and $\mathcal{D}$. Moreover, these types are mutually disjoint.*

Proof. We first prove, using Theorem 3.4, that all members of $\mathcal{I} \cup \mathcal{L} \cup \mathcal{D}$ are subdirectly irreducible. For type $\mathcal{I}$, the claim is easy and for type $\mathcal{L}$ the claim follows from the definition of type $\mathcal{L}$ and from Theorem 3.4. For type $\mathcal{D}$, if $(\mathbf{A}, \tau)$ is diagonal, say, $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ with $\mathbf{B} \subseteq \mathbf{C}$, $\mathbf{C}$ is subdirectly irreducible and τ is diagonal, we have that $\mathbf{F}_\tau(\mathbf{A})$ consists of all pairs $(1, c)$ with $c \in C$, and hence it is isomorphic (as a Wajsberg hoop) to $\mathbf{C}$. Since $\mathbf{C}$ is subdirectly irreducible, so is $\mathbf{F}_\tau(\mathbf{A})$. Finally, $\tau(\mathbf{A})$ consists of all pairs of the form (b, b) with $b \in B$. Now if $(b, b) \vee (1, c) = (1, 1)$, then either $(b, b) = (1, 1)$ or $(1, c) = (1, 1)$. Hence, $\tau(\mathbf{A})$ and $\mathbf{F}_\tau(\mathbf{A})$ have the disjunction property, and by Theorem 3.4, $(\mathbf{A}, \tau)$ is subdirectly irreducible.

For the converse, we use Theorem 4.1. It is clear that condition (i) in Theorem 4.1 corresponds to type $\mathcal{I}$. For case (ii) the additional conditions that $\mathbf{F}_\tau(\mathbf{A})$ is subdirectly irreducible and $\mathbf{F}_\tau(\mathbf{A})$ and $\tau(\mathbf{A})$ have the disjunction property follows from Theorem 3.4 and the additional condition that $(\mathbf{A}, \tau)$ is subdiagonal follows from Theorem 3.9.

Now, suppose (iii) is the case. Identifying $\mathbf{B}$ with its isomorphic copy $h(\mathbf{B})$, we can rephrase the definition of τ as $\tau(b, c) = (b, b)$, and hence $(\mathbf{A}, \tau)$ is of type $\mathcal{D}$.

Finally, types $\mathcal{I}$, $\mathcal{L}$ and $\mathcal{D}$ are mutually disjoint, because if $(\mathbf{A}, \tau)$ is of type $\mathcal{I}$, then $\mathbf{F}_\tau(\mathbf{A})$ is trivial, while if $(\mathbf{A}, \tau)$ is of type $\mathcal{L}$ or $\mathcal{D}$, then $\mathbf{F}_\tau(\mathbf{A})$ is non-trivial. Moreover, the MV-reduct of a diagonal SMMV-algebra has two maximal filters, and hence it cannot be a local MV-algebra. This finishes the proof. □

There is yet another type of subdirectly irreducible SMMV-algebras, namely, type $\mathcal{K}$ (killing infinitesimals), which is described as follows:

Definition 4.4. *An SMMV-algebra $(\boldsymbol{A}, \tau)$ is said to be of type $\mathcal{K}$ if $\boldsymbol{A}$ is of type $\mathcal{L}$ and is linearly ordered.*

The next example shows that the class of SMMV-algebras of type $\mathcal{K}$ is properly contained in the class of SMMV-algebras of type $\mathcal{L}$.

Example 4.5. *Let $\boldsymbol{C}_1$ be the Chang MV-algebra. Let $\boldsymbol{A}$ be the subalgebra of $\mathbf{C}_1 \times \mathbf{C}_1$ generated by $Rad(\mathbf{C}_1) \times Rad(\mathbf{C}_1)$, i.e., $A = (Rad(\mathbf{C}_1) \times Rad(\mathbf{C}_1)) \cup (Rad_1(\mathbf{C}_1) \times Rad_1(\mathbf{C}_1))$. We define $\tau : A \to A$ via $\tau(x, y) = (x, x)$. Then τ is a state morphism operator on $\boldsymbol{A}$ such that $(\boldsymbol{A}, \tau)$ is a subdirectly irreducible SMMV-algebra, $\boldsymbol{F}_\tau(\boldsymbol{A}) = \{1\} \times Rad_1(\boldsymbol{C}_1)$, τ is not faithful, $\boldsymbol{A}$ has no non-trivial Boolean elements, but it is not linearly ordered. We note that $Rad_1(\boldsymbol{A}) = Rad_1(\boldsymbol{C}_1) \times Rad_1(\boldsymbol{C}_1)$ is the unique maximal filter.*

5. Varieties of SMMV-algebras and their generators

We describe the varieties of SMMV-algebras and their generators. In particular, we answer in the positive to an open question from [7] that the diagonalization of the real interval [0, 1] generates the variety of SMMV-algebras.

Given a variety $\mathcal{V}$ of MV-algebras, $\mathcal{V}_{SMMV}$ will denote the class of SMMV-algebras whose MV-reduct is in $\mathcal{V}$. Clearly, $\mathcal{V}_{SMMV}$ is a variety.

Definition 5.1. *For every MV-algebra $\boldsymbol{A}$ we set $D(\boldsymbol{A}) = (\boldsymbol{A} \times \boldsymbol{A}, \tau_A)$, where τ_A is defined, for all $a, b \in A$, by $\tau_A(a, b) = (a, a)$. For every class $\mathcal{K}$ of MV-algebras, we set $\mathsf{D}(\mathcal{K}) = \{D(\boldsymbol{A}) : \boldsymbol{A} \in \mathcal{K}\}$.*

As usual, given a class $\mathcal{K}$ of algebras of the same type, $\mathsf{I}(\mathcal{K})$, $\mathsf{H}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$ and $\mathsf{P}(\mathcal{K})$ and $\mathsf{P}_\mathsf{U}(\mathcal{K})$ will denote the class of isomorphic images, of homomorphic images, of subalgebras, of direct products and of ultraproducts of algebras from $\mathcal{K}$, respectively. Moreover, $\mathsf{V}(\mathcal{K})$ will denote the variety generated by $\mathcal{K}$.

Lemma 5.2. (1) *Let $\mathcal{K}$ be a class of MV-algebras. Then $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_{SMMV}$.*
(2) *Let $\mathcal{V}$ be any variety of MV-algebras. Then $\mathcal{V}_{SMMV} = \mathsf{ISD}(\mathcal{V})$.*

Proof. (1) We have to prove that every MV-reduct of an algebra in $\mathsf{VD}(\mathcal{K})$ is in $\mathsf{V}(\mathcal{K})$. Let $\mathcal{K}_0$ be the class of all MV-reducts of algebras in $\mathsf{D}(\mathcal{K})$. Then since the MV-reduct of $D(\mathbf{A})$ is $\mathbf{A} \times \mathbf{A}$, and since $\mathbf{A}$ is a homomorphic image (under the projection map) of $\mathbf{A} \times \mathbf{A}$, $\mathcal{K}_0 \subseteq \mathsf{P}(\mathcal{K})$ and $\mathcal{K} \subseteq \mathsf{H}(\mathcal{K}_0)$. Hence, $\mathcal{K}_0$ and $\mathcal{K}$ generate the same variety. Moreover, MV-reducts of subalgebras (homomorphic images, direct products respectively) of algebras from $\mathsf{D}(\mathcal{K})$ are subalgebras (homomorphic images, direct products respectively) of the corresponding MV-reducts. Therefore, the MV-reduct of any algebra in $\mathsf{VD}(\mathcal{K})$ is in $\mathsf{HSP}(\mathcal{K}_0) = \mathsf{HSP}(\mathcal{K}) = \mathsf{V}(\mathcal{K})$, and claim (1) is proved.

(2) Let $(\mathbf{A}, \tau) \in \mathcal{V}_{SMMV}$. Then the map $\Phi : a \mapsto (\tau(a), a)$ is an embedding of $(\mathbf{A}, \tau)$ into $D(\mathbf{A})$. Conversely, the MV-reduct of any algebra in $\mathsf{D}(\mathcal{V})$ is in $\mathcal{V}$, (being a direct product of algebras in $\mathcal{V}$), and hence the MV-reduct of any member of $\mathsf{ISD}(\mathcal{V})$ is in $\mathsf{IS}(\mathcal{V}) = \mathcal{V}$. Hence, any member of $\mathsf{ISD}(\mathcal{V})$ is in $\mathcal{V}_{SMMV}$. □

Lemma 5.3. *Let $\mathcal{K}$ be a class of MV-algebras. Then:*
(1) $\mathsf{DH}(\mathcal{K}) \subseteq \mathsf{HD}(\mathcal{K})$.
(2) $\mathsf{DS}(\mathcal{K}) \subseteq \mathsf{ISD}(\mathcal{K})$.
(3) $\mathsf{DP}(\mathcal{K}) \subseteq \mathsf{IPD}(\mathcal{K})$.
(4) $\mathsf{VD}(\mathcal{K}) = \mathsf{ISD}(\mathsf{V}(\mathcal{K}))$.

Proof. (1) Let $D(\mathbf{C}) \in \mathsf{DH}(\mathcal{K})$. Then there are $\mathbf{A} \in \mathcal{K}$ and a homomorphism h from $\mathbf{A}$ onto $\mathbf{C}$. Let for all $a, b \in A$, $h^*(a, b) = (h(a), h(b))$. We claim that h^* is a homomorphism from $D(\mathbf{A})$ onto $D(\mathbf{C})$. That h^* is an MV-homomorphism is clear. We verify that h^* is compatible with τ_A. We have $h^*(\tau_A(a, b)) = h^*(a, a) = (h(a), h(a)) = \tau_C(h(a), h(b)) = \tau_C(h^*(a, b))$. Finally, since h is onto, given $(c, d) \in C \times C$, there are $a, b \in A$ such that $h(a) = c$ and $h(b) = d$. Hence, $h^*(a, b) = (c, d)$, h^* is onto, and $D(\mathbf{C}) \in \mathsf{HD}(\mathcal{K})$.

(2) Almost trivial.

(3) Let $\mathbf{A} = \prod_{i \in I}(\mathbf{A}_i) \in \mathsf{P}(\mathcal{K})$, where each $\mathbf{A}_i$ is in $\mathcal{K}$. Then the map

$$\Phi : ((a_i : i \in I), (b_i : i \in I)) \mapsto ((a_i, b_i) : i \in I)$$

is an isomorphism from $D(\mathbf{A})$ onto $\prod_{i \in I} D(\mathbf{A}_i)$. Indeed, it is clear that Φ is an MV-isomorphism. Moreover, denoting the state morphism of $\prod_{i \in I} D(\mathbf{A}_i)$ by τ^*, we get:

$$\Phi(\tau_A((a_i : i \in I), (b_i : i \in I))) = \Phi((a_i : i \in I), (a_i : i \in I))$$
$$= ((a_i, a_i) : i \in I) = (\tau_{A_i}(a_i, b_i) : i \in I) = \tau^*(\Phi((a_i : i \in I), (b_i : i \in I))),$$

and hence Φ is an SMMV-isomorphism.

(4) By (1), (2) and (3), $\mathsf{DV}(\mathcal{K}) = \mathsf{DHSP}(\mathcal{K}) \subseteq \mathsf{HSPD}(\mathcal{K}) = \mathsf{VD}(\mathcal{K})$, and hence $\mathsf{ISDV}(\mathcal{K}) \subseteq \mathsf{ISVD}(\mathcal{K}) = \mathsf{VD}(\mathcal{K})$. Conversely, by Lemma 5.2(1), $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_{SMMV}$, and by Lemma 5.2(2), $\mathsf{V}(\mathcal{K})_{SMMV} = \mathsf{ISDV}(\mathcal{K})$. This settles the claim. □

Theorem 5.4. (1) *For every MV-algebra* $\boldsymbol{A}$, $\mathsf{V}(D(\boldsymbol{A})) = \mathsf{V}(\boldsymbol{A})_{SMMV}$.
(2) *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be MV-algebras. Then* $\mathsf{V}(D(\boldsymbol{A})) = \mathsf{V}(D(\boldsymbol{B}))$ *iff* $\mathsf{V}(\boldsymbol{A}) = \mathsf{V}(\boldsymbol{B})$.
(3) *The variety of all SMMV-algebras is generated by* $D([0, 1]_{MV})$ *as well as by any* $D(\boldsymbol{A})$ *such that* $\boldsymbol{A}$ *generates the variety of MV-algebras.*
(4) *Let* $\mathbf{C}_1$ *be Chang's algebra and let* $\mathcal{C}$ *be the variety generated by it. Then* $\mathcal{C}_{SMMV}$ *is generated by* $D(\mathbf{C}_1)$.

Proof. (1) By Lemma 5.3(4), $\mathsf{VD}(\mathbf{A}) = \mathsf{V}(D(\mathbf{A})) = \mathsf{ISD}(\mathsf{V}(\mathbf{A}))$. Moreover, by Lemma 5.2(2), $\mathsf{V}(\mathbf{A})_{SMMV} = \mathsf{ISDV}(\mathbf{A})$. Hence, $\mathsf{V}(D(\mathbf{A})) = \mathsf{V}(\mathbf{A})_{SMMV}$.

(2) We have $\mathsf{V}(D(\mathbf{A})) = \mathsf{V}(\mathbf{A})_{SMMV}$ and $\mathsf{V}(D(\mathbf{B})) = \mathsf{V}(\mathbf{B})_{SMMV}$. Clearly, $\mathsf{V}(\mathbf{A}) = \mathsf{V}(\mathbf{B})$ implies $\mathsf{V}(\mathbf{A})_{SMMV} = \mathsf{V}(\mathbf{B})_{SMMV}$, and hence $\mathsf{V}(D(\mathbf{A})) = \mathsf{V}(D(\mathbf{B}))$. Conversely, $\mathsf{V}(D(\mathbf{A})) = \mathsf{V}(D(\mathbf{B}))$ implies $\mathsf{V}(\mathbf{A})_{SMMV} = \mathsf{V}(\mathbf{B})_{SMMV}$. But *any* algebra $\mathbf{C} \in \mathsf{V}(\mathbf{A})$ is the MV-reduct of an algebra in $\mathsf{V}(\mathbf{A})_{SMMV}$, namely, of $(\mathbf{C}, \mathrm{Id}_C)$, where Id_C is the identity on C.

It follows that, if $\mathsf{V}(\mathbf{A})_{SMMV} = \mathsf{V}(\mathbf{B})_{SMMV}$, then the classes of MV-reducts of $\mathsf{V}(\mathbf{A})_{SMMV}$ and of $\mathsf{V}(\mathbf{B})_{SMMV}$ coincide, and hence $\mathsf{V}(\mathbf{A}) = \mathsf{V}(\mathbf{B})$.

(3) Since $\mathsf{V}([0, 1]_{MV})$ is the variety $\mathcal{MV}$ of all MV-algebras, $\mathsf{V}(D([0, 1]_{MV}))$ is $\mathcal{MV}_{SMMV}$, that is, the variety of all SMMV-algebras. The same argument holds if we replace $[0, 1]_{MV}$ by any MV-algebra which generates the whole variety $\mathcal{MV}$.

(4) Completely parallel to (3). □

Another consequence is the decidability of the variety $\mathcal{SMMV}$ of all SMMV-algebras.

Theorem 5.5. $\mathcal{SMMV}$ *is decidable.*

Proof. We associate to every term $t(x_1, \ldots, x_n)$ of SMMV-algebras a pair of terms t^1, t^2 whose variables are among $x_1^1, x_1^2, \ldots, x_n^1, x_n^2$ by induction as follows: If t is a variable, say, $t = x_i$, then $t^1 = x_i^1$ and $t^2 = x_i^2$; if $t = 0$, then $t^1 = t^2 = 0$. If $t = \neg s$, then $t^1 = \neg s^1$ and $t^2 = \neg s^2$; if $t = s \oplus u$, then $t^1 = s^1 \oplus u^1$ and $t^2 = s^2 \oplus u^2$. Finally, if $t = \tau(s)$, then $t^1 = t^2 = s^1$. The following lemma is straightforward.

Lemma 5.6. *Let* $a_1^1, a_1^2, \ldots, a_n^1, a_n^2, b^1, b^2 \in [0, 1]$ *and let* $t(x_1, \ldots, x_n)$ *be a term. Then the following are equivalent:*
(1) $t((a_1^1, a_1^2), \ldots, (a_n^1, a_n^2)) = (b^1, b^2)$ *holds in* $D([0, 1]_{MV})$.
(2) $t^i(a_1^1, a_1^2, \ldots, a_n^1, a_n^2) = b^i$*, for* $i = 1, 2$ *holds in* $[0, 1]_{MV}$.

As a consequence, we obtain that an equation $t = s$ holds identically in $D([0, 1]_{MV})$ iff $t^1 = s^1$ and $t^2 = s^2$ hold identically in $[0, 1]_{MV}$. Since validity of an equation in $[0, 1]_{MV}$ is decidable, the equational logic of $D([0, 1]_{MV})$ is decidable, and since $D([0, 1]_{MV})$ generates the whole variety of SMMV-algebras, the claim follows. □

6. Every subdirectly irreducible SMMV-algebra is subdiagonal

We are in a position to prove Theorem 3.9, stating that every subdirectly irreducible SMMV-algebra is subdiagonal (subalgebra of a diagonal SMMV-algebra). We start from some easy facts.

First of all, any linearly ordered SMMV-algebra $(\mathbf{A}, \tau)$ is subdiagonal, being isomorphic to a subalgebra of $(\tau(\mathbf{A}) \times \mathbf{A}, \tau^*)$, with $\tau^*(\tau(a), a) = (\tau(a), \tau(a))$. Next we prove that the variety of SMMV-algebras has CEP.

Lemma 6.1. *The variety of SMMV-algebras has Congruence Extension Property.*

Proof. Let $(\mathbf{A}, \tau) \subseteq (\mathbf{B}, \tau)$ be SMMV-algebras and θ a congruence on $(\mathbf{A}, \tau)$. Thus, $1/\theta$ is a τ-filter of $(\mathbf{A}, \tau)$. By monotonicity of τ the upward closure (in $\mathbf{B}$) of $1/\theta$ is a τ-filter of $(\mathbf{B}, \tau)$, which restricts to $1/\theta$ on $(\mathbf{A}, \tau)$. This proves the claim. □

The next lemma is also easy:

Lemma 6.2. *The class of subdiagonal SMMV-algebras is closed under subalgebras and ultraproducts.*

Proof. Closure under S is definitional. Closure under P_U follows from the following facts:

(1) For every class $\mathcal{K}$ of algebras of the same type $\mathsf{P}_\mathsf{U}\mathsf{S}(\mathcal{K}) \subseteq \mathsf{S}\mathsf{P}_\mathsf{U}(\mathcal{K})$ (this is a well-known fact of Universal Algebra).

(2) Every ultraproduct $(\prod_{i\in I}(\mathbf{B}_i \times \mathbf{C}_i, \tau_i))/U$ of diagonal SMMV-algebras is isomorphic to the diagonal SMMV-algebra $((\prod_{i\in I} \mathbf{B}_i)/U \times (\prod_{i\in I} \mathbf{C}_i)/U, \tau_U))$, where $\tau_U((b_i : i \in I)/U, (c_i : i \in I)/U) = ((b_i : i \in I)/U, (b_i : i \in I)/U)$, with respect to the isomorphism $((b_i, c_i) : i \in I)/U \mapsto ((b_i : i \in I)/U, (c_i : i \in I)/U)$. □

To deal with homomorphic images we need the following definition:

Definition 6.3. *An SMMV-algebra $(\boldsymbol{A}, \tau)$ is said to be skew diagonal if it has the form $(\boldsymbol{B} \times \boldsymbol{C}/\varphi, \tau)$, where $\boldsymbol{B}$ and $\boldsymbol{C}$ are MV-chains, $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{C}$, φ is a congruence of $\boldsymbol{C}$ and τ is defined $\tau(b, c/\varphi) = (b, b/\varphi)$ for all $b \in B$ and $c \in C$.*

The projection onto the first coordinate is a homomorphism from the skew-diagonal algebra $(\mathbf{B} \times \mathbf{C}/\varphi, \tau)$ onto $(\mathbf{B}, \mathrm{Id}_B)$. Compatibility with τ is proved as follows: $\pi_1\tau(b, c/\varphi) = \pi_1(b, b/\varphi) = b = \mathrm{Id_B}\pi_1(\mathrm{b}, \mathrm{c})$.

Lemma 6.4. *Let $(\boldsymbol{A}, \tau)$ be a subdiagonal algebra with $\boldsymbol{A} \subseteq \boldsymbol{B} \times \boldsymbol{C}$, and θ a congruence on $(\boldsymbol{A}, \tau)$. Then there are MV-chains $\boldsymbol{D} \subseteq \boldsymbol{E}$, and a congruence φ on $\boldsymbol{E}$ such that $(\boldsymbol{A}, \tau)/\theta$ is subdirectly embedded into a skew-diagonal algebra $(\boldsymbol{D} \times \boldsymbol{E}/\varphi, \tau)$.*

Proof. Clearly, we may assume that the natural identity embedding $\mathbf{A} \subseteq \mathbf{B} \times \mathbf{C}$ is subdirect. By CEP, the congruence θ extends to a congruence ψ on $(\mathbf{B} \times \mathbf{C}, \tau)$. Of course, ψ is also a congruence on the MV-reduct $\mathbf{B} \times \mathbf{C}$. By congruence distributivity, all congruences of finite products are product congruences, so $\psi = \psi_B \times \psi_C$ for some congruences ψ_B on $\mathbf{B}$ and ψ_C on $\mathbf{C}$.

The congruences ψ_B and ψ_C are defined as follows: $(b, b') \in \psi_B$ iff there are $c, c' \in C$ such that $((b, c), (b', c')) \in \psi$, and $(c, c') \in \psi_C$ iff there are $b, b' \in B$ such that $((b, c), (b', c')) \in \psi$. Denoting by θ_1 and θ_2 the congruences associated to the projection maps, and using congruence distributivity, we have: $((b, c), (b', c')) \in \psi$ iff $((b, c), (b', c')) \in (\psi \vee \theta_1) \wedge (\psi \vee \theta_2)$ iff $(b, b') \in \psi_B$ and $(c, c') \in \psi_C$, and $\psi = \psi_B \times \psi_C$. It follows:

$$(\mathbf{B} \times \mathbf{C})/\psi = \mathbf{B}/\psi_B \times \mathbf{C}/\psi_C$$

and moreover, since ψ is compatible with τ we obtain

$$\tau(b, c)/\psi = (b, b)/\psi = (b/\psi_B, b/\psi_C).$$

Furthermore, $((b, 1), (1, 1)) \in \psi$ implies $(\tau(b, 1), \tau(1, 1)) = ((b, b), (1, 1)) \in \psi$. It follows that $(b, 1) \in \psi_B$ implies $(b, 1) \in \psi_C$. Let χ be the congruence of $\mathbf{C}$ generated by ψ_B. Then $\chi \subseteq \psi_C$, and by the CEP, $\psi_B = \chi \cap B^2$. Now let $\mathbf{D} = \mathbf{B}/\psi_B$, $\mathbf{E} = \mathbf{C}/\chi$, $\varphi = \chi/\psi_C$. Note that $\mathbf{D}$ and $\mathbf{E}$ are MV-chains. Moreover, by construction we have $\mathbf{D} \subseteq \mathbf{E}$, and hence

$$\mathbf{A}/\theta \subseteq (\mathbf{B} \times \mathbf{C})/\psi = \mathbf{B}/\psi_B \times \mathbf{C}/\psi_C = \mathbf{D} \times \mathbf{E}/\varphi$$

proving the claim for the MV-reducts of the appropriate algebras. In particular, the embedding is subdirect. Furthermore,

$$\tau(b, c)/\psi = (b/\psi_B, b/\psi_C) = (b/\psi_B, (b/\chi)/\varphi)$$

and the embedding lifts to the full type of SMMV. □

Lemma 6.5. *Let $(\boldsymbol{A}, \tau)$ be a subdirectly irreducible SMMV-algebra, and suppose that $(\boldsymbol{A}, \tau)$ is a subalgebra of a skew diagonal SMMV-algebra $(\boldsymbol{B} \times \boldsymbol{C}/\varphi, \tau^*)$, and that the identity MV-embedding of $\boldsymbol{A}$ into $(\boldsymbol{B} \times \boldsymbol{C}/\varphi)$ is subdirect. Then $(\boldsymbol{A}, \tau)$ is subdiagonal.*

Proof. If for all $b \in B$, $(b, 1) \in \varphi$ implies $b = 1$, then the map $b \mapsto b/\varphi$ is one-one and $\mathbf{B}$ is (isomorphic to) a subalgebra of $\mathbf{C}/\varphi$. Hence, $\mathbf{C}/\varphi$ is an MV-chain and $\mathbf{B}$ is a subchain of $\mathbf{C}/\varphi$. It follows that $(\mathbf{B} \times \mathbf{C}/\varphi, \tau^*)$ is diagonal and $(\mathbf{A}, \tau)$ is subdiagonal. Now suppose that $(b, 1) \in \varphi$ for some $b \in B \setminus \{1\}$. Since $\mathbf{A}$ is a subdirect product of $\mathbf{B} \times \mathbf{C}/\varphi$, there is $c \in C$ such that $(b, c/\varphi) \in A$. Moreover, $\tau(b, c/\varphi) = (b, b/\varphi) = (b, 1/\varphi) \in \tau(A)$.

Now if $(1, c/\varphi) \in A$, then $\tau(1, c/\varphi) = (1, 1/\varphi)$ and hence $(1, c/\varphi) \in F_\tau(A)$. Clearly, $(1, c/\varphi) \vee (b, 1/\varphi) = (1, 1/\varphi)$, and since $\tau(\mathbf{A})$ and $\mathbf{F}_\tau(\mathbf{A})$ have the disjunction property, we must have $c/\varphi = 1/\varphi$. Now $F_\tau(A)$ consists of all elements of the form $(1, c/\varphi)$, and hence it is the singleton of $(1, 1/\varphi)$. On the other hand, $F_\tau(A)$ is the filter associated to the homomorphism τ, and hence τ is an embedding and $\mathbf{A}$ is isomorphic to $\tau(\mathbf{A})$, which is in turn isomorphic to $\mathbf{B}$ via the map $b \mapsto (b, b/\varphi)$. Since $\mathbf{B}$ is linearly ordered, $\mathbf{A}$ is linearly ordered and hence subdiagonal. □

We can conclude the proof of Theorem 3.9.

Proof. Let **A** be subdirectly irreducible. Since the variety of SMMV-algebras is generated by $\mathsf{D}([0,1]_{MV})$, and since SMMV-algebras are congruence distributive, by Jónsson's lemma **A** belongs to $\mathsf{HSP_U}(\mathsf{D}([0,1]_{MV}))$. Thus, **A** is a homomorphic image of some $\mathbf{B} \in \mathsf{SP_U}(\mathsf{D}([0,1]_{MV}))$.

Now $\mathsf{D}([0,1]_{MV})$ is subdiagonal, and by Lemma 6.4 subdiagonal SMMV-algebras are closed under S and $\mathsf{P_U}$, so **B** is subdiagonal as well. Then, since **A** is subdirectly irreducible, Lemma 6.5 applies, and we conclude that **A** is subdiagonal. Hence, every subdirectly irreducible SMMV-algebra is subdiagonal. □

We end this section with an example showing that the class of subdiagonal SMMV-algebras is not closed under homomorphic images. Indeed, our example shows that not even the class of subdirectly irreducible subdiagonal SMMV-algebras is closed under homomorphic images. Consider the diagonal algebra $\mathbf{A} = (\mathbf{C}_1 \times \mathbf{C}_1, \tau_{C_1})$. Here again $\mathbf{C}_1$ stands for Chang's algebra. The set $F = \{1\} \times Rad_1(\mathbf{C}_1)$ is a τ-filter of **A**. It is easy to see that the congruence θ_F corresponding to F is the smallest non-trivial congruence on **A**, so **A** is subdirectly irreducible. It is not difficult to see that the MV-reduct of the quotient algebra $\mathbf{A}/\theta_F$ is isomorphic to $\mathbf{C}_1 \times \mathbf{2}$, where **2** is the two-element Boolean algebra. The operation τ on this algebra is given by

$$\tau(c,1) = \tau(c,0) = \begin{cases} (c,1) & \text{if } c \in Rad_1(\mathbf{C}_1) \\ (c,0) & \text{if } c \notin Rad_1(\mathbf{C}_1). \end{cases}$$

Lemma 6.6. *The algebra* $\mathbf{A}/\theta_F$ *is not subdiagonal.*

Proof. If A/θ_F is subdiagonal then there exist linearly ordered MV-algebras **D** and **E** such that $\mathbf{C}_1 \subseteq \mathbf{D}$, $\mathbf{2} \subseteq \mathbf{E}$ and either $(\mathbf{D} \times \mathbf{E}, \tau)$ is diagonal, or $(\mathbf{E} \times \mathbf{D}, \tau)$ is diagonal. Now, if $(\mathbf{D} \times \mathbf{E}, \tau)$ is diagonal, we have $\tau(d,e) = (d,d)$ for all $(d,e) \in D \times E$. In particular, $(c,z) = (c,c)$ for any $(c,z) \in C_1 \times 2$. This fails for any $c \notin \{0,1\}$. Then, if $(\mathbf{E} \times \mathbf{D}, \tau)$ is diagonal, we have $\tau(e,d) = (e,e)$ for all $(e,d) \in E \times D$. In particular, $(z,c) = (z,z)$ for any $(z,c) \in 2 \times C_1$. This again fails for any $c \notin \{0,1\}$. Thus, A/θ_F is not subdiagonal. □

7. Varieties of SMMV-algebras

When studying a variety of universal algebras, an interesting problem is the investigation of the lattice of its subvarieties. In the case of SMMV-algebras, we have a unique atom (above the trivial variety), namely, the variety $\mathcal{BI}$ of Boolean algebras equipped with the identical endomorphism. This variety is generated by the two element Boolean algebra equipped with the identity map. Since this algebra is a subalgebra of any non-trivial SMMV-algebra, $\mathcal{BI}$ is contained in any non-trivial variety of SMMV-algebras.

Other varieties of SMMV-algebras are obtained as follows: let $\mathcal{V}$ be a variety of MV-algebras, let $\mathcal{V}_{SMMV}$ denote the class of algebras whose MV-reduct is in $\mathcal{V}$, and $\mathcal{VI}$ denote the class of SMMV-algebras $(\mathbf{A}, \mathrm{Id_A})$, where $\mathrm{Id_A}$ is the identity on A and $\mathbf{A} \in \mathcal{V}$. The following problem arises: given a variety $\mathcal{V}$ of MV-algebras, investigate the varieties of SMMV-algebras between $\mathcal{VI}$ and $\mathcal{V}_{SMMV}$. To begin with, besides $\mathcal{VI}$ and $\mathcal{V}_{SMMV}$, we will discuss two more kinds of subvarieties, namely, the subvariety generated by all SMMV-chains in $\mathcal{V}_{SMMV}$ (representable SMMV-algebras) and the subvariety generated by all algebras in $\mathcal{V}_{SMMV}$ whose MV-reduct is a local MV-algebra. The above classes will be denoted by $\mathcal{VR}$ and $\mathcal{VL}$ respectively. We consider $\mathcal{V}_{SMMV}$ and $\mathcal{VI}$ first. The following result is straightforward.

Theorem 7.1. (1) $\mathcal{V}_{SMMV}$ *is axiomatized over the axioms of SMMV-algebras by the defining equations of* $\mathcal{V}$.
(2) $\mathcal{VI}$ *is axiomatized over* $\mathcal{V}_{SMMV}$ *by the identity* $\tau(x) = x$.
(3) $\mathcal{VI} \subseteq \mathcal{VR}$, *and the inclusion is proper if and only if* $\mathcal{V}$ *is not finitely generated.*
(4) *The maps* $\mathcal{V} \mapsto \mathcal{VI}$ *and* $\mathcal{V} \mapsto \mathcal{V}_{SMMV}$ *are embeddings of the lattice of MV-varieties into the lattice of SMMV-varieties.*

Proof. Claims (1) and (2) are immediate.

As regards to (3), since subdirectly irreducible algebras of type $\mathcal{I}$ are linearly ordered we have that $\mathcal{VI} \subseteq \mathcal{VR}$. If $\mathcal{V}$ is finitely generated, then $\mathcal{VI} = \mathcal{VR}$, because every MV-chain in $\mathcal{V}$ is finite, and its only endomorphism is the identity. Finally, if $\mathcal{V}$ is not finitely generated, then it contains Chang's algebra, $\mathbf{C}_1$. Let τ be defined for all $x \in C_1$, by $\tau(x) = 0$ if x is infinitesimal and $\tau(x) = 1$ otherwise. Then $(\mathbf{C}_1, \tau) \in \mathcal{VR} \setminus \mathcal{VI}$, and the inclusion $\mathcal{VI} \subseteq \mathcal{VR}$ is proper.

Finally, claim (4) is almost immediate (using Theorem 5.4). □

Now we concentrate ourselves on $\mathcal{VR}$.

Theorem 7.2. *Representable SMMV-algebras constitute a proper subvariety of the variety of SMMV-algebras, which is characterized by the equation*

$$(lin_\tau) \qquad \tau(x) \vee (x \to (\tau(y) \leftrightarrow y)) = 1.$$

Proof. We have to prove that a subdirectly irreducible SMMV-algebra $(\mathbf{A}, \tau)$ satisfies (lin_τ) iff it is linearly ordered. Thus, let $(\mathbf{A}, \tau)$ be a subdirectly irreducible SMMV-algebra.

Suppose first that $(\mathbf{A}, \tau)$ satisfies (lin_τ). We start from the following observation. Let $z, u \in A$. Then $z \to (\tau(u) \leftrightarrow u) \in F_\tau(A)$. Since $\tau(\mathbf{A})$ and $\mathbf{F}_\tau(\mathbf{A})$ have the disjunction property, we have that either $\tau(z) = 1$ or $z \leq \tau(u) \leftrightarrow u$. Now every element $u \in F_\tau(A)$ is equal to $\tau(u) \leftrightarrow u$, and vice versa every element of the form $\tau(u) \leftrightarrow u$ is in $F_\tau(A)$. It follows that if $\tau(z) < 1$, then z is a lower bound of $F_\tau(A)$.

Now assume, by way of contradiction, that $x, y \in A$ are incomparable with respect to the order. We distinguish three cases.

(i) If $x \to y \in F_\tau(A)$ and $y \to x \in F_\tau(A)$, then since $\mathbf{F}_\tau(\mathbf{A})$ is linearly ordered and $(x \to y) \vee (y \to x) = 1$, we must have either $x \to y = 1$ or $y \to x = 1$, a contradiction.

(ii) If $x \to y \notin F_\tau(A)$ and $y \to x \notin F_\tau(A)$, then they are both lower bounds of $F_\tau(A)$, and hence $1 = (x \to y) \vee (y \to x)$ is a lower bound of $F_\tau(A)$. But then $\mathbf{A}$ would be isomorphic to $\tau(\mathbf{A})$, and hence it would be linearly ordered, a contradiction.

(iii) Finally, suppose $x \to y \in F_\tau(A)$ and $y \to x \notin F_\tau(A)$ (or vice versa). Then $y \to x$ is a lower bound of $F_\tau(A)$, and hence $y \to x \leq x \to y$. But in any MV-algebra this is the case iff $x \leq y$, and again a contradiction has been obtained.

Hence, $(\mathbf{A}, \tau)$ is linearly ordered. Conversely, if $(\mathbf{A}, \tau)$ is linearly ordered, then for all x, z such that $\tau(x) < 1$ and $\tau(z) = 1$, we cannot have $z < x$, and hence we must have $x \leq z$. Taking $z = \tau(y) \leftrightarrow y$, we obtain that for all x either $\tau(x) = 1$ or $x \leq z$, and (lin_τ) holds.

Finally, representable SMMV-algebras constitute a proper subvariety of the variety of SMMV-algebras, because any subdirectly irreducible SMMV-algebra of type $\mathcal{D}$ is not linearly ordered. □

Remark 7.3. According to [8, Prop. 3.6], if $(\mathbf{A}, \tau)$ is an SMV-algebra such that $\mathbf{A}$ is a chain, then $(\mathbf{A}, \tau)$ is an SMMV-algebra. Hence, the class of all representable SMV-algebras satisfies (lin_τ). We do not know whether every subdirectly irreducible SMV-algebra satisfying (lin_τ) has a linearly ordered MV-reduct.

Theorem 7.4. *$\mathcal{VR} \subseteq \mathcal{VL}$, and the inclusion is proper if and only if $\mathcal{V}$ is not finitely generated.*

Proof. Since every linearly ordered SMMV-algebra is local, the inclusion follows. Moreover, every local and finite MV-algebra is linearly ordered, and hence for finitely generated MV-varieties the opposite inclusion also holds. On the other hand, if $\mathcal{V}$ is not finitely generated, then it contains Chang's algebra $\mathbf{C}_1$, and the subalgebra of $D(\mathbf{C}_1)$ described in Example 4.5, is a local subdirectly irreducible SMMV-algebra in $\mathcal{V}_{SMMV}$ which is not linearly ordered. Hence, the inclusion $\mathcal{VR} \subseteq \mathcal{VL}$ is proper. □

Next, we discuss varieties of the form $\mathcal{VL}$.

Theorem 7.5. (1) *The variety $\mathcal{VL}$ is axiomatized over $\mathcal{V}_{SMMV}$ by the equation*

$$(loc_\tau) \qquad \neg(\tau(x) \leftrightarrow x) \leq (\tau(x) \leftrightarrow x).$$

(2) *For any non-trivial variety $\mathcal{V}$ of MV-algebras, $\mathcal{VL}$ is a proper subvariety of $\mathcal{V}_{SMMV}$.*

Proof. We start from the following lemma:

Lemma 7.6. *Let $\mathbf{A}$ be a local MV-algebra and M be its only maximal filter. Then for every $m \in M$, $\neg m \leq m$.*

Proof. The claim follows from [4], where it is shown that if $\mathbf{A}$ is a non-trivial BL-algebra and $a, b \in Rad(\mathbf{A})$, then $a \leq \neg b$. □

We continue the proof of Theorem 7.5. In order to prove claim (1), it suffices to prove that an SMMV-algebra is subdirectly irreducible iff it satisfies (loc_τ). Now in every SMMV-algebra we have $\tau(\tau(x) \leftrightarrow x) = 1$, and hence $\tau(x) \leftrightarrow x \in F_\tau(A) \subseteq M$, where M denotes the unique maximal filter of $\mathbf{A}$. Then Lemma 7.6 implies that every subdirectly irreducible local SMMV-algebra satisfies (loc_τ). Before proving the converse, we prove claim (2).

Let $\mathbf{A}$ be a non-trivial chain in $\mathcal{V}$. Then (loc_τ) is invalidated in $D(\mathbf{A})$, taking $x = (1, 0)$. We have $\tau(x) = (1, 1)$, $\tau(x) \leftrightarrow x = (1, 0)$, and

$$\neg(\tau(x) \leftrightarrow x) = (0, 1) \not\leq (1, 0) = \neg(\tau(x) \leftrightarrow x).$$

This settles the claim.

In order to prove the opposite direction of claim (1), note that every subdirectly irreducible SMMV-algebra is either of type $\mathcal{I}$ (in which case it is local) or of type $\mathcal{L}$ (in which case, once again it is local) or of type $\mathcal{D}$. In the last case the proof of (2) shows that it does not satisfy (loc_τ). Hence if a subdirectly irreducible SMMV-algebra satisfies (loc_τ) it is local. □

Another interesting problem in the study of the lattice of subvarieties of a variety is the investigation of covers of a given subvariety (if any). For instance, one may wonder what are the covers of $\mathcal{BI}$. We note that, for any variety defined by a finite set of equations (such as $\mathcal{BI}$), every variety which properly contains it it contains a cover of it. A partial answer to this question is provided by the following theorem:

Theorem 7.7. *Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of MV-algebras. If $\mathcal{W}$ is a cover of $\mathcal{V}$, then $\mathcal{WI}$ is a cover of $\mathcal{VI}$. Hence, if $\mathcal{W}$ is generated either by $\mathbf{S}_p$ for some prime number p or by Chang's algebra $\mathbf{C}_1$, then $\mathcal{WI}$ is a cover of $\mathcal{BI}$.*

Proof. If $(\mathbf{A}, \tau) \in \mathcal{WI} \setminus \mathcal{VI}$, then since τ is forced to be the identity, we must have $\mathbf{A} \in \mathcal{W} \setminus \mathcal{V}$, and since $\mathcal{W}$ is a cover of $\mathcal{V}$, the variety generated by $\{\mathbf{A}\} \cup \mathcal{V}$ is $\mathcal{W}$, and hence the variety generated by $\{(\mathbf{A}, \tau)\} \cup \mathcal{VI}$ is $\mathcal{WI}$, and the claim follows. □

Remark 7.8. Varieties $\mathcal{VI}$, where $\mathcal{V}$ is a cover of the Boolean variety $\mathcal{B}$, do not exhaust the covers of $\mathcal{BI}$. Another cover is $\mathcal{B}_{SMMV}$. Indeed, any subdirectly irreducible SMMV-algebra $(\mathbf{A}, \tau)$ in $\mathcal{B}_{SMMV} \setminus \mathcal{BI}$ must have a Boolean reduct and cannot be of type $\mathcal{I}$ or $\mathcal{L}$, otherwise τ would be identical. Hence, it must be of type $\mathcal{D}$ and $D(\mathbf{S}_1)$ is a subalgebra of $(\mathbf{A}, \tau)$. Therefore, $(\mathbf{A}, \tau)$ generates the whole variety $\mathcal{B}_{SMMV}$.

Theorem 7.7 suggests the following problem:

Problem 2. Let $\mathcal{V}$ be a variety of MV-algebras and let $\mathcal{V}'$ be a cover of $\mathcal{V}$. Is it true that $\mathcal{V}'_{SMMV}$ is a cover of $\mathcal{V}_{SMMV}$? Or, equivalently, is $\mathsf{V}D(\mathcal{V})$ a cover of $\mathsf{V}D(\mathcal{V}')$?

The answer to these questions is no, in general. Here is a sample of counterexamples.

(1) Let $\mathcal{V}$ be the variety of Boolean algebras and $\mathcal{V}'$ be the variety generated by Chang's algebra. Then $\mathcal{V}'$ is a cover of $\mathcal{V}$. However, there is an intermediate variety between $\mathcal{V}_{SMMV}$ and $\mathcal{V}'_{SMMV}$, namely, the subvariety $\mathcal{V}''_{SMMV}$ of $\mathcal{V}'_{SMMV}$ axiomatized by the equation

$$(*) \qquad \tau(x) \vee \tau(\neg x) = 1.$$

Indeed, clearly the equation $(*)$ holds in any Boolean SMMV-algebra. Moreover, there is an algebra in $\mathcal{V}'_{SMMV}$ which satisfies $(*)$ and its reduct is not a Boolean algebra, namely, Chang's algebra $\mathbf{C}_1$ with τ defined by $\tau(x) = 0$ if $x \in Rad(\mathbf{C}_1)$ and $\tau(x) = 1$ otherwise.

Finally, there is an algebra in $\mathcal{V}'_{SMMV}$ which does not satisfy $(*)$, namely, the diagonalization, $D(\mathbf{C}_1)$, of Chang's algebra. Indeed, if $c \in Rad(\mathbf{C}_1) \setminus \{0\}$, then $\tau(c, c) = (c, c)$ and $\tau(\neg(c, c)) = (\neg c, \neg c)$. Hence, $\tau(c, c) \vee \tau(\neg(c, c)) = (\neg c, \neg c) < 1$.

(2) Let $\mathcal{V} = \mathsf{V}(\mathbf{S}_{i_1}, \ldots, \mathbf{S}_{i_n})$ and $\mathcal{V}' = \mathsf{V}(\mathbf{S}_{i_1}, \ldots, \mathbf{S}_{i_n}, \mathbf{C}_1)$ for some integers $1 \leq i_1 < \cdots < i_n$. Then $\mathcal{V}'$ is a cover variety of $\mathcal{V}$. Define $\mathcal{V}''_{SMMV}$ as the class of all $(\mathbf{A}, \tau) \in \mathcal{V}'$ such that $\tau(\mathbf{A}) \in \mathcal{V}$.

Then $\mathcal{V}_{SMMV} \subseteq \mathcal{V}''_{SMMV} \subseteq \mathcal{V}'$. But if τ is as in (1), then $(\mathbf{C}_1, \tau) \in \mathcal{V}''_{SMMV} \setminus \mathcal{V}_{SMMV}$ and $D(\mathbf{C}_1) \in \mathcal{V}'_{SMMV} \setminus \mathcal{V}''_{SMMV}$.

(3) Define on $\mathbf{C}_n \times \mathbf{C}_n$ a map $\tau_n(i, j) = (i, 0)$ for all $(i, j) \in \mathbf{C}_n$, then $(\mathbf{C}_n, \tau_n)$ is an SMMV-algebra.

Let $1 = i_1 < \cdots < i_n$ and $1 = j_1 < \cdots < j_k$ with $k \geq 2$ be finite sets of integers such that every j_s does not divide any j_t with $1 < j_s < j_t$ and fix an index $j_0 \in J := \{j_1, \ldots, j_k\}$ with $j_0 \geq 2$ such that $j_0 \in I := \{i_1, \ldots, i_k\}$.

Let $\mathcal{V}' = \mathsf{V}(\{\mathbf{S}_i, \mathbf{C}_j : i \in I, j \in J\})$ and $\mathcal{V} = \mathsf{V}(\{\mathbf{S}_i, \mathbf{C}_j : i \in I, j \in J \setminus \{j_0\}\})$. Set $\mathcal{V}''_{SMMV}$ as the class of $(\mathbf{A}, \tau) \in \mathcal{V}'_{SMMV}$ such that $\tau(\mathbf{A}) \in \mathcal{V}$. Then $(\mathbf{C}_{j_0}, \tau_{j_0}) \in \mathcal{V}''_{SMMV} \setminus \mathcal{V}_{SMMV}$ and $D(\mathbf{C}_{j_0}) \in \mathcal{V}'_{SMMV} \setminus \mathcal{V}''_{SMMV}$.

(4) Let $\mathcal{V}' = \mathsf{V}(\mathbf{S}_{i_1}, \ldots, \mathbf{S}_{i_n})$, where $1 = i_1 < \cdots < i_n$, $n \geq 2$ and every i_s does not divide any i_t with $1 < i_s < i_t$. Let $i_0 \in \{i_2, \ldots, i_n\}$ be fixed and let $\mathcal{V} = \mathsf{V}(\mathbf{S}_i : i \in \{i_1, \ldots, i_n\} \setminus i_0)$. Then $\mathcal{V}'$ is a cover of $\mathcal{V}$. Let $\mathcal{V}''$ be the variety generated by $\mathcal{V}_{SMMV}$ and $(\mathbf{S}_{i_0}, \mathrm{Id}_{S_{i_0}})$. Then $\mathcal{V}_{SMMV} \subset \mathcal{V}'' \subset \mathcal{V}'_{SMVV}$ because $(\mathbf{S}_{i_0}, \mathrm{Id}_{S_{i_0}}) \in \mathcal{V}'' \setminus \mathcal{V}_{SMMV}$ and $D(\mathbf{S}_{i_0}) \in \mathcal{V}'_{SMMV} \setminus \mathcal{V}''$.

(5) Let $\mathcal{V}' = \mathsf{V}(\mathbf{C}_{j_1}, \ldots, S_{j_k})$, where $1 = i_1 < \cdots < i_k$, $k \geq 2$ and every j_s does not divide any j_t with $1 < j_s < j_t$. Let $j_0 \in \{j_2, \ldots, j_n\}$ be fixed and let $\mathcal{V} = \mathsf{V}(\mathbf{C}_j : j \in \{j_1, \ldots, j_k\} \setminus j_0)$. Let $\mathcal{V}''$ be the variety generated by $\mathcal{V}_{SMMV}$ and $(\mathbf{S}_{i_0}, \tau)$. Then $\mathcal{V}_{SMMV} \subset \mathcal{V}'' \subset \mathcal{V}'_{SMVV}$ because $(\mathbf{C}_{j_0}, \mathrm{Id}_{C_{j_0}}) \in \mathcal{V}'' \setminus \mathcal{V}_{SMMV}$ and $D(\mathbf{C}_{j_0}) \in \mathcal{V}'_{SMMV} \setminus \mathcal{V}''$.

The above examples offer several interesting methods for obtaining intermediate varieties. But the fact that if $\mathcal{W}$ is an MV-cover of $\mathcal{V}$, then $\mathcal{W}_{SMMV}$ need not be a cover of $\mathcal{V}_{SMMV}$ can be strengthened:

Theorem 7.9. *If $\mathcal{W}$ properly contains $\mathcal{V}$, then the join, $\mathcal{V}_{SMMV} \vee \mathcal{WI}$, of $\mathcal{V}_{SMMV}$ and $\mathcal{WI}$, is a proper extension of $\mathcal{V}_{SMMV}$ and a proper subvariety of $\mathcal{W}_{SMMV}$. Hence, $\mathcal{W}_{SMMV}$ can never be a cover of $\mathcal{V}_{SMMV}$.*

Proof. Inclusions are clear. Moreover, if $\mathbf{A} \in \mathcal{W} \setminus \mathcal{V}$, then $(\mathbf{A}, \mathrm{Id}_\mathrm{A}) \in (\mathcal{WI} \vee \mathcal{V}_{\mathrm{SMMV}}) \setminus \mathcal{V}_{\mathrm{SMMV}}$, and hence the first inclusion is proper. In order to prove that also the inclusion $(\mathcal{WI} \vee \mathcal{V}_{SMMV}) \subseteq \mathcal{W}_{SMMV}$, consider an MV-identity $\eta(x) = 1$ which axiomatizes $\mathcal{V}$ over $\mathcal{W}$, and set

$$(\epsilon_V) \qquad \eta(x) \vee (\tau(y) \leftrightarrow y) = 1.$$

Clearly, (ϵ_V) holds both in $\mathcal{V}_{SMMV}$ and in $\mathcal{WI}$, and hence it holds in $\mathcal{V}_{SMMV} \vee \mathcal{WI}$. Now take a subdirectly irreducible MV-algebra $\mathbf{A} \in \mathcal{W} \setminus \mathcal{V}$. Then $D(\mathbf{A}) \in \mathcal{W}_{SMMV}$, but it is readily seen that (ϵ_V) is not valid in $D(\mathbf{A})$, and also the inclusion $(\mathcal{V}_{SMMV} \vee \mathcal{WI}) \subseteq \mathcal{W}_{SMMV}$ is proper. □

It follows that Problem 2 should be replaced by the following:

Problem 3. Suppose that $\mathcal{W}$ is an MV-cover of $\mathcal{V}$. Is it true that $\mathcal{WI} \vee \mathcal{V}_{SMMV}$ is a cover of $\mathcal{V}_{SMMV}$?

According to Komori, [5, Theorem 8.4.4], the lattice of subvarieties of the variety of MV-algebras is countable. Now we investigate the number of varieties of SMMV-algebras, and we prove that there are uncountably many of them. Let $[0, 1]^*$ be an ultrapower of the MV-algebra on $[0, 1]$, and let us fix a positive infinitesimal $\varepsilon \in [0, 1]^*$. For every set X of prime numbers, we denote by $\mathbf{A}(X)$ the subalgebra of $[0, 1]^*$ generated by ε and by the set of all rational numbers $\frac{n}{m}$ with $0 \leq n \leq m$, and $m > 0$ such that:

(1) either $n = 0$ or $\gcd(n, m) = 1$;

(2) for all $p \in X$, p does not divide m.

Note that for all $x \in A(X)$, the standard part of x is a rational number $\frac{n}{m}$ satisfying (1) and (2). Indeed the set of rational numbers satisfying (1) and (2) is closed under all MV-operations.

On $\mathbf{A}(X)$ we define $\tau(x)$ to be the standard part of x. Note that τ is an idempotent homomorphism from $\mathbf{A}(X)$ into itself, and hence $(\mathbf{A}(X), \tau)$ is a linearly ordered SMMV-algebra.

Lemma 7.10. *If X and Y are distinct sets of primes, then $\boldsymbol{A}(X)$ and $\boldsymbol{A}(Y)$ generate different varieties.*

Proof. Without loss of generality, we may assume that there is a prime p such that $p \in X \setminus Y$. Consider the equations:

(a_p) $(p-1)x \leftrightarrow \neg x = 1$

(b_p) $\tau((p-1)x) \leftrightarrow \tau(\neg x) = 1$

(c_p) $(\tau((p-1)x) \leftrightarrow \tau(\neg x))^2 \leq ((p-1)x \leftrightarrow \neg x)$.

The following claims are easy to prove, recalling that $\frac{1}{p} \in A(Y) \setminus A(X)$:

Claim 1. *Eq. (a_p) has no solution in $(\boldsymbol{A}(X), \tau)$, and its only solution in $(\boldsymbol{A}(Y), \tau)$ is $\frac{1}{p}$.*

Claim 2. *Eq. (b_p) has no solution in $(\boldsymbol{A}(X), \tau)$, and its solutions in $(\boldsymbol{A}(Y), \tau)$ are precisely those real numbers in $A(Y)$ whose standard part is $\frac{1}{p}$.*

Claim 3. *In both $(\boldsymbol{A}(X), \tau)$ and $(\boldsymbol{A}(Y), \tau)$, for every x, $\tau((p-1)x) \leftrightarrow \tau(\neg x)$ is the standard part of $(p-1)x \leftrightarrow \neg x$.*

Now consider the equation (c_p).

Claim 4. *Eq. (c_p) is valid in $(\boldsymbol{A}(X), \tau)$ and it is not valid in $(\boldsymbol{A}(Y), \tau)$.*

Proof of Claim 4. *Let $x \in A(X)$, let $\alpha = \tau((p-1)x) \leftrightarrow \tau(\neg x)$ and $\beta = (p-1)x \leftrightarrow \neg x$. By Claims 2 and 3, α is a real number strictly less than 1, and differs from β by an infinitesimal. Hence, α^2 is either 0 or a real strictly smaller than α, and hence it is smaller than β. It follows that (c_p) holds in $(\boldsymbol{A}(X), \tau)$.*

Now we prove that equation (c_p) is not valid in $(\boldsymbol{A}(Y), \tau)$. Let $x = \frac{1}{p} + \varepsilon$. Then $x \in A(Y)$. Moreover, by Claim 2, $\tau((p-1)x) \leftrightarrow \tau(\neg x) = (\tau((p-1)x) \leftrightarrow \tau(\neg x))^2 = 1$, and by Claim 1,

$$(p-1)x \leftrightarrow \neg x = (\frac{1}{p} - (p-1)\varepsilon) + (1 - \frac{1}{p} - \varepsilon) = 1 - p\varepsilon < 1.$$

Thus, Eq. (c_p) is not valid in $\boldsymbol{A}(Y)$. This concludes the proof of Claim 4, and hence of Lemma 7.10. □

We can say more:

Theorem 7.11. *Let $\mathcal{MV}$ denote the variety of all MV-algebras. Then there are uncountably many varieties between $\mathcal{MVI}$ and $\mathcal{MVR}$.*

Proof. Consider, for every set X of prime numbers, the variety $\mathcal{V}(X)$ axiomatized by (lin_τ) and by all equations (c_p) with $p \in X$. Clearly, $\mathbf{A}(X) \in \mathcal{V}(X)$ for every set X of primes. By Lemma 7.10, different sets of primes originate different varieties, and hence there is a continuum of varieties of the form $\mathcal{V}(X)$. Moreover, both equations (lin_τ) and (c_p) hold in all SMMV-algebras of type $\mathcal{I}$, and hence $\mathcal{MVI} \subseteq \mathcal{V}(X)$ for any set X of primes. Finally, since (lin_τ) is an axiom of every $\mathcal{V}(X)$, we have $\mathcal{V}(X) \subseteq \mathcal{MVR}$. □

Corollary 7.12. *There are varieties of representable SMMV-algebras which are not recursively axiomatizable, and hence not finitely axiomatizable.*

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