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Nordhaus–Gaddum results for restrained domination and total restrained domination in graphs

Johannes H. Hattingh[a], Elizabeth Jonck[b], Ernst J. Joubert[b], Andrew R. Plummer[a]

[a]Department of Mathematics and Statistics, University Plaza, Georgia State University, Atlanta, GA 30303, USA
[b]Department of Mathematics, University of Johannesburg, P.O. Box 524, Auckland Park 2006, South Africa

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Abstract

Let $G=(V,E)$ be a graph. A set $S\subseteq V$ is a total restrained dominating set if every vertex is adjacent to a vertex in S and every vertex of $V-S$ is adjacent to a vertex in $V-S$. A set $S\subseteq V$ is a restrained dominating set if every vertex in $V-S$ is adjacent to a vertex in S and to a vertex in $V-S$. The total restrained domination number of G (restrained domination number of G, respectively), denoted by $\gamma_{\mathrm{tr}}(G)$ ($\gamma_{\mathrm{r}}(G)$, respectively), is the smallest cardinality of a total restrained dominating set (restrained dominating set, respectively) of G. We bound the sum of the total restrained domination numbers of a graph and its complement, and provide characterizations of the extremal graphs achieving these bounds. It is known (see [G.S. Domke, J.H. Hattingh, S.T. Hedetniemi, R.C. Laskar, L.R. Markus, Restrained domination in graphs, Discrete Math. 203 (1999) 61–69.]) that if G is a graph of order $n\geqslant 2$ such that both G and $\overline{G}$ are not isomorphic to P_3, then $4\leqslant\gamma_{\mathrm{r}}(G)+\gamma_{\mathrm{r}}(\overline{G})\leqslant n+2$. We also provide characterizations of the extremal graphs G of order n achieving these bounds.

Keywords: Restrained; Total; Domination; Nordhaus–Gaddum

1. Introduction

In this paper, we follow the notation of [1]. Specifically, let $G=(V,E)$ be a graph with vertex set V and edge set E. A set $S\subseteq V$ is a *dominating set*, denoted DS, of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [1] includes a chapter on domination. A thorough study of domination appears in [6,7].

In this paper, we continue the study of two variations of the domination theme, namely that of restrained domination [4,3,5,8] and total restrained domination [2,11].

A set $S\subseteq V$ is a *total restrained dominating set*, denoted TRDS, if every vertex is adjacent to a vertex in S and every vertex in $V-S$ is also adjacent to a vertex in $V-S$. Every graph without isolated vertices has a total restrained dominating set, since $S=V$ is such a set. The *total restrained domination number* of G, denoted by $\gamma_{\mathrm{tr}}(G)$, is the minimum cardinality of a TRDS of G.

E-mail address: jhhattingh@gsu.edu (J.H. Hattingh).

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A set $S \subseteq V$ is a *restrained dominating set*, denoted RDS, if every vertex in $V - S$ is adjacent to a vertex in S and a vertex in $V - S$. Every graph has a restrained dominating set, since $S = V$ is such a set. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a RDS of G. If u, v are vertices of G, then the distance between u and v will be denoted by $d(u, v)$.

Nordhaus and Gaddum present best possible bounds on the sum of the chromatic number of a graph and its complement in [10]. The corresponding result for the domination number is presented by Jaeger and Payan in [9]: If G is a graph of order $n \geqslant 2$, then $\gamma(G) + \gamma(\overline{G}) \leqslant n + 1$. A best possible bound on the sum of the restrained domination numbers of a graph and its complement is obtained in [3]:

Theorem 1. *If G is a graph of order $n \geqslant 2$ such that both G and $\overline{G}$ are not isomorphic to P_3, then $4 \leqslant \gamma_r(G) + \gamma_r(\overline{G}) \leqslant n + 2$.*

A best possible bound on the sum of the total restrained domination numbers of a graph and its complement is obtained in [2]:

Theorem 2. *If G is a graph of order $n \geqslant 2$ such that neither G nor $\overline{G}$ contains isolated vertices or has diameter two, then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leqslant n + 4$.*

Let K be the graph obtained from K_3 by matching the vertices of $\overline{K}_2$ to distinct vertices of K_3. Note that K is self-complementary, K nor $\overline{K}$ contains isolated vertices or has diameter two, while $\gamma_{tr}(K) + \gamma_{tr}(\overline{K}) = 2 \times 5 = 10 > n(K) + 4$. Thus, Theorem 2 is incorrect.

We will show, in Section 2, that if G is a graph of order $n \geqslant 2$ such that neither G nor $\overline{G}$ contains isolated vertices or is isomorphic to K, then $4 \leqslant \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leqslant n + 4$. Moreover, we will characterize the graphs G of order n for which $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$ and also characterize those graphs G for which $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4$. In Section 3, we characterize the graphs G of order n for which $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$ as well as those graphs G for which $\gamma_r(G) + \gamma_r(\overline{G}) = 4$.

2. Total restrained domination

In this section, we provide bounds on the sum of the total restrained domination numbers of a graph and its complement, and provide characterizations of the extremal graphs achieving these bounds.

Let $n \geqslant 5$ be an integer and suppose $\{x, y, u, v\}$ and X are disjoint sets of vertices such that $|X| = n - 4$. Let $\mathscr{L}$ be the family of graphs G of order n where $V(G) = \{x, y, u, v\} \cup X$ and with the following properties:

(P1) x and y are non-adjacent, while u and v are adjacent;
(P2) each vertex in $\{x, y\} \cup X$ is adjacent to some vertex of $\{u, v\}$;
(P3) each vertex in $\{u, v\} \cup X$ is non-adjacent to some vertex of $\{x, y\}$;
(P4) each vertex in $\{x, y\} \cup X$ is adjacent to some vertex of $\{x, y\} \cup X$;
(P5) each vertex in $\{u, v\} \cup X$ is non-adjacent to some vertex of $\{u, v\} \cup X$.

Theorem 3. *If G is a graph of order $n \geqslant 2$ such that neither G nor $\overline{G}$ contains isolated vertices, then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4$ if and only if $G \in \mathscr{L}$.*

Proof. Suppose G is a graph such that neither G nor $\overline{G}$ contains isolated vertices, and suppose $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4$. Then $\gamma_{tr}(G) = \gamma_{tr}(\overline{G}) = 2$. Let $S = \{u, v\}$ ($S' = \{x, y\}$, respectively) be a TRDS of G ($\overline{G}$, respectively). Then x is non-adjacent to y, while u is adjacent to v, and Property (P1) holds. Clearly, $S \neq S'$. Suppose $u = x$ with $v \neq y$. Since $\{u, v\}$ is a DS of G and y is non-adjacent to $x = u$, the vertex y must be adjacent to v. But then v is not dominated by S' in $\overline{G}$, which is a contradiction. Thus, $S \cap S' = \emptyset$. Let $X = V(G) - \{x, y, u, v\}$. Then $|X| = n - 4$, and since S (S', respectively) is a TRDS of G ($\overline{G}$, respectively), Properties (P2)–(P5) hold for G. Thus, $G \in \mathscr{L}$. The converse clearly holds as $\{u, v\}$ ($\{x, y\}$, respectively) is a TRDS of G ($\overline{G}$, respectively). $\square$

Let diam (G) denote the diameter of G, and let u, v be two vertices of G such that $d(u, v) = \mathrm{diam}(G)$. The set of vertices at distance i from u, $0 \leqslant i \leqslant \mathrm{diam}(G)$, will be denoted by V_i, and the sets $V_0, \ldots, V_{\mathrm{diam}(G)}$ will then be called the *level decomposition of G with respect to u*.

Let $\mathscr{U} = \{G | G$ is a graph of order n which can be obtained from a P_4 with consecutive vertices labeled u, v_1, v_2, v by joining vertices v_1 and v_2 to each vertex of K_{n-4} where $n \geqslant 6\}$.

Theorem 4. *Let G be a graph of order $n \geqslant 2$ such that neither G nor $\overline{G}$ contains isolated vertices or is isomorphic to K. Then $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) \leqslant n + 4$. Moreover, $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) = n + 4$ if and only if $G \in \mathscr{U}$ or $\overline{G} \in \mathscr{U}$ or $G \cong P_4$.*

Proof. If G is disconnected, then $\gamma_{\mathrm{tr}}(\overline{G}) = 2$. Hence $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) \leqslant n + 2$. Thus, without loss of generality, assume both G and $\overline{G}$ are connected. Let u and v be vertices such that $d(u, v) = \mathrm{diam}(G)$ and let $V_0, \ldots, V_{\mathrm{diam}(G)}$ be the level decomposition of G with respect to u.

We consider the following cases:

Case 1: $\mathrm{diam}(G) \geqslant 5$.

We claim that $\{u, v\}$ is a TRDS of $\overline{G}$. The vertex u is non-adjacent to all vertices in V_i where $2 \leqslant i \leqslant \mathrm{diam}(G)$, while the vertex v is non-adjacent to all vertices in V_i where $0 \leqslant i \leqslant \mathrm{diam}(G) - 2$. Moreover, every vertex in $V(G) - \{u, v\}$ is non-adjacent to some vertex of $V(G) - \{u, v\}$. Thus, $\gamma_{\mathrm{tr}}(\overline{G}) = 2$, and so $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) \leqslant n + 2$.

Case 2: $\mathrm{diam}(G) = 4$.

Suppose u, v_1, v_2, v_3, v is a diametrical path. If $|V_4| \geqslant 2$, then $\{u, v\}$ is a TRDS of $\overline{G}$, and the result follows.

Thus, $V_4 = \{v\}$. Let $V_{21} = \{x \in V_2 |$ there exists a vertex in $V_1 \cup V_2 \cup V_3$ that is not adjacent to $x\}$ and let $V_{22} = V_2 - V_{21}$. The set $\{u, v\} \cup V_{22}$ is a TRDS of $\overline{G}$. So we have that $\gamma_{\mathrm{tr}}(\overline{G}) \leqslant 2 + |V_{22}|$. If $|V_{22}| \leqslant 1$, then $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) \leqslant n + 3$.

Hence $|V_{22}| \geqslant 2$. Let $t \in V_{22}$ such that $t \neq v_2$. Suppose $|V_1 \cup V_{21} \cup V_3| \geqslant 4$. Let $s \in V_1 \cup V_{21} \cup V_3 - \{v_1, v_2, v_3\}$. Then $V_1 \cup V_{21} \cup V_3 \cup \{u, v, t\} - \{s\}$ is a TRDS of G and so $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) \leqslant n - (|V_{22}| - 1) - 1 + |V_{22}| + 2 \leqslant n + 2$. Hence $|V_1| = 1$, $|V_{21}| \leqslant 1$ and $|V_3| = 1$. Therefore, $V(G) - V_{22}$ is a TRDS of G and so $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) \leqslant n - |V_{22}| + 2 + |V_{22}| \leqslant n + 2$.

Case 3: $\mathrm{diam}(G) = 3$.

Let u, v_1, v_2, v be a diametrical path. Suppose $t \in V_3 - \{v\}$. We define $V_{21} = \{x \in V_2 |$ there exists a vertex in $V_1 \cup V_2 \cup V_3 - \{t\}$ that is not adjacent to $x\}$ and let $V_{22} = V_2 - V_{21}$. The set $\{u, t\} \cup V_{22}$ is a TRDS of $\overline{G}$ and so $\gamma_{\mathrm{tr}}(\overline{G}) \leqslant 2 + |V_{22}|$. If $|V_{22}| = 1$, then surely $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) \leqslant n + 3$. Hence $|V_{22}| \geqslant 2$. The vertex t is adjacent to some vertex $s \in V_2$. If $s \in V_{22}$, then the set $\{u, s\} \cup V_1 \cup V_{21} \cup V_3 - \{v\}$ is a TRDS of G. If $s \notin V_{22}$, then the set $\{u, w\} \cup V_1 \cup V_{21} \cup V_3 - \{v\}$is a TRDS of G, where $w \in V_{22}$. In both cases, $\gamma_{\mathrm{tr}}(G) \leqslant n - |V_{22}|$, and so $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) \leqslant n - |V_{22}| + 2 + |V_{22}| = n + 2$.

Thus, $V_3 = \{v\}$. Define $V_{11} = \{x \in V_1 |$ there exists a vertex in $V_1 \cup V_2$ that is not adjacent to $x\}$ and let $V_{12} = V_1 - V_{11}$. Moreover, let $V_{21} = \{x \in V_2 |$ there exists a vertex in $V_1 \cup V_2$ that is not adjacent to $x\}$ and let $V_{22} = V_2 - V_{21}$. Then $\{u, v\} \cup V_{12} \cup V_{22}$ is a TRDS of $\overline{G}$, whence $\gamma_{\mathrm{tr}}(\overline{G}) \leqslant 2 + |V_{12}| + |V_{22}|$.

Case 3.1: $|V_{12}| + |V_{22}| \leqslant 2$.

Clearly $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) \leqslant n + 4$. We now investigate when, in this case, $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) = n + 4$. As $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) = n + 4$, we must have that $|V_{12}| + |V_{22}| = 2$.

We first show that $\deg(u) = \deg(v) = 1$. Suppose, to the contrary, $\{v_1, w\} \subseteq N(u)$, and let $t \in V_{12} \cup V_{22} - \{w\}$. Then t is adjacent to every vertex of $V_1 \cup V_2$, and so $V(G) - \{u, w\}$ is a TRDS of G. It now follows that $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) \leqslant n - 2 + 4 = n + 2$, which is a contradiction. Thus, $\deg(u) = 1$, and $\deg(v) = 1$ follows similarly.

Hence $V_1 = V_{12} = \{v_1\}$, and the set V_{22} consists of exactly one vertex, say w. Suppose $w \neq v_2$. If $|V_2| = 2$, then $G \cong K$, which is not allowable. So, let $w' \in V_2 - \{v_2, w\}$. Then w and w' are adjacent, and $V(G) - \{w, w'\}$ is a TRDS of G. As before, we obtain a contradiction.

We conclude $w = v_2$. If $V_{21} = \emptyset$, then $G \cong P_4$. If $V_{21} \neq \emptyset$, then surely $|V_{21}| \geqslant 2$. If two vertices, say t and t', of V_{21} are adjacent in G, then $V(G) - \{t, t'\}$ is a TRDS of G, and we obtain a contradiction as before. Thus, V_{21} is independent, and so $\overline{G} \in \mathscr{U}$.

Case 3.2: $|V_{12}| + |V_{22}| \geqslant 3$.

If we can show that G has a TRDS of size at most $s := n - |V_{12}| - |V_{22}| + 1$, then $\gamma_{\mathrm{tr}}(G) + \gamma_{\mathrm{tr}}(\overline{G}) \leqslant n - |V_{12}| - |V_{22}| + 1 + 2 + |V_{12}| + |V_{22}| = n + 3$.

First consider the case when $v_1 \in V_{11}$. Choose $w = v_2$ if $v_2 \in V_{22}$, otherwise choose $w \in V_{12} \cup V_{22}$. In both situations, $\{u, v, w\} \cup V_{11} \cup V_{21}$ is a TRDS of G of size s. Thus, $v_1 \notin V_{11}$. If $v_2 \in V_{21}$, then $\{u, v_1, v\} \cup V_{11} \cup V_{21}$ is a TRDS of G of size s. Thus, $v_2 \notin V_{21}$.

We conclude that $v_1 \in V_{12}$, while $v_2 \in V_{22}$.

Suppose u is adjacent to a vertex w which is distinct from v_1. If $w \in V_{12}$, then $\{v_1, v_2, v\} \cup V_{11} \cup V_{21}$ is a TRDS of size s. If $w \in V_{11}$, then $\{v_1, v_2, v\} \cup (V_{11} - \{w\}) \cup V_{21}$ is a TRDS of size $s-1$. Thus, $\deg(u) = 1$, and $\deg(v) = 1$ follows similarly.

Suppose $V_{22} = \{v_2\}$. If $V_{21} = \emptyset$, then $G \cong P_4$ and $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$. If $V_{21} \neq \emptyset$, then surely $|V_{21}| \geqslant 2$. If two vertices, say t and t', of V_{21} are adjacent in G, then $\{u, v_1, v_2, v\} \cup (V_{21} - \{t, t'\})$ is a TRDS of G of size $s-1$. Thus, V_{21} is independent, $\overline{G} \in \mathscr{U}$ and $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$.

Thus, $|V_{22}| \geqslant 2$. If $V_{21} = \emptyset$, then V_{22} induces a clique. If $|V_{22}| = 2$, then $G \cong K$, which is not allowable. If $|V_{22}| \geqslant 3$, then $G \in \mathscr{U}$ and $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n+4$. Thus, $V_{21} \neq \emptyset$, and so $|V_{21}| \geqslant 2$. Let $\{t, t'\} \subseteq V_{21}$. Then $\{u, v_1, v_2, v\} \cup (V_{21} - \{t, t'\})$ is a TRDS of G of size $s-1$.

Case 4: $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$.

Note that $\delta(G) \geqslant 2$ and $\delta(\overline{G}) \geqslant 2$, since otherwise G or $\overline{G}$ will have isolated vertices.

Case 4.1: $\delta(G) = 2$ or $\delta(\overline{G}) = 2$.

Without loss of generality, assume $\delta(G) = 2$ and suppose u is a vertex of minimum degree in G. Let $N(u) = \{v, w\}$. Let $N_{v,w} = \{x \in V(G) - \{u, v, w\} | x$ is adjacent to both v and $w\}$, let $N_{v,\overline{w}} = \{x \in V(G) - \{u, v, w\} | x$ is adjacent to v but not to $w\}$, and let $N_{w,\overline{v}} = \{x \in V(G) - \{u, v, w\} | x$ is adjacent to w but not to $v\}$. Moreover, let $N_1 = \{x \in N_{u,v} | N(x) = \{v, w\}\}$ and let $N_2 = N_{v,w} - N_1$.

If $N_1 = \emptyset$, then $\{u, v, w\}$ is a TRDS of G and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leqslant n + 3$. Thus, $N_1 \neq \emptyset$. If $N_{v,\overline{w}} = \emptyset$ ($N_{w,\overline{v}} = \emptyset$, respectively), then $\{u, w\}$ ($\{u, v\}$, respectively) is a TRDS of G, whence $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leqslant n + 2$. Thus, $N_{v,\overline{w}} \neq \emptyset$ and $N_{w,\overline{v}} \neq \emptyset$.

The set $\{u, v, w\} \cup N_1$ is a TRDS of G. Let $Y = V(G) - \{u\} - N_1$. Since all vertices in $N_{v,\overline{w}}$ dominate all vertices in $N_1 \cup \{u\}$ in $\overline{G}$, and since $N_1 \cup \{u\}$ is a clique in $\overline{G}$, we have that Y is a RDS of $\overline{G}$. If Y is total, we have that $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leqslant 3 + |N_1| + n - 1 - |N_1| = n + 2$ and we are done.

Assume, therefore, that Y is not total. As w (v, respectively) is non-adjacent to every vertex of $N(v, \overline{w})$ ($N(w, \overline{v})$, respectively), the set $N_2 \neq \emptyset$, since otherwise Y is a TRDS of $\overline{G}$. Moreover, Y will also be a TRDS of $\overline{G}$ if every vertex of N_2 is non-adjacent to some vertex of Y. Hence, there exists a vertex $y \in N_2$ which is adjacent to every vertex of $Y - \{y\}$.

The set $\{v, y\}$ is a TDS of G. If $\{v, y\}$ is also a RDS, we have that $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leqslant n + 2$. The set $\{w, y\}$ is also a TDS of G and if it is a RDS, we are done. Thus, there exist vertices $v' \in N_{v,\overline{w}}$ and $w' \in N_{w,\overline{v}}$ such that $N(v') = \{v, y\}$ and $N(w') = \{w, y\}$.

We now show that $Z = \{u, v', w'\}$ is a TRDS of $\overline{G}$. We show first that Z is a TDS of $\overline{G}$. The vertex v' dominates w in $\overline{G}$, the vertex w' dominates v in $\overline{G}$, while the vertex u dominates $V(G) - \{u, v, w, v', w'\}$ in $\overline{G}$. Moreover, the vertex u dominates $\{v', w'\}$ in $\overline{G}$.

Suppose, to the contrary, that Z is not a RDS of $\overline{G}$. Hence, there exists a vertex $z \notin Z$ such that z is adjacent to every vertex of $V(G) - Z - \{z\}$ in G. As $\deg(\overline{G}) \geqslant 2$, the vertex z is adjacent in $\overline{G}$ to at least two vertices of Z. We consider the following cases:

Case 4.1.1: The vertex z is adjacent in $\overline{G}$ to u and at least one of the vertices v' and w'.

Without loss of generality assume that z is adjacent in $\overline{G}$ to the vertex v'. As z is non-adjacent to u in G, it follows that $z \notin \{v, w\}$. As z is adjacent to both of the vertices v and w in G, we have $z \in N_1 \cup N_2$. If $z \in N_1$, then it is not adjacent to y in G, which contradicts the fact that z is adjacent to every vertex of $V(G) - Z - \{z\}$. If $z \in N_2$, then since $N_1 \neq \emptyset$, there exists a vertex $z' \in N_1$ such that z is not adjacent to z' in G, which is again a contradiction.

Case 4.1.2: The vertex z is adjacent in $\overline{G}$ to v' and w', but not to u.

In this case, $z \in \{v, w\}$. Without loss of generality, assume $z = v$. Then v is adjacent in $\overline{G}$ to both v' and w', which is a contradiction.

Therefore, the set $Z = \{u, v', w'\}$ is a TRDS of $\overline{G}$ and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leqslant n + 3$.

Case 4.2: $\delta(G) \geqslant 3$ and $\delta(\overline{G}) \geqslant 3$.

Let u be a vertex of minimum degree in G. Suppose $N(u) = \{u_1, \ldots, u_\delta\}$ where $\delta = \delta(G)$.

Suppose the sets $N[u]$ and $N[u] - \{u_i\}$ for $i \in \{1, \ldots, \delta\}$ are not total restrained dominating sets of G. Let $N_1 = \{x \in V(G) - N[u] | N(x) = N(u)\}$ and let $N_2 = V(G) - N[u] - N_1$. As $N[u]$ is a TDS of G, but not a RDS of G, the set $N_1 \neq \emptyset$. If $N_2 = \emptyset$, then $\{u, u_1\}$ is a TRDS of G, whence $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leqslant 2 + n$. Thus, $N_2 \neq \emptyset$.

Suppose $N[u]-\{u_i\}$ is a DS for some $i \in \{1, \ldots, \delta\}$. If a vertex $x \in N_2$ is adjacent to vertices in $N(u)-\{u_i\}$ only, then $\deg(x) \leqslant \delta - 1$, which is impossible. Thus, $N[x]-\{u_i\}$ is a TRDS of G, which is contrary to our assumption. Hence, for each $i \in \{1, \ldots, \delta\}$, there exists $u_i' \in N_2$ such that $N(u_i') \cap N(u) = \{u_i\}$.

We claim that $X = \{u, u_1', u_2'\}$ is a TRDS of $\overline{G}$. The vertex u_1' dominates all vertices in $N(u)-\{u_1\}$ in $\overline{G}$. Similarly, u_2' dominates all vertices in $N(u)-\{u_2\}$ in $\overline{G}$. The vertex u dominates all vertices in $V(G)-N[u]$ in $\overline{G}$, and so X is a TDS. Suppose X is not a RDS of $\overline{G}$. Thus, there exists a vertex $x \notin X$ such that x is adjacent in G to each of the vertices in $V(G)-X-\{x\}$. As $\delta(\overline{G}) \geqslant 3$, the vertex x is not adjacent to each of the vertices in X. Hence, $x \in N_1 \cup N_2$. If $x \in N_1$, then since $|N_2| \geqslant \delta \geqslant 3$, there exists a vertex $x' \in N_2 - \{u_1', u_2'\} \subset V(G)-X-\{x\}$ such that x is not adjacent to x' in G, which is a contradiction. Similarly, if $x \in N_2 - \{u_1', u_2'\}$, then, since $N_1 \neq \emptyset$, there exists a vertex $x' \in N_1 \subset V(G)-X-\{x\}$ such that x is not adjacent to x' in G, which is a contradiction. Hence X is a TRDS of $\overline{G}$ and so $\gamma_{\rm tr}(G) + \gamma_{\rm tr}(\overline{G}) \leqslant n+3$.

We may therefore assume that $N_G[u]$ or $N_G[u]-\{u_i\}$ is a TRDS of G for some $i \in \{1, \ldots, \delta\}$. Similarly, if v is a minimum degree vertex in $\overline{G}$ and $N_{\overline{G}}(v) = \{v_1, \ldots, v_{\delta(\overline{G})}\}$, we assume that $N_{\overline{G}}[v]$ or $N_{\overline{G}}[v]-\{v_j\}$ is a TRDS of $\overline{G}$ for some $j \in \{1, \ldots, \delta(\overline{G})\}$. Hence $\gamma_{\rm tr}(G) + \gamma_{\rm tr}(\overline{G}) \leqslant \delta(G) + 1 + \delta(\overline{G}) + 1 = \delta(G) + 1 + n - \Delta(G) - 1 + 1 = n + \delta(G) - \Delta(G) + 1 \leqslant n+1$.

Clearly, if $G \in \mathscr{U}$ or $\overline{G} \in \mathscr{U}$ or $G \cong P_4$, then $\gamma_{\rm tr}(G) + \gamma_{\rm tr}(\overline{G}) = n+4$. $\square$

3. Restrained domination

In this section, we provide bounds on the sum of the restrained domination numbers of a graph and its complement, and provide characterizations of the extremal graphs achieving these bounds.

Let $\mathscr{H}$ be the family of graphs G of order n where G or $\overline{G}$ is one of the following four types:

Type 1: $V(G) = \{x, y, z\} \cup X$. Moreover:

(P1.1) x is adjacent to each vertex of $\{y, z\} \cup X$;
(P1.2) each vertex of $\{y, z\} \cup X$ is adjacent to some vertex of $\{y, z\} \cup X$;
(P1.3) each vertex of X is non-adjacent to some vertex of $\{y, z\}$ and non-adjacent to some vertex in X.

Type 2: $V(G) = \{x, y\} \cup X$. Moreover:

(P2.1) each vertex of X is adjacent to exactly one vertex of $\{x, y\}$ and also non-adjacent to exactly one vertex of $\{x, y\}$;
(P2.2) each vertex of X is non-adjacent to some vertex of X;
(P2.3) each vertex of X is adjacent to some vertex of X.

Type 3: $V(G) = \{u, v, y\} \cup X$. Moreover:

(P3.1) each vertex of $X \cup \{y\}$ is adjacent to some vertex of $\{u, v\}$;
(P3.2) each vertex of $X \cup \{u\}$ is non-adjacent to some vertex of $\{v, y\}$;
(P3.3) each vertex of $X \cup \{y\}$ is adjacent to some vertex of $X \cup \{y\}$;
(P3.4) each vertex of $X \cup \{u\}$ is non-adjacent to some vertex of $X \cup \{u\}$.

Type 4: $V(G) = \{x, y, u, v\} \cup X$. Moreover:

(P4.1) each vertex in $\{x, y\} \cup X$ is adjacent to some vertex of $\{u, v\}$;
(P4.2) each vertex in $\{u, v\} \cup X$ is non-adjacent to some vertex of $\{x, y\}$;
(P4.3) each vertex in $\{x, y\} \cup X$ is adjacent to some vertex of $\{x, y\} \cup X$;
(P4.4) each vertex in $\{u, v\} \cup X$ is non-adjacent to some vertex of $\{u, v\} \cup X$.

Theorem 5. *If G be a graph of order $n \geqslant 2$, then $\gamma_{\rm r}(G) + \gamma_{\rm r}(\overline{G}) = 4$ if and only if G or $\overline{G} \in \mathscr{H}$.*

Proof. Suppose G is a graph such that $\gamma_{\rm r}(G) + \gamma_{\rm r}(\overline{G}) = 4$. Then $\gamma_{\rm r}(G) = 1$ and $\gamma_{\rm r}(\overline{G}) = 3$ or $\gamma_{\rm r}(\overline{G}) = 1$ and $\gamma_{\rm r}(G) = 3$ or $\gamma_{\rm r}(G) = \gamma_{\rm r}(\overline{G}) = 2$.

Case 1: $\gamma_r(G)=1$ and $\gamma_r(\overline{G})=3$ or $\gamma_r(\overline{G})=1$ and $\gamma_r(G)=3$.

Suppose $\gamma_r(G)=1$ and $\gamma_r(\overline{G})=3$. Let $\{x\}$ be a RDS of G. Then x is adjacent to every other vertex of G, and so x is isolated in $\overline{G}$ and is therefore in every RDS of $\overline{G}$—let $\{x,y,z\}$ be a RDS of $\overline{G}$. Let $X=V(G)-\{x,y,z\}$. It now follows that Properties (P1.1)–(P1.3) hold for G. Thus, G is a graph of Type 1.

If $\gamma_r(\overline{G})=1$ and $\gamma_r(G)=3$, then $\overline{G}$ is also of Type 1.

Case 2: $\gamma_r(G)=2$ and $\gamma_r(\overline{G})=2$.

Let $\{u,v\}$ ($\{x,y\}$, respectively) be a RDS of G ($\overline{G}$, respectively). Let $X=V(G)-\{u,v,x,y\}$.

Case 2.1: Suppose $u=x$ and $v=y$.

If some vertex $w\in X$ is adjacent to both u and v, then w is not dominated by $\{u,v\}$ in $\overline{G}$, which is a contradiction. As $\{u,v\}$ is a DS of G, each vertex $w\in X$ is adjacent to at least one vertex in $\{u,v\}$. Thus, G satisfies Property (P2.1). Moreover, Properties (P2.2) and (P2.3) hold for G. Thus, G is a graph of Type 2.

Case 2.2: Suppose $u\neq y$ and $x=v$.

Clearly, in this case G is a graph of Type 3.

Case 2.3: $\{u,v\}\cap\{x,y\}=\emptyset$.

It is easy to see, that (P4.1)–(P4.4) hold, so G is a graph of Type 4.

For the converse, suppose $G\in\mathscr{H}$. For a graph of Type 1 we have $\gamma_r(G)=1$ and $\gamma_r(\overline{G})\leqslant 3$. For Types 2, 3 or 4 we obtain $\gamma_r(G)\leqslant 2$ and $\gamma_r(\overline{G})\leqslant 2$. Hence, in all cases $\gamma_r(G)+\gamma_r(\overline{G})\leqslant 4$. It is known (see [3]) that $\gamma_r(G)+\gamma_r(\overline{G})\geqslant 4$. Therefore, $\gamma_r(G)+\gamma_r(\overline{G})=4$. □

As before, the sets $V_0,\dots,V_{\mathrm{diam}(G)}$ will denote the level decomposition of G with respect to u.

Let $\mathscr{B}=\{P_3,\overline{P}_3\}$, and let $\mathscr{G}=\{G|G$ or $\overline{G}$ is a galaxy of non-trivial stars$\}$.

Let $\mathscr{S}=\{G|G$ or $\overline{G}\cong K_1\cup S$ where S is a star and $|S|\geqslant 3\}$.

Lastly, let $\mathscr{E}=\mathscr{G}\cup\mathscr{S}$.

Lemma 6. *If $G\in\mathscr{E}-\mathscr{B}$, then $\gamma_r(G)+\gamma_r(\overline{G})=n+2$.*

Proof. Suppose $G\in\mathscr{G}$ has order n and, without loss of generality, suppose G is a galaxy of non-trivial stars $S_1,S_2,\dots,S_k$, for $k\geqslant 2$. Then $\gamma_r(G)=n$. Let $s\in V(S_1)$ and $t\in V(S_2)$. Since S_i is non-trivial for $i\in\{1,\dots,k\}$, it follows that $R=\{s,t\}$ is a RDS of $\overline{G}$. Suppose $\{v\}$ is a RDS of $\overline{G}$. Then $\deg_G(v)=0$, which is a contradiction. Hence $\gamma_r(G)+\gamma_r(\overline{G})=n+2$. Now, suppose $k=1$. That is, G is a non-trivial star S such that $S\neq P_3$. The result follows immediately if $|S|=2$. Thus we may assume $|S|\geqslant 4$. Then $\gamma_r(G)=n$. Let s be the center of S and let $t\in N_G(s)$. Notice that $\langle V(G)-\{s\}\rangle\cong K_{n-1}$in $\overline{G}$. Thus $R=\{s,t\}$ is a RDS of $\overline{G}$. Suppose $\{v\}$ is a RDS of $\overline{G}$. Then $\deg_G(v)=0$, which is a contradiction.

Suppose $G\in\mathscr{S}$ and, without loss of generality, let $G=K_1\cup S$ where S is a star and $|S|\geqslant 3$. Then $\gamma_r(G)=n$. Let s be the center of S and let $\langle u\rangle$ be the second component of G. Then $R=\{s,u\}$ is a RDS of $\overline{G}$. Suppose $\{v\}$ is a RDS of $\overline{G}$. Then $\deg_G(v)=0$, and $v=u$, which is a contradiction as $\{u\}$ is not a RDS of $\overline{G}$. Hence $\gamma_r(G)+\gamma_r(\overline{G})=n+2$. □

Theorem 7. *Let $G=(V,E)$ be a graph of order $n\geqslant 2$ such that $G\notin\mathscr{B}$. Then $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n+2$. Moreover, $\gamma_r(G)+\gamma_r(\overline{G})=n+2$ if and only if $G\in\mathscr{E}$.*

Proof. Let $G=(V,E)$ be a graph of order n such that $G\notin\mathscr{B}$. Notice that either G or $\overline{G}$ must be connected. Without loss of generality, suppose $\overline{G}$ is connected. Note that G may also be connected. Let G be comprised of the components $G_1,G_2,\dots,G_\ell$ with ℓ possibly equal to one. Without loss of generality, let G_1 be a component of G with longest diameter. □

Claim 1. *If G_1 contains a path uv_1v_2v and $\ell\geqslant 3$, then $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n$.*

Proof. Let uv_1v_2v be a path in G_1. Notice that $V(G)-\{v_1,v_2\}$ is a RDS of G. Hence $\gamma_r(G)\leqslant n-2$. Let $x\in V(G_1)$ and $w\in V(G_2)$. Since $\ell\geqslant 3$ it follows that $\{x,w\}$ is a RDS of $\overline{G}$ and $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n-2+2=n$. □

Claim 2. *If $\ell\geqslant 3$ and there exists $i\in\{1,\dots,\ell\}$ such that $G_i\cong K_1$, then $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n+1$.*

Proof. Trivial. □

By Claim 1, for cases in which $\mathrm{diam}(G_1)\geqslant 3$, we may immediately assume that $\ell\leqslant 2$. Note that for the following two cases $V(G_2)$ may or may not be empty.

Suppose $\mathrm{diam}(G_1)\geqslant 5$. Let $uv_1v_2\ldots v_{\mathrm{diam}(G_1)}$ be a diametrical path in G_1. Notice that $V(G)-\{v_1,v_2\}$ is a RDS of G. Hence $\gamma_r(G)\leqslant n-2$. Moreover, notice that $R'=\{u,v_5\}$ is a RDS of $\overline{G}$, as R' is clearly a dominating set of $\overline{G}$, $v_1\in V(\overline{G})-R'$ is adjacent to $V_3\cup V_4\cup\ldots\cup V_{\mathrm{diam}(G)}$, and $v_4\in V(\overline{G})-R'$ is adjacent to $V_1\cup V_2\cup V(G_2)$. Hence $\gamma_r(\overline{G})\leqslant 2$ and we have that $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n-2+2=n$.

Now, suppose $\mathrm{diam}(G_1)=4$. Let $uv_1v_2v_3v_4$ be a diametrical path in G_1. Notice that $V(G)-\{v_1,v_2\}$ is a RDS of G. Hence $\gamma_r(G)\leqslant n-2$. Suppose $|V_4|\geqslant 2$. Then there exists a vertex $t\in V_4-\{v_4\}$. Notice that $R'=\{u,v_4\}$ is a RDS of $\overline{G}$, as R' is clearly a dominating set of $\overline{G}$, $v_1\in V(\overline{G})-R'$ is adjacent to $V_3\cup V_4$, and $t\in V(\overline{G})-R'$ is adjacent to $V_1\cup V_2\cup V(G_2)$. Hence $\gamma_r(\overline{G})\leqslant 2$ and we have that $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n-2+2=n$.

Thus we may assume that $|V_4|=1$. Let $V_{21}=\{x\in V_2|$ there exists $y\in V_1\cup V_2\cup V_3$ such that $xy\notin E(G_1)\}$ and let $V_{22}=V_2-V_{21}$. Consider $R'=\{u,v_4\}\cup V_{22}$. Notice that R' is a dominating set of $\overline{G}$,$v_1\in V(\overline{G})-R'$ is adjacent to V_3,and $v_3\in V(\overline{G})-R'$ is adjacent to $V_1\cup V(G_2)$. If $V_{21}=\emptyset$,then $V_2=V_{22}\subseteq R'$ and R' is a RDS of $\overline{G}$. If $V_{21}\neq\emptyset$,then by definition,for each $x\in V_{21}$ there exists a $y\in V_1\cup V_{21}\cup V_3$ such that $xy\notin E(G_1)$. Hence R' is a RDS of$\overline{G}$. In either case we have that $\gamma_r(\overline{G})\leqslant 2+|V_{22}|$.

If $|V_{22}|\leqslant 1$, then $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n-2+2+|V_{22}|\leqslant n+1$. Thus we may assume that $|V_{22}|\geqslant 2$. Hence there exists a vertex $t\in V_{22}-\{v_2\}$. Then $R=\{u,v_4,t\}\cup V(G_2)$ is a RDS of G, as R clearly dominates G, and a vertex $w\in V_{22}-\{t\}$ is adjacent to every vertex of $V(G)-R$. Thus, $\gamma_r(G)\leqslant 3+|V(G_2)|$ and so $\gamma_r(G)+\gamma_r(\overline{G})\leqslant 3+|V(G_2)|+2+|V_{22}|=1+(4+|V_{22}|+|V(G_2)|)=1+(|\{u,v_1,v_3,v_4\}|+|V_{22}|+|V(G_2)|)=1+|\{u,v_1,v_3,v_4\}\cup V_{22}\cup V(G_2)|\leqslant 1+|V(G)|=1+n$.

Now, suppose $\mathrm{diam}(G_1)=3$. Let $uv_1v_2v_3$ be a diametrical path in G_1. Notice that $V(G)-\{v_1,v_2\}$ is a RDS of G. Suppose that $V(G_2)\neq\emptyset$. If $V(G_2)=\{v\}$, then $\{v\}$ is a RDS of $\overline{G}$, whence $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n-2+1=n-1$. Thus we may assume that $|V(G_2)|\geqslant 2$. Let $v\in V(G_2)$. Then $\{u,v\}$ is a RDS of $\overline{G}$ and so $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n-2+2=n$.

Thus $V(G_2)=\emptyset$ and both $G_1=G$ and $\overline{G}$ are connected. Suppose $|V_3|\geqslant 2$ and let $t\in V_3-\{v_3\}$. Let $V_{21}=\{x\in V_2|$ there exists $y\in(V_1\cup V_2\cup V_3)-\{t\}$ such that $xy\notin E(G)\}$ and let $V_{22}=V_2-V_{21}$. Consider $R'=\{u,t\}\cup V_{22}$. By reasoning similar to that in the case for $\mathrm{diam}(G_1)=4$, R' is a RDS of $\overline{G}$ and $\gamma_r(\overline{G})\leqslant 2+|V_{22}|$. If $|V_{22}|\leqslant 1$, then $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n-2+2+|V_{22}|\leqslant n+1$.

Thus we may assume that $|V_{22}|\geqslant 2$. Hence there exists a vertex $z\in V_{22}-\{v_2\}$. Consider $R=\{u,t,z\}$. By reasoning similar to that in the case for $\mathrm{diam}(G_1)=4$, R is a RDS of G and so $\gamma_r(G)+\gamma_r(\overline{G})\leqslant 3+2+|V_{22}|=1+(4+|V_{22}|)=1+(|\{u,v_1,v_3,t\}|+|V_{22}|)=1+|\{u,v_1,v_3,t\}\cup V_{22}|\leqslant 1+|V(G)|=1+n$.

So we may assume that $|V_3|=1$. Let $V_{11}=\{x\in V_1|$ there exists $y\in V_1\cup V_2$ such that $xy\notin E(G)\}$ and let $V_{12}=V_1-V_{11}$. Also, let $V_{21}=\{x\in V_2|$ there exists $y\in V_1\cup V_2$ such that $xy\notin E(G)\}$ and let $V_{22}=V_2-V_{21}$. Then $\{u,v_3\}\cup V_{12}\cup V_{22}$ is a RDS of $\overline{G}$ and $\gamma_r(\overline{G})\leqslant 2+|V_{12}|+|V_{22}|$.

If $|V_{12}|+|V_{22}|\leqslant 1$, then $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n-2+2+|V_{12}|+|V_{22}|\leqslant n+1$.

So we may assume that $|V_{12}|+|V_{22}|\geqslant 2$. Since $v_1v_3uv_2$ is a path in $\overline{G}$, it follows that $V(\overline{G})-\{v_3,u\}$ is a RDS of $\overline{G}$, whence $\gamma_r(\overline{G})\leqslant n-2$.

Now, suppose $|V_{12}|\geqslant 2$ and let $z\in V_{12}-\{v_1\}$. Then $\{z,v_3\}$ is a RDS of G, and so $\gamma_r(G)+\gamma_r(\overline{G})\leqslant 2+n-2=n$. Thus $|V_{12}|\leqslant 1$.

Suppose $V_{12}=\{z\}$. Then $\{u,v_3,z\}$ is a RDS of G except when $G=P_4$, in which case $\{u,v_3\}$ is a RDS of G. In both cases $\gamma_r(G)\leqslant 3$. Hence, $\gamma_r(G)+\gamma_r(\overline{G})\leqslant 3+n-2=n+1$.

Thus $V_{12}=\emptyset$ and so $|V_{22}|\geqslant 2$. Let $z\in V_{22}-\{v_2\}$. Then $\{u,v_3,z\}$ is a RDS of G. Therefore, $\gamma_r(G)\leqslant 3$. Hence, $\gamma_r(G)+\gamma_r(\overline{G})\leqslant 3+n-2=n+1$.

Thus we may assume $\mathrm{diam}(G_1)\leqslant 2$, and by a similar argument, $\mathrm{diam}(\overline{G})\leqslant 2$.

As $n\geqslant 2$, $\mathrm{diam}(\overline{G})\geqslant 1$. Suppose $\mathrm{diam}(\overline{G})=1$. Then $\overline{G}\cong K_i$ for some $i\geqslant 2$. If $i\geqslant 3$, then $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n+1$. Thus, $\overline{G}\cong K_2$, and so $G\in\mathscr{G}$ and $\gamma_r(G)+\gamma_r(\overline{G})=n+2$.

Thus, $\mathrm{diam}(\overline{G})=2$.

Suppose $\mathrm{diam}(G_1)=0$. Then $G\cong nK_1$ and $\overline{G}\cong K_n$, which is a contradiction as $\mathrm{diam}(\overline{G})=2$.

Suppose $\mathrm{diam}(G_1)=1$. Then $G_1\cong K_i$ where $2\leqslant i\leqslant n$. Since we assumed that $\overline{G}$ is connected, $\ell\neq 1$. Suppose $\ell=2$. If $G_2\cong K_1$, then $i\neq 2$, as $G\notin\mathscr{B}$. Thus $i\geqslant 3$, so $G\in\mathscr{G}$ and $\gamma_r(G)+\gamma_r(\overline{G})=n+2$. Thus $G_2\cong K_j$ where $2\leqslant j\leqslant n-i$. If $i=j=2$, then $G\in\mathscr{G}$ and we are done. Without loss of generality, suppose $i\geqslant 3$. Let $V(G_1)=\{v_1,v_2,\ldots,v_i\}$ and let $z\in V(G_2)$. Since $i\geqslant 3$, $V(G)-\{v_2,v_3\}$ is a RDS of G and $\{v_1,z\}$ is a RDS of $\overline{G}$. Hence $\gamma_r(G)+\gamma_r(\overline{G})\leqslant n-2+2=n$. Thus $\ell\geqslant 3$. By Claim 2, $G_k\not\cong K_1$ for all $k\in\{1,\ldots,\ell\}$. Suppose $G_k\cong K_2$ for all k. Then $G\in\mathscr{G}$ and we are done. Thus, by relabeling if necessary, we may assume that $G_1 \cong K_i$ for $i \geqslant 3$. Let $V(G_1) = \{v_1, v_2, \ldots, v_i\}$ and let $z \in V(G_2)$. Since $i \geqslant 3$, $V(G) - \{v_2, v_3\}$ is a RDS of G and $\{v_1, z\}$ is a RDS of $\overline{G}$. Hence $\gamma_r(G) + \gamma_r(\overline{G}) \leqslant n - 2 + 2 = n$.

Thus we may assume $\mathrm{diam}(G_1) = 2$. Suppose $\ell \geqslant 3$. By Claim 2, $G_k \not\cong K_1$ for all $k \in \{1, \ldots, \ell\}$. If G is a galaxy of non-trivial stars, then $G \in \mathscr{G}$, and we are done. Thus at least one component, say G_1, contains a cycle containing an edge v_1v_2, say. Let $z \in V(G_2)$. Then $V(G) - \{v_1, v_2\}$ is a RDS of G, while $\{v_1, z\}$ is a RDS of $\overline{G}$, whence $\gamma_r(G) + \gamma_r(\overline{G}) \leqslant n - 2 + 2 = n$.

Suppose $\ell = 2$ and first suppose $G_2 \not\cong K_1$. If G_1 and G_2 are stars, then $G \in \mathscr{G}$ and we are done. Thus at least one component contains a cycle containing the edge v_1v_2. Let z be an arbitrary vertex in the other component of G. Then $V(G) - \{v_1, v_2\}$ is a RDS of G, while $\{v_1, z\}$ is a RDS of $\overline{G}$, whence $\gamma_r(G) + \gamma_r(\overline{G}) \leqslant n - 2 + 2 = n$.

So we may assume that $G_2 \cong K_1$. Let $V(G_2) = \{z\}$. If $\Delta(G_1) \leqslant n-3$, then $\{z\}$ is a RDS of $\overline{G}$ and so $\gamma_r(G) + \gamma_r(\overline{G}) \leqslant n+1$. Thus $\Delta(G_1) = n - 2$, and there exists a vertex $u \in V(G_1)$ such that $\deg(u) = n - 2$. Let L be the set of leaves in G_1 and let $X = N(u) - L$. If $L = \emptyset$, then $\{u, z\}$ is a RDS of G. Since $\mathrm{diam}(G_1) = 2$, there exist non-adjacent vertices $x, y \in V(G_1)$. Then $V(\overline{G}) - \{x, y\}$ is a RDS of $\overline{G}$ and $\gamma_r(G) + \gamma_r(\overline{G}) \leqslant n - 2 + 2 = n$. Thus $L \neq \emptyset$. Let $v \in L$ and consider $\{u, v\}$. Since $\mathrm{diam}(G_1) = 2$, it follows that $\deg(u) \geqslant 2$. Thus $\{u, v\}$ is a RDS of $\overline{G}$. Suppose $X \neq \emptyset$ and let $s \in X$. Since $s \notin L$, s is adjacent to a vertex $t \in N(v)$. Hence $t \notin L$, so $t \in X$ and thus $|X| \geqslant 2$. Moreover, $V(G) - X$ is a RDS of G, and so $\gamma_r(G) + \gamma_r(\overline{G}) \leqslant n - 2 + 2 = n$. Thus $X = \emptyset$ and so G_1 is a non-trivial star of order $n - 1 \geqslant 3$. Therefore $G \in \mathscr{S}$ and we are done.

Thus $G \cong G_1$, and $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$. Let uv_1v_2 be a diametrical path in G. If v_2 is a leaf of G, then every vertex $v \in V_1 - \{v_1\}$ is adjacent to v_1, whence $\deg(v_1) = n - 1$, which is a contradiction as $\overline{G}$ is connected. Moreover, if some vertex $v \in V_1$ is a leaf, then $\mathrm{diam}(G) \geqslant d(v, v_2) = 3$, which is a contradiction. Lastly, if u is a leaf, then v_1 is adjacent to every vertex of V_2, whence $\deg(v_1) = n - 1$, which is a contradiction. Thus we may assume that $\delta(G) \geqslant 2$. A similar argument shows that $\delta(\overline{G}) \geqslant 2$. Let $\mathscr{F}$ be the collection of graphs described in [5]. It is known (see [5]) that if $G \notin \mathscr{F}$ is a connected graph with order $n \geqslant 3$ and $\delta(G) \geqslant 2$, then $\gamma_r(G) \leqslant (n - 1)/2$. It follows immediately that $\gamma_r(G) + \gamma_r(\overline{G}) \leqslant n - 1$, provided that $G, \overline{G} \notin \mathscr{F}$. Without loss of generality, suppose $G \in \mathscr{F}$. It is easily verified that $\gamma_r(G) + \gamma_r(\overline{G}) \leqslant n + 1$ and we are done.

Finally, recounting the argument, we have that $\gamma_r(G) + \gamma_r(\overline{G}) \leqslant n + 1$ in all cases, save when $G \in \mathscr{E}$. Hence, if $\gamma_r(G) + \gamma_r(\overline{G}) = n+2$ it follows that $G \in \mathscr{E}$. This observation together with Lemma 6 implies that $\gamma_r(G) + \gamma_r(\overline{G}) = n+2$ if and only if $G \in \mathscr{E}$. □

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