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Penrose patterns are almost entirely determined by two points

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Abstract

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It is shown that for any Penrose pattern π and for any positive number ε we can find two vertices P and Q of π such that in any large circular disk all but a fraction of at most ε of the vertices is common to all Penrose patterns (with the same pieces, in the same directions) that have P and Q as vertices.

1. Introduction

A Penrose pattern is a tiling of the plane by means of infinitely many copies of two particular arrowed rhombs. The rhombs are shown in Fig. 1: the *thick* rhomb (with an acute angle of 72°) and the *thin* rhomb (with an acute angle of 36°). The edges are provided with two kinds of arrows: single arrows and double arrows. The copies of the rhombs have to be fitted together according to the rule that every edge of every rhomb should be pasted to an edge of the same kind in the same direction. The surprising discovery of Penrose (see [7]) was that there are infinitely many ways (actually the cardinality is the one of the continuum) to get a full tiling of the plane with these arrowed pieces and that none of these tilings is periodic.

The tilings became generally known through Gardner's article [5], who used the *kites* and *darts* as basic pieces instead of the rhombs, but passing from the one set of pieces to the other is a matter of simple local transformation (see [1, Section 2]).

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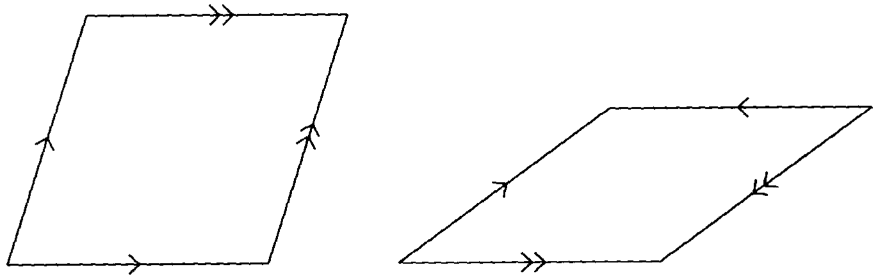

Fig. 1. Penrose's arrowed rhombs.

Gardner explained in [5] that a configuration consisting of a few pieces sometimes enforces the tiling in quite a big region (the term 'empire' was used). He only considered connected regions, but Grünbaum and Shephard [6, Section 10.6] showed that much more can be forced than such a connected empire.

In [3] infinite disconnected empires were derived for Beatty sequences, a particular kind of zero–one sequences that form a one-dimensional analog of the Penrose patterns. In Section 8.5 of that paper it was indicated how just two entries of such a sequence sometimes enforce an infinite set of other entries. It is even more than just an infinite empire: it is almost the whole world! That is, the asymptotic density of the index places *not* belonging to the empire is small. Actually it can be made arbitrarily small by suitable selection of the two entries that determine the empire.

In the present paper we shall show the same thing for the Penrose patterns, but the method will not be the same as in [3]. Instead of using the method of updown generation (that method was carried out for Penrose patterns too, in [4]), we shall use the method of [1], in particular in the version of [1, Section 8].

2. Preliminaries

The plane in which we describe our Penrose patterns will be the complex plane. We make two restrictions:

(i) All rhomb edges have length 1.

(ii) The angles that the rhomb edges are forming with the real axis are multiples of 36°.

A Penrose pattern is completely determined by its vertex set, i.e., the set of all vertices of all rhombs. Once we know that a set is the vertex set of a tiling with arrowed rhombs, it is unique what the edges are, and it is also unique what the directions and the types (single or double) of the arrows have to be. The uniqueness of the edges follows from the remark that in a Penrose pattern two vertices are connected by an edge if and only if they have distance 1. The uniqueness of the arrows is easily established by inspecting the vertex types in Fig. 7 of [1]. Only for the vertex types S and S5 we have to look into the situation at one of the direct neighbours. For the other types (K, Q, D, J, S3, S4) the arrowing follows by inspecting the rhombs around the vertex itself.

Sometimes two Penrose patterns are very much alike. We shall formulate a notion of 'being equal up to ε', where ε is any real number >0.

First we explain a notion about point sets in the plane which have at most a finite number of points in every bounded region. If S_1 is such a set, and if $\varepsilon > 0$, then we say that a subset S_2 of S_1 is an *ε-complete* subset of S_1 if there exists a positive number R such that for every circular disk with radius $>R$ we have $D_2 > (1-\varepsilon)D_2$, where D_i is the number of points of S_i in the disk ($i = 1, 2$).

Now consider Penrose patterns π_1 and π_2. Let S_1, S_2 be their sets of vertices. We shall say that π_1 and π_2 are *equal up to* ε if the intersection $S_1 \cap S_2$ is *ε-complete* with respect to the union $S_1 \cup S_2$.

As an example we mention that two patterns which we get from one and the same singular pentagrid (see [1, Section 12]) are equal up to ε for every positive ε.

The notion of Penrose patterns being equal up to ε refers to the Penrose patterns *themselves,* and not to equivalence classes where two patterns are called equivalent if they can be obtained from each other by shifts. If we would take the notion in the sense of such equivalence classes we would have to admit that for every $\varepsilon > 0$ *all* Penrose patterns are equal up to ε. Actually it is not hard to derive (from Lemma 2 of Section 4 below) that if π_1 and π_2 are Penrose patterns and $\varepsilon > 0$ then there is a shift that transforms π_2 into π_2^* such that π_1 and π_2^* are equal up to ε.

3. The result

Theorem. *For any Penrose pattern π and for any positive number ε we can find two vertices P and Q of π such that there is a set E which is an ε-complete subset of the vertex set of every Penrose pattern that has both P and Q as vertices. In particular this means that every Penrose pattern that has P and Q as vertices is equal to π up to ε.*

The proof will use the projection method of which some details will be explained in Section 4.

The proof will be given with total neglect of singular tilings. In order to give the singular tilings the extra attention they need, one might use a result like this: for any singular Penrose pattern π and any positive number δ there is a nonsingular Penrose pattern π_1 such that π and π_1 are equal up to δ. That can be proved again by the projection method.

4. Preparations

The fifth root of unity $\exp(2\pi i/5)$ will be denoted by ζ. By $\mathscr{Z}[\zeta]$ we denote the set of all numbers of the form

$$\sum_{j=0}^{4} k_j\zeta^j \tag{4.1}$$

with integers $k_0, \ldots, k_4$.

We shall distinguish the notions 'Penrose pattern', 'AR-pattern' and 'AR*-pattern'. An AR-pattern (the AR stands for 'arrowed rhombus') is a Penrose pattern with all its vertices in the set $\mathscr{Z}[\zeta]$. Actually, if one of its vertices is in that set, then all of them are, because of the condition of length and direction of the edges in a Penrose pattern.

Any Penrose pattern can be turned into an AR-pattern by means of shift. Taking any vertex z_0 and subtracting it from all other vertices we get an AR-pattern.

It was shown [1, Section 6], that for any AR-pattern the sum $k_0 + \cdots + k_4$ takes its values from a set of four consecutive residue classes mod 5. If this set is the set $\{1, 2, 3, 4\}$ then the pattern is called an AR*-pattern. For any AR-pattern there is an integer q such that a horizontal shift over a distance q turns it into an AR*-pattern. This q is uniquely determined mod 5. If we just take any q which is not in that set of four consecutive residue classes, and subtract it from all the vertices of the AR-pattern, we get an AR*-pattern.

The k_j are not uniquely determined by the value of (4.1). If we add 1 to each one of $k_0, \ldots, k_4$ simultaneously, there is no effect on the value of the sum (4.1), and that is all the freedom we have.

According to [1, Section 15], any regular AR*-pattern can be described by means of a pentagrid with a set of five real parameters $\gamma_0, \ldots, \gamma_4$ satisfying $\gamma_0 + \cdots + \gamma_4 = 0$. The vertices of the pattern all have the form (4.1), where the set of all admitted integral vectors $(k_0, \ldots, k_4)$ is obtained as follows. Writing $k = k_0 + \cdots + k_4$, the vector is to be restricted by $k \in \{1, 2, 3, 4\}$ and

$$\sum_{j=0}^{4} (k_j - \gamma_j)\zeta^{2j} \in V_k \quad (1 \leqslant k \leqslant 4), \tag{4.2}$$

where the $V_1, \ldots, V_4$ are pentagon-shaped disks in the complex plane. V_1 is the interior of the pentagon with vertices $1, \zeta, \zeta^2, \zeta^3, \zeta^4$. Furthermore $V_2 = (\zeta^2 + \zeta^3)V_1$ (that means that the elements of V_2 are obtained from those of V_1 upon multiplication by $\zeta^2 + \zeta^3$), and similarly $V_3 = -(\zeta^2 + \zeta^3)V_1$, $V_4 = -V_1$.

This description of the AR*-pattern is called the *projection* method since it can be interpreted as projecting all integral vectors $(k_0, \ldots, k_4)$ of the five-dimensional space R^5 onto a certain 3-dimensional subspace S, and taking only those vectors $(k_0, \ldots, k_4)$ of R^5 for which the projection falls into a particuar subset of S. That subset (in general it is called the *window*) is a polytope W

obtained by projection of the interior of a unit cube in R^5 onto R. In particular this projection has the effect that each hyperplane of constant sum $k_0 + \cdots + k_4$ is projected into a 2-dimensional subspace of S. We have to consider only four values of that sum, so all the projections that fall into W actually fall into one of four parallel planar slices through W, and these have the form of pentagons. We refer to those as 'pancakes'. We can rotate S such that the pancakes become horizontal and that the horizontal projections of the pancakes become the pentagons $V_1, \ldots, V_4$.

With fixed $\gamma_0, \ldots, \gamma_4$ (with zero sum) the projections are everywhere dense in the pancakes. In other words, for every k $(1 \leq k \leq 4)$ the points $\sum_{j=0}^{4} (k_j - \gamma_j)\zeta^{2j}$ lie everywhere dense in V_k. This was stated without proof at the end of Section 8 in [1], but a proof is simple. If q is any integer, we show that the set of all $\sum_{j=0}^{4} k_j \zeta^{2j}$ with $k_0 + \cdots + k_4 = q$ lies everywhere dense in the complex plane. If we subtract q from each one of those points, we get the set of all integral linear combinations of $1 - \zeta^2$, $1 - \zeta^4$, $1 - \zeta^6$, $1 - \zeta^8$. Therefore it suffices to remark that the real numbers $1 - \zeta^2 + 1 - \zeta^8$ and $1 - \zeta^4 + 1 - \zeta^6$ are incommensurable, and that the same thing holds for the purely imaginary numbers $(1 - \zeta^2) - (1 - \zeta^8)$ and $(1 - \zeta^4) - (1 - \zeta^6)$.

But the projections are not just everywhere dense, they are also *uniformly distributed* over the pancakes. We shall explain what that means.

Let ϕ be any function which is continuous over the whole complex plane. We assume it has a bounded carrier, i.e., that there is a positive number M such that $\phi(z) = 0$ for all z with $|z| > M$. Let $I(\phi)$ be the integral of ϕ over the complex plane. And let q be some integer. With this ϕ and q fixed, we have the following lemma.

Lemma 1. *For any positive number ε there exists a positive number R such that for any circular disk D with radius $>R$ and area $A(D)$ we have*

$$\sum_{(k_0, \ldots, k_4)} \phi\Big(\sum_{j=0}^{4} (k_j - \gamma_j)\zeta^{2j}\Big) - 4 \cdot 5^{-\frac{5}{2}} \cdot A(D) I(\phi) \,\Big| < \varepsilon A(D)$$

where the outer summation runs over all vectors $(k_0, \ldots, k_4)$ with

$$\sum_{j=0}^{4} k_j \zeta^j \in D, \quad k_0 + \cdots + k_4 = q.$$

It would be beyond the scope of this paper to give more than an outline of a proof. Apart from some integration theory it uses the following result on the number of lattice points inside some big regions.

Take integers $m > 0$, $n > 0$, and $m + n$ linearly independent vectors $v_1, \ldots, v_{m+n}$ in the $(m+n)$-dimensional space R^{m+n}. Let Δ be the absolute value of the determinant of those v's. R^{m+n} is the Cartesian product of R^m and R^n. The projections onto R^m and R^n are denoted by Π_1 and Π_2, respectively.

The sums $k_1v_1 + \cdots + k_{m+n}v_{m+n}$ with integer coefficients $k_1, \ldots, k_{m+n}$ form a lattice Λ. We assume that the vectors v are such that the projections

$$\Pi_2(k_1v_1 + \cdots + k_{m+n}v_{m+n})$$

are everywhere dense in R^n. Let Q be a parallelopiped in R^n with volume $V_n(Q) > 0$. With Λ and Q fixed, we can now state that for every positive number ε there exists a positive number R such that for any sphere S in R^m with radius $>R$ and volume $V_m(S)$ we have

$$|N(S \times Q) - V_m(S)V_n(Q)/\Delta| < \varepsilon V_m(S),$$

where $N(S \times Q)$ stands for the number of points of Λ inside the Cartesian product $S \times Q$.

We need Lemma 1 in the proof of Lemma 2.

Lemma 2. *There is a positive constant β such that the following is true. Let $\pi^{(1)}$ be an AR^*-pattern generated by $\gamma_0^{(1)}, \ldots, \gamma_4^{(1)}$ (with $\gamma_0^{(1)} + \cdots + \gamma_4^{(1)} = 0$). Then for every $\varepsilon > 0$ there is a subset E of the vertex set of $\pi^{(1)}$ such that for all $\gamma_0^{(2)}, \ldots, \gamma_4^{(2)}$ with $\gamma_0^{(2)} + \cdots + \gamma_4^{(2)} = 0$ the condition $|\xi^{(1)} - \xi^{(2)}| < \beta\varepsilon$ (where $\xi^{(\nu)} = \sum_{j=1}^4 \gamma_j^{(\nu)}\zeta^{2j}$) guarantees that E is an ε-complete subset of the vertex set of the AR^*-pattern $\pi^{(2)}$ generated by $\gamma_0^{(2)}, \ldots, \gamma_4^{(2)}$.*

Here is a sketch of a proof. Let r be the radius of the inscribed circle of V_1, and require that β anyway satisfies $\beta\varepsilon < r$. Now assuming $|\xi^{(1)} - \xi^{(2)}| < \beta\varepsilon$ we know that if $w - \xi^{(1)}$ lies in $(1 - \beta\varepsilon/r)V_k$ then $w - \xi^{(2)}$ lies in V_k, and if $w - \xi^{(2)}$ lies in V_k then $w - \xi^{(1)}$ lies in $(1 + \beta\varepsilon/r)V_k$.

If ξ is a complex number and λ a positive real number, we define $H(\xi, \lambda)$ as the set of all numbers $\sum_{j=0}^4 k_j\zeta^j$ with

$$\left(\sum_{j=0}^{4} k_j\zeta^{2j}\right) - \xi \in \lambda V_k,$$

where $k = k_0 + \cdots + k_4$. So (4.2) says that $H(\xi^{(j)}, 1)$ is the vertex set of $\pi^{(j)}$ $(j = 1, 2)$. We define E by $E = H(\xi^{(1)}, 1 - \beta\varepsilon/r)$. The vertex set S_2 of $\pi^{(2)}$ obviously satisfies $E \subset S_2 \subset H(\xi^{(1)}, 1 + \beta\varepsilon/r)$.

Denote by V_k^* the set of points which are both inside $(1 + \beta\varepsilon/r)V_k$ and outside $(1 - \beta\varepsilon/r)V_k$. The area of V_k^* is small if ε is small. So we can find a continuous function ϕ which equals 1 on V_k^*, is nonnegative everywhere else, and which has a small integral. Therefore it is not hard to see that we can select a suitable value of β such that Lemma 1 leads to a proof of Lemma 2.

An alternative proof can be given on the basis of the notion of the *tolerance* of a mesh in a pentagrid (see the beginning of Section 17 in [2]). Small changes in the parameters of a pentagrid affect the topology of the grid only as far as meshes with small tolerance are concerned, and in every large circle the number of such meshes is small compared to the total number of meshes.

5. Proof of the theorem

In the union of the sets $V_1, \ldots, V_4$ the leftmost point is $\zeta^2+\zeta^3$ (it lies in V_2, but not in V_1, V_3, V_4) and the rightmost one is $-(\zeta^2+\zeta^3)$ (in V_3, but not in V_1, V_2, V_4). So if two points in that union have a distance close to $1+\sqrt{5}$ (the value of $-2(\zeta^2+\zeta^3)$) then we know they belong to V_2 and V_3, respectively. We express this as a lemma.

Lemma 3. *Assume η satisfies $0<\eta<(1+\sqrt{5})/2$. Let $a \in \{1, 2, 3, 4\}$, $b \in \{1, 2, 3, 4\}$, and let the complex numbers $w^{(1)}$, $w^{(2)}$ satisfy $w^{(1)} \in V_a$, $w^{(2)} \in V_b$. Finally assuming that*

$$\operatorname{Re}(w^{(1)}-w^{(2)})>1+\sqrt{5}-\eta \tag{5.1}$$

we have $a=3$, $b=2$ and

$$|w^{(1)}+\zeta^2+\zeta^3|<\eta/(\cos 54°), \qquad |w^{(2)}-\zeta^2-\zeta^3|<\eta/(\cos 54°). \tag{5.2}$$

We can now start proving the theorem. Any Penrose pattern can be turned into an AR*-pattern by a shift. And if two Penrose patterns have a vertex P in common, then a shift that turns one of them into an AR*-pattern will turn the other one at least into an AR-pattern. So it suffices to show that for any AR*-pattern π and for every positive number ε we can find two vertices P and Q of π such that for all AR-patterns π_1 we have: if π_1 has both P and Q as vertices then π and π_1 are equal up to ε.

According to the end of Section 3 we shall restrict ourselves to nonsingular π and π_1.

We abbreviate $\eta=\beta\varepsilon(\cos 54°)/2$, where β is the number mentioned in Lemma 2.

According to [1] Section 15, π is obtained from a pentagrid. Let $\gamma_0, \ldots, \gamma_4$ be the parameters. Since the projections are everywhere dense in the pancakes, we can find integral vectors $(k_0^{(1)}, \ldots, k_4^{(1)})$, $(k_0^{(2)}, \ldots, k_4^{(2)})$ with $k_0^{(1)}+\cdots+k_4^{(1)}=3$ and $(k_0^{(2)}+\cdots+k_4^{(2)})=2$ such that $\operatorname{Re}(z^{(1)}-z^{(2)})>1+\sqrt{5}-\eta$ holds with

$$z^{(1)}=\sum_{j=0}^{4}(k_j^{(1)}-\gamma_j)\zeta^{2j}, \qquad z^{(2)}=\sum_{j=0}^{4}(k_j^{(2)}-\gamma_j)\zeta^{2j},$$

and $z^{(1)} \in V_3$, $z^{(2)} \in V_2$. The points P and Q mentioned in the theorem will be taken as

$$P=\sum_{j=0}^{4}k_j^{(1)}\zeta^j, \qquad Q=\sum_{j=0}^{4}k_j^{(2)}\zeta^j.$$

Let π_1 be any AR-pattern that has both P and Q as vertices. We have to show that π and π_1 are equal up to ε. We first show that π_1 is an AR*-pattern. According to Section 4, we can find an integer q such that a horizontal shift over a distance q turns π_1 into an AR*-pattern π_1^*. Let $\gamma_0^*, \ldots, \gamma_4^*$ be parameters for π_1^*. The points $P+q$ and $Q+q$ are described by vectors $(h_0^{(1)}, \ldots, h_4^{(1)})$,

$(h_0^{(2)}, \ldots, h_4^{(2)})$ where

$$h_j^{(1)} = k_j^{(1)} + q\delta_{j,0}, \qquad h_j^{(2)} = k_j^{(2)} + q\delta_{j,0},$$

and $\delta_{j,0} = 1$ or 0 according to $j = 0$ or $\neq 0$. With

$$w^{(1)} = \sum_{j=0}^{4} (h_j^{(1)} - \gamma_j^*)\zeta^{2j}, \qquad w^{(2)} = \sum_{j=0}^{4} (h_j^{(2)} - \gamma_j^*)\zeta^{2j},$$

we have $\mathrm{Re}(w^{(1)} - w^{(2)}) = \mathrm{Re}(z^{(1)} - z^{(2)}) > 1 + \sqrt{5} - \eta$. Moreover, $w^{(1)} \in V_a$, $w^{(2)} \in V_b$ with $a = h_0^{(1)} + \cdots + h_4^{(1)}$, $b = h_0^{(2)} + \cdots + h_4^{(2)}$. Lemma 3 now shows that $a = 3$, which leads to $q = 0$. It follows that π_1 is an AR*-pattern.

Applying (5.2) twice we find $|z^{(1)} + \zeta^2 + \zeta^3| < \eta/(\cos 54°)$, $|w^{(1)} + \zeta^2 + \zeta^3| < \eta/(\cos 54°)$. So with $\xi = \sum_{j=0}^4 \gamma_j \zeta^{2j}$, $\xi^* = \sum_{j=0}^4 \gamma_j^* \zeta^{2j}$ we have $|\xi - \xi^*| < 2\eta/(\cos 54°) = \beta\varepsilon$. The theorem now follows from Lemma 2.

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