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Hopf algebras of dimension pq, II

Siu-Hung Ng [1]

Department of Mathematics, Iowa State University, Ames, IA 50011, USA

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Abstract

Let H be a Hopf algebra of dimension pq over an algebraically closed field of characteristic zero, where p, q are odd primes with $p < q \leqslant 4p + 11$. We prove that H is semisimple and thus isomorphic to a group algebra, or the dual of a group algebra.

Keywords: Hopf algebra; Frobenius algebra; Projective cover

Introduction

In recent years, there has been some progress on the classification of finite-dimensional Hopf algebras of dimension pq over an algebraically closed field $\Bbbk$ of characteristic 0, where p, q are prime numbers. The case for $p = q$ has been settled completely in [N1,N3] and [Ma]. They are group algebras and Taft algebras of dimension p^2 (cf. [T]). However, the classification for the case $p \neq q$ remains open in general.

In the works [EG1] and [GW], the semisimple case of the problem has been solved; namely, any semisimple Hopf algebra of dimension pq is isomorphic to a group algebra or the dual of a group algebra. It is natural to ask whether Hopf algebras of dimension pq, where $p < q$ are prime numbers, are always semisimple. The question has been answered affirmatively in some specific low dimensions. Williams settled the dimensions 6 and 10 in [W], Andruskiewitsch and Natale did dimensions 15, 21, and 35 [AN], and Beattie and Dăscălescu did dimensions 14, 55, 65, and 77 [BD].

E-mail address: rng@iastate.edu.

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More recently, the author has proved that if p, q are twin primes, or $p = 2$, then Hopf algebras of dimension pq are semisimple (cf. [N2,N4]). Meanwhile, Etingof and Gelaki proved the same result for $2 < p < q \leqslant 2p + 1$ by considering the indecomposable projective modules over these Hopf algebras (cf. [EG2]).

In this paper, we will study the indecomposable modules over these Hopf algebras. We prove that if p, q are odd primes and $p < q \leqslant 4p + 11$, then every Hopf algebra of dimension pq over $\Bbbk$ is semisimple (Theorem 4.1). The result covers all odd dimensions listed above.

The organization of the paper is as follows: we begin with some notations and preliminary results for modules over a finite-dimensional Hopf algebra in Section 1. In Section 2, we obtain a lower bound for the dimensions of certain indecomposable modules over a non-semisimple Hopf algebra H of dimension pq. In Section 3, we further assume H is not unimodular, and consider the action of the group $G(H^*)$ of all group-like elements of H^* on the set $\mathrm{Irr}(H)$ of isomorphism classes of simple H-modules. We obtain some lower bounds for the number of $G(H^*)$-orbits in $\mathrm{Irr}(H)$. We finally prove our main result in Section 4.

Throughout this paper, the base field $\Bbbk$ is always assumed to be algebraically closed of characteristic zero and the tensor product $\otimes$ means $\otimes_{\Bbbk}$, unless otherwise stated. The notation introduced in Section 1 will continue to be used in the remaining sections. The readers are referred to [Mo] and [S] for elementary properties of Hopf algebras.

1. Notation and preliminaries

Let H be a finite-dimensional Hopf algebra over $\Bbbk$ with antipode S, counit ϵ. There are natural actions $\rightharpoonup$ and $\leftharpoonup$ of H^* on H given by

$$f \rightharpoonup a = a_1 f(a_2) \quad \text{and} \quad a \leftharpoonup f = f(a_1)a_2$$

where $\Delta(a) = a_1 \otimes a_2$ is Sweedler's notation with the summation suppressed. For $f \in H^*$, we define the $\Bbbk$-linear endomorphism $L(f)$ and $R(f)$ on H by

$$L(f)(a) = f \rightharpoonup a \quad \text{and} \quad R(f)(a) = a \leftharpoonup f.$$

A non-zero element a of H is said to be group-like if $\Delta(a) = a \otimes a$, and we denote by $G(H)$ the set of all group-like elements of H. For $\beta \in G(H^*)$, β is an algebra epimorphism from H onto $\Bbbk$. The associated maps $L(\beta)$, $R(\beta)$ are algebra automorphisms of H, and they commute with S^2. Moreover, each $\beta \in G(H^*)$ is a degree 1 irreducible character of H. We will denote by $\Bbbk_\beta$ the 1-dimensional H-module which affords the character β. In particular, $\Bbbk_\epsilon$ is the trivial H-module $\Bbbk$.

Let Λ be a non-zero left integral of H. The distinguished group-like element α of H^* is defined by $\Lambda a = \alpha(a)\Lambda$ for $a \in H$. Similarly, if $\lambda \in H^*$ is a non-zero right integral of H^*, the distinguished group-like element g of H is defined by $f * \lambda = f(g)\lambda$ for all $f \in H^*$. In this convention, the celebrated Radford formula [R1] is given by

$$S^4(h) = g\big(\alpha \rightharpoonup h \leftharpoonup \alpha^{-1}\big)g^{-1} \quad \text{for } h \in H. \tag{1.1}$$

The non-zero right integral λ defines a non-degenerate associative bilinear form on H, and so H is a Frobenius algebra. By [R2],

$$\lambda(ab) = \lambda\big(\theta(b)a\big) \quad \text{for } a, b \in H, \tag{1.2}$$

where $\theta(b) = S^2(b \leftharpoonup \alpha)$. Therefore, θ^{-1} is the associated *Nakayama automorphism*.

Recall that the *left dual* $V^\vee$ of an H-module V is the left H-module with the underlying space $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ and the H-action given by

$$(hf)(x) = f\big(S(h)x\big) \quad \text{for } x \in V,\ f \in V^*.$$

Similarly, the *right dual* ${}^\vee V$ of V is the left H-module with the same underlying space V^* and the H-action given by

$$(hf)(x) = f\big(S^{-1}(h)x\big) \quad \text{for } x \in V,\ f \in V^*.$$

Given an algebra automorphism σ on H, one can *twist* the action of an H-module V by σ to obtain another H-module ${}_\sigma V$. More precisely, ${}_\sigma V$ is an H-module on V with the action given by

$$h \cdot_\sigma v = \sigma(h)v \quad \text{for } h \in H,\ v \in V.$$

${}_\sigma(-)$ defines a $\Bbbk$-linear equivalence on the category H-$\mathbf{mod}_{\mathsf{fin}}$ of finite-dimensional left H-modules. In particular, if σ is an inner automorphism of H, then ${}_\sigma(-)$ is $\Bbbk$-linearly equivalent to the identity functor.

Using the above notation, one can easily see that $\rho : \Bbbk_\beta \otimes V^{\vee\vee} \to {}_{S^2 \circ R(\beta)}V$ defined by

$$\rho(1 \otimes \hat{v}) = v \tag{1.3}$$

is a natural isomorphism of H-modules for $V \in H$-$\mathbf{mod}_{\mathsf{fin}}$ and $\beta \in G(H^*)$, where $\hat{v} \in V^{**}$ denotes the natural image of the element $v \in V$. In particular, we have the H-module isomorphisms

$$V^{\vee\vee} \cong {}_{S^2}V \quad \text{and} \quad \Bbbk_\alpha \otimes V^{\vee\vee} \cong {}_\theta V.$$

Similarly, one also has the H-module isomorphisms

$${}^{\vee\vee}V \cong {}_{S^{-2}}V \quad \text{and} \quad \Bbbk_{\alpha^{-1}} \otimes {}^{\vee\vee}V \cong {}_{\theta^{-1}}V.$$

By Radford's antipode formula (1.1), we also have

$$\Bbbk_\alpha \otimes V^{\vee\vee\vee\vee} \otimes \Bbbk_{\alpha^{-1}} \cong \Bbbk_\alpha \otimes {}_{S^4}V \otimes \Bbbk_{\alpha^{-1}} \cong V. \tag{1.4}$$

It follows from a property of Frobenius algebras that an H-module is projective if, and only if, it is injective. Each indecomposable projective H-module P is isomorphic to He for some primitive idempotent $e \in H$. Moreover, the socle $\mathrm{Soc}(P)$ and the head $\mathrm{Head}(P) = P/JP$ of P are simple (cf. [CR, IX]). Let us denote the projective cover and the injective envelope of an H-module V by $P(V)$ and $E(V)$ respectively. If V is a simple H-module, then $\overline{V} = \mathrm{Soc}(P(V))$ is also simple, and so $P(V) \cong E(\overline{V})$. The assignment of simple H-module $V \mapsto \mathrm{Soc}(P(V))$ defines a permutation on a complete set of non-isomorphic simple H-modules, and its inverse π

is called the *Nakayama permutation* (cf. [La, §16A]), i.e. $P(\pi(V)) \cong E(V)$. By [La, §16C], if $P(V) \cong He$ for some primitive idempotent $e \in H$, we have

$$P\big(\pi(V)\big) \cong H\theta^{-1}(e) \tag{1.5}$$

and so $V \cong \mathrm{Soc}(H\theta^{-1}(e))$. This allows us to rewrite the Nakayama permutation as in the following lemma.

Lemma 1.1. *Let H be a finite-dimensional Hopf algebra over $\Bbbk$ with distinguished group-like element $\alpha \in H^*$, and V a simple H-module. Then $\pi(V) \cong {}_\theta V$ and hence*

$$\Bbbk_{\alpha^{-1}} \otimes {}^{\vee\vee}V \cong \mathrm{Soc}\big(P(V)\big) \quad \textit{and} \quad V \cong \mathrm{Soc}\big(P\big(\Bbbk_\alpha \otimes V^{\vee\vee}\big)\big). \tag{1.6}$$

Moreover, V is projective if, and only if, $\dim P(V) < 2\dim V$. *In this case, $V \cong \Bbbk_\alpha \otimes V^{\vee\vee}$ as H-modules.*

Proof. Let e be a primitive idempotent of H such that $\mathrm{Head}(He) \cong V$. Note that ${}_\theta(He) \cong H\theta^{-1}(e)$ as H-modules under the map $he \mapsto \theta^{-1}(he)$. By (1.5),

$$\pi(V) \cong \mathrm{Head}\, {}_\theta(He) \cong {}_\theta V.$$

Since ${}_\theta V \cong \Bbbk_\alpha \otimes V^{\vee\vee}$, the second isomorphism of (1.6) follows immediately from the definition of Nakayama permutation. Since $\Bbbk_{\alpha^{-1}} \otimes {}^{\vee\vee}(\Bbbk_\alpha \otimes V^{\vee\vee}) \cong V$, we have $\pi^{-1}(V) \cong \Bbbk_{\alpha^{-1}} \otimes {}^{\vee\vee}V$ and hence $\Bbbk_{\alpha^{-1}} \otimes {}^{\vee\vee}V \cong \mathrm{Soc}(P(V))$.

If V is not projective, then

$$\dim P(V) \geqslant \dim \mathrm{Soc}\big(P(V)\big) + \dim \mathrm{Head}\big(P(V)\big) = 2\dim V.$$

Obviously, if V is projective, then $P(V) = V$ and hence $\dim P(V) < 2\dim V$. In this case, $\Bbbk_\alpha \otimes V^{\vee\vee}$ is also projective. Therefore, we have

$$V \cong \mathrm{Soc}\, P\big(\Bbbk_\alpha \otimes V^{\vee\vee}\big) \cong \Bbbk_\alpha \otimes V^{\vee\vee}. \qquad \square$$

Lemma 1.2. *Let $V \in H$-$\mathbf{mod}_{\mathsf{fin}}$ such that $V \cong \Bbbk_\beta \otimes V^{\vee\vee}$ for some non-trivial $\beta \in G(H^*)$. Then* $\mathrm{Tr}(\tau) = 0$ *for $\tau \in \mathrm{Hom}_H(V, {}_\sigma V)$, where $\sigma = S^2 \circ R(\beta)$.*

Proof. Note that the evaluation map $\mathrm{ev} : V^\vee \otimes V \to \Bbbk$ and the diagonal basis map $\mathrm{db} : \Bbbk \to V \otimes V^\vee$ defined by $\mathrm{db}(1) = \sum_i x_i \otimes x^i$ are H-module homomorphisms, where $\{x_i\}$ is a basis for V and $\{x^i\}$ is the dual basis for V^*. Consider the composition

$$\Bbbk \xrightarrow{\mathrm{db}} V \otimes V^\vee \xrightarrow{\tau\otimes\mathrm{id}} {}_\sigma V \otimes V^\vee \xrightarrow{\rho^{-1}\otimes\mathrm{id}} \Bbbk_\beta \otimes V^{\vee\vee} \otimes V^\vee \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} \Bbbk_\beta \tag{1.7}$$

of H-module maps, where ρ is defined in (1.3). The composition is a scalar given by

$$\sum_i x^i\big(\tau(x_i)\big) = \mathrm{Tr}(\tau).$$

Since $\mathbb{k}$ and $\mathbb{k}_\beta$ are not isomorphic as H-modules, the composition is a zero map and hence

$$\mathrm{Tr}(\tau) = 0. \qquad \square$$

The composition displayed in (1.7) may not be zero if β is trivial. However, if H is not semisimple and V is projective, then such composition must be zero. For otherwise, $\mathrm{db} : \mathbb{k} \to V \otimes V^\vee$ is a split embedding and hence $\mathbb{k}$ is a direct summand of the projective H-module $V \otimes V^\vee$. This implies $\mathbb{k}$ is projective. However, $\mathbb{k}$ is not projective if H is not semisimple (cf. [EG2, Lemma 2.2]). The reason for this observation has been presented in the proofs of [EG2, Lemma 2.11], [EO, Theorem 2.16] and [Lo, Theorem 2.3 (b)]. We state this conclusion as

Lemma 1.3. *Let V be a projective module over a non-semisimple finite-dimensional Hopf algebra H such that $V \cong V^{\vee\vee}$. Then for all homomorphism $\phi : V \to {}_{S^2}V$ of H-modules,* $\mathrm{Tr}(\phi) = 0$.

We close this section with a generalization of [EG2, Proposition 2.5].

Lemma 1.4. *Let β be a group-like element of H^*. Suppose that $V \in H$-$\mathbf{mod}_{\mathsf{fin}}$ is indecomposable and $\mathbb{k}_\beta \otimes V \cong V$. Then* $\mathrm{ord}(\beta) \mid \dim V$.

Proof. Note that $\mathbb{k}_\beta \otimes V \cong {}_{R(\beta)}V$ as H-modules. Let $n = \mathrm{ord}(\beta)$ and $\eta : {}_{R(\beta)}V \to V$ an isomorphism of H-modules. Since $R(\beta)^n = \mathrm{id}_H$, $R(\beta)$ is diagonalizable. Suppose $h \in H$ is an eigenvector of $R(\beta)$. Then $R(\beta)(h) = \omega h$ for some nth root of unity $\omega \in \mathbb{k}$, and

$$\omega\eta(hv) = \eta\big(R(\beta)(h)v\big) = h\eta(v)$$

for all $v \in V$. Thus, η^n is an H-module automorphism on V. Since V is finite-dimensional and indecomposable, $\mathrm{End}_H(V)$ is a finite-dimensional local algebra over $\mathbb{k}$ (cf. [P]). Since $\mathbb{k}$ is algebraically closed, $\eta^n = c \cdot \mathrm{id}_V$ for some $c \in \mathbb{k}$. By dividing η with an nth root of c, one may assume $\eta^n = \mathrm{id}_V$. Then V is a left $\mathbb{k}[\beta]$-module with the action given by

$$\beta v = \eta(v), \quad v \in V.$$

Define the right H^*-comodule structure $\rho : V \to V \otimes H^*$, $\rho(v) = \sum v_0 \otimes v_1$ by the equation

$$hv = \sum v_0 v_1(h) \quad \text{for all } h \in H.$$

It is straightforward to check that $\rho(\beta v) = \beta\rho(v)$ for $v \in V$, and so $V \in {}_{\mathbb{k}[\beta]}\mathcal{M}^{H^*}$. By the Nichols–Zoeller theorem, V is a free $\mathbb{k}[\beta]$-module. In particular, $\mathrm{ord}(\beta)$ divides $\dim V$. $\square$

2. Non-semisimple Hopf algebras of dimension pq

Throughout the remaining discussion, we will assume H to be a non-semisimple Hopf algebra over $\mathbb{k}$ of dimension pq, where p, q are primes and $2 < p < q$. The antipode of H will continue to be denoted by S. By [LR1], H^* is also a non-semisimple Hopf algebra of dimension pq with antipode S^*.

In this section, we will obtain a lower bound for the dimensions of certain indecomposable H-modules V which satisfy $V \cong \Bbbk_\beta \otimes V^{\vee\vee}$ for some $\beta \in G(H^*)$ (Corollary 2.3). We begin with the following lemma:

Lemma 2.1. *H, p, q, S as above. Then*:

(i) *H or H^* is not unimodular.*
(ii) $|G(H)| = 1$ *or* p.
(iii) $\operatorname{ord} S^2 = 2p$.
(iv) *Let C be a subcoalgebra of H invariant under S^{2p} and $C \nsubseteq \Bbbk G(H)$. Then $S^{2p}|_C \neq \mathrm{id}_C$.*
(v) *Let V be an indecomposable H-module with* $\dim V > 1$. *Then there exist $h \in H$, $v \in V$ such that $hv \neq S^{2p}(h)v$.*
(vi) *There exists a simple H-module V such that*

$$\dim V > 1 \quad \textit{and} \quad \dim P(V) \geqslant 2 \dim V.$$

Proof. (i) Since $\dim H$ is odd, it follows from [LR2, Theorem 2.2] that H or H^* is not unimodular.

(ii) follows immediately from [N1, Proposition 5.2].

(iii) By [N1, Proposition 5.2 and Theorem 5.4], $\operatorname{ord} S^4 = p$ and $\operatorname{Tr}(S^{2p}) = p^2 d$ for some odd integer d. If $S^{2p} = \mathrm{id}$, then $pd = q$ which contradicts that p, q are distinct primes. Therefore, $\operatorname{ord} S^2 = 2p$ and this proves (iii).

(iv) Let B be the subalgebra of H generated by C. Then B is a bialgebra and hence a Hopf subalgebra of H. In particular, $\dim B = 1, p, q$ or pq. If $\dim B \in \{1, p, q\}$, then B is a group algebra (cf. [Z]) and hence $C \subseteq B \subseteq \Bbbk G(H)$. The assumption on C forces $\dim B = pq$ or equivalently $B = H$. If $S^{2p}|_C = \mathrm{id}_C$, then $S^{2p}|_B = \mathrm{id}_B$ and this contradicts (iii). Therefore, $S^{2p}|_C \neq \mathrm{id}_C$.

(v) Suppose that $hv = S^{2p}(h)v$ for all $h \in H$ and $v \in V$. Let $I \subseteq H$ be the annihilator of V, and $\pi : H \to H/I$ the natural surjection. Then $S^{2p}(h) - h \in I$ for $h \in H$, and so $\pi \circ S^{2p} = \pi$. Hence we have the commutative diagram of coalgebra maps:

$$\begin{tikzcd} & H^* \arrow[dd, "(S^{2p})^*=(S^*)^{2p}."] \\ (H/I)^* \arrow[ur, "\pi^*"] \arrow[dr, "\pi^*"'] & \\ & H^* \end{tikzcd}$$

Let $C = \pi^*(H/I)^*$. Then the commutative diagram implies that $(S^*)^{2p}(C) = C$ and $(S^*)^{2p}|_C = \mathrm{id}_C$. Since V is an indecomposable H/I-module and $\dim V > 1$, H/I is not a commutative semisimple algebra. Hence C is not cosemisimple cocommutative. In particular, $C \nsubseteq \Bbbk G(H^*)$. However, this contradicts (iv).

(vi) Suppose the statement is false. Then, by Lemma 1.1, every simple H-module V with $\dim V > 1$ is projective. Thus the composition factors of $P(\Bbbk)$ are all 1-dimensional. By [EG2, Lemma 2.3], $P(\Bbbk)$ is a direct sum of 1-dimensional H-modules. Hence $\Bbbk = P(\Bbbk)$. However, this contradicts [EG2, Lemma 2.2] that $\Bbbk$ is not projective. $\square$

Lemma 2.2. *Let $\sigma = S^2 \circ R(\beta)$ for some $\beta \in G(H^*)$, and let $V \in H$-$\mathbf{mod}_{\mathsf{fin}}$ be indecomposable with* $\dim V > 1$ *and $V \cong {}_\sigma V$. Then*:

(i) *For any isomorphism $\phi : V \to {}_\sigma V$ of H-modules, $\phi^p \neq \mathrm{id}$.*
(ii) *There exists an H-module isomorphism $\psi : V \to {}_\sigma V$ such that $\psi^{2p} = \mathrm{id}$ and $\mathrm{Tr}(\psi^p)$ is a non-negative integer.*

Proof. (i) By Lemma 2.1(ii), $\sigma^p = S^{2p}$. Suppose there exists an H-module map $\phi : V \to {}_\sigma V$ such that $\phi^p = \mathrm{id}$. Then for all $h \in H$ and $v \in V$,

$$hv = \phi^p(hv) = \sigma^p(h)\phi^p(v) = \sigma^p(h)v = S^{2p}(h)v.$$

This contradicts Lemma 2.1(v).

(ii) Let $\phi : V \to {}_\sigma V$ be an isomorphism of H-modules. Then for all $h \in H$ and $v \in V$, we have

$$hv = \phi^{2p}(hv) = \sigma^{2p}(h)\phi^{2p}(v).$$

By Lemma 2.1(iii), $\sigma^{2p} = S^{4p} = \mathrm{id}$, so ϕ^{2p} is an H-module automorphism on V. Since V is indecomposable, $\mathrm{End}_H(V)$ is a finite-dimensional local $\Bbbk$-algebra (cf. [P]). In particular, $\phi^{2p} = c\,\mathrm{id}_V$ for some non-zero $c \in \Bbbk$. Let $t \in \Bbbk$ be a $2p$th root of c and $\overline{\phi} = t^{-1}\phi$. Then $\overline{\phi}$ is also an isomorphism of H-module from V to ${}_\sigma V$ and $(\overline{\phi})^{2p} = \mathrm{id}$. In particular, $\mathrm{Tr}(\overline{\phi}^p)$ is an integer. Set $\psi = -\overline{\phi}$ if $\mathrm{Tr}(\overline{\phi}^p) < 0$, and $\psi = \overline{\phi}$ otherwise. Then ψ is a required isomorphism. □

We close this section with the following corollary which is an enhanced result of [EG2, Lemma 2.11].

Corollary 2.3. *Let V be an indecomposable H-module of odd dimension which satisfies one of the following conditions*:

(I) *$V \cong \Bbbk_\beta \otimes V^{\vee\vee}$ for some non-trivial element $\beta \in G(H^*)$,*
(II) *$V \cong V^{\vee\vee}$ and V is projective.*

Then $\dim(V) \geqslant p + 2$.

Proof. Notice that any 1-dimensional H-module does not satisfy (I) or (II). Therefore, if V satisfies condition (I) or (II), then $\dim V > 1$. Since $V \cong \Bbbk_\beta \otimes V^{\vee\vee}$, $V \cong {}_\sigma V$ where $\sigma = S^2 \circ R(\beta)$. By Lemma 2.2, there exists an H-module isomorphism $\psi : V \to {}_\sigma V$ such that $\psi^{2p} = \mathrm{id}_V$ and $\mathrm{Tr}(\psi^p) \geqslant 0$. Since V satisfies (I) or (II), it follows from Lemmas 1.2 and 1.3 that $\mathrm{Tr}(\psi) = 0$. By a linear algebra argument, $\mathrm{Tr}(\psi^p) = pd$ for some non-negative odd integer d (cf. [N1, Lemma 1.3]). This forces $\dim V \geqslant p$. Suppose $\dim V = p$. Then $\mathrm{Tr}(\psi^p) = p$ and hence $\psi^p = \mathrm{id}_V$. However, this contradicts Lemma 2.2(i). Therefore, $\dim V > p$. □

3. Orbits of simple modules

We continue to assume that H is a non-semisimple Hopf algebra of dimension pq with p, q prime and $2 < p < q$. By duality and Lemma 2.1(i), *we can further assume that the distinguished* *group-like element $\alpha \in H^*$ is non-trivial*. Hence, by Lemma 2.1(ii), $\langle\alpha\rangle = G(H^*)$ and $\operatorname{ord}\alpha = |G(H^*)| = p$.

Let us denote $[V]$ for the isomorphism class of an H-module V, and $\operatorname{Irr}(H)$ the set of all isomorphism classes of simple H-modules. One can define a left action of $G(H^*)$ on $\operatorname{Irr}(H)$ as follows: $\beta[V] = [\Bbbk_\beta \otimes V]$ for $[V] \in \operatorname{Irr}(H)$. A right action of $G(H^*)$ on $\operatorname{Irr}(H)$ can be defined similarly. The orbits of $[V] \in \operatorname{Irr}(H)$ under these $G(H^*)$-actions are respectively denoted by $O_l(V)$ and $O_r(V)$. A simple H-module V is called *regular* if $O_l(V) = O_r(V)$.

Following the terminology of [EG2], a simple H-module V is called *left stable* if $\Bbbk_\alpha \otimes V \cong V$, i.e. $O_l(V)$ is a singleton, or otherwise *left unstable*. We can define *right stable* (respectively *right unstable*) H-modules similarly. The set of all left (respectively right) $G(H^*)$-orbits in $\operatorname{Irr}(H)$ is denoted by $\mathfrak{O}_l(H)$ (respectively $\mathfrak{O}_r(H)$). It is easy to see that $\Bbbk_\beta$ is left unstable and regular for $\beta \in G(H^*)$.

In this section, we obtain in Corollary 3.5 that $|\mathfrak{O}_l(H)| \geqslant 3$. We also show in Proposition 3.6 that if every simple H-module is regular and left unstable, then $|\mathfrak{O}_l(H)| \geqslant 4$.

Remark 3.1. Since

$$O_r\big(V^\vee\big) = \big\{\big[W^\vee\big] \bigm| [W] \in O_l(V)\big\}$$

for $[V] \in \operatorname{Irr}(H)$, we find $|\mathfrak{O}_l(H)| = |\mathfrak{O}_r(H)|$. Moreover, $P(\Bbbk_\beta \otimes V) \cong \Bbbk_\beta \otimes P(V)$ and $P(V \otimes \Bbbk_\beta) \cong P(V) \otimes \Bbbk_\beta$. Therefore, $\dim P(W) = \dim P(V)$ for all $[W] \in O_l(V) \cup O_r(V)$. In particular, $\dim P(\Bbbk) = \dim P(\Bbbk_\beta)$ for all $\beta \in G(H^*)$.

There is a lower bound for the dimensions of the left stable indecomposable projective H-modules.

Lemma 3.2. *If V is a left stable simple H-module, then* $\dim P(V) \geqslant 2p$.

Proof. By Lemma 1.4, $\dim V = np$ for some positive integer n. If $\dim P(V) \geqslant 2\dim V$ then $\dim P(V) \geqslant 2p$. Now we assume $\dim P(V) < 2\dim V$. By Lemma 1.1, $P(V) = V$ and hence $V \cong \Bbbk_\alpha \otimes V^{\vee\vee}$. By Corollary 2.3, $\dim V \neq p$ and hence $\dim P(V) = \dim V \geqslant 2p$. □

Recall that if Q is a projective H-module, then Q is a direct sum of indecomposable projective H-modules. More precisely,

$$Q \cong \bigoplus_{[V]\in\operatorname{Irr}(H)} N_V^Q \cdot P(V) \tag{3.1}$$

where the multiplicity $N_V^Q = \dim \operatorname{Hom}_H(Q, V)$. Note that if V, W are simple, then

$$\dim \operatorname{Hom}_H\big(P(V), W\big) = \delta_{[V],[W]}. \tag{3.2}$$

For all $X, Y \in H\text{-}\mathbf{mod}_{\mathsf{fin}}$, $X \otimes Q$ and $Q \otimes Y$ are projective and we have the natural isomorphisms of $\Bbbk$-linear spaces

$$\operatorname{Hom}_H\big(X^\vee \otimes Q, Y\big) \cong \operatorname{Hom}_H(Q, X \otimes Y) \cong \operatorname{Hom}_H\big(Q \otimes {}^\vee Y, X\big). \tag{3.3}$$

Lemma 3.3. *For* $[V] \in \mathrm{Irr}(H)$ *and* $\beta \in G(H^*)$, *we have*

$$\dim \mathrm{Hom}_H\big(P(V) \otimes {}^{\vee}V, \Bbbk_\beta\big) \cong \begin{cases} \delta_{\epsilon,\beta} & \text{if } V \text{ is left unstable,} \\ 1 & \text{if } V \text{ is left stable.} \end{cases}$$

Proof. By (3.3) and (3.2), we have

$$\begin{aligned} \dim \mathrm{Hom}_H\big(P(V) \otimes {}^{\vee}V, \Bbbk_\beta\big) &= \dim \mathrm{Hom}_H\big(P(V), \Bbbk_\beta \otimes V\big) \\ &= \begin{cases} \delta_{\epsilon,\beta} & \text{if } V \text{ is left unstable,} \\ 1 & \text{if } V \text{ is left stable.} \end{cases} \qquad \square \end{aligned}$$

The following corollary gives a lower bound for the dimension contributed by a left or right $G(H^*)$-orbit.

Corollary 3.4. *For all simple* H*-module* V, *we have*

$$\sum_{[W] \in O(V)} \dim W \dim P(W) \geqslant p \cdot \dim P(\Bbbk)$$

where $O(V) = O_l(V)$ *or* $O_r(V)$.

Proof. If V is left unstable, then Lemma 3.3 implies that $P(\Bbbk)$ is a summand of $P(V) \otimes {}^{\vee}V$ and so

$$\sum_{[W] \in O_l(V)} \dim W \dim P(W) = p \cdot \dim\big(P(V) \otimes {}^{\vee}V\big) \geqslant p \cdot \dim P(\Bbbk).$$

If V is left stable, then $\bigoplus_{\beta \in G(H^*)} P(\Bbbk_\beta)$ is a summand of $P(V) \otimes {}^{\vee}V$. Hence, by Remark 3.1,

$$\sum_{[W] \in O_l(V)} \dim W \dim P(W) = \dim\big(P(V) \otimes {}^{\vee}V\big) \geqslant p \cdot \dim P(\Bbbk).$$

Since $O_r(V) = \{[W^{\vee}] \mid [W] \in O_l({}^{\vee}V)\}$, we have

$$\begin{aligned} \sum_{[W] \in O_r(V)} \dim W \dim P(W) &= \sum_{[W] \in O_l({}^{\vee}V)} \dim W^{\vee} \cdot \dim P\big(W^{\vee}\big) \\ &= \sum_{[W] \in O_l({}^{\vee}V)} \dim W \cdot \dim P(W) \geqslant p \cdot \dim P(\Bbbk). \qquad \square \end{aligned}$$

Let $\{V_0, \ldots, V_\ell\}$ be a set of simple H-modules such that $O_l(V_0), \ldots, O_l(V_\ell)$ are all the disjoint orbits in $\mathrm{Irr}(H)$ with $V_0 = \Bbbk$. By Remark 3.1, $\dim P(V_i) = \dim P(W)$ for $[W] \in O_l(V_i)$. Let us simply denote d_i and D_i respectively for $\dim V_i$ and $\dim P(V_i)$. Obviously, $D_i \geqslant d_i \geqslant 2$ for $i \neq 0$. Since H is a Frobenius algebra, we have

$$\dim H = \sum_{[V]\in \mathrm{Irr}(H)} \dim V \cdot \dim P(V)$$
$$= p \sum_{\text{unstable } V_i} d_i D_i + \sum_{\text{stable } V_i} d_i D_i \tag{3.4}$$

(cf. [CR, 61.13]). Now we can show that $\ell \geqslant 2$.

Corollary 3.5. $|\mathfrak{O}_l(H)|, |\mathfrak{O}_r(H)| \geqslant 3$.

Proof. By Remark 3.1, it suffices to show $|\mathfrak{O}_l(H)| \geqslant 3$. By Lemma 2.1(vi), there exists a simple H-module V with $\dim V > 1$. Therefore, $|\mathfrak{O}_l(H)| \geqslant 2$. Suppose there are exactly two orbits $O_l(V_0)$ and $O_l(V_1)$. By Lemma 2.1(vi), $D_1 \geqslant 2d_1 \geqslant 4$. If V_1 is left stable, then (3.4) becomes

$$pq = pD_0 + d_1 D_1.$$

Lemma 3.3 and (3.1) imply

$$d_1 D_1 = \dim\bigl(P(V_1) \otimes {}^{\vee}V_1\bigr) = pD_0 + n_1 D_1$$

for some non-negative integer $n_1 < d_1$. By eliminating D_0, we find

$$q = (2d_1 - n_1)\frac{D_1}{p}.$$

Since $P(V_1)$ is also left stable, D_1 is divisible by p. This forces $2d_1 - n_1 = q$ and $D_1 = p$ which contradicts Lemma 3.2. Therefore, V_1 must be left unstable. Now, (3.4) becomes

$$pq = pD_0 + pd_1 D_1.$$

Lemma 3.3 and (3.1) imply

$$d_1 D_1 = \dim\bigl(P(V_1) \otimes {}^{\vee}V_1\bigr) = D_0 + n_1 D_1$$

for some non-negative integer $n_1 < d_1$. By eliminating D_0, we find

$$q = (2d_1 - n_1) D_1.$$

Since $2d_1 - n_1 > 1$ and $D_1 \geqslant 4$, the above equality contradicts that q is a prime. Therefore, $|\mathfrak{O}_l(H)| \neq 2$ and hence $|\mathfrak{O}_l(H)| \geqslant 3$. □

We close this section with the following proposition.

Proposition 3.6. *If every simple H-module is left unstable and regular, then $|\mathfrak{O}_l(H)| \geqslant 4$.*

Proof. By Corollary 3.5, it suffices to show $|\mathfrak{O}_l(H)| \neq 3$. Suppose that $|\mathfrak{O}_l(H)| = 3$. By (3.1), (3.4) and Lemma 3.3, we have the equations:

$$pq = pD_0 + pd_1 D_1 + pd_2 D_2, \tag{3.5}$$

$$d_1 D_1 = D_0 + N_{11}^1 D_1 + N_{11}^2 D_2, \tag{3.6}$$

$$d_2 D_2 = D_0 + N_{22}^1 D_1 + N_{22}^2 D_2, \tag{3.7}$$

where $N_{ii}^j = \sum_{\beta \in G(H^*)} \dim(\mathrm{Hom}_H(P(V_i) \otimes {}^\vee V_i, \Bbbk_\beta \otimes V_j))$. On the other hand, if $i \neq j$, then $[V_i] \notin O_r(V_j)$ since V_j is regular. Therefore,

$$\mathrm{Hom}_H\big(V_j^\vee \otimes P(V_i), \Bbbk_\beta\big) \cong \mathrm{Hom}_H\big(P(V_i), V_j \otimes \Bbbk_\beta\big) = 0$$

for all $\beta \in G(H^*)$, and so we have

$$d_2 D_1 = M_{21}^1 D_1 + M_{21}^2 D_2, \tag{3.8}$$

$$d_1 D_2 = M_{12}^1 D_1 + M_{12}^2 D_2, \tag{3.9}$$

where $M_{ji}^k = \sum_{\beta \in G(H^*)} \dim(\mathrm{Hom}_H(V_j^\vee \otimes P(V_i), \Bbbk_\beta \otimes V_k))$. By (3.3), we find

$$\begin{aligned} M_{ji}^k &= \sum_{\beta \in G(H^*)} \dim\big(\mathrm{Hom}_H\big(P(V_i) \otimes {}^\vee V_k, V_j \otimes \Bbbk_\beta\big)\big) \\ &= \sum_{\beta \in G(H^*)} \dim\big(\mathrm{Hom}_H\big(P(V_i) \otimes {}^\vee V_k, \Bbbk_\beta \otimes V_j\big)\big) = N_{ik}^j. \end{aligned}$$

The second equality is a consequence of the regularity of V_j. Now, (3.8) and (3.9) become

$$d_2 D_1 = N_{11}^2 D_1 + M_{21}^2 D_2, \tag{3.10}$$

$$d_1 D_2 = M_{12}^1 D_1 + N_{22}^1 D_2. \tag{3.11}$$

Equations (3.5)–(3.7) imply

$$q = \big(2d_1 - N_{11}^1\big) D_1 + \big(d_2 - N_{11}^2\big) D_2, \tag{3.12}$$

$$q = \big(d_1 - N_{22}^1\big) D_1 + \big(2d_2 - N_{22}^2\big) D_2, \tag{3.13}$$

$$1 < 2d_i - N_{ii}^i \quad \text{and} \quad q > D_i \quad \text{for } i = 1, 2. \tag{3.14}$$

In particular, D_1, D_2 are relatively prime. It follows from (3.10) and (3.11) that

$$D_1 \mid d_1 - N_{22}^1, \quad D_2 \mid d_2 - N_{11}^2, \quad d_1 \geqslant N_{22}^1, \quad d_2 \geqslant N_{11}^2.$$

By Lemma 2.1(vi), $D_i > d_i$ for some $i = 1, 2$. If $D_1 > d_1$, then $d_1 = N_{22}^1$. Hence, by (3.13), $q = (2d_2 - N_{22}^2) D_2$. Similarly, if $D_2 > d_2$, then $q = (2d_1 - N_{11}^1) D_1$. However, both of these conclusions contradict that q is a prime number. □

4. The case $q \leqslant 4p + 11$

In this section, we prove our main result:

Theorem 4.1. *Every Hopf algebra of dimension pq over $\Bbbk$, where p, q are odd primes with $p < q \leqslant 4p + 11$, is trivial.*

By [EG1], it suffices to show that Hopf algebras of these dimensions are semisimple. We proceed to prove that by contradiction. Suppose there exists a non-semisimple Hopf algebra H of these dimensions. By duality and Lemma 2.1(i), we can further assume the distinguished group-like element $\alpha \in H^*$ is not trivial.

Let D_0 denote $\dim P(\Bbbk)$. Since the composition factors of $P(\Bbbk)$ cannot be all 1-dimensional (cf. [EG2, Lemma 2.3]), $D_0 \geqslant 4$. It follows from Corollary 3.4 that

$$\sum_{[W] \in O(V)} \dim W \dim P(W) \geqslant pD_0 \geqslant 4p \tag{4.1}$$

for all simple H-modules V.

Lemma 4.2. *A simple H-module V is left stable if, and only if, V is right stable. In this case, $\dim V \geqslant 2p$.*

Proof. Let V be a left stable simple H-module. By Lemma 1.4, $\dim V = np$ for some positive integer n. It follows from Lemma 3.2 that $\dim V \cdot \dim P(V) \geqslant 2p^2$.

Notice that W is left stable for $[W] \in O_r(V)$. By Corollary 3.5, there exists a right $G(H^*)$-orbit different from $O_r(V)$, $O_r(\Bbbk)$. If V is not right stable, then by Corollary 3.4 and (4.1),

$$\begin{aligned}\dim H &\geqslant \sum_{[W] \in O_r(V)} \dim W \dim P(W) + pD_0 + pD_0 \\ &\geqslant p\bigl(2p^2\bigr) + 8p > 4p^2 + 11p.\end{aligned}$$

Therefore, V is also right stable. Conversely, if V is right stable, then ${}^\vee V$ is left stable. Hence, by the first part of the proof, ${}^\vee V$ is also right stable. Therefore, $V \cong ({}^\vee V)^\vee$ is left stable.

Now let V be a left stable simple H-module. Since V is also right stable, $V^{\vee\vee\vee\vee} \cong \Bbbk_{\alpha^{-1}} \otimes V \otimes \Bbbk_\alpha \cong V$. Consider the set $\mathcal{A} = \{[V], [V^\vee], [V^{\vee\vee}], [V^{\vee\vee\vee}]\}$. Since $[V^{\vee\vee\vee\vee}] = [V]$ and $S^{4p} = \mathrm{id}$, the cyclic group $C_{4p} = \langle x \rangle$ of order $4p$ acts transitively on $\mathcal{A}$ with

$$x \cdot [V] = \bigl[V^\vee\bigr].$$

Thus $|\mathcal{A}| = 1, 2$ or 4. If $|\mathcal{A}| = 4$, then

$$\dim H \geqslant 4 \dim V \dim P(V) + pD_0 \geqslant 8p^2 + 4p > 4p^2 + 11p.$$

Therefore, $|\mathcal{A}| \leqslant 2$, and so $[V^{\vee\vee}] = x^2 \cdot [V] = [V]$. Thus we obtain $\Bbbk_\alpha \otimes V^{\vee\vee} \cong V$. By Corollary 2.3, $\dim V \neq p$ and hence $\dim V \geqslant 2p$. $\square$

Lemma 4.3. *H has no left or right stable simple H-module.*

Proof. By Lemma 4.2, it suffices to show that left stable simple H-modules do not exist. Suppose there is a left stable simple H-module V. Then

(1) *V is projective*. For otherwise, by Lemmas 1.1 and 4.2, $P(V) \geqslant 2\dim V \geqslant 4p$ and so

$$\dim H \geqslant \dim V \dim P(V) + pD_0 \geqslant 8p^2 + 4p > 4p^2 + 11p.$$

(2) *V is the unique left stable simple H-module.* If there exists a left stable simple H-module W not isomorphic to V, then $\dim V, \dim W \geqslant 2p$ and so

$$\dim H \geqslant \dim V \dim P(V) + \dim W \dim P(W) + pD_0 \geqslant 8p^2 + 4p.$$

(3) Since V is projective, by Lemma 2.1(vi), there exists a simple H-module W such that $\dim P(W) \geqslant 2\dim W \geqslant 4$. Obviously, $O_l(W)$ is different from $O_l(V)$, $O_l(\Bbbk)$. By (2), W is left unstable. Therefore,

$$\sum_{[W'] \in O_l(W)} \dim W' \dim P(W') = p \dim W \dim P(W) \geqslant p(2 \cdot 4) = 8p$$

and hence

$$\dim H \geqslant \dim V \dim P(V) + 8p + pD_0 \geqslant 4p^2 + 12p,$$

a contradiction! □

Lemma 4.4. $|\mathfrak{O}_l(H)| \geqslant 4$ *and* $D_0 \leqslant p + 2$.

Proof. To show the first inequality, by Proposition 3.6 and Lemma 4.3, it suffices to prove that every simple H-module is regular. Suppose there exists a simple H-module V such that $O_r(V) \neq O_l(V)$. Then $\dim V \geqslant 2$ and the set

$$\mathcal{A} = \left\{[\Bbbk_\beta \otimes V \otimes \Bbbk_{\beta'}] \,\middle|\, \beta, \beta' \in G(H^*)\right\}$$

contains more than p elements. Obviously, the group $G(H^*) \times G(H^*)$ acts transitively on $\mathcal{A}$. Therefore, $|\mathcal{A}| = p^2$. If V is not projective, then $\dim V \cdot \dim P(V) \geqslant 2 \cdot 4$ and hence

$$\begin{aligned}\dim H &\geqslant \sum_{[W] \in \mathcal{A}} \dim W \dim P(W) + pD_0 \\ &= p^2 \dim V \dim P(V) + pD_0 \geqslant 8p^2 + 4p > 4p^2 + 11p.\end{aligned}$$

Therefore V is projective, and hence $\Bbbk_\beta \otimes V \otimes \Bbbk_{\beta'}$ are projective for $\beta, \beta' \in G(H^*)$. By Lemma 2.1(vi), there exists a simple H-module W such that $\dim W \geqslant 2$ and $\dim P(W) \geqslant 2\dim W$. Thus,

$$\dim H \geqslant p^2 \dim V \dim P(V) + p \dim W \dim P(W) + pD_0 \geqslant 4p^2 + 8p + 4p,$$

which is a contradiction! Therefore, every simple H-module is regular.

By Corollary 3.4 and the first inequality, we have

$$4p^2 + 11p \geqslant \dim H \geqslant 4pD_0.$$

Therefore, $D_0 \leqslant p + 2$. □

Lemma 4.5. *If V is a simple H-module such that $V \cong \Bbbk_{\alpha^{-1}} \otimes V \otimes \Bbbk_\alpha$, then* $\dim P(V) \geqslant 2 \dim V$.

Proof. Suppose V is a simple H-module such that $\Bbbk_{\alpha^{-1}} \otimes V \otimes \Bbbk_\alpha \cong V$ and $\dim P(V) < 2 \dim V$. By Lemma 1.1, V is projective and $V \cong \Bbbk_\alpha \otimes V^{\vee\vee}$. Since $V^{\vee\vee\vee\vee} \cong \Bbbk_{\alpha^{-1}} \otimes V \otimes \Bbbk_\alpha$ (cf. (1.4)), we have

$$V^{\vee\vee} \cong \Bbbk_\alpha \otimes V^{\vee\vee\vee\vee} \cong V \otimes \Bbbk_\alpha,$$

and hence

$$V \cong V^{\vee\vee\vee\vee} \cong V^{\vee\vee} \otimes \Bbbk_\alpha \cong V \otimes \Bbbk_{\alpha^2}.$$

Since $\operatorname{ord}(\alpha) = p$, V is right stable, but this contradicts Lemma 4.3. □

Using an argument similar to [EG2, Lemma 2.8], we find $V^{\vee\vee\vee\vee} \cong V$ for all $V \in H$-$\mathbf{mod}_{\mathsf{fin}}$ in the following lemma.

Lemma 4.6. *For all simple H-module V, $V^{\vee\vee\vee\vee} \cong V$ and* $\dim P(V) \geqslant 2 \dim V$.

Proof. By [EG2, Lemma 2.3], $P(\Bbbk)$ has a simple constituent U with $\dim U > 1$. Since $P(\Bbbk) \cong \Bbbk_{\alpha^{-1}} \otimes P(\Bbbk) \otimes \Bbbk_\alpha$, $\Bbbk_{\beta^{-1}} \otimes U \otimes \Bbbk_\beta$ are simple constituents of $P(\Bbbk)$ for $\beta \in G(H^*)$. By Lemma 4.4, $\dim P(\Bbbk) \leqslant p + 2 < 2p + 2$. Therefore, U must be invariant under the conjugation by $\Bbbk_\alpha$.

Let E be the full subcategory of H-$\mathbf{mod}_{\mathsf{fin}}$ consisting of those H-modules with composition factors invariant under the conjugation by $\Bbbk_\alpha$. Let V, W be simple objects in E. Suppose $V \otimes W \notin E$. Then $V \otimes W$ has a simple constituent X which is not invariant under conjugation by $\Bbbk_\alpha$. Then, $[X] \notin O_l(\Bbbk)$ and $\Bbbk_{\beta^{-1}} \otimes X \otimes \Bbbk_\beta$ are composition factors of $V \otimes W$ for $\beta \in G(H^*)$. In particular, $\dim X > 1$, and hence $\dim V \otimes W \geqslant p \dim X \geqslant 2p$. Without loss of generality, we may assume that $\dim V \geqslant \sqrt{2p}$. By Lemma 4.5, $\dim P(V) \geqslant 2 \dim V \geqslant 2\sqrt{2p}$. Hence, by Lemma 4.4 and Corollary 3.4,

$$\dim H \geqslant p \dim V \dim P(V) + 3pD_0 \geqslant 4p^2 + 12p, \quad \text{a contradiction!}$$

Therefore, $V \otimes W \in E$ and so E forms a tensor subcategory of H-$\mathbf{mod}_{\mathsf{fin}}$. There exists a quotient Hopf algebra H' of H such that E is tensor equivalent to H'-$\mathbf{mod}_{\mathsf{fin}}$. If $H' \ncong H$, then $\dim H' = 1$, p or q, and hence H' is an abelian group algebra (cf. [Z]). Thus every simple H-module in E is one-dimensional. This contradicts that $U \in E$. Therefore, $H' = H$ and hence every H-module is invariant under the conjugation by $\Bbbk_\alpha$. By (1.4), $V^{\vee\vee\vee\vee} \cong V$ for all $V \in H$-$\mathbf{mod}_{\mathsf{fin}}$. The second assertion follows immediately from Lemma 4.5. □

Remark 4.7. For all simple H-modules V with $\dim V \geqslant 2$,

$$\sum_{[W]\in O_l(V)} \dim W \dim P(W) \geqslant p \dim V \dim P(V) \geqslant 8p.$$

Lemma 4.8. *D_0 is an even integer.*

Proof. Suppose D_0 is odd. Since $P(\Bbbk) \cong P(\Bbbk)^{\vee\vee}$, by Corollary 2.3, $D_0 \geqslant p+2$. Recall from Corollary 3.4 that $\dim V \dim P(V) \geqslant D_0$ for all simple H-modules V. There exists a simple V such that $\dim V \dim P(V) > D_0$ for otherwise we have

$$pq = \dim H = p \cdot \left|\mathfrak{O}_l(H)\right| \cdot D_0$$

which contradicts that q is a prime. By Lemma 3.3, there is a simple H-module W with $\dim W > 1$ such that $P(W) \oplus P(\Bbbk)$ is a direct summand of $P(V) \otimes {}^{\vee}V$. It follows from Lemma 4.6 that

$$\dim V \dim P(V) \geqslant D_0 + \dim P(W) \geqslant D_0 + 4$$

and so

$$\dim H \geqslant 3pD_0 + p \dim V \dim P(V) \geqslant p(4D_0 + 4) \geqslant 4p^2 + 12p. \qquad \square$$

Lemma 4.9. *For any simple H-module V, $V^{\vee\vee} \in O_l(V)$ if, and only if, $V^{\vee\vee} \cong V$.*

Proof. If $V^{\vee\vee} \in O_l(V)$, then $V^{\vee\vee} \cong \Bbbk_\beta \otimes V$ for some $\beta \in G(H^*)$. Since $V^{\vee\vee\vee\vee} \cong V$, we have

$$V \cong (\Bbbk_\beta \otimes V)^{\vee\vee} \cong \Bbbk_\beta \otimes V^{\vee\vee} \cong \Bbbk_{\beta^2} \otimes V.$$

By Lemma 4.3, V is left unstable. Therefore, $\beta^2 = \epsilon$ and hence $\beta = \epsilon$. Thus, we have $V^{\vee\vee} \cong V$. The converse of the statement is obvious. $\square$

Let S be a complete set of representatives of the left $G(H^*)$-orbits in $\mathrm{Irr}(H)$ with $\Bbbk \in S$. Then we have

$$\begin{aligned} pq = \dim H &= \sum_{V\in S} p \dim V \dim P(V) \\ &= p \sum_{\substack{V\in S \\ V^{\vee\vee}\ncong V}} \dim V \dim P(V) + pD_0 + p \sum_{\substack{V\in S\setminus\{\Bbbk\} \\ V^{\vee\vee}\cong V}} \dim V \dim P(V). \end{aligned}$$

By Lemma 4.9, the first term of the last expression is an even integer. The second term is also even by Lemma 4.8. Therefore, the third term must be odd. There exists $X \in S \setminus \{\Bbbk\}$ such that $X \cong X^{\vee\vee}$ and $\dim X \dim P(X)$ is odd. In particular, $\dim X$ and $\dim P(X)$ are odd integers > 1. It follows from Corollary 2.3 that $\dim P(X) \geqslant p+2$. If $\dim X > 3$, then

$$\dim H \geqslant 3pD_0 + p\dim X \dim P(X) \geqslant p\big(12 + 5(p+2)\big) > 4p^2 + 11p.$$

Therefore, $\dim X = 3$. Notice that $X \otimes P(\Bbbk)$ has even dimension and it contains a summand isomorphic to $P(X)$. Since $\dim P(X)$ is odd and $X \otimes P(\Bbbk) \cong (X \otimes P(\Bbbk))^{\vee\vee}$, $X \otimes P(\Bbbk)$ must have another odd-dimensional indecomposable projective summand Q such that $Q^{\vee\vee} \cong Q$. By Corollary 2.3, $\dim Q \geqslant p+2$ and hence

$$3D_0 = \dim\big(X \otimes P(\Bbbk)\big) \geqslant \dim P(X) + \dim Q \geqslant 2(p+2).$$

Therefore, $D_0 \geqslant (2p+4)/3$ and so

$$\dim H \geqslant p\big(3D_0 + \dim X \dim P(X)\big) \geqslant p\big(2p + 4 + 3(p+2)\big) > 4p^2 + 11p.$$

This is again a contradiction! That means non-semisimple Hopf algebras of dimension pq, where p and q are distinct odd primes with $p < q \leqslant 4p + 11$, do not exist. This completes the proof of Theorem 4.1.

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