Positive Sextics and Schur's Inequalities

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1. INTRODUCTION

This paper is devoted to the study of ternary forms, or more specifically, symmetric ternary forms, over the real numbers. We say that a ternary form $f(x, y, z)$ is positive semidefinite (psd) if $f(a, b, c) \geq 0$ for all $(a, b, c) \in \mathbb{R}^3$. Clearly, such a form must have an even degree $2d$. As usual, we can identify a form with the ordered tuple of its coefficients, so we can

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think of a form as a point in some Euclidean space. In this way, the set of all psd ternary forms of a fixed degree $2d$ forms a closed and convex cone (in $\mathbb{R}^{(d+1)(2d+1)}$), which we shall denote by $P_{3,2d}$. It is also of interest to study ternary forms $g(x, y, z)$ with the weaker property that $g(a, b, c) \geq 0$ for all nonnegative $a, b, c \in \mathbb{R}$. We shall refer to this property by saying that $g$ is "copositive" (in the terminology of Hall and Newman [HN]), or simply that $g$ is psd on
\[ \mathbb{R}_+^3 = \{(a, b, c) \in \mathbb{R}^3 : a, b, c \geq 0\}. \]

These forms $g$ need no longer be of even degree. For a fixed positive number $d$, let $P_{3,d}^+$ be the cone of copositive ternary forms of degree $d$. We then have $P_{3,2d} \subseteq P_{3,d}^+$ for every even degree $2d$. Note that every $g \in P_{3,d}^+$ gives rise to an even form $f(x, y, z) = g(x^2, y^2, z^2) \in P_{3,2d}$. And indeed, every form in $P_{3,2d}$ arises uniquely in this manner. Thus, for all intents and purposes, the study of $P_{3,d}^+$ is equivalent to the study of the subcone of $P_{3,2d}$ consisting of the even forms.

Most of the time, we shall be dealing with symmetric forms. Thus, we focus attention on the subcones
\[ \text{Sym } P_{3,2d} \subseteq P_{3,2d}, \quad \text{Sym } P_{3,d}^+ \subseteq P_{3,d}^+ \tag{1.1} \]
consisting of the symmetric forms, respectively, in $P_{3,2d}$ and $P_{3,d}^+$. For instance, from the Arithmetic–Geometric Inequality, we have $(x + y + z)^3 - 27xyz \in P_{3,3}^+$. Aside from forms of this kind which arise from the AG-Inequality, one of the earliest nontrivial examples of a symmetric copositive form is the following ternary cubic mentioned in a 1820 textbook of Lehmus (see [Cox]):
\[ \Gamma(x, y, z) = xyz - (y + z - x)(z + x - y)(x + y - z) = \sum x^3 - \sum x^2y + 3xyz \in \text{Sym } P_{3,3}^+. \tag{1.2} \]
The associated even symmetric psd sextic
\[ S(x, y, z) := \Gamma(x^2, y^2, z^2) = x^2y^2z^2 - (y^2 + z^2 - x^2)(z^2 + x^2 - y^2)(x^2 + y^2 - z^2) = \sum x^6 - \sum x^4y^2 + 3x^2y^2z^2 \tag{1.3} \]
was rediscovered by R. M. Robinson in 1969. In [R], Robinson showed that $S$ is not a sum of squares of cubic forms, and that $S$ has exactly 10 real zeros in the projective 2-space. In [CL], two of the present authors showed that the Robinson form $S$ is, in fact, extremal in the cone $P_{3,6}$, i.e.,
if $S = f_1 + f_2$ where $f_i \in P_{3,6}$, then we must have $f_i = \lambda_i S$ for suitable (non-negative) real scalars $\lambda_1, \lambda_2$. In particular, it follows that the "Lehmus form" $I$ is extremal in $P_{3,3}^+$. (This can also be deduced from Rigby's result in [Ri1], or the more general results in [CLR].)

In this paper, we shall report the discovery of several new symmetric forms in $P_{3,6}$, including

$$2F(x, y, z) = (y + z - x)^2(z + x - y)^2(x + y - z)^2 - (y^2 + z^2 - x^2)(z^2 + x^2 - y^2)(x^2 + y^2 - z^2) \in \text{Sym} P_{3,6},$$

$$G(x, y, z) = \sum x^4(y - z)^2 - (x - y)^2(y - z)^2(z - x)^2 \in \text{Sym} P_{3,6}.$$ (1.4)

These forms are no longer even, but, like $S$ itself, they are both extremal in $P_{3,6}$ (and à fortiori in Sym $P_{3,6}$). Thus, we have the following remarkable new symmetric "extremal" inequalities:

$$G(x, y, z) = 2 \sum x^4(y - z)^2 - (x - y)^2(y - z)^2(z - x)^2 \in \text{Sym} P_{3,6}.$$ (1.5)

$$\sum x^4(y - z)^2 \geq \frac{1}{2}(x - y)^2(y - z)^2(z - x)^2,$$

holding for all real numbers $x, y, z$. The forms $F$ and $G$ both have exactly seven real zeros (in projective 2-space), and are related to one another by a linear change of variables:

$$G(x, y, z) = F((y + z - x)/2, (z + x - y)/2, (x + y - z)/2),$$

$$F(x, y, z) = 4G((y + z - x)/2, (z + x - y)/2, (x + y - z)/2).$$ (1.6)

As is clear from (1.3) and (1.4), the form $F$ is closely related to Robinson's form $S(x, y, z) = x^2y^2z^2 - (y^2 + z^2 - x^2)(z^2 + x^2 - y^2)(x^2 + y^2 - z^2)$. We discovered this new form $F$ by certain considerations in euclidean geometry, which will be presented in Section 3. This part of our work was very much inspired by the celebrated classical geometry of the nine-point circle which goes back to Euler, Chapple, and Feuerbach.

The extremality of $S$, $F$, and $G$ means that, for any one of these forms $f$, we cannot have $f(x, y, z) \geq g(x, y, z) \geq 0$ for all $x, y, z \in \mathbb{R}$ unless the polynomial $g$ is in fact a scalar multiple of $f$. If, however, we restrict our attention to $(x, y, z) \in \mathbb{R}_{+}^3$, the situation is quite different; it turns out that, for such $x, y, z$, we have the nontrivial inequalities

$$S(x, y, z) \geq 8(x - y)^2(y - z)^2(z - x)^2,$$

$$F(x, y, z) \geq 4(x - y)^2(y - z)^2(z - x)^2,$$

$$G(x, y, z) \geq (x - y)^2(y - z)^2(z - x)^2.$$ (1.10)
In particular, these imply that \( S, F, \) and \( G \) are no longer extremal in \( P_{3,6}^+ \). In fact, since \((x - y)^2(y - z)^2(z - x)^2\) is symmetric, we can even conclude that \( S, F, \) and \( G \) are not extremal in \( \text{Sym} P_{3,6}^+ \). It is, therefore, of interest to study how these three forms decompose into sums of extremal forms in \( \text{Sym} P_{3,6}^+ \). (It is well known that, in a finite dimensional closed and convex cone, every ray is a finite sum of extremal rays.) In Sections 6 and 7, we obtain such decompositions explicitly. In doing so, we discover several interesting new extremal forms in the cone \( \text{Sym} P_{3,6}^+ \), which in turn lead to many new symmetric ternary sextic inequalities holding in \( \mathbb{R}^3_+ \). To mention a few, we have for all \((x, y, z) \in \mathbb{R}_+^3\),

\[
\sum x^4(x - y)(x - z) \geq 5(x - y)^2(y - z)^2(z - x)^2,
\]

\[
\sum x^3(y + z)(x - y)(x - z) \geq 3(x - y)^2(y - z)^2(z - x)^2,
\]

\[
\sum x^2(y - z)^4 \geq (x - y)^2(y - z)^2(z - x)^2,
\]

\[
\Gamma(y + z, z + x, x + y)^2 \geq (x - y)^2(y - z)^2(z - x)^2,
\]

\[
\Gamma(x^2, y^2, z^2)^2 \geq \Gamma(x, y, z)^2,
\]

\[
S(x, y, z) \geq F(x, y, z).
\]

The first of these, for instance, is an improvement of a classical inequality of Schur [HLP, p. 64] in degree 6. The others may be regarded as extensions or refinements of Schur's and Lehmus' inequalities. Finally, in Section 9, we make a systematic study of Schur's inequalities in any degree, and give a complete determination of when these inequalities are extremal (in the sense of this paper).

This work is to be viewed as a contribution to the general theory of polynomial inequalities. For related works in the literature, the reader may consult [H1, H2, M1, M2, R, CL1, CL2, CLR1, CLR2, R1, R2, R3], etc., and the standard books [HLP, Mi, B]. Hilbert's pioneering work [H1] (which appeared exactly 100 years ago) already showed that ternary sextics and quaternary quartics are two of the most interesting classes of forms for the investigation of polynomial inequalities. The case of symmetric quartics will be studied in detail elsewhere [CLR3]. In this paper, we focus our attention on the symmetric ternary sextics which are psd either on \( \mathbb{R}^3 \) or on \( \mathbb{R}_+^3 \). It is hoped that the discovery of the hitherto unknown positive and copositive sextics in this paper will lay the groundwork for a complete determination of the two cones \( \text{Sym} P_{3,6} \) and \( \text{Sym} P_{3,6}^+ \) in the future.
2. SOME BASIC FACTS

In this section, we shall assemble a few basic facts and notations so that we can refer to them freely in the balance of this paper. For the convenience of the reader, we shall include most of the relevant proofs.

First, let us point out that the Lehmus form $\Gamma \in \mathcal{P}_{3,3}$ (defined in (1.2)) has a fairly well-known generalization to higher degree forms. In fact, for any $d \geq 2$, we have the following Schur form of degree $d$:

$$\Gamma_d(x, y, z) = x^{d-2}(x - y)(x - z) + y^{d-2}(y - x)(y - z) + z^{d-2}(z - x)(z - y),$$

which for $d = 3$ boils down to the Lehmus form $\Gamma$ after a direct expansion. Schur's Inequality (an exercise in [HLP, p. 641]) says precisely that, for any $d \geq 2$, $\Gamma_d \in \mathcal{P}_{3,d}$. To see this, take real numbers $z \geq y \geq x \geq 0$. Then obviously

$$\Gamma_d(x, y, z) = x^{d-2}(y - x)(z - x) + (z - y)[z^{d-2}(z - x) - y^{d-2}(y - x)] \geq 0$$

since $z^{d-2} \geq y^{d-2} \geq 0$ and $z - x \geq y - x \geq 0$. By symmetry, it follows that $\Gamma_d(x, y, z) \geq 0$ for all $(x, y, z) \in \mathbb{R}^3_+$. We say that a form $f(x, y, z)$ is $\vert \cdot \vert$-convex if $f(x, y, z) \geq f(\lvert x \rvert, \lvert y \rvert, \lvert z \rvert)$ for all $(x, y, z) \in \mathbb{R}^3$. This is an interesting property, for, if $f$ has this property, then we will have

$$f \in \mathcal{P}_{3,d} \iff f \in \mathcal{P}_{3,d}^+. $$

Any even form is obviously $\vert \cdot \vert$-convex; however, and $\vert \cdot \vert$-convex form need not be even (e.g., $(x - y)^2$, or, to stick to three variables, $(x - y)^2(y - z)^2(z - x)^2$).

**Lemma 2.2.** If $d$ is an even integer, then the Schur form $\Gamma_d$ is $\vert \cdot \vert$-convex. In particular, $\Gamma_d \in \mathcal{P}_{3,d}$.

**Proof.** In order to show that $\Gamma_d(x, y, z) \geq \Gamma_d(\lvert x \rvert, \lvert y \rvert, \lvert z \rvert)$, we may assume, of course, that at least one of $x, y, z$ is negative. By symmetry, and by the fact that $\Gamma_d$ is a form of even degree, it is enough to treat the case when $x \leq 0 \leq y \leq z$. In this case,

$$\Gamma_d(x, y, z) - \Gamma_d(\lvert x \rvert, \lvert y \rvert, \lvert z \rvert) = \Gamma_d(x, y, z) - \Gamma_d(-x, y, z) = x^{d-2}(-2yx - 2zx) + y^{d-2}(-2yx + 2zx) + z^{d-2}(2xy - 2xz) = -2x^{d-1}(y + z) + 2x(y^{d-2} - z^{d-2})(y - z).$$
This is clearly nonnegative, since $z - y \geq 0$, $x(y^2 - z^2) \geq 0$, and the fact that $d$ is even implies that $-2x^{d-1} \geq 0$. Q.E.D.

For later reference, let us define

$$S_d(x, y, z) = \Gamma_d(x^2, y^2, z^2) = x^{2d-4}(x^2 - y^2)(x^2 - z^2) + \ldots$$

$$H_d(x, y, z) = \Gamma_d(yz, xz, xy) = (xyz)^d \Gamma_d(1/x, 1/y, 1/z)$$

$$= y^{d-1}z^{d-1}(x - y)(x - z) + \ldots.$$ (2.3)

(2.4)

We have then $S_d \in P_{3,2d}$, $H_d \in P_{3,2d}^+$, and, when $d$ is even, $H_d \in P_{3,2d}$. We shall continue to write $\Gamma$ and $S$ for $\Gamma_3$ and $S_3$, and, in the same vein, we shall write $H$ for $H_3$.

Next, we shall define two basic transformations on ternary forms. For three independent (commuting) indeterminates $x, y, z$, let us introduce the new indeterminates $a = y + z$, $b = z + x$, and $c = x + y$. Solving these equations for $x, y, z$, we have $x = (b + c - a)/2$, $y = (c + a - b)/2$, and $z = (a + b - c)/2$. In particular, we see that

$$(x, y, z) \in \mathbb{R}_+^3 \iff \left\{ \begin{array}{l} b + c \geq a \\ c + a \geq b \\ a + b \geq c \end{array} \right. .$$

The RHS condition here is of geometric interest since it amounts to the fact that $a, b, c$ (which are necessarily nonnegative) form the three sides of a triangle. (We allow, of course, "degenerate triangles" whose three vertices are collinear.) Now let $P_{3,d}$ (resp. Sym $P_{3,d}$) be the cone of ternary forms (resp. symmetric ternary forms) $g$ of degree $d$ such that $g(a, b, c) \geq 0$ whenever $a, b, c$ are the three sides of a triangle. Then $P_{3,d}^+$ (resp. Sym $P_{3,d}^+$) contains $P_{3,d}$ (resp. Sym $P_{3,d}$) as a subcone.

For any ternary form $f$, define two forms $f^+$ and $f^d$ of the same degree, as follows:

$$f^+(x, y, z) = f(y + z, z + x, x + y),$$

$$f^d(a, b, c) = f((b + c - a)/2, (c + a - b)/2, (a + b - c)/2).$$ (2.5) (2.6)

By the observations made above, we have then

$$f \in P_{3,d}^+ \iff f^+ \in P_{3,d}^+,$$ (2.7)

$$f \in P_{3,d}^+ \iff f^d \in P_{3,d}^d.$$ (2.8)

and moreover, for any $f$,

$$(f^d)^+ = f, \quad (f^+)^d = f.$$ (2.9)
In particular, the transformations $f \mapsto f^+$ and $g \mapsto g^d$ define mutually inverse linear isomorphisms between the cones $P^d_{3,d}$ and $P^+_{3,d}$ (resp. $\text{Sym } P^d_{3,d}$ and $\text{Sym } P^+_d$). Therefore, for all intents and purposes, the studies of $P^d_{3,d}$ and $\text{Sym } P^d_{3,d}$ are equivalent to those of $P^+_{3,d}$ and $\text{Sym } P^+_d$. In general, dealing with the latter cones seems to involve simpler notations, while dealing with the former cones enables one to invoke more geometric intuition involving triangles.

From (2.7), it is clear that $f \mapsto f^+$ takes $P^+_{3,d}$ into itself; however, $g \mapsto g^d$ does not (it "expands" $P^+_{3,d}$ into $P^d_{3,d}$ instead). As a simple example, for $d$ odd, although the Schur form $\Gamma_d$ is psd on $\mathbb{R}^d_+$, it is not difficult to see that the transform $\Gamma_d^d$ is not, so $\Gamma_d^d \not\in P^+_{3,d}$. Nevertheless, for the form $H_d$, we have the following rather surprising fact:

**Proposition 2.10.** For any $d \geq 2$, $H_d^d \in P^+_{3,2d}$.

**Proof.** If $d$ is even, then, as we have observed before, $H_d \in P_{3,2d}$. In this case, we have then $H_d^d \in P_{3,2d} \subseteq P^+_{3,2d}$. Now assume $d$ is odd. To show that $H_d^d \in P^+_d$, it suffices to show that $H_d^d(x, y, z) \geq 0$ whenever $z \geq y \geq x \geq 0$. In the expression

$$H_d^d(x, y, z) = \left(\frac{x + z - y}{2}\right)^{d-1} \left(\frac{x + y - z}{2}\right)^{d-1} (x - y)(x - z) + \cdots,$$

the first term is clearly $\geq 0$ since $d - 1$ is even. The other two terms are (up to a positive scalar factor)

$$(y - x)(y - z)(y + z - x)^{d-1}(y + x - z)^{d-1}$$

$$+ (z - x)(z - y)(z + y - x)^{d-1}(z + x - y)^{d-1}$$

$$= (z - y)(y + z - x)^{d-1}[(z - x)(x - y + z)^{d-1} - (y - x)(x + y - z)^{d-1}].$$

This expression is also $\geq 0$ since $d - 1$ is even, $y - x \geq 0$, $z - x \geq y - x \geq 0$, and

$$x - y + z = |x| + |y - z| \geq |x + y - z| \geq 0.$$

**Q.E.D.**

**Remark.** Although the proposition proved above will be very useful in this paper, the fact that $H_d^d \in P^+_{3,2d}$ remains something of an accident. One can show by direct computation that, for odd $d \geq 3$, $H_d^{dd}$ is no longer in $P^+_{3,2d}$. For instance, taking $d$ to be 3, we have

$$H^{dd}(1, 2, 4) = H(-3/4, 1/4, 9/4) = 4^{-6} (9, -27, -3) = -1161/256.$$

For the reader's convenience, we shall recall here a few basic facts concerning extremal positive and copositive forms. These facts will often be
used implicity in the paper. The proofs of these facts are all easy, and so will not be presented here.

If \( L(x_1, \ldots, x_n) \) is a nonzero linear form, and \( f(x_1, \ldots, x_n) \) is any psd \( n \)-ary \( m \)-ic, then \( L^2f \) is extremal iff \( f \) is extremal. In particular, if \( L_1, \ldots, L_k \) are any \( n \)-ary linear forms, then \( L_1^2 \cdots L_k^2 \) is always extremal as a psd \( n \)-ary \( 2k \)-ic. (2.12)

In the cone \( P_{2,m} \) of all psd binary \( m \)-ics where \( m = 2k \), the extremal forms are precisely \( L_1^2 \cdots L_k^2 \), where \( L_i(x, y) = a_i x - b_i y \), with \( a_i, b_i \in \mathbb{R} \). (2.13)

In the cone \( P_{2,m}^+ \) of all copositive binary \( m \)-ics (where \( m \) can be odd or even), the extremal forms are precisely \( x^r y^s L_1^2 \cdots L_k^2 \), where \( L_i(x, y) = a_i x - b_i y \) with \( a_i, b_i \in \mathbb{R}_+ \), \( r, s, k \geq 0 \), and \( r + s + 2k = m \). (2.14)

In the following sections, various special forms will be introduced, and these will be the principal objects of our study in this paper. For easy reference, a glossary of these forms is given in Section 10.

3. Euclidean Geometry and the Form \( F \)

This section is entirely devoted to geometric considerations and can be read largely independently of Section 2. The goal here is to explain how the form \( F \) in (1.4) arises, and to prove that \( F \) is positive semidefinite. Our main tool is the classical euclidean geometry of triangles. Let us begin by pointing out a trigonometric interpretation of the Robinson form \( S \) (in (1.3)) in terms of the cosines of the three angles of a triangle.

**Lemma 3.1.** Let \( a, b, c \) be the three sides of a triangle \( ABC \). Then

\[
1 - 8 \cos A \cos B \cos C = S(a, b, c)/a^2b^2c^2.
\]

**Proof.** By the Law of Cosines, we have \( \cos A = (b^2 + c^2 - a^2)/2bc \), etc. Therefore,

\[
1 - 8 \cos A \cos B \cos C
= 1 - 8 \frac{(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)}{8a^2b^2c^2}
= S(a, b, c)/a^2b^2c^2.
\] Q.E.D.
In view of this lemma, the fact that $S \in P_{3,6}^A$ amounts to the trigonometric assertion that
\[
\cos A \cos B \cos C \leq 1/8
\]
for the three angles of a triangle. (This inequality is fairly well known; see, e.g., [B, p.25; MP].) There is also another slightly different interpretation in terms of the squares of the cosines. By an elementary calculation, one can prove the following trigonometric identity for any $\triangle ABC$:
\[
\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos B \cos C. \tag{3.3}
\]
Therefore, we can transform (3.1) into
\[
S(a, b, c)/a^2b^2c^2 = 4(\cos^2 A + \cos^2 B + \cos^2 C) - 3, \tag{3.4}
\]
and so $S \in P_{3,6}^A$ also amounts to
\[
\cos^2 A + \cos^2 B + \cos^2 C \geq 3/4. \tag{3.5}
\]

For a given triangle $\triangle ABC$ with sides $a$, $b$, and $c$, we shall use the following standard notations in euclidean geometry:
- $R =$ radius of the circumscribed circle of $\triangle ABC$, centered at $O$;
- $r =$ radius of the inscribed circle of $\triangle ABC$, centered at $I$;
- $H =$ orthocenter of $\triangle ABC$ (intersection of the three altitudes);
- $N =$ (center of the nine-point circle of $\triangle ABC$) = midpoint of $OH$;
- $\Delta =$ area of the triangle $\triangle ABC$.

It is well known [Cox$_1$, pp. 12–13] that
\[
R = \frac{abc}{4\Delta}, \quad r = \frac{2\Delta}{(a + b + c)}, \tag{3.6}
\]
and, according to Heron's formula,
\[
16\Delta^2 = (a + b + c)(b + c - a)(c + a - b)(a + b - c). \tag{3.7}
\]
Furthermore, the radius of the nine-point circle is $R/2$ and we have Feuerbach's famous identity, $IN = R/2 - r$. The geometric interpretation of this is one of the most fascinating facts in euclidean geometry: the inscribed circle of $\triangle ABC$ lies inside the nine-point circle, and is tangential to it. The fact that $R \geq r/2$ was known before Feuerbach, for Chapple and Euler had already shown earlier that $OI^2 = R(R - 2r)$. Using (3.6) and (3.7), we compute that
\[ R - 2r = \frac{abc}{4A} - \frac{4A}{a + b + c} \]
\[ = \frac{abc(a + b + c) - 16A^2}{4A(a + b + c)} \]
\[ = \frac{abc - (b + c - a)(c + a - b)(a + b - c)}{4A} \]
\[ = \frac{\Gamma(a, b, c)}{4A}. \]

Therefore we get

\[ IN = R/2 - r = \frac{\Gamma(a, b, c)}{8A}, \quad (3.8) \]
\[ OI = \sqrt{abc/4A} \sqrt{\Gamma(a, b, c)/4A} = \sqrt{abc\Gamma(a, b, c)/4A}, \quad (3.9) \]

which give two different geometric interpretations of the Lehmus form \( \Gamma \).

Next, note that

\[ R^2 - (2r)^2 = \frac{a^2 b^2 c^2}{16A^2} - \frac{16A^2}{(a + b + c)^2} \]
\[ = \frac{a^2 b^2 c^2 - (b + c - a)^2(c + a - b)^2(a + b - c)^2}{16A^2}. \]

This leads us to define the following ternary sextic:

\[ S'(x, y, z) = x^2y^2z^2 - (y + z - x)^2(z + x - y)^2(x + y - z)^2 \]
\[ = \Gamma(x, y, z)[xyz + (y + z - x)(z + x - y)(x + y - z)], \quad (3.10) \]

which clearly belongs to \( P_{3,6}^d \). Using this new symmetric form \( S' \), we have then

\[ R^2 - (2r)^2 = \frac{S'(a, b, c)}{16A^2}. \quad (3.11) \]

Next, let us recall some classical expressions for \( OH^2 \) and \( IH^2 \). According to [Ho, pp. 199–200],

\[ OH^2 = R^2(1 - 8 \cos A \cos B \cos C), \quad (3.12) \]
\[ IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C. \quad (3.13) \]
In view of Lemma 3.1, these can both be expressed in terms of the form $S$. We first do this for $OH^2$:

$$OH^2 = R^2 \frac{S(a, b, c)}{a^2b^2c^2} = \frac{S(a, b, c)}{16\Delta^2}. \quad (3.14)$$

Secondly, eliminating $\cos A \cos B \cos C$ from (3.12) and (3.13), we get

$$OH^2 - 2IH^2 = R^2 - (2r)^2 = S'(a, b, c)/16\Delta^2.$$

Therefore,

$$IH^2 = \left[ \frac{OH^2 - (R^2 - (2r)^2)}{2} \right] = \frac{1}{2} \left[ \frac{S(a, b, c)}{16\Delta^2} - \frac{S'(a, b, c)}{16\Delta^2} \right] = \frac{F(a, b, c)}{16\Delta^2}, \quad (3.15)$$

where $F := (S - S')/2$. By (3.15), $F \in \text{Sym } P^4_{3,6}$, and $S = S' + 2F$ gives a non-trivial decomposition of $S$ in the cone $\text{Sym } P^4_{3,6}$. (Thus, $S$ is not extremal in $\text{Sym } P^4_{3,6}$.) Note that $F$ is a very nice-looking symmetric sextic:

$$2F(a, b, c) = [a^2b^2c^2 - (b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)] - [a^2b^2c^2 - (b + c - a)^2(c + a - b)^2(a + b - c)^2] = (b + c - a)^2(c + a - b)^2(a + b - c)^2 - (b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2). \quad (3.16)$$

By direct expansion, one can check that

$$S'(a, b, c) = -\sum a^6 + 2\sum a^5b + \sum a^4b^2 + 6\sum a^4bc + 4\sum a^3b^3 + 4\sum a^3b^3c - 9a^2b^2c^2, \quad (3.17)$$

$$F(a, b, c) = \sum a^6 - \sum a^5b - \sum a^4b^2 + 3\sum a^4bc + 2\sum a^3b^3 - 2\sum a^3b^3c + 6a^2b^2c^2. \quad (3.18)$$

Note that all "non-even" terms cancel out in $S' + 2F$ to give back Robinson's form $S = \sum a^6 - \sum a^4b^2 + 3a^2b^2c^2$. Also note that the presence of the term $-\sum a^6$ in $S'(a, b, c)$ implies that, although $S'$ is in $P^4_{3,6}$, it is not in $P^+_{3,6}$. Summarizing the above calculations, we have the following (where $\Gamma$ means $\Gamma(a, b, c)$, etc.):
Here, our expression for $IH^2$ in terms of the new form $F$ happens to be different from that given by Euler. According to Euler (see [Cox$_2$]),

$$IH^2 = 4(R+r)^2 - q,$$

where $q := ab + bc + ca$. The fact that we have two expressions for $IH^2$ enables us to say something useful about $F(a, b, c)$. In fact, we now have

$$\frac{F(a, b, c)}{16\Delta^2} = 4 \left( \frac{abc}{4\Delta} + \frac{2\Delta}{a+b+c} \right)^2 - q$$

$$= 4 \left( \frac{2abc(a+b+c) + 16\Delta^2}{8\Delta(a+b+c)} \right)^2 - q$$

$$= 4 \left( \frac{2abc + (b+c-a)(c+a-b)(a+b-c)}{8\Delta} \right)^2 - q$$

$$= \frac{(\sum a^3 - \sum a^2b)^2}{16\Delta^2} - q.$$ 

Therefore, we obtain the following new expression for $F$:

$$F(a, b, c) = \left( \sum a^3 - \sum a^2b \right)^2$$

$$- (ab + bc + ca)(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

(3.20)
Since this holds for all $a, b, c$ which form the three sides of a triangle, it follows that it holds already as a polynomial equation. (Of course this can be checked explicitly by a direct expansion, but it is more interesting to show an actual derivation especially since it made nice use of the classical formula of Euler.) The new expression (3.20) for $F$ has the following immediate application:

**Proposition 3.21.** $F(x, y, z) \in P_{3,6}^+.$

**Proof.** Consider any $(a, b, c) \in \mathbb{R}_+^3.$ If

$$b + c - a \geq 0, \quad c + a - b \geq 0, \quad a + b - c \geq 0,$$

(3.22)

then $a, b, c$ form the three sides of a (possibly degenerate) triangle. In this case, it follows from (3.15) that $F(a, b, c) \geq 0.$ On the other hand, if the inequalities in (3.22) do not all hold, then exactly one of $b + c - a, c + a - b, a + b - c$ is negative, in which case

$$-(ab + bc + ca)(a + b + c)(b + c - a)(c + a - b)(a + b - c) \geq 0,$$

and hence $F(a, b, c) \geq 0$ by (3.20). Q.E.D.

Next, we go back to the original definition of $F$ in (3.16) to make the following observation:

**Proposition 3.23.** $F(x, y, z)$ is $|\cdot|$-convex.

**Proof.** To check that $F(x, y, z) \geq F(|x|, |y|, |z|)$ for all $(x, y, z) \in \mathbb{R}^3,$ we may assume that $a, b, c$ are not all $\geq 0.$ By symmetry, and by the fact that $F$ is a form of even degree, it suffices to treat the case when $x \leq 0$ and $y, z \geq 0.$ In this case,

$$2[F(x, y, z) - F(|x|, |y|, |z|)]$$

$$= (y + z - x)^2(z + x - y)^2(x + y - z)^2 - (|y| + |z| - |x|)^2(|z| + |x| - |y|)^2(|x| + |y| - |z|)^2$$

$$= (z + x - y)^2(x + y - z)^2[(y + z - x)^2 - (y + z + x)^2]$$

$$= (z + x - y)^2(x + y - z)^24|z|(y + z) \geq 0.$$ Q.E.D.

Combining Propositions 3.21 and 3.23, we now reach the following conclusion:

**Theorem 3.24.** $F(x, y, z) \in \text{Sym } P_{3,6}.$

Note that the equation $S = S' + 2F$ does not contradict the extremality of $S$ in $P_{3,6}$ (proved in [CL$_2$]), since, although we have just shown that $F$
belongs to $P_{3,6}$, $S'$ does not. To close this section, we derive below an interesting relation between the three forms $S$, $F$, and $\Gamma$:

**Proposition 3.25.** $S + \Gamma^2 = 2(F + xyz\Gamma')$.

**Proof.** From (3.10), we have

$$S'(x, y, z) = \Gamma(x, y, z)(2xyz - \Gamma(x, y, z)).$$

Equating this with $S - 2F$, we get the desired equation. Q.E.D.

Note that (3.25) has a nice geometric interpretation. In fact, when evaluated on $(a, b, c)$ which are the three sides of a triangle, this equation expresses precisely the parallelogram law for the parallelogram with $OI$ and $IH$ as two of its sides in the picture (3.19). Clearly, this remark is also sufficient to give a geometric proof of (3.25) as a polynomial identity.

### 4. The Extremal Property of $F$

We begin this section by explicitly determining all the real zeros of $F$. Since $F$ is a form, we should work in the projective space $\mathbb{RP}^2$ when we count the zeros of $F$. Bearing this in mind, we have:

**Theorem 4.1.** The form $F$ has exactly the following seven real zeros:

$$(0, 1, \pm 1), \quad (\pm 1, 0, 1), \quad (1, \pm 1, 0), \quad \text{and} \quad (1, 1, 1).$$

**Proof.** By direct substitution, we can check the following:

$$F(x, y, y) = x^4(x - y)^2, \quad (4.2)$$
$$F(y + z, y, z) = 4y^2z^2(y + z)^2, \quad (4.3)$$
$$F(-y - z, y, z) = 36y^2z^2(y + z)^2, \quad (4.4)$$
$$F(0, y, z) = (y^2 - z^2)^2[(y - z)^2 + y^2 + z^2]/2. \quad (4.5)$$

From (4.2), (4.5), and symmetry, it is immediately clear that the seven points listed in the theorem are indeed zeros of $F$. Now consider any real zero $(a, b, c) \neq (0, 0, 0)$ of $F$. If $a = 0$, then in view of (4.5), we must have $b = \pm c$. Thus we may assume that $abc \neq 0$ in the following.

**Case 1.** $a, b, c > 0$. In this case, $a, b, c$ must form the three sides of a nondegenerate $\Delta ABC$. (For, if not, then we have, say $b + c - a \leq 0$, and
hence \( c + a - b > 0, \ a + b - c > 0 \). But from (4.3), we must have \( a \neq b + c \).

Therefore,

\[
(b + c - a)(c + a - b)(a + b - c) < 0,
\]

and (3.20) gives \( F(a, b, c) > 0 \). By the geometric interpretation (3.15) of \( F \), we see that \( F(a, b, c) = 0 \) means that the orthocenter \( H \) of \( \triangle ABC \) coincides with the center \( I \) of its inscribed circle. Clearly, this can happen only for an equilateral triangle, so we have \( a = b = c \).

After treating Case 1, we are left essentially with the following case:

**Case 2.** \( a < 0, \ b, c > 0 \). By the \( 1 \)-convexity of \( F, \ F(a, b, c) = 0 \Rightarrow F(|a|, |b|, |c|) = 0 \), so by Case 1, we must have \( -a = b = c \). By (4.2) this is impossible, so Case 2 actually never arises. Q.E.D.

Another somewhat easier, but less geometric, method for determining the zeros of the form \( F \) will be mentioned briefly in Section 5.

Using the specialization formulas (4.2)-(4.4) above, we shall now establish the following main result in this section:

**Theorem 4.6.** The form \( F \) is extremal in the cone \( P_{3,6} \) (and therefore also in the cone \( \text{Sym} \ P_{3,6} \)).

**Proof.** Let \( f \in P_{3,6} \) be such that \( F \geq f \). Since, by (4.2), (4.3), and (4.4), \( F(x, y, y), F(y + z, y, z), \) and \( F(-y - z, y, z) \) are extremal as binary forms, we must have

\[
\begin{align*}
f(x, y, y) &= aF(x, y, y), \\
f(y + z, y, z) &= bF(y + z, y, z), \\
f(-y - z, y, z) &= cF(-y - z, y, z),
\end{align*}
\]

for suitable constants \( a, b, c \). Evaluating \( f \) at \((2, 1, 1)\) using (4.7) and (4.8), we get \( f(2, 1, 1) = aF(2, 1, 1) = bF(2, 1, 1) \). Cancelling \( F(2, 1, 1) = 16 \), we get \( a = b \). Similarly, evaluating \( f \) at \((-2, 1, 1)\) using (4.7) and (4.9), we get \( f(-2, 1, 1) = aF(-2, 1, 1) = cF(-2, 1, 1) \). Cancelling \( F(-2, 1, 1) = 144 \), we get \( c = a(= b) \). Therefore, \( f - cF \) vanishes on the planes \( y = z \) and \( x = y + z \) as well as on \( x + y + z = 0 \). Similarly, we can argue that \( f - cF \) also vanishes on the planes \( z = x, \ y = z + x \) and \( x = y, \ z = x + y \). This means that \( f - cF \) is divisible by the seven linear forms \( x + y + z, \ y - z, \ z - x, \ x - y \) and \(-x + y + z, \ x - y + z, \ x + y - z \). Since \( f - cF \) is a sextic, we conclude that \( f = cF \). Q.E.D.

The method given above for proving that \( F \) is extremal is in fact a special case of a graph-theoretic method for testing the extremality of psd forms of
any degree and in any number of variables. For the details of this more general method, we refer the reader to [CLR].

We shall now end this section by obtaining two other expressions for the form $F$. At this point, we introduce the following basic form $P$ which will play a very important role in the rest of the paper:

$$P(x, y, z) = (x - y)^2(y - z)^2(z - x)^2 = \sum x^4y^2 - 2 \sum x^4yz - 2 \sum x^3y^3z + 2 \sum x^3y^2z - 6x^3y^3z.$$  (4.10)

**Lemma 4.11.** Let $f$ and $g$ be two symmetric ternary sextics which coincide on the plane $\pi_1 = \{ y = z \}$. Then $g - f = \beta P$ for some real constant $\beta$. If $f$ and $g$ also coincide on a plane $\pi$ other than $\pi_1 = \{ y = z \}$, $\pi_2 = \{ z = x \}$, and $\pi_3 = \{ x = y \}$, then $f = g$.

**Proof.** The fact that $f$ and $g$ coincide on $\pi_1$ means that the symmetric form $h := g - f$ satisfies $h(x, y, y) = 0$. Therefore, $h = (y - z)^2 h_1$ for some form $h_1$. From $h(x, y, z) = h(x, z, y)$, we get $(y - z) h_1(x, y, z) = (z - y) h_1(x, z, y)$. Thus, $h_1(x, y, z) = -h_1(x, z, y)$. In particular, we must have $h_1(x, y, y) = 0$, so $h_1 = (y - z)^2 h_2$ for some form $h_2$. We have now seen that $h$ is divisible by $(y - z)^2$. By symmetry, $h$ must also be divisible by $(z - x)^2$ and $(x - y)^2$. Therefore, $h = \beta P$ for some $\beta \in \mathbb{R}$. The last part of the lemma follows easily from this since $P$ has no linear factors other than scalar multiples of $y - z, z - x, \text{ and } x - y$. Q.E.D.

Using this lemma, it is now remarkably easy to get other new expressions for the form $F$. From (4.2) we have $F(x, y, y) = x^4(x - y)^2$. On the other hand, the Schur form $\Gamma_6$ also has the property that $\Gamma_6(x, y, y) = x^4(x - y)^2$. Thus, Lemma 4.11 implies that $\Gamma_6 - F = \beta P$ for some $\beta \in \mathbb{R}$. Comparing the coefficients of $x^4y^2$ on both sides, we see that $\beta = 1$. Therefore, we have proved the following (which, of course, can also be checked directly using the expansions (2.1), (3.18), and (4.10)):

**Proposition 4.12.** $F = \Gamma_6 - P$.

This is a somewhat surprising conclusion, since it shows that the Schur form $\Gamma_6$ fails to be extremal already in the cone $\text{Sym } P_{3,6}$, a fact which was not previously known. In Section 9, we shall come back to discuss the extremal properties of the other Schur forms $\Gamma_d$ ($d \neq 6$).

Next, consider the symmetric form $g = \sum (y + z - x)^4(y - z)^2$, for which we clearly have $g(x, y, y) = 2x^4(x - y)^2$. Applying the lemma once more, we see that $F - (1/2) g = \beta' P$ for some $\beta' \in \mathbb{R}$. By comparing coefficients again, or by evaluating this equation at any point which is not a zero of $P$, we see that $\beta' = -4$. Therefore, we have proved:
PROPOSITION 4.13. \( F = \frac{1}{2} \sum (y+z-x)^4(y-z)^2 - 4P. \)

Of course, it is also possible to check the last two propositions by direct expansion. But our presentations above served to explain on general ground why they are natural and inevitable conclusions. Note that by applying the same method, we can also derive, for instance, new expressions for the square of the Lehmus form, \( \Gamma(x, y, z)^2 \), and for the Robinson form \( S(x, y, z) = \Gamma(x^2, y^2, z^2) \). In fact, from

\[
\Gamma(x, y, y)^2 = x^2(x - y)^4
\]

and

\[
S(x, y, y) = \Gamma(x^2, y^2, y^2) = x^2(x^2 - y^2)^2,
\]

we can deduce, just as above, that:


(1) \( \Gamma(x, y, z)^2 = \sum x^2(x - y)^2(x - z)^2 - 2P(x, y, z) \)

\[
= \frac{1}{4} \sum (y + z - x)^2(y - z)^4 - P(x, y, z);
\]

(2) \( S(x, y, z) := \sum x^2(x^2 - y^2)(x^2 - z^2) \)

\[
= \frac{1}{4} \sum (y + z - x)^2(y^2 - z^2)^2 - P(x, y, z).
\]

Note that, in (1) above, the first expression for \( \Gamma(x, y, z)^2 \) is just the direct expansion of \( (\sum x(x - y)(x - z))^2 \), by using the formula \( (a + b + c)^2 = \sum a^2 + 2 \sum ab \).

Lemma 4.11 is also very useful for checking the extremality of forms in the two cones Sym \( P_{3,6} \) and Sym \( P_{3,6}^+ \). We shall return to this theme in Section 8.

5. THE FORM G AND THE FORMS kQ - P

We start by recalling the fact (Theorem 4.1) that the form \( F \) has exactly the following seven real zeros projectively:

\[
(0, 1, \pm 1), \quad (\pm 1, 0, 1), \quad (1, \pm 1, 0), \quad \text{and} \quad (1, 1, 1). \quad (5.1)
\]

If we apply the invertible linear transformation

\[
(x, y, z) \mapsto (y + z, z + x, x + y), \quad (5.2)
\]
the seven points above, viewed in the range space, have preimages

\[(1, 0, 0), \quad (0, -1, 1),\]
\[(0, 1, 0), \quad (1, 0, -1),\]
\[(0, 0, 1), \quad (-1, 1, 0), \quad \text{and}\]
\[(1/2, 1/2, 1/2).\]

Therefore, it will be advantageous to consider the transformed form (using the notation of Section 2),

\[G(x, y, z) := F^+(x, y, z)/4 = F(y + z, z + x, x + y)/4 \in \text{Sym } P_{3,6}, \quad (5.3)\]

since, by the above observations, \(G\) has exactly the seven real zeros listed in (5.3), which include the three unit vectors. This latter fact means that \(G\) has no \(\sum x^6\) term, and therefore also no \(\sum x^5y\) term, since \(G\) is psd. For this reason, it is often easier to work with the form \(G\) than with the form \(F\). Moreover, applying the \("+\"\)-transform to the expression for \(F\) in (4.13), we get the following very simple expression for \(G\):

\[G(x, y, z) = 2 \sum x^4(y - z)^2 - P(x, y, z), \quad (5.5)\]

where \(P\) is defined as in (4.10). Using the expansion for \(P\) in (4.10), we get immediately the expansion for \(G\) as follows:

\[G(x, y, z) = \sum x^4y^2 - 2 \sum x^4yz + 2 \sum x^3y^3 - 2 \sum x^3yz^2 + 6x^2y^2z^2. \quad (5.6)\]

Since \(G = F^+/4\) and \(F = 4G^d\) (cf. Section 2), the fact that \(F\) is extremal in \(P_{3,6}\) implies that \(G\) is also extremal in \(P_{3,6}\). However, the \(\|\|\)-convexity of \(F\) does not imply the \(\|\|\)-convexity of \(G\). In fact, we have \(G(0, -1, 1) = 0\), but \(G(|0|, |-1|, |1|) = G(0, 1, 1) = 4\), so \(G\) is not \(\|\|\)-convex.

Perhaps not surprisingly, our next result is:

**Theorem 5.7.** \(G(x, y, z)\) is not a sum of squares of cubic forms (and therefore the same is true of \(F(x, y, z)\)).

**Proof:** Since \(G(x, y, z)\) does not have a \(\sum x^6\) term and therefore no \(\sum x^5y\) term, it is easy to apply the \("term inspection"\) method of [CL2] to show that \(G\) is not a sum of squares of cubic forms. On the other hand, assuming everything we have said about \(G\) so far, we can also reach this conclusion directly as follows. If \(G\) is a sum of squares of cubic forms, then, by the extremal property of \(G\), it must be the square of a cubic form \(C\). This is impossible since \(G\) has only seven real zeros, while \(C\), as an
indefinite form, has infinitely many real zeros projectively (see, e.g., [CLR1, Proposition 2.5]). Q.E.D.

The expression (5.5) for $G(x, y, z)$ calls to attention the symmetric sextic

$$Q = Q(x, y, z) = \sum x^4(y - z)^2. \quad (5.8)$$

The two sextics $P$ and $Q$ can be expressed in terms of the three cubics

$$U = x^2(y - z), \quad V = y^2(z - x), \quad W = z^2(x - y), \quad (5.9)$$

as

$$P = (U + V + W)^2, \quad (5.10)$$

$$Q = U^2 + V^2 + W^2. \quad (5.11)$$

Using these expressions, we shall investigate the properties of the forms $kQ - P$ for $k = 1, 2, 3$ in the balance of this section. This will enable us to compare various other forms with the basic form $P$ (on $\mathbb{R}^3$ and on $\mathbb{R}_+^3$) in Section 6.

**Proposition 5.12.** (1) $Q - P = -2(VW + WU + UV) \in P_3,^+$.

(2) $2Q - P = U^2 + V^2 + W^2 - 2(VW + WU + UV) \in P_3,^+$.

(3) $3Q - P = 2(U^2 + V^2 + W^2 - VW - WU - UV) \in \Sigma_3,^+$, the cone of ternary sextics which are sums of squares of cubics.

**Proof.** The expressions for $kQ - P$ ($k = 1, 2, 3$) in terms of $U, V,$ and $W$ follow immediately from (5.10) and (5.11). For the rest, we proceed as follows:

(1) To show that $Q(x, y, z) \geq P(x, y, z)$ for $(x, y, z) \in \mathbb{R}_+^3$, we may assume, by symmetry, that $x \geq y$ and $x \geq z$. Then $x^4 \geq (x - y)^2(x - z)^2$ and so

$$Q(x, y, z) \geq x^4(y - z)^2 \geq (x - y)^2(y - z)^2(z - x)^2 = P(x, y, z). \quad (5.13)$$

(2) $2Q - P$ is just the form $G$ by (5.5), so if we assume the knowledge that $G$ is psd, the desired conclusion follows. Alternatively, we can transform the expression for $2Q - P$ in (5.12)(2) into the following somewhat less symmetrical form:

$$2Q - P = -4VW + (U - V - W)^2$$

$$= 4y^2z^2(x - y)(x - z) + (y - z)^2(x(x + y + z) - yz)^2. \quad (5.14)$$

If $x \geq y$ and $x \geq z$, this shows immediately that $(2Q - P)(x, y, z) \geq 0$. By symmetry, it follows that $(2Q - P)(x, y, z) \geq 0$ for all $(x, y, z) \in \mathbb{R}^3$. There-
fore, we have now a second proof for the fact that $G \in P_{3,6}$ (and also $F \in P_{3,6}$, since $F(x, y, z) = 4G^4(x, y, z)$ in the notation of Section 2).

(3) This follows upon noting that
\[2(U^2 + V^2 + W^2 - VW - WV - UW) = (U - V)^2 + (V - W)^2 + (W - U)^2.\]
Q.E.D.

Remarks 5.15. (a) The form $Q - P$ in (1) above is just $2H$ where $H$ is the form $H_3(x, y, z) = \Gamma(yz, zx, xy)$ defined in Section 2. In fact,
\[Q - P = -2(VW + WU + UV)\]
\[= 2[y^2z^2(x - y)(x - z) + z^2x^2(y - z)(y - x) + x^2y^2(z - x)(z - y)]\]
\[= 2\Gamma(yz, zx, xy)\]
\[= 2H(x, y, z).\]

(b) The form $4Q - P$ has also an interesting expression as a sum of squares of three cubic forms. In fact, if we start with (5.14), and add it to the two similar equations obtained by cyclic permutations of $x, y, z$, we get
\[3(2Q - P) = 4 \sum y^2z^2(x - y)(x - z) + \sum (y - z)^2(x(x + y + z) - yz)^2\]
\[= 2(Q - P) + \sum (y - z)^2(x(x + y + z) - yz)^2\]
by using (a). By transposition, we have the desired expression:
\[4Q - P = \sum (y - z)^2(x(x + y + z) - yz)^2. \quad (5.16)\]

There are at least a few more useful applications of the expression (5.14) for $G = 2Q - P$. For instance, it is a relatively easy matter to determine the real zeros of $G$ (and therefore also the real zeros of $F$) from it. Since we already know the structure of these zero sets, we won't repeat ourselves here. Instead, we shall use (5.14) to derive an explicit expression of $G(x, y, z)$ as a sum of squares of rational functions. (Such an expression is guaranteed to exist by Artin's solution of Hilbert's Seventeenth Problem.) In fact, using (5.14) three times as above, we have
\[(x^2 + y^2 + z^2) G(x, y, z)\]
\[= \sum x^2G(x, y, z)\]
\[= \sum 4x^2y^2z^2(x - y)(x - z) + \sum x^2(y - z)^2(x(x + y + z) - yz)^2\]
\[= 2x^2y^2z^2 \sum (y - z)^2 + \sum x^2(y - z)^2(x(x + y + z) - yz)^2. \quad (5.17)\]
After multiplying this by $x^2 + y^2 + z^2$ and using the 4-square identity, we then obtain an expression of $G(x, y, z)$ as a sum of eight squares of rational functions. A different (and better) expression can also be obtained by multiplying $G(x, y, z)$ by $U^2 + V^2 + W^2$ instead. From (5.14) (and its cyclic permutations), we have

$$
(U^2 + V^2 + W^2) G(x, y, z) = -4UVW \sum U + \sum U^2(U - V - W)^2. \tag{5.18}
$$

Since $UVW = -x^3y^2z^2(U + V + W)$, we have from (5.10) and (5.11),

$$
QG = 4x^2y^2z^2(U + V + W)^2 + \sum U^2(U - V - W)^2
= 4x^2y^2z^2P + \sum x^4(y - z)^4(x + y + z) - yz)^2. \tag{5.19}
$$

Multiplying this by $Q = U^2 + V^2 + W^2$, we get an expression of $G$ as a sum of four squares of rational functions.

Next we shall investigate the extremal properties of the forms $kQ - P$ $(k = 1, 2, 3)$ in the respective cones indicated in (5.12). For any cone $C$, let us write $\mathcal{E}(C)$ for the set of extremal elements of $C$. We shall continue to use the notation $\Sigma_{3,6}$ introduced in (5.12)(3), and shall write $\text{Sym } \Sigma_{3,6}$ for the subcone of $\Sigma_{3,6}$ consisting of its symmetric forms.

**Theorem 5.20.** (1) $2H = Q - P \in \mathcal{E}(P_{3,6}^+)$,
(2) $G = 2Q - P \in \mathcal{E}(P_{3,6})$,
(3) $2K := 3Q - P \in \mathcal{E}(\text{Sym } \Sigma_{3,6})$.

Since we have already observed the extremality of the form $G$ (in the paragraph following (5.6)), we need only prove (1) and (3). In order to prove (3), we shall need the following lemma.

**Lemma 5.21.** Suppose $f(x, y, z)$ is a cubic form such that $Q(x, y, z) \geq f(x, y, z)^2$ on $\mathbb{R}^3$. Then $f = aU + bV + cW$ for suitable real numbers $a, b,$ and $c$.

**Proof.** Since $Q(x, y, z) = x^4(y - z)^2 + \text{(terms of degree } \leq 3 \text{ in } x)$, and $(y - z)^2$ is extremal as a binary form, $Q \geq f^2$ implies that

$$
f(x, y, z) = ax^2(y - z) + \text{(terms of degree } \leq 1 \text{ in } x)
$$

for some $a \in \mathbb{R}$. By symmetry, we see that there exist further real constants $b, c, d$ such that

$$
f(x, y, z) = ax^2(y, z) + by^2(z - x) + cz^2(x - y) + dxyz.
$$

Since $f(1, 1, 1) = 0$, we have $d = 0$ and so $f = aU + bV + cW$. Q.E.D.
We can now present the

**Proof of (5.20)(3).** Suppose $2K(x, y, z) = \sum_i f_i(x, y, z)$, where $f_i \in \text{Sym} \Sigma_{3,6}$, say $f_i = \sum_j f_{ij}^2$. Then $3Q = P + \sum_i \sum_j f_{ij}^2$ implies that $3Q \geq f_{ij}^2$ for all $i, j$ and so by (5.21), $f_{ij}(x, y, z) = a_{ij}U + b_{ij}V + c_{ij} W$ for suitable constants $a_{ij}, b_{ij},$ and $c_{ij}$. Therefore,

$$2K = 3Q - P = (U - V)^2 + (V - W)^2 + (W - U)^2$$

$$= \sum_i \sum_j (a_{ij}U + b_{ij}V + c_{ij} W)^2.$$ 

It is easy to check that $U^2, V^2, W^2, UV, UW,$ and $VW$ are linearly independent. (In fact, $U, V,$ and $W$ are algebraically independent.) Therefore, by comparing the coefficients of $U^2, V^2, \ldots$, etc. in the preceding equation, we get

$$2 = \sum_i \sum_j a_{ij}^2 = \sum_i \sum_j b_{ij}^2 = \sum_i \sum_j c_{ij}^2,$$

$$-1 = \sum_i \sum_j a_{ij}b_{ij} - \sum_i \sum_j a_{ij}c_{ij} - \sum_i \sum_j b_{ij}c_{ij}.$$ 

Thus, $\sum_i \sum_j (a_{ij} + b_{ij} + c_{ij})^2 = 2 + 2 + 2 - 2 - 2 - 2 = 0$, and so $c_{ij} = -(a_{ij} + b_{ij})$ for all $i, j$. On the other hand, it is easy to see that the only quadratic forms in $U, V, W$ which are symmetric in $x, y, z$ are of the shape $\alpha(U^2 + V^2 + W^2) + \beta(UV + UW + VW)$. Thus,

$$f_i = \sum_j (a_{ij}U + b_{ij}V - (a_{ij} + b_{ij}) W)^2$$

$$= \alpha_i(U^2 + V^2 + W^2) + \beta_i(UV + UW + VW)$$

for suitable $\alpha_i, \beta_i \in \mathbb{R}$. Comparing coefficients once more, we get $\sum_i a_{ij}^2 = \sum_j b_{ij}^2 = \sum_j (a_{ij} + b_{ij})^2 = \alpha_i$ and so $\beta_i = \sum_j 2a_{ij}b_{ij} = -\alpha_i$. Hence $f_i = \alpha_i K$ for all $i$, as desired. Q.E.D.

Finally, let us now give the

**Proof of (5.20)(1).** We shall deduce $H = (Q - P)/2 \in \mathcal{E}(P_{3,6}^+)$ from $I \in \mathcal{E}(P_{3,3}^+)$). The latter is equivalent to the fact that the Robinson form $S$ is extremal among the even forms in $P_{3,6}$. This is fact is weaker than $S \in \mathcal{E}(P_{3,6})$; for an easy direct proof, see [CL2, Lemma (3.9)] (cf. also [Ri1, CLR2]). Note that

$$H(x, y, z) = I'(yz, zx, xy) = x^3y^3z^3I'(1/x, 1/y, 1/z).$$
Replacing \( x, y, z \) by their inverses, we get
\[
\Gamma(x, y, z) = x^3y^3z^3H(1/x, 1/y, 1/z).
\]

Now suppose \( H \geq H' \geq 0 \) on \( \mathbb{R}^3_+ \), where \( H' \) is a ternary sextic. Since the degree of \( H \) in \( x \) (resp. \( y, z \)) is 3, the degree of \( H' \) in \( x \) (resp. \( y, z \)) is \( \leq 3 \). Therefore, \( x^3y^3z^3H'(1/x, 1/y, 1/z) \) is a ternary cubic. From
\[
\Gamma(x, y, z) - x^3y^3z^3H(1/x, 1/y, 1/z) \geq x^3y^3z^3H'(1/x, 1/y, 1/z)
\]
for \( x, y, z > 0 \) (and continuity), we see that \( x^3y^3z^3H'(1/x, 1/y, 1/z) = a\Gamma(x, y, z) \) for some \( a \in \mathbb{R} \), and therefore \( H'(x, y, z) = ax^3y^3z^3\Gamma(1/x, 1/y, 1/z) = aH(x, y, z) \). Q.E.D.

Let us now record some consequences of the extremal properties of \( kQ - P \) proved above.

**Corollary 5.22.** Let \( f_k(x, y, z) = kQ - P \) where \( k \in \mathbb{R} \). Then (1) \( f_k + P_{3, 6}^+ \) iff \( k \geq 1 \), (2) \( f_k \in P_{3, 6} \) iff \( k \geq 2 \), and (3) \( f_k \in \Sigma_{3, 6} \) iff \( k \geq 3 \).

**Proof.** The “if” part is clear from (5.12). For the “only if” part, let us prove it for (3); the two other cases are similar. Assume that \( f_k \in \Sigma_{3, 6} \), but that \( k < 3 \). Write \( k = 3 - \varepsilon \) where \( \varepsilon > 0 \). Then \( f_k = (3 - \varepsilon)Q - P \Rightarrow 2K = 3Q - P = f_k + \varepsilon Q \), which contradicts our proven result \( 2K \in \delta(\text{Sym } \Sigma_{3, 6}) \). Hence we must have \( k \geq 3 \). (Note that this part actually gives a generalization of Theorem 5.7.) Q.E.D.

6. **Comparison of Various Sextics with \( P \)**

In this section, we shall compare various sextic forms \( f \) (in \( \text{Sym } P_{3, 6} \) or in \( \text{Sym } P_{3, 6}^+ \)) with the basic form \( P = P(x, y, z) = (x - y)^2(y - z)^2(z - x)^2 \). This is done by proving inequalities of the sort \( f(x, y, z) \geq \alpha P(x, y, z) \), for \((x, y, z) \in \mathbb{R}^3_+ \), or as the case may be, \((x, y, z) \in \mathbb{R}^3_+ \), where \( \alpha \) is a suitable positive constant. Although it will not be stated explicitly, it will be understood in the following that all such inequalities obtained are actually “the best possible,” in the very strong sense that \( \alpha \) is always chosen such that \( f - \alpha P \) is extremal in \( P_{3, 6} \), or, as the case may be, extremal in \( P_{3, 6}^+ \). In particular, this implies that in all cases \( \alpha \) will indeed be as large as possible for the inequalities to hold. For many of the forms \( f \in \text{Sym } P_{3, 6} \) treated below, we shall also be able to compare \( f \) with the function \( \beta P([x], [y], [z]) \), by proving inequalities of the sort \( f(x, y, z) \geq \beta P([x], [y], [z]) \) for all \((x, y, z) \in \mathbb{R}^3_+ \), where \( \beta \) is a constant \( > \alpha \). (Recall that \( P(x, y, z) \) is \([\cdot]\)-convex.) All inequalities of this type will also be chosen the best possible, in the strong sense that \( f - \beta P \) is extremal in \( P_{3, 6}^+ \). In particular, \( \beta \) is also as large as possible in all cases. There is, of course, a good reason
why so many symmetric ternary sextics can be “compared” with the special form \(P\). We shall come back to this point in Section 8 after we develop the right machinery for giving the explanation. As far as we can determine, all inequalities obtained in this section are hitherto unknown.

Before we state any inequalities, let us first introduce the very useful notion of the dual of a sextic form. Let \(f(x, y, z)\) be any sextic such that \(\deg_x f, \deg_y f, \deg_z f\) are all \(\leq 4\). (This is the case, for instance, if \(f\) is psd and vanishes on \((1, 0, 0), (0, 1, 0), \) and \((0, 0, 1)\).) Then \(x^4y^4z^4f(1/x, 1/y, 1/z)\) is clearly a polynomial in \((x, y, z)\); in fact it is also a sextic form. We shall call this sextic the dual of \(f\), and denote it by \(f^*\). Clearly, we have again \(\deg_x f^*, \deg_y f^*, \deg_z f^* \leq 4\), so it makes sense to form the double dual \(f^{**}\). The following properties are all easy to verify, and will be assumed in the balance of this paper.

\[
\begin{align*}
f^{**} &= f. \quad \text{(6.0)} \\
f \in P_{3,6} \text{ (resp. } P^+_{3,6} \text{)} &\iff f^* \in P_{3,6} \text{ (resp. } P^+_{3,6} \text{)}. \quad \text{(6.1)}
\end{align*}
\]

\[
\begin{align*}
f \in \mathcal{S}(P_{3,6}) \text{ (resp. } \mathcal{S}(P^+_{3,6}) \text{)} &\iff f^* \in \mathcal{S}(P_{3,6}) \text{ (resp. } \mathcal{S}(P^+_{3,6}) \text{)}. \quad \text{(6.2)}
\end{align*}
\]

\[
\begin{align*}
f \in \Sigma_{3,6} &\iff f^* \in \Sigma_{3,6}. \quad \text{(6.3)}
\end{align*}
\]

\[
\begin{align*}
f \text{ is symmetric iff } f^* \text{ is symmetric.} \quad \text{(6.4)}
\end{align*}
\]

The analogues of (6.2) and (6.3) for symmetric forms.

\[
\begin{align*}
f^{**} &= f. \quad \text{(6.5)}
\end{align*}
\]

Since we shall be interested mostly in symmetric forms in this paper, it will be worthwhile to figure out more explicitly how the duality operator works on the class of symmetric ternary sextics. The class of such sextics for which the dual is defined consists of

\[f = \alpha \sum x^4y^2 + \beta \sum x^4yz + \gamma \sum x^3y^3 + \delta \sum x^3y^2z + \varepsilon x^2y^2z^2.\]

By a direct calculation, we see that \(\sum x^4y^2, \sum x^3y^2z, \text{ and } x^2y^2z^2\) are self-dual, while \(\sum x^4yz\) and \(\sum x^3y^3\) are dual to each other. Therefore,

\[f^* = \alpha \sum x^4y^2 + \gamma \sum x^4yz + \beta \sum x^3y^3 + \delta \sum x^3y^2z + \varepsilon x^2y^2z^2.\]

Thus, if we write symbolically \(f = (\alpha, \beta, \gamma, \delta, \varepsilon)\), then \(f^* = (\alpha, \gamma, \beta, \delta, \varepsilon)\). In particular, \(f\) is self-dual iff \(\beta = \gamma\). A good example of a self-dual symmetric form is \(P(x, y, z) = (x - y)^2(y - z)^2(z - x)^2\), which will play a crucial role in this section. (We note incidentally that we also have \(P^+ = P = P^d\), in the notation of Section 2.) Another interesting example of a self-dual symmetric form is \(\sum x^2(y^2 - z^2)^2\), which will also be examined later in this section. The form \(H(x, y, z) = \sum y^2z^2(x - y)(x - z) = (xyz)^3\Gamma(1/x, 1/y, 1/z)\) has dual \(xyz\Gamma(x, y, z)\).
We shall now begin our comparison of various sextics with \( P(x, y, z) = (x - y)^2(y - z)^2(z - x)^2 \). From the results in Section 5, we first derive the following.

**Proposition 6.6.** For all \((x, y, z) \in \mathbb{R}^3\),

1. \( \sum x^4(y - z)^2 \geq (1/2) P(x, y, z) \);
2. \( \sum x^4(y - z)^2 \geq P(|x|, |y|, |z|) \).

**Proof.** (1) follows from (5.5) and the fact that \( G \) is psd (see also the direct argument in the proof of (5.12)(2)). For (2), note first that \( Q(x, y, z) = \sum x^4(y - z)^2 \geq P(x, y, z) \) for all \( x, y, z \geq 0 \) by (5.12)(1). Since \( Q \) is obviously \( |\cdot| \)-convex, (2) follows for general \( x, y, z \in \mathbb{R} \). Q.E.D.

By taking duals, we deduce immediately the following related inequalities.

**Proposition 6.7.** For all \((x, y, z) \in \mathbb{R}^3\),

1. \( \sum y^2z^2(y - z)^2 \geq (1/2) P(x, y, z) \);
2. \( \sum y^2z^2(y - z)^2 \geq P(|x|, |y|, |z|) \).

Next we shall compare the forms \( F \) and \( G \) with \( P \). Note that the inequality below do not contradict the fact that \( G, F \in \mathcal{E}(P_{3,6}) \).

**Proposition 6.8.** (1) \( G(x, y, z) \geq P(x, y, z) \) \( \forall (x, y, z) \in \mathbb{R}^3_+ \);

2. \( F(x, y, z) \geq 4P(|x|, |y|, |z|) \) \( \forall (x, y, z) \in \mathbb{R}^3 \).

**Proof.** Using the notations of Section 5, we have

\[
G = 2Q - P = 2(Q - P) + P = 4H + P. \tag{6.9}
\]

Since \( H \geq 0 \) on \( \mathbb{R}^3_+ \), (1) follows. Applying the "\( \Delta \)-"transform on (6.9), we get \( G^d = 4H^d + P^d \). Since \( G = (1/4)F^+ \) and \( P^d = P \), this gives \( (1/4)F = 4H^d + P \), or

\[
F = 16H^d + 4P. \tag{6.10}
\]

But \( H^d \) is psd on \( \mathbb{R}^3_+ \) by (2.10), so \( F(x, y, z) \geq 4P(x, y, z) \) for \( x, y, z \geq 0 \). This now implies (2) since \( F \) is \( |\cdot| \)-convex by (3.23). Q.E.D.

According to (5.20)(1), \( H \in \mathcal{E}(P_{3,6}^+) \). Thus, (6.9) gives a decomposition of \( G \) into a sum of two extremal forms in \( P_{3,6}^+ \). But

\[
H \in \mathcal{E}(P_{3,6}^+) \Rightarrow H^d \in \mathcal{E}(P_{3,6}^d), \quad \text{and we have} \quad H^d \in P_{3,6}^+; \\
P \in \mathcal{E}(P_{3,6}^+) \Rightarrow P^d \in \mathcal{E}(P_{3,6}^d), \quad \text{and we have} \quad P^d = P.
\]
Thus, (6.10) gives a decomposition of $F$ into a sum of two extremal forms in $P^d_{3,6}$ as well as in $P^+_{3,6}$.

Exploiting the fact that $H^d \in \mathcal{S}(P^+_{3,6})$ some more, we get the following inequalities:

**Proposition 6.11.** (1) \[ \sum (y + z - x)^4(y - z)^2 \geq 8P(x, y, z) \forall (x, y, z) \in \mathbb{R}^3; \]

(2) \[ \sum (y + z - x)^4(y - z)^2 \geq 16P(x, y, z) \forall (x, y, z) \in \mathbb{R}^3_+ \]

**Proof.** Applying the "\(\Delta\)"-operator to \(2H = Q - P\), we get \(2H^d(x, y, z) = \sum ((y + z - x)/2)^4(y - z)^2 - P(x, y, z)\). This implies (2). On the other hand, (1) follows from (4.13) and the fact that $F$ is psd. Q.E.D.

Next, we shall compare the Schur from $\Gamma_6(x, y, z) \in \text{Sym} P_{3,6}$ with $P(x, y, z)$ and $P(|x|, |y|, |z|)$. This is made possible by our earlier result (4.12) which relates $\Gamma_6$ to the forms $P$ and $F$. Note that (1) and (2) below give explicit decompositions of $\Gamma_6$ and $\Gamma_6^+$ into sums of two extremal (symmetric) forms in $P_{3,6}$ and $P^+_{3,6}$.

**Proposition 6.12.** (1) $\Gamma_6 = F + P = 16H^d + 5P$.

(2) $\Gamma_6^+ = 4G + P = 16H + 5P$.

**Proof.** (1) follows from (4.12) and (6.10); (2) follows by applying the "\(+\)"-operator to (1). Q.E.D.

**Corollary 6.13.** (1) \[ \sum x^4(x - y)(x - z) \geq P(x, y, z) \forall (x, y, z) \in \mathbb{R}^3; \]

(2) \[ \sum x^4(x - y)(x - z) \geq 5P(|x|, |y|, |z|) \forall (x, y, z) \in \mathbb{R}^3; \]

(3) \[ \sum (y + z)^4(x - y)(x - z) \geq P(x, y, z) \forall (x, y, z) \in \mathbb{R}^3; \]

(4) \[ \sum (y + z)^4(x - y)(x - z) \geq 5P(x, y, z) \forall (x, y, z) \in \mathbb{R}^3_+. \]

**Proof.** Part (1) follows from (4.12) and the fact that $F$ is psd. Part (2) follows from $\Gamma_6 = 16H^d + 5P$, (2.10), and the fact (2.2) that $\Gamma_6$ is \(\|\|\)-convex. Parts (3) and (4) follow similarly upon noting that $\Gamma_6^+(x, y, z) = \sum (y + z)^4(x - y)(x - z)$. Q.E.D.

Let us now consider the Lehmus form $\Gamma = \Gamma_3$ (as defined in (1.2)). As a cubic form, $\Gamma$ is indefinite on $\mathbb{R}^3$, and it is easily checked to be irreducible. By [CKLR, Theorem 5.1], it follows that $\Gamma^2$ is extremal in $P_{3,6}$. Using the two expressions for $\Gamma^2$ in (4.14)(1), we have therefore the following inequalities, and they are the "best possible" in the technical sense of this section:
Proposition 6.14. For all \((x, y, z) \in \mathbb{R}^3:\)

\begin{enumerate}
\item \(\sum x^2(y - z)^2(x - z)^2 \geq 2P(x, y, z);\)
\item \(\sum (y + z - x)^2(y - z)^4 \geq 2P(x, y, z).\)
\end{enumerate}

Finally, we study the even form \(f_0 = \sum x^2(y^2 - z^2)^2.\) According to the results in \([CLR_2,]\) \(f_0, x^2y^2z^2,\) and Robinson's form \(S(x, y, z)\) are (up to positive multiples) exactly all the extremal forms in the cone of all even forms in \(\text{Sym} \ P_{3,6} .\) Here, both \(x^2y^2z^2\) and \(S\) are actually extremal in \(P\). However, it turns out that \(f_0\) fails to be extremal already in \(\text{Sym} \ P_{3,6} .\) This can be proved most efficiently by using Lemma 4.11 as follows. Since \(G^*(x, y, z) = 2 \sum y^2z^2(y - z)^2 - P(x, y, z),\) we have \(G^*(x, y, y) = 4x^2y^2(x - y)^2.\) Thus, \((G^*)^+(x, y, y) = G^*(2y, x + y, x + y) = 16y^2(x^2 - y^2)^2.\) On the other hand, \(f_0(x, y, y) = 2y^2(xz - y^2)^2.\) Therefore, by (4.11), \(8f_0 - (G^*)^+ = \beta P\) for some constant \(\beta.\) By comparing coefficients, or by evaluating this equation at any point which is not a zero of \(P,\) we see that \(\beta = 3.\) Thus, we get the following explicit decomposition of \(f_0\) into a sum of two (symmetric) extremal forms in \(P_{3,6} :\)

\(8f_0 = (G^*)^+ + 3P.\) (6.15)

From (6.15), we can derive yet another expression for the form \(G!\) First, applying the "\(A\)"-transform to (6.15), we get (after a direct calculation)

\(G^* = 2 \sum (y - z)^2(xy + xz - x^2)^2 - 3P(x, y, z).\) (6.16)

Taking the dual then yields

\(G(x, y, z) = 2 \sum (y - z)^2(xy + xz - yz)^2 - 3P(x, y, z).\) (6.17)

We can easily double-check this new expression for \(G\) by using the three cubics \(U, V, W\) introduced in (5.9), for the RHS of (6.17) is

\(2 \sum (V + W)^2 - 3(U + V + W)^2 = U^2 + V^2 + W^2 - 2(UV + VW + WU),\)

and this is equal to \(G\) by (5.5) and (5.12)(2). Alternatively, (6.16) and (6.17) can also be checked directly by using Lemma 4.11.

Summarizing the above, we have proved:

Proposition 6.18. For all \((x, y, z) \in \mathbb{R}^3:\)

\begin{enumerate}
\item \(\sum x^2(y^2 - z^2)^2 \geq (3/8) P(x, y, z);\)
\item \(\sum (y - z)^2(xy + xz - x^2)^2 \geq (3/2) P(x, y, z);\)
\item \(\sum (y - z)^2(xy + xz - yz)^2 \geq (3/2) P(x, y, z).\)
\end{enumerate}
7. Decompositions of the Robinson Form in $\text{Sym}^+_3$.

In this section, we shall study the Robinson form $S$ as a member of $P^+_3$, or $\text{Sym}^+_3$. Not surprisingly at this point, it turns out that $S$ is not extremal in $\text{Sym}^+_3$; in fact, we shall prove below the inequality

$$S(x, y, z) \geq 8P(x, y, z), \quad \forall x, y, z \geq 0. \quad (7.1)$$

The nature of this inequality is, however, somewhat different from that of our earlier inequalities obtained in Section 6. Although the constant 8 is indeed the largest possible for an inequality of this kind to hold, the difference $S - 8P$ is not yet an extremal form in $\text{Sym}^+_3$; rather, it breaks up further into the sum of three extremal forms in $\text{Sym}^+_3$. Therefore, $S$ can be decomposed into a sum of four extremal forms in $\text{Sym}^+_3$. By working differently, we also obtain a decomposition of $S$ into the sum of three extremal forms in $\text{Sym}^+_3$. A new extremal member of $\text{Sym}^+_3$ which emerges from this analysis is the interesting form

$$C(x, y, z) = x(y-z)^2(y+z-x)^3 - 5x^3y^2z + 18x^2y^2z^2. \quad (7.2)$$

In this section, we shall first use this form $C$ to derive the two aforementioned decompositions of the Robinson form $S$ in $\text{Sym}^+_3$. The proofs for the extremality of the forms used in these decompositions will be given in Section 8.

We begin our considerations by explicitly computing the difference

$$S(x, y, z) - xyz\Gamma(x, y, z)$$

$$= \sum x^2(x^2 - y^2)(x^2 - z^2) - xyz \sum x(x - y)(x - z)$$

$$= \sum x^2[(x + y)(x + z) - yz](x - y)(x - z)$$

$$= (x + y + z) \Gamma_5(x, y, z). \quad (7.3)$$

Thus, $S = xyz\Gamma + (x + y + z) \Gamma_5$ is indeed not extremal in $\text{Sym}^+_3$. In this decomposition, $xyz\Gamma$ is extremal in $P^+_3$ (and hence in $\text{Sym}^+_3$), since $\Gamma$ is extremal in $P^+_3$. But $(x + y + z) \Gamma_5$ can be further decomposed as a symmetric form as follows:

$$(x + y + z) \Gamma_5(x, y, z) = \sum [x^4 + x^3(y + z)](x - y)(x - z)$$

$$= I_6'(x, y, z) + C'(x, y, z), \quad (7.4)$$
where $C'(x, y, z) := \sum x^3(y + z)(x - y)(x - z)$. The next result shows that $C'$ is copositive, and gives an explicit decomposition of it into a sum of two extremal forms in Sym $P^{+}_{3,6}$.

**Proposition 7.5.** Let $C(x, y, z)$ be as in (7.2), and $C'$ be as above. Then

1. $C' = C + 3P$;
2. $C$ is copositive; and
3. $C$ is extremal in $\text{Sym} P^{+}_{3,6}$.

**Proof.** The decomposition $C' = C + 3P$ is checked as usual by noting that $C(x, y, y) = 2x^3y(x - y)^2 = C'(x, y, y)$. To prove that $C$ is copositive, it suffices (by symmetry) to show that $C(x, y, z) \geq 0$ whenever $0 < x < y < z$. Under the latter condition, it is easy to see (cf. (2.11)) that $z + x - y > |x + y - z|$, and hence that

$$ (z + x - y)^3 \geq -(x + y - z)^3. \tag{7.6} $$

On the other hand, since $y(z - x)^2 - z(x - y)^2 = (yz - x^2)(2 - y) > 0$, we have

$$ y(z - x)^2 \geq z(x - y)^2 \geq 0. \tag{7.7} $$

Multiplying (7.6) with (7.7), we see that

$$ y(z - x)^2(z + x - y)^3 + z(x - y)^2(x + y - z)^3 \geq 0. $$

Adding this to $x(y - z)^2(y + z - x)^3 \geq 0$, we see that $C(x, y, z) \geq 0$ as claimed. The proof of part (3) will be postponed to Section 8 (see proof of (8.5)(c)).

Q.E.D.

Note that from the equation $C' = C + 3P$ we can get the expansion of $C$ displayed in (7.2) from the much easier expansion of $C'$ without having to expand the cubes in $C$.

We have now the following decomposition of $S$ in Sym $P^{+}_{3,6}$:

$$ S = xyz\Gamma + \Gamma_6 + C' $$

$$ = xyz\Gamma + (F + P) + (C + 3P) \quad \text{(see (4.12))} $$

$$ = xyz\Gamma + 16H^4 + C + 8P \quad \text{(see (6.10))}, \tag{7.8} $$

where, in the last equation, every form is extremal in Sym $P^{+}_{3,6}$.

If we recall an earlier relation (stated in (3.25)) between the forms $S$, $F$, and $\Gamma^2$, it is also possible to arrive at an expression of $S$ as a sum of three extremal forms in Sym $P^{+}_{3,6}$. In fact, from the middle equation in (7.8), we have

$$ 2S = 2(xyz\Gamma + F) + 2C + 8P = S + \Gamma^2 + 2C + 8P, $$
according to (3.25). Thus, we get

$$S = I^2 + 2C + 8P,$$

(7.9)

where, again, each form on the RHS is extremal in \(\text{Sym} \ P_{3,6}^+\) (by results we shall prove in Section 8). From the above work, we deduce the following new collection of interesting inequalities:

**Theorem 7.10.** For any \(x, y, z \geq 0\), we have:

1. \(\sum x^3(y + z)(x - y)(x - z) \geq 3P(x, y, z);\)
2. \(S(x, y, z) \geq 8P(x, y, z);\)
3. \(S(x, y, z) = I(x^2, y^2, z^2) \geq I(x, y, z)^2;\)
4. \(S(x, y, z) \geq F(x, y, z).\)

Note that in (2) above, the constant 8 is the largest possible choice for the inequality to hold in \(\mathbb{R}^3_+\). In fact, suppose \(S(x, y, z) \geq kP(x, y, z) \forall x, y, z \geq 0\). Setting \(z = 0\), we have

$$0 \leq S(x, y, 0) - kP(x, y, 0)$$

$$= (x^2 - y^2)(x^4 - y^4) - kx^2y^2(x - y)^2$$

$$= (x - y)^2[(x + y)^2(x^2 + y^2) - kx^2y^2] \quad \forall x, y \geq 0.$$

It follows that \((x + y)^2(x^2 + y^2) \geq kx^2y^2\) on \(\mathbb{R}^2_+\). Setting \(x = y = 1\), we see that \(k \leq 8\).

**Corollary 7.11.** We have

1. \(\sum (y + z - x)^2(y^2 - z^2)^2 \geq 2P(x, y, z) \forall (x, y, z) \in \mathbb{R}^3;\)
2. \(\sum (y + z - x)^2(y^2 - z^2)^2 \geq 18P(x, y, z) \forall (x, y, z) \in \mathbb{R}^3_+.

**Proof.** (1) follows since the summation on the LHS equals \(2P(x, y, z) + 2S(x, y, z)\) (according to (4.14)(2)), and \(S\) is psd. Using this, (2) now follows from (7.10)(2). Q.E.D.

In the above, we have obtained two different decompositions of the Robinson form \(S\) into sums of extremal forms in the cone Sym \(P_{3,6}^+\). It is also possible to derive for \(S\) such a decomposition in the larger cone \(P_{3,6}^+\). To do this, recall from (7.3) that \(S = xyz\Gamma + x\Gamma_5 + y\Gamma_5 + z\Gamma_5\). Here, \(xyz\Gamma\) is already known to be extremal in \(P_{3,6}^+\). Thus, our job is reduced to that of decomposing \(\Gamma_5\) into a sum of extremal forms in \(P_{3,5}^+\). (It is easy to see that \(f \in \mathcal{E}(P_{3,5}^+) \Rightarrow xf, yf, zf \in \mathcal{E}(P_{3,6}^+).\)) This will be carried out in Section 9.
The goal of this section will be to give proofs for the extremality of various forms in the cone $\text{Sym} P^*_3$. These extremal forms will then lead to new symmetric sextic inequalities on $\mathbb{R}^3_+$. Since the forms we use will generally not extremal in $P^*_3$, the inequalities which ensue will only be "extremal" among symmetric (sextic) inequalities.

The basic form $(x - y)^2(y - z)^2(z - x)^2$ will continue to be denoted by $P$.

We shall now prove a theorem and its corollary below which help explain why so many symmetric positive and copositive sextic forms can be "compared" with this form $P$.

**THEOREM 8.1.** Suppose $f(x, y, z) \in \text{Sym} P^*_3$ (resp. $\text{Sym} P^*_3$) is not a scalar multiple of $P$. Assume that $f(x, y, y)$ lies in $\delta(P^*_3)$ (resp. $\delta(P^*_3)$). Then $f \in \delta(\text{Sym} P^*_3)$ (resp. $\delta(\text{Sym} P^*_3)$) iff there is no positive constant $k$ such that $f \geq kP$ on $\mathbb{R}^3_+$ (resp. on $\mathbb{R}^3$).

**Proof:** We shall only treat the case of copositive forms here, since the proof in the other case is completely similar. Also, the "only if" part is clear, so we need only prove the "if" part. Thus, let assume that $f \notin \delta(\text{Sym} P^*_3)$. This means that there exists a form $f' \in \text{Sym} P^*_3$ which is not a scalar multiple of $f$, but for which $f \geq f'$ on $\mathbb{R}^3_+$. From $f(x, y, y) \geq f'(x, y, y) \geq 0$ and the assumption that $f(x, y, y) \in \delta(P^*_3)$, we see that $f'(x, y, z) = af(x, y, z) + BP(x, y, z)$ for some $b \in \mathbb{R}$, which is necessarily nonzero. If $b > 0$, then $f \geq f' = af + BP \geq BP$ on $\mathbb{R}^3_+$, and we are done. If $b < 0$, then for $f' = -\beta > 0$, we have $af - \beta P = f' \geq 0$ on $\mathbb{R}^3_+$. This implies, in particular, that $\alpha \neq 0$ so $f \geq (\beta/\alpha)P$ on $\mathbb{R}^3_+$, and we are done again. Q.E.D.

Note that the "if" part above is not true in general if we did not impose the extremal condition on the binary form $f(x, y, y)$. For instance, recall the decomposition $S = \Gamma^2 + 2C + 8P$ for the Robinson form $S$ in $\text{Sym} P^*_3$ (see (7.9)). Here, the form $f = \Gamma^2 + 2C$ is not extremal in $\text{Sym} P^*_3$, but since the constant 8 is the largest choice for the inequality $S \geq 8P$ to hold in $\mathbb{R}^3_+$, it follows that the symmetric form $f$ does not satisfy $f \geq kP$ on $\mathbb{R}^3_+$ for any $k > 0$. Note that here $f(x, y, y) = S(x, y, y) = x^2(x - y)^2(x + y)^2$ is not extremal in $P^*_3$; it breaks up into the sum of $\Gamma(x, y, y)^2 = x^2(x - y)^4$ and $2C(x, y, y) = 4x^2y(x - y)^2$.

The following consequence of Theorem 8.1 says that once we impose the condition that $f(x, y, y) \in \delta(P^*_3)$, then the symmetric form $f$ is in fact "very close to" being extremal in $\text{Sym} P^*_3$. The reader will no doubt recognize that most of the results in Section 6 comparing various forms with the form $P$ were manifestations of the corollary below.
**Corollary 8.2.** Let $f \in \text{Sym} P_{3,6}^+$ (resp. $\text{Sym} P_{3,6}$) be such that $f(x, y, y)$ lies in $\mathcal{E}(P_{2,6}^+)$ (resp. in $\mathcal{E}(P_{2,6})$). Then $f = g + kP$ where $k \geq 0$ and $g$ is extremal in $\text{Sym} P_{3,6}^+$ (resp. in $\text{Sym} P_{3,6}$).

**Proof.** Again, we shall only deal with the copositive case. We may clearly assume that $f$ is neither extremal in $\text{Sym} P_{3,6}^+$, nor a scalar multiple of $P$. Then by the theorem, we have $f \geq kP$ on $\mathbb{R}_+^3$ for some constant $k > 0$. Clearly, such a constant $k$ must be bounded. By the closedness of the cone $\text{Sym} P_{3,6}^+$, we see that there exists a largest choice for $k$, say $k_0$. We are done if we can show that $g := f - k_0P$ is extremal in $\text{Sym} P_{3,6}^+$. But if $g$ is not, then, since $g(x, y, y) = f(x, y, y) \in \mathcal{E}(P_{2,6}^+)$, the theorem would give $g \geq \varepsilon P$ on $\mathbb{R}_+^3$ for some $\varepsilon > 0$. But then $f = g + k_0P \geq (k_0 + \varepsilon)P$ on $\mathbb{R}_+^3$ would give a contradiction. Hence we must have $g \in \mathcal{E}(\text{Sym} P_{3,6}^+)$. Q.E.D.

We shall now apply Theorem 8.1 to the proof of the extremality of various forms in $\text{Sym} P_{3,6}^+$. According to this theorem, if $f$ is any form in this cone, we can conclude that $f$ is extremal once we check that $f$ satisfies the following two sufficient conditions:

$$f(x, y, y) \in \mathcal{E}(P_{2,6}^+)$$ (8.3)

and

$$f \geq kP \geq 0 \quad \text{on} \quad \mathbb{R}_+^3 \implies k = 0.$$ (8.4)

In the following choice of examples, (8.3) will be obvious since $f(x, y, y)$ splits into linear factors in all cases, and (8.4) usually follows easily by specializing the variable $z$ to zero.

**Theorem 8.5.** The following forms are extremal in $\text{Sym} P_{3,6}^+$:

(a) $\Gamma(x, y, z) = (\sum x(x - y)(x - z))^2$;

(b) $T(x, y, z) = xyz\Gamma^+(x, y, z) = xyz \sum (y + z)(x - y)(x - z) = xyz \sum (x - z)^2$;

(c) $C(x, y, z) =\sum x(y - z)^2(y + z - x)^2$;

(d) $D(x, y, z) = \sum xy(x - y)^4$;

(e) $H_{\lambda, \mu}(x, y, z) = \sum yz(\lambda y - \mu x)(\lambda z - \mu x)(x - y)(x - z)$, where $\lambda, \mu \geq 0$.

**Proof.** The property (8.3) is checked easily since

$$\Gamma(x, y, y) = x^2(x - y)^4, \quad T(x, y, y) = 2xy^2(x - y)^2,$$

$$C(x, y, y) = 2x^3y(x - y)^2, \quad D(x, y, y) = 2xy(x - y)^4,$$

$$H_{\lambda, \mu}(x, y, y) = y^2(\lambda y - \mu x)^2(x - y)^2.$$
The hypotheses $\lambda, \mu \geq 0$ in (e) are used here to ensure that $H_{\lambda, \mu}(x, y, z)$ is extremal in $P_{3,6}^+$. Similarly, we can check (8.4) by noting that $P(x, y, 0) = x^2 y^2 (x - y)^2$, while

$$I^2(x, y, 0) = (x + y)^2 (x - y)^4, \quad T(x, y, 0) = 0,$$

$$C(x, y, 0) = xy(x - y)^4, \quad D(x, y, 0) = xy(x - y)^4,$$

$$H_{\lambda, \mu}(x, y, 0) = \lambda^2 x^2 y^2.$$

For the sake of completeness, we should mention here that the form

$$16H^4 = \sum (x - y + z)^2 (x + y - z)^2 (x - y)(x - z)$$

has the specializations $16H^4(x, y, z) = x^4(x - y)^2$ and $16H^4(x, y, 0) = (x - y)^4(x^2 + 3xy + y^2)$. Thus, the method of proof above also shows that $H^4$ is extremal in $\text{Sym} P_{3,6}^+$. However, we already knew, in fact, that $H^4$ is extremal in the larger cone $P_{3,6}$ (see the discussion in the paragraph following (6.10)). This is why we did not include $H^4$ in the statement of the theorem. Q.E.D.

In the above, we have omitted the proof for the fact that the forms $H_{\lambda, \mu}$ are all copositive. The reason for our omission is that the copositivity of the $K_{\lambda, \mu}$'s can be reduced to the earlier work of Rigby [Ri2] (see also [Ri3]). Recalling that $H(x, y, z) = \sum y^2 z^2(x - y)(x - z)$ and that $H^*(x, y, z) = xyz(x, y, z)$, we see by direct expansion of (8.5)(e) that

$$H_{\lambda, \mu}(x, y, z)$$

$$= \lambda^2 \sum y^2 z^2(x - y)(x - z) - \mu \lambda x y z \sum (y + z)(x - y)(x - z)$$

$$+ \mu^2 x y z \sum x(x - y)(x - z)$$

$$= \lambda^2 H(x, y, z) - \mu \lambda T(x, y, z) + \mu^2 H^*(x, z).$$

In this latter form, the family $\{H_{\lambda, \mu}\}$ has been defined by Rigby. In fact, Rigby studied the cone $\mathcal{C}$ consisting of the forms in $\text{Sym} P_{3,6}^+$ which are without the $\sum x^6$ and $\sum x^2 y$ terms, and which also vanish at the point $(1, 1, 1)$. Although Rigby did not use the terminology of extremal forms, his main result on the cone $\mathcal{C}$ can be translated as follows:

**Theorem 8.6** [Ri2]. *The extremal forms of the cone $\mathcal{C}$ are precisely $P$, $T$, and the forms in the family $\{H_{\lambda, \mu} = \lambda^2 H - \lambda \mu T + \mu^2 H^*: \lambda, \mu \geq 0\}$.*

The two forms $H_{1,0} = H$ and $H_{0,1} = H^*$ are, of course, extremal already in $P_{3,6}^+$. But the form $H_{1,1} = H - T + H^*$ is not, since the expression $H_{1,1} = \sum yz(x - y)^2(x - z)^2$ shows that it is the sum of three extremal
forms in $P^+_{3,6}$. This form has made an earlier appearance in [CL$_1$, p. 401], where it was noticed that it has the following remarkable factorization:

$$H_{1,1} = (x^2y + y^2z + z^2x - 3xyz)(xy^2 + yz^2 + zx^2 - 3xyz). \quad (8.7)$$

Some other interesting expressions for $H_{1,1}$ are given by

$$4H_{1,1} = \left( \sum x(y - z)^2 \right)^2 - P(x, y, z) = \Gamma^+(x, y, z)^2 - P(x, y, z), \quad (8.8)$$

$$4H_{1,1} = \sum (y + z)^2(x - y)^2(x - z)^2 - 3P(x, y, z), \quad (8.9)$$

$$2H_{1,1} = \sum x^2(y - z)^4 - P(x, y, z). \quad (8.10)$$

The first of these is checked by noting that $\Gamma^+(x, y, y)^2 - P(x, y, y) = 4y^2(x - y)^4 = 4H_{1,1}(x, y, y)$, and that $\Gamma^+(x, y, 0)^2 - P(x, y, 0) = 4x^3y^3 = 4H_{1,1}(x, y, 0)$ (see Lemma (4.11)). The second and the third one are checked similarly. (Alternatively, these can also be deduced from (8.8) by applying the “$+$”-operator to the two expressions for $\Gamma^+$ in (4.14).)

**Corollary 8.11.**

1. $(\sum x(y - z)^2)^2 \geq P(x, y, z) \forall (x, y, z) \in \mathbb{R}_+^3$;
2. $\sum (y + z)^2(x - y)^2(x - z)^2 \geq 2P(x, y, z) \forall (x, y, z) \in \mathbb{R}_+^3$;
3. $\sum (y + z)^2(x - y)^2(x - z)^2 \geq 3P(x, y, z) \forall (x, y, z) \in \mathbb{R}_+^3$;
4. $\sum x^2(y - z)^4 \geq (1/2) P(x, y, z) \forall (x, y, z) \in \mathbb{R}_+^3$;
5. $\sum x^2(y - z)^4 \geq P(x, y, z) \forall (x, y, z) \in \mathbb{R}_+^3$.

**Proof.** Parts (1), (3), and (5) follow respectively from (8.8), (8.9), (8.10), and the fact that $H_{1,1} \in P^+_{3,6}$. Parts (2) and (4) follow by applying the “$+$”-operator to the two expressions for $\Gamma^+$ in (4.14). Q.E.D.

By doing some extra work, it is possible to get a family of symmetric extremal forms in $P_{3,6}$ which is, in some sense, analogous to the Rigby family $\{H_{i,\nu}\}$. However, for lack of space, we will not present the details here.

We finish now by recording some alternative expressions for the two forms $C$ and $D$ in Theorem 8.5. These expressions can be verified, as usual, by a direct application of Lemma 4.11:

$$C(x, y, z) = \sum yz(y + z - x)^2(y - z)^2 - 4P(x, y, z), \quad (8.12)$$

$$D(x, y, z) = \sum x(y + z)(x - y)^2(x - z)^2 - 2P(x, y, z). \quad (8.13)$$

From these, we get two more extremal symmetric inequalities on $\mathbb{R}_+^3$ of the form $f(x, y, z) \geq kP(x, y, z)$. 
9. Extremality of the Schur Forms (Or the Lack of It)

In the older literature, a number of papers have been written on Schur's inequalities by various authors. For a survey of the work in this area, we refer the reader to Section 2.17 in Mitrinović's book [Mi]. For the most part, these papers dealt with different ways of generalizing the classical inequalities of Schur. The question most natural from the viewpoint of this paper, namely, whether the Schur forms $\Gamma_d$ are extremal in the various cones to which they belong seemed to have remained unanswered. In this section, we shall study this question and answer it completely. It turns out that the Schur forms are rarely extremal in the cones studied in this paper; however, to determine precisely which special Schur forms are indeed extremal does require a certain amount of work. Our main result can be stated in two parts, as follows.

**Theorem 9.1.** Let $d \geq 2$. Then

1. $\Gamma_d$ is extremal in $P_{3,d}^+$ iff $d = 3$;
2. $\Gamma_d$ is extremal in $\text{Sym} \ P_{3,d}^+$ iff $d = 2, 3, 4,$ or $5$.

**Theorem 9.2.** Let $d$ be an even integer $\geq 2$. Then

1. $\Gamma_d$ is never extremal in $P_{3,d}$;
2. $\Gamma_d$ is extremal in $\text{Sym} \ P_{3,d}^+$ iff $d = 2$ or $4$.

To begin the proof, we shall first establish all the affirmative cases claimed in the two theorems above. We have already mentioned several times the fact (proved in [Ri, CLJ] that $\Gamma_3$ is extremal in $P_{3,3}^+$ (and hence also in $\text{Sym} \ P_{3,3}^+$). Next, let us prove that $\Gamma_d$ is extremal in $\text{Sym} \ P_{3,d}^+$ for $d \leq 5$. (9.3)

Suppose $\Gamma_d \geq g$ on $\mathbb{R}_+^3$, where $g \in \text{Sym} \ P_{3,d}^+$. Specializing to $y = z$, we have $x^{d-2}(x - y)^2 = \Gamma_d(x, y, y) \geq g(x, y, y) \ \forall x, \ y \geq 0$. Since $x^{d-2}(x - y)^2$ is extremal in $P_{2,d}^+$, this implies that $g(x, y, y) = \alpha \Gamma_d(x, y, y)$ for some $\alpha \in \mathbb{R}$. If $d \leq 5$, then, since $g$ and $\Gamma_d$ are both symmetric, the proof of Lemma 4.11 shows that $g = \alpha \Gamma_d$. This proves the claim (9.3), and the same argument can be used to show that $\Gamma_2$ and $\Gamma_4$ are extremal respectively in $\text{Sym} \ P_{3,2}$ and $\text{Sym} \ P_{3,4}$.

Having disposed of all the affirmative cases in (9.1) and (9.2), we now begin to treat the remaining cases. First, we observe that, for $d \leq 5$, we have an identity

$$\Gamma_d(x, y, z) = \frac{1}{d} \sum (y - z)^2(y + z)(x)^{d-2}. \quad (9.4)$$
In fact, when \( y = z \), both sides are equal to \( x^{d-2}(x - y)^2 \), so if \( d \leq 5 \), the identity follows as above by the proof of (4.11). For \( d = 2, 4 \), this shows that \( I_d \) is not extremal in \( P_{3,d}^+ \), and therefore also not extremal in \( P_{3,d} \). In fact, in these two cases, (9.4) shows that \( I_d \) is a sum of squares of forms. (The identity (9.4) in the case \( d = 4 \) was first noted in [K, p. 141], and independently in [CL2, p. 61].) Also, it is worth noting that, for \( d = 6 \), we have the following analogue of (9.4):

\[
I_d(x, y, z) = \frac{1}{2} \sum (y - z)^2(y + z - x)^4 - 3P(x, y, z),
\]

(9.5)

where \( P \) is as in (4.10). This follows from (4.12) and (4.13), or directly from (4.11).

Our next goal is to show that, although \( I_5 \) is extremal in \( \text{Sym} P_{3,5}^+ \), it is not extremal in \( P_{3,5}^+ \). Let

\[
E(x, y, z) = (y - z)^2(y + z - x)^3 + (z - x)^2(z + x - y)^3,
\]

(9.6)

which is a form symmetric with respect to \( \{x, y\} \) (but not with respect to \( \{x, y, z\} \)). Then, by (9.4) for \( d = 5 \), we have

\[
4I_5(x, y, z) = E(x, y, z) + E(y, z, x) + E(z, x, y).
\]

(9.7)

In view of this equation, the following lemma will clearly show that \( I_5 \) is not extremal in \( P_{3,5}^+ \).

**Lemma 9.8.** \( E(x, y, z) \in P_{3,5}^+ \).

**Proof.** By the symmetry in \( x \) and \( y \), it is sufficient to show that \( E(x, y, z) \geq 0 \) when \( y \geq x \geq 0 \) and \( z \geq 0 \). We consider the following three cases:

1. \( x + z \geq y \geq 0 \). In this case, we have both \( y + z - x \geq 0 \) and \( z + x - y \geq 0 \), and therefore clearly \( E(x, y, z) \geq 0 \).

2. \( y \geq x + z \geq 0 \) and \( z \geq x \geq 0 \). Here we have \( y - z \geq y - z - x \geq 0 \), \( y + z - x \geq y - z - x \geq 0 \), and \( y + z - x \geq z - x \geq 0 \). Therefore, we have

\[
(y - z)^2(y + z - x)^3 = (y - z)^2(y - z - x)(y + z - x)^2
\]

\[
\geq (y - z - x)^2(y - z - x)(z - x)^2
\]

\[
= -(z + x - y)^2(z - x)^2,
\]

so transposition gives \( E(x, y, z) \geq 0 \).
(3) \( y \geq x + z \geq 0 \) and \( x \geq z \geq 0 \). Here we have \( y - z \geq x - z \geq 0 \) and \( y + z - x \geq y - x - z \geq 0 \). Therefore, we have
\[
(y - z)^2(y + z - x)^3 \geq (x - z)^2(y - x - z)^3,
\]
so again transposition gives \( E(x, y, z) \geq 0 \). Q.E.D.

It now remains to treat the Schur forms \( I^*_d \) for \( d \geq 6 \). From the decomposition \( I^*_6 = F + P \) in (4.12), we know that \( I^*_6 \) is not extremal in \( \text{Sym} \ P_{3,6} \), so \( I^*_6 \) is also not extremal in \( P_{3,6}^+, P_{3,6}^+ \), and \( \text{Sym} \ P_{3,6}^+ \). Unfortunately, this method does not extend to \( d > 6 \). Therefore, we have to come up with another method to ascertain the lack of extremality of \( I^*_d \) (for \( d > 6 \)) in the various cones.

**Proposition 9.9.** Let \( d \geq 6 \). Then there exists a positive constant \( \alpha \) (which, for instance, can be taken to be \( 1/48 \)) such that
\[
I^*_d(x, y, z) \geq \alpha P(x, y, z)(x^{d-6} + y^{d-6} + z^{d-6})
\]
for all \( (x, y, z) \in \mathbb{R}_+^3 \), and also for all \( (x, y, z) \in \mathbb{R}^3 \) in case \( d \) is even.

Here, \( P(x, y, z) \) is the form \( (x - y)^2(y - z)^2(z - x)^2 \) introduced in (4.10). Of course, it follows from this proposition that, for \( d \geq 6 \): (1) \( I^*_d \) is never extremal in \( P_{3,d}^+ \) and \( \text{Sym} \ P_{3,d}^+ \), and in case \( d \) is even, (2) \( I^*_d \) is never extremal in \( P_{3,d} \) and \( \text{Sym} \ P_{3,d} \). This would then complete the proof of Theorems 9.1 and 9.2.

**Proof:** To fix ideas, let us first work in the case when \( d (\geq 6) \) is even. In order to prove the asserted inequality for all \( (x, y, z) \in \mathbb{R}_+^3 \), we may assume, by symmetry, that \( x \leq y \leq z \) and that \( z \geq |x| \). (Note that, for \( d \) even, \( I^*_d(-x, -y, -z) = I^*_d(x, y, z) \)). Then \( x^{d-2} \geq 0 \), \( z^{d-2} \geq y^{d-2} \geq 0 \), and we have
\[
I^*_d(x, y, z) = x^{d-2}(z - x)(y - x) + (z^{d-2} - y^{d-2})(z - y)(y - x)
+ z^{d-2}(z - y)^2
\geq z^{d-2}(z - y)^2.
\]
But from \( |y - x| \leq |y| + |x| \leq 2z \) and \( |z - x| \leq |z| + |x| \leq 2z \), we also have \( (y - x)^2 \leq 4z^2 \) and \( (z - x)^2 \leq 4z^2 \), and so \( P(x, y, z) \leq 16z^4(z - y)^2 \). Therefore, from the above,
\[
48I^*_d(x, y, z) \geq 48z^{d-2}(z - y)^2
= 16z^4(z - y)^2 \cdot 3z^{d-6}
\geq P(x, y, z) \cdot (x^{d-6} + y^{d-6} + z^{d-6}),
\]
as claimed. If \( d \) is not assumed to be even, the same argument still works for all \((x, y, z) \in \mathbb{R}_+^3\). Here, we may assume as before that \( z \geq y \geq x \); then \( z \geq |x| \) is automatic, and we have \( z^m \geq y^m \geq x^m \geq 0 \) for any \( m \geq 0 \), so the proof works just as before.

Q.E.D.

In retrospect, the most subtle point in Theorem 9.1 may very well be the fact that \( \Gamma_5 \) is extremal in \( \text{Sym} P_{3,5}^+ \) but not in \( P_{3,5}^+ \). In (9.7), we have obtained a decomposition of \( \Gamma_5 \) into a sum of three forms in \( P_{3,5}^+ \). It turns out that the three summands there are each extremal in \( P_{3,5}^+ \), so (9.7) gives indeed a decomposition of \( \Gamma_5 \) into a sum of (three) extremal forms in \( P_{3,5}^+ \). To see this, it clearly suffices to prove that:

**Proposition 9.10.** \( E(x, y, z) \) is extremal in \( P_{3,5}^+ \).

**Proof.** Suppose \( E \geq g \) on \( \mathbb{R}_+^3 \), where \( g \in P_{3,5}^+ \). We wish to show that \( g = aE \) for some \( a \in \mathbb{R} \). Let \( a = g(1, 0, 0) \geq 0 \), \( b = g(0, 1, 0) \geq 0 \), and let

\[
h(x, y, z) = g(x, y, z) - a(x - 2)^2(x - y + 2)^3 - b(y - z)^2(-x + y + z)^3.
\]

For \( x \geq y \geq 0 \), we have

\[
0 \leq g(x, y, y) \leq E(x, y, y) = x^3(x - y)^2,
\]

\[
0 \leq g(x, y, x - y) \leq E(x, y, x - y) = 8y^2(x - y)^3.
\]

Therefore, \( g(x, y, y) = a_1 x^3(x - y)^2 \) and \( g(x, y, x - y) = 8a_2 y^2(x - y)^3 \) for some constants \( a_1, a_2 \in \mathbb{R} \). Evaluating these on \((1, 0, 0)\) and \((2, 1, 1)\), we get \( a = g(1, 0, 0) = a_1 = (1/8)g(2, 1, 1) = a_2 \), and so \( h(x, y, y) = 0 \) and \( h(x, y, x - y) = 0 \). These imply that \( h \) is divisible by \( y - z \) and \( -x + y + z \).

Similarly, we can see that \( h \) is also divisible by \( x - z \) and \( x - y + z \). Thus, we have

\[
g(x, y, z) = a(x - z)^2(x - y + z)^3 + b(y - z)^2(-x + y + z)^3 - (cx + dy + ez)(x - z)(y - z)(-x + y + z),
\]

where \( c, d, e \in \mathbb{R} \). For \( y, z \geq 0 \), we then have

\[
E(y, y, z) \geq g(y, y, z) = (a + b)(y - z)^2z^3 - (cy + dy + ez)(y - z)^2z^2
\]

\[
= (y - z)^2z^2[(a + b - e)z - (c + d)y].
\]

Since \( E(y, y, z) = 2(y - z)^2z^3 \) is extremal in \( P_{2,5}^+ \), this implies that \( c + d = 0 \). On the other hand, for \( x, y \geq 0 \), we also have

\[
E(x, y, 0) \geq g(x, y, 0)
\]

\[
= ax^2(x - y)^3 + by^2(-x + y)^3 - (cx + dy)(xy)(x - y)(-x + y)
\]

\[
= (x - y)^3(ax^2 - by^2 + cxy).
\]
Since \( E(x, y, 0) = (x + y)(x - y)^4 \), we get
\[
(x + y)(x - y)^2 \geq (x - y)(ax^2 - hy^2 + cxy)
\]
for all \( x, y \geq 0 \), or
\[
(x - y)[x^2 - y^2 - (ax^2 - hy^2 + cxy)] \geq 0,
\]
for all \( x, y \geq 0 \). Clearly, this implies that \( x^2 - y^2 - (ax^2 - hy^2 + cxy) \) must vanish on \((x, y) = (1, 1)\), and so we have \( a - b + c = 0 \). Finally, we let \( x = 1 \), \( y = e^2(1 + \varepsilon) \), and \( z = 1 - e^3 \), where \( |\varepsilon| \to 0 \). Then, \((x, y, z) \to (1, 0, 1)\) in \( \mathbb{R}_+^3 \).

By a straightforward computation, we have
\[
E(1, e^2(1 + \varepsilon), 1 - e^3) = 9\varepsilon^6 + \text{higher degree terms},
\]
\[
g(1, e^2(1 + \varepsilon), 1 - e^3) = 2(\varepsilon + e) \varepsilon^5 + \text{higher degree terms}.
\]

Since \( g(x, y, z) \leq E(x, y, z) \), this implies that \( c + e = 0 \). Similarly (by interchanging \( x \) and \( y \)), we see that \( d + e = 0 \). Recalling that \( d = -c \) and \( a - b + c = 0 \), we now have \( c = d = e = 0 \), and \( a = b \). Therefore, \( g = aE \), as desired.

Q.E.D.

Note that, by plugging (9.7) into the equation \( S = xyz\Gamma + (x + y + z)\Gamma_5 \) (cf. Section 7) and expanding, we get a decomposition of the Robinson form \( S \) into a sum of ten extremal forms in \( P^+_{3,6} \).

10. A Glossary of Special Forms

For ease of reference, we shall compile here a glossary of the various special forms introduced and studied in this paper. All forms listed below are symmetric ternary forms.

We begin by recalling that, for any ternary form \( f \), we defined in Section 2:
\[
f^+(x, y, z) = f(y + z, z + x, x + y),
\]
\[
f^d(x, y, z) = f((y + z - x)/2, (z + x - y)/2, (x + y - z)/2).
\]

The Lehmus form is by definition
\[
\Gamma(x, y, z) = \sum x^3 - \sum x^2y + 3xyz \in \mathcal{S}(P^+_{3,3}),
\]
with
\[
\Gamma^+(x, y, z) = \sum x^2y - 6xyz \in P^+_{3,3}.
\]
Now we come to ternary sextics. If a symmetric ternary sextic $f$ has the form

$$
\alpha \sum x^6 + \beta \sum x^5y + \gamma \sum x^4y^2 + \delta \sum x^3yz + e \sum x^2yz^2 + f \sum xyz^3,
$$

we shall write $f = [\alpha, \beta] + (\gamma, \delta, e)$ for short. If $\alpha = \beta = 0$, we shall simply write $f = (\gamma, \delta, e)$. For a form $f$ of the latter type, we define the dual of $f$ to be the form $f^*(x, y, z) = x^4y^4z^4f(1/x, 1/y, 1/z)$, so that $f^* = (\gamma, \delta, e)$ (see Section 6). The following are some of the key forms studied in this paper, with the appropriate cross-references:

$I(x, y, z)^2 = [1, -2] + (-1, 8, 4, -6, 15) \in \mathcal{S}(\text{Sym} \ P_{2,6}^+)$

$(4.14)(1), (8.5)(a))$, 

$I^+(x, y, z)^2 = (1, 2, 2, -10, 42) \in \mathcal{S}(P_{2,6})$ (8.8), 

$I_6(x, y, z) = (1, -1) + (0, 1, 0, 0, 0) \in P_{2,6}$ ((2.2), (4.12), (7.8)).

$P(x, y, z) = (x - y)^2(y - z)^2(z - x)^2$

$$
= (1, -2, -2, 2, -6) \in \mathcal{S}(P_{2,6}^+)$ (4.10),

$S(x, y, z) = I(x^2, y^2, z^2) = [1, 0] + (-1, 0, 0, 3) \in \mathcal{S}(P_{2,6})$

((1.3), (4.14)(2), (7.8)-(7.10)),

$S^+ = (9, -14, 18, -10, -6) \in \mathcal{S}(P_{2,6}),$

$\frac{1}{2}S^d = (1, -1, 2, -5, 21) \in \mathcal{S}(P_{2,6}),$

$F(x, y, z) = \frac{1}{2}[(\prod (y + z - x)^2 - \prod (y^2 + z^2 - x^2)]$

$= [1, -1] + (-1, 3, 2, -2, 6) \in \mathcal{S}(P_{2,6})$ (Sections 3, 4),

$Q(x, y, z) = \sum x^4(y - z)^2 = (1, -2, 0, 0, 0) \in \Sigma_{2,6}$ ((5.8), (5.11), (5.21)),

$G = \frac{1}{2}F^+ - 2Q - P = (1, -2, 2, -2, 6) \in \mathcal{S}(P_{2,6})$ (Section 5),

$(G^*)^+ = (5, 6, 6, -6, -30) \in \mathcal{S}(P_{2,6})$ (6.15),

$K = 2(3Q - P) - (1, -2, 1, -1, 3) \in \mathcal{S}(\text{Sym} \ \Sigma_{2,6})$ ((5.20)(3)),

$H(x, y, z) = I( yz, zx, xy) = \frac{1}{2}(Q - P)$

$= (0, 0, 1, -1, 3) \in \mathcal{S}(P_{2,6}^+)$ ((2.4), (5.20)(1), (8.6)),

$H^*(x, y, z) = xyz \Gamma(x, y, z) = (0, 1, 0, -1, 3) \in \mathcal{S}(P_{2,6})$ (8.6).

$16H^d = [1, -1] + (-5, 11, 10, -10, 30) \in \mathcal{S}(P_{2,6}^+)$ ((2.10), (6.12)(1)).
\[ f_0 = \sum x^2(y^2 - z^2)^2 \]

\[ = (1, 0, 0, 0, -6) \quad (6.15), \text{ (extremal among even forms in Sym } P_{3,0}). \]

\[ T(x, y, z) = xyz \Gamma^+(x, y, z) \]

\[ = (0, 0, 0, 1, -6) \in \mathcal{E}(\text{Sym } P_{3,0}^+) \quad ((8.5)(b), (8.6)), \]

\[ C(x, y, z) = \sum x(y - z)^2(y + z - x)^3 \]

\[ = [0, 1] + (-4, 4, 6, -5, 18) \in \mathcal{E}(\text{Sym } P_{3,6}^+) \quad ((7.5), (7.9), (8.5)), \]

\[ D(x, y, z) = \sum xyz(x - y)^d \]

\[ = [0, 1] + (-4, 0, 6, 0, 0) \in \mathcal{E}(\text{Sym } P_{3,6}^+) \quad ((8.5), (8.13)), \]

\[ H_{\lambda, \mu}(x, y, z) = \sum yz(\lambda y - \mu x)(\lambda z - \mu x)(x - y)(x - z) \]

\[ = (0, \mu^2, \lambda^2, -(\lambda^2 + \lambda \mu + \mu^2), 3(\lambda + \mu)^2) \in \mathcal{E}(\text{Sym } P_{3,6}^+) \quad ((8.5), (8.6)-(8.10)). \]

Some basic identities among these forms are

\[ \Gamma_6 = F + P \quad (4.12), \]

\[ S + \Gamma^2 = 2(F + xyz \Gamma) \quad (3.25), \]

\[ G = 4H + P \quad (6.9), \]

\[ F = 16H^4 + 4P \quad (6.10), \]

\[ 8f_0 = (G^*)^+ + 3P \quad (6.15), \]

\[ S = xyz \Gamma + 16H^4 + C + 8P \quad (7.8), \]

\[ S = \Gamma^7 + 2C + 8P \quad (7.9), \]

\[ H_{\lambda, \mu} = \lambda^2 H - \lambda \mu T + \mu^2 H^* \quad \text{ (before (8.6)).} \]

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