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THE MALGRANGE–MATHER DIVISION THEOREM

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§1. INTRODUCTION

THE GENERALIZATION of the Weierstrass preparation Theorem [13, 17] to the case of C^∞ functions was conjectured by R. Thom as a tool for the theory of singularities he had discovered (see [16]). Malgrange gave the first proof of this generalization in [5] and [6] and proved many consequences of it. It is a rather technical proof which involves a detailed study of the local geometrical structure of analytic sets. A stronger result was obtained by a direct analytic argument by Mather [7]. Four other proofs [2, 4, 8, 11] use an extension of the function $f(t, x)$, $t \times x \in \mathbf{R} \times \mathbf{R}^n$, to the complex variable t. Extensions of the preparation and division theorems to real C^m functions were proved by M. G. Lassale [3]. Let

$$P^d(t, \lambda) = t^d + \sum_{j=1}^{d} \lambda_j t^{d-j}, \quad \lambda = (\lambda_1 \ldots \lambda_d) \tag{1}$$

be a generic monic polynomial in t with constant coefficients λ_j. It is well known the usual preparation and division theorems follow quite easily from the Malgrange–Mather Theorem of division by $P^d(t, \lambda)$. In this paper we give a short, rather elementary proof of the Malgrange–Mather Division Theorem, which applies to analytic, C^∞ or C^m functions. Our proof is based on the following.

Definition 1. Let $V^d \subset \mathbf{R}^{d+1}$ be the set of zeroes of $P^d(t, \lambda)$, $\lambda \in \mathbf{R}^d$ and $t \in \mathbf{R}$, and $\pi_d: \mathbf{R}^{d+1} \to \mathbf{R}^d$ be a natural projection; i.e., $\pi_d: (t, \lambda) \mapsto \lambda$. The mapping $\hat{\pi}_d: \mathbf{R}^d \to V^d$ defined by the formula $\hat{\pi}_d(t, \lambda_1, \ldots, \lambda_{d-1}) = (t, \lambda_1, \ldots, \lambda_{d-1}, \lambda_d)$, where $\lambda_d = \lambda_d(t, \lambda_1, \ldots, \lambda_{d-1}) = -t^d - \sum_{j=1}^{d-1} \lambda_j t^{d-j}$, is an uniformization of the algebraic hypersurface $V^d \subset \mathbf{R}^{d+1}$; i.e., $(t, \lambda_1, \ldots, \lambda_{d-1}) = \hat{\pi}_d^{-1}(t, \lambda_1, \ldots, \lambda_{d-1}, \lambda_d) \in \mathbf{R}^d$ are global coordinates on V^d. By C^∞, C^m, analytic or polynomial function on V^d we mean respectively a C^∞, C^m, analytic or polynomial function in global coordinates on V^d.

THEOREM 1. *Let $C_\pi^\infty(V^d \times \mathbf{R}^n)$ be the subspace of $C^\infty(V^d \times \mathbf{R}^n)$ consisting of all functions constant on the fibres $\pi_d^{-1}(\lambda)$. Then there exists a continuous linear operator*

$$J: C_\pi^\infty(V^d \times \mathbf{R}^n) \to C^\infty(\mathbf{R}^d \times \mathbf{R}^n),$$

$$\textit{such that for every } (t, \lambda) \in V^d \textit{ and } x \in \mathbf{R}^n \tag{2}$$

$$(J\phi)(\lambda, x) = \phi(t, \lambda, x).$$

Remark 1. Let C_π^m be the subspace of C^m, A_π be the subspace of the space A of all analytic functions on $V^d \times \mathbf{R}^n$ and P be the subspace of the space P of all polynomials on $V^d \times \mathbf{R}^n$ consisting of functions which are constant on the fibres $\pi_d^{-1}(\lambda) \cap V^d$ of π_d. A statement analogous to Theorem 1 holds

(a) in the C^m case with an operator $J: C_\pi^m(V^d \times \mathbf{R}^n) \to C^{[\frac{m}{2}]}(R^d \times R^n)$, where $\left[\frac{m}{2}\right]$ is the integral part of $\frac{m}{2}$;

(b) in the analytic case with an operator $J: A_\pi(V^d \times \mathbf{R}^n) \to A(\pi_d(V^d) \times \mathbf{R}^n)$;

(c) in the polynomial case with an operator $J: P_\pi(V^d \times \mathbf{R}^n) \to P(V^d \times \mathbf{R}^n)$.

Remark 2. Let us define $(J\phi)(\lambda, x)$ for $\lambda \in \pi_d(V^d)$ and $x \in \mathbf{R}^n$ by (2). The main idea of the proof is

(a) The function $(J\phi)(\lambda, x)$ is a C^1 function for $x \in \mathbf{R}^n$ and λ from an open dense subset $\tilde{U} = \{\lambda \in \mathbf{R}^d | \exists t \in \mathbf{R}$ such that $P^d(t, \lambda) = 0$ and $(\partial/\partial t P^d)(t, \lambda) \neq 0\} \subset \pi_d(V^d)$. The function $\phi^{\text{new}}(t, \lambda, x) = (d_\lambda J\phi)(\pi_d(t, \lambda), x)$ as a function on $V^d \times \mathbf{R}^n$ coincides with d-tuple of C^∞ functions.

(b) The pair of functions $(J\phi, J\phi^{\text{new}})$ is a Whitney function of class C^1 on $\pi_d(V^d) \times \mathbf{R}^n$ (see p. 3 in [6]).

Repeating the procedure for the function $\phi^{\text{new}}(t, \lambda, x)$ we obtain a linear continuous operator from $C_\pi^\infty(V^d \times \mathbf{R}^n)$ into Whitney fields on $\pi_d(V^d) \times \mathbf{R}^n \subset \mathbf{R}^d \times \mathbf{R}^n$, namely: $\phi(t, \lambda, x) \mapsto \{(d_\lambda{}^n J\phi)(\lambda, x)\}_{n \in \mathbf{Z}^+}$. For odd d the proof is now finished because $\pi_d(V^d) = \mathbf{R}^d$. For even d, due to the convexity of the region $\mathbf{R}^d \backslash \pi_d(V^d)$, we may apply the Stein extension theorem (see Ex. 2, p. 189[14]) which completes the proof of Theorem 1. The necessary extension result also follows from the extension of C^∞ functions defined in a half space (see [10] or [12]) and Corollary 2 of [1].

Remark 3. Let us consider functions on $\mathbf{R}^{d+1} \times \mathbf{R}^n$ *in coordinates* $(t, \lambda_1, \ldots, \lambda_{d-1}, p, x)$ where $p = P^d(t, \lambda)$ and $\lambda \in \mathbf{R}^d$, $t \in \mathbf{R}$, $x \in \mathbf{R}^n$. A function $f \in C_\pi^\infty(V^d \times \mathbf{R}^n)$ in coordinates $(t, \tilde{\lambda}, p, x)$, where $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{d-1})$, is defined for $p = 0$ and divides $(\partial/\partial t P^d)(t, \lambda)$ in the space of C^∞ functions (see the proof of statement (a) from Remark 2, §3).

Theorem 1 is equivalent to the following Cauchy problem.

If there exists a continuous linear division of $f(t; \lambda_1 \ldots \lambda_{d-1}; x)$ on $(\partial/\partial t P^d)(t, \lambda)$ and the function $(f \circ \hat{\Pi}_d{}^{-1})(t, \lambda, x)$ where $(t, \lambda) \in V^d$ and $x \in \mathbf{R}^n$, is constant by t, then function $f(t; \lambda_1 \ldots \lambda_{d-1}; x)$ may be extended in a linear continuous way to a function $F(t; \lambda_1 \ldots \lambda_{d-1}; p; x)$, i.e. $F(t; \lambda_1 \ldots \lambda_{d-1}; 0; x) = f(t, \lambda_1, \ldots \lambda_{d-1}; x)$, such that

$$0 = \frac{\partial F}{\partial t}(t, \lambda_1 \ldots \lambda_{d-1}; p; x) + \left(\frac{\partial}{\partial t} P^d\right)(t, \lambda) \cdot \frac{\partial F}{\partial p}(t, \lambda_1 \ldots \lambda_{d-1}; p; x)$$

or in other words such that the function $F(t, \lambda_1 \ldots \lambda_{d-1}; P^d(t, \lambda); x)$ is constant by t.

THE MALGRANGE–MATHER DIVISION THEOREM. *There exist continuous linear operators "quotient"* $q^d: C^\infty(\mathbf{R} \times \mathbf{R}^n) \to C^\infty(\mathbf{R}^{d+1} \times \mathbf{R}^n)$ *and "residue"* $r^d = \sum_{j=1}^{d} t^{d-j} \cdot r_j^d$, *where*

$$r_j^d: C^\infty(\mathbf{R} \times \mathbf{R}^n) \to C^\infty(\mathbf{R}^d \times \mathbf{R}^n) \qquad \textit{such that}$$

$$f(t, x) = q_f{}^d(t, \lambda, x) \cdot P^d(t, \lambda) + \sum_{j=1}^{d} r_{jf}^d(\lambda, x) \cdot t^{d-j} \tag{3}$$

Remark 4. We have found it easier to prove the following more precise form of the Malgrange–Mather theorem.

Let $\lambda^d: \mathbf{R} \times \mathbf{R}^{d-1} \to \mathbf{R}^d$ be a mapping defined by identity of polynomials in z

$$P^d(z, \lambda^d(s, \mu)) = (z - s) \cdot P^{d-1}(z, \mu), (s, \mu) \in \mathbf{R} \times \mathbf{R}^{d-1}, \tag{4}$$

i.e.

$$\lambda_j = \mu_j - \mu_{j-1} \cdot s \quad \text{for } 2 \leq j \leq d-1 \qquad \text{and} \quad \begin{matrix} \lambda_1 = \mu_1 - s, \\ \lambda_d = -\mu_{d-1} \cdot s. \end{matrix} \tag{5}$$

Then in addition to assertion (3) the following holds

$$q_f{}^d(t, \lambda^d(s, \mu)) = \frac{q_f{}^{d-1}(t, \mu) - q_f{}^{d-1}(s, \mu)}{t - s}. \tag{6}$$

Remark 5. Analogous theorems hold with $q^d: C^m \to C^{l_q(m)}$ and $r_j^d: C^m \to C^{l_r(m)}$ in the C^m case and in analytic case (only local version). About the best possible choice of $l_q(m)$ and $l_r(m)$ see [3].

In §2 we deduce the Malgrange–Mather Division Theorem from Theorem 1 and construct extension of the function $f(t, x)$, $t \times x \in \mathbf{R} \times \mathbf{R}^n$, to the complex variable t (see also [2, 4, 8, 11]).

In §3 we reduce Theorem 1 to a lemma about linear continuous extension of C^∞ functions from $\pi_d(V^d) \times \mathbf{R}^n$ to C^∞ functions on $\mathbf{R}^d \times \mathbf{R}^n$. For odd d the proof is now finished because $\pi_d(V^d) = \mathbf{R}^d$.

Remark 6. In the analytic case the extension is trivial and for C^m case (or C^∞ case, but without linearity of quotient and residue) the Whitney extension theorem [18] finishes the proof. In the C^∞ case, due to the convexity of the region $\mathbf{R}^d \setminus \pi_d(V^d)$, we may apply the Stein extension theorem (see Ex. 2, p. 189 [14]).

Some results similar to Theorem 1 may be found in [9, 15].

During the research for this paper the author studied the subject through lectures of Bierstone on differential analysis and many helpful discussions with him and Ivan Kupka. Kupka made a number of useful observations on the original manuscript; in particular he simplified our proof of boundedness of operator $\nabla_1: C^m \to C^{m-1}$ (see (8)) and brought to our attention that ideas of [19] are already sufficient for differentiability of $(J\phi)(\lambda)$ at 0 (see (23)).

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Notations. Let E be a Fréchet space over field $\mathbf{R}$. In §2 and §3, we drop the parameter $x \in \mathbf{R}^n$, but assume the functions depending on the parameter x to be E-valued C^∞ functions (respectively C^m or analytic with E a Banach space). If D is a domain of definition, we denote these spaces by $C^\infty(D; E)$ (or $C^m(D; E)$). By $\mathscr{A}_\rho(D; E)$ we mean the Banach space of convergent power series in polydisk D with radii $\rho = (\rho_1, \rho_2, \ldots, \rho_q)$, $q = \dim D$ and $\rho_i > 0$, i.e.

$$\left|\!\left|\!\left| \sum_j a_j t^j \right|\!\right|\!\right|_\rho = \sum_j \|a_j\|_E \cdot \rho^j, \tag{7}$$

where j is a multi-index $j = (j_1, \ldots, j_q)$ and $\rho^j = \rho_1^{j_1} \cdot \ldots \cdot \rho_q^{j_q}$.

§2

Here we assume that Theorem 1 holds and prove the Malgrange–Mather Division Theorem.

Proof. Induction on d. Let $P^0(t, \lambda) = 1$, $q_f^0(t) = f(t)$ and $r_f^0(t, \lambda) = 0$. An operator $\nabla_1: f(t) \to \dfrac{f(t) - f(s)}{t - s}$ continuous from $C^m(\mathbf{R}, E)$ into $C^{m-1}(\mathbf{R} \times \mathbf{R}, E)$ and bounded in $|||\cdot|||_\rho$ norms because of equality

$$(\nabla_1 f)(t, s) = \int_0^1 f'(\epsilon \cdot t + (1 - \epsilon) \cdot s)\, d\epsilon, \tag{8}$$

boundedness of d/dt and isometry of $T: \phi(t) \to (T\phi)(t, s) = \int_0^1 \phi(\epsilon \cdot t + (1 - \epsilon)s)\, d\epsilon$ in $|||\cdot|||_\rho$ norms. Therefore the theorem is true for $d = 1$, namely:

$$f(t) = (\nabla_1 f)(t, s)(t - s) + f(s) \tag*{$(3)^{d=1}$}$$

and

$$q_f(t, s) = (\nabla_1 f)(t, s) = \frac{f(t) - f(s)}{t - s} = \frac{q_f^0(t) - q_f^0(s)}{t - s}. \tag*{$(6)^{d=1}$}$$

Assuming the theorem has been proved for $d - 1$ we wish to prove it for d. For the sake of convenience, let $\lambda \in \mathbf{R}^d$, $\mu \in \mathbf{R}^{d-1}$, $\nu \in \mathbf{R}^{d-2}$ and $P(t, \lambda) = P^d(t, \lambda)$, $P(t, \mu) = P^{d-1}(t, \mu)$, $P(t, \nu) = P^{d-2}(t, \nu)$. Also (see (4) and (5)) we use the notations $\lambda(s, \mu) = \lambda^d(s, \mu)$ and $\mu(\tau, \nu) = \lambda^{d-1}(\tau, \nu)$. In particular we have $P(z, \lambda(s, \mu(\tau, \nu))) = (z - s) \cdot (z - \tau) \cdot P(z, \nu)$, i.e. $\lambda(s, \mu(\tau, \nu))$ is symmetric in $(s, \tau) \in \mathbf{R} \times \mathbf{R}$.

Equality (3) (holds for $d - 1$) implies

$$f(t) = \frac{q_f^{d-1}(t, \mu) - q_f^{d-1}(s, \mu)}{t - s} \cdot P(t, \lambda(s, \mu)) + \sum_{k=1}^{d} r_{kf}(s, \mu) t^{d-k}, \tag{*}$$

where

$$\sum_{k=1}^{d} r_{kf}(s, \mu) \cdot t^{d-k} = q_f^{d-1}(s, \mu) \cdot P(t, \mu) + r_f^{d-1}(t; \mu).$$

Equality (6) (holds for $d - 1$) implies

$$(\nabla_1 q_f^{d-1})(t,s;\mu(\tau,\nu)) = \frac{\dfrac{q_f^{d-2}(t,\nu)-q_f^{d-2}(\tau,\nu)}{t-\tau} - \dfrac{q_f^{d-2}(s,\nu)-q_f^{d-2}(\tau,\nu)}{s-\tau}}{t-s}. \qquad (**)$$

Therefore $(\nabla_1 q_f^{d-1})(t,s;\mu(\tau,\nu)) = (\nabla_1 q_f^{d-1})(t,\tau;\mu(s,\nu))$ and due to the symmetry of $\lambda(s,\mu(\tau,\nu))$ in (s,τ), using (*) we have

$$r_{kf}(s,\mu(\tau,\nu)) = r_{kf}(\tau,\mu(s,\nu)), \quad 1 \le k \le d. \qquad (9)$$

The algebraic change of coordinates $(s,\mu) \mapsto (s,\tilde{\lambda}(s,\mu))$, where $\tilde{\lambda} = (\lambda_1,\dots\lambda_{d-1})$, defined by (5) is invertible. Let $(s,\tilde{\lambda}) \mapsto (s,\eta(s,\tilde{\lambda}))$, $\mu = \eta(s,\tilde{\lambda})$, be the inverse change of coordinates. We may consider functions

$$\tilde{r}_{kf}(s,\tilde{\lambda}) = r_{kf}(s,\eta(s,\tilde{\lambda})), \qquad (10)$$

as functions on V^d because $(s,\tilde{\lambda})$ are global coordinates on V^d.

Assume that $s \ne \tau$, $(s,\tilde{\lambda})$ and $(\tau,\tilde{\lambda}) \in \pi_d^{-1}(\lambda) \cap V^d$. Then there exists $\nu \in \mathbf{R}^{d-2}$ such that $P^d(t,\lambda) = (t-s)(t-\tau)\cdot P^{d-2}(t,\nu)$ and $\lambda = \lambda(s,\mu(\tau,\nu))$. Therefore, using the symmetry of $\lambda(s,\mu(\tau,\nu))$ in (s,τ) and equality (9), it follows that $\eta(s,\tilde{\lambda}) = \mu(\tau,\nu)$, $\eta(\epsilon,\tilde{\lambda}) = \mu(s,\nu)$ and hence $\tilde{r}_{kf}(s,\tilde{\lambda}) = r_{kf}(s,\mu(\tau,\nu)) = r_{kf}(\tau,\mu(s,\nu)) = \tilde{r}_{kf}(\tau,\tilde{\lambda})$. Applying Theorem 1 we obtain

$$r_{kf}^d(\lambda) = (J\tilde{r}_{kf})(\lambda) \qquad \text{such that}$$

$$r_{kf}^d(\lambda(s,\mu)) = (J\tilde{r}_{kf})(\lambda(s,\mu)) = \tilde{r}_{kf}(s,\tilde{\lambda}(s,\mu)) = r_{kf}(s,\mu). \qquad (11)$$

It is clear that operators $r_k^d : f \mapsto r_{kf}^d$ are continuous and linear. Moreover $P^d(t,\lambda) = 0$ implies $\lambda = \lambda(t,\mu)$ for some $\mu \in \mathbf{R}^{d-1}$ and due to (*) we obtain

$$f(t) - \sum_{k=1}^{d} r_{kf}^d(\lambda)\cdot t^{d-k} = f(t) - \sum_{k=1}^{d} r_{kf}(t,\mu)\cdot t^{d-k} = 0. \qquad (12)$$

Let

$$f(t,\lambda) \overset{\text{def.}}{=\!=} f(t) - \sum_{k=1}^{d} r_{kf}^d(\lambda)\cdot t^{d-k}. \qquad (13)$$

By an algebraic change of coordinates $(t,\lambda) \mapsto (t,\tilde{\lambda},P^d(t,\lambda))$, where $\tilde{\lambda} = (\lambda_1,\dots\lambda_{d-1})$, we obtain

$$\phi(t,\tilde{\lambda},p) \overset{\text{def.}}{=\!=} f\left(t;\tilde{\lambda};p - t^d - \sum_{k=1}^{d-1}\lambda_k\cdot t^{d-k}\right). \qquad (14)$$

Using (12) we have $\phi(t,\tilde{\lambda},0) = 0$. Therefore $\phi(t,\tilde{\lambda},p)$ is divisible by p and $f(t,\lambda)$ is divisible by $P^d(t,\lambda)$. Hence

$$q : f(t,\lambda) \mapsto \frac{f(t,\lambda)}{P^d(t,\lambda)} \qquad (15)$$

is a continuous map from the subspace of $C^m(\mathbf{R}^{d+1},E)$ of all functions vanishing on $V^d \subset \mathbf{R}^{d+1}$ into the space $C^{m-1}(\mathbf{R}^{d+1},E)$, and bounded in the $|||\cdot|||_p$ norms. We have finally

$$f(t) = q_f^d(t,\lambda)\cdot P^d(t,\lambda) + \sum_{k=1}^{d} r_{kf}^d(\lambda)\cdot t^{d-k} \qquad (3)^d$$

with

$$q_f^d(t,\lambda) = \frac{f(t) - \sum_{k=1}^{d} r_{kf}^d(\lambda)t^{d-k}}{P^d(t,\lambda)}.$$

Also for $\lambda = \lambda(s,\mu)$ with $P^d(t,\lambda(s,\mu)) \ne 0$ using (*) we obtain

$$q_f^d(t,\lambda(s,\mu)) = \frac{q_f^{d-1}(t,\mu) - q_f^{d-1}(s,\mu)}{t-s}. \qquad (6)^d$$

Operators $q^d : C^\infty(\mathbf{R},E) \to C^\infty(\mathbf{R}\times\mathbf{R}^d,E)$ and $r_k^d : C^\infty(\mathbf{R},E) \to C^\infty(\mathbf{R}^d,E)$ are continuous (as well as in the C^m or analytic case) and this completes the reduction to Theorem 1.

Remark 8. *Extension to the complex variable.* Let $f \in C^\infty(\mathbf{R}, E)$. We wish to define a continuous linear operator Ext. from $C^\infty(\mathbf{R}, E)$ into the space of all polynomials of degree less or equal d whose coefficients are functions in $C^\infty(\mathbf{R}^d, E)\otimes_{\mathbf{R}}\mathbf{C}$ and such that $f(t) = (\text{Ext.} f)(t, \lambda)$ when t is real and $P^d(t, \lambda) = 0$.

Construction. From $(3)^{d-1}$ we obtain

$$f(t) = q_f^{d-1}(t, \mu) \cdot t^{d-1} + \sum_{k=1}^{d-1} \{q_f^{d-1}(t, \mu) \cdot \mu_k + r_{kf}^{d-1}(\mu)\} \cdot t^{d-k-1}. \tag{16}$$

As in (*) we denote $r_{1f}(s, \mu) = q_f^{d-1}(s, \mu)$ and $r_{kf}(s, \mu) = q_f^{d-1}(s, \mu) \cdot \mu_{k-1} + r_{k-1f}^{d-1}(\mu)$, $1 < k \leq d$. Using Theorem 1 we have obtained (see (11)) functions $r_{kf}^d(\lambda)$ such that $r_{kf}^d(\lambda(t, \mu)) = r_{kf}(t, \mu)$, $1 \leq k \leq d$. Therefore

$$f(t) = r_{1f}^d(\lambda(t, \mu)) \cdot t^{d-1} + \sum_{k=1}^{d-1} r_{k+1f}^d(\lambda(t, \mu)) \cdot t^{d-k-1}. \tag{17}$$

(17) also immediately follows from $(3)^d$ letting $\lambda = \lambda(t, \mu)$. Let us define an operator Ext. by

$$(\text{Ext.} f)(z, \lambda) \overset{\text{def.}}{=\!=\!=} \sum_{k=1}^{d} r_{kf}^d(\lambda) z^{d-k}, \quad z \in \mathbf{C} \text{ and } \lambda \in \mathbf{R}^d \tag{18}$$

(see extension lemmas in [2, 4, 8, 11]). An operator Ext. gives us the desired extension.

§3. PROOF OF THEOREM 1

Induction on d. Recall the notations: $V^d = \{(t, \lambda) \in \mathbf{R} \times \mathbf{R}^d$ such that $P^d(t, \lambda) = 0\}$, the mapping $\pi_d\colon (t, \lambda) \mapsto \lambda$ is the projection $\mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$ and $C_\pi^\infty(V^d, E)$ (or $C_\pi^m(V^d, E)$ respectively) consists of C^∞ (or C^m) functions depending of $(t, \lambda_1 \ldots \lambda_{d-1})$ as local coordinates on V^d and constant on the fibres of π_d. We want to find a linear continuous operator $J\colon C_\pi^\infty(V^d, E) \to C^\infty(\mathbf{R}^d, E)$ such that

$$(J\phi)(\lambda) = \phi(t, \lambda), \text{ for every } (t, \lambda) \in V^d. \tag{2}$$

Let us define $(J\phi)(\lambda)$ for $\lambda \in \pi_d(V^d)$ by (2). The projection $\pi_d\colon \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$ restricted on V^d is a proper map and local diffeomorphism at the set U of points $(t, \lambda) \in V^d$ and such that $P_t'(t, \lambda) \neq 0$. Hence the operator J maps $C_\pi^0(V^d) \xrightarrow{\text{into}} C^0(\pi_d(V^d))$ and is continuous. Also $(d_\lambda J\phi)(\lambda)$ exists for $\lambda \in \tilde{U} = \pi_d(U)$; i.e., $P(t, \lambda) = 0$ and $(\partial/\partial t)P(t, \lambda) \neq 0$ for some $t \in \mathbf{R}$. Let us mention, that for odd d the set $\pi_d(V^d) = \mathbf{R}^d$ and for even d the set $\mathbf{R}^d \setminus \pi_d(V^d)$ is a convex subset of $\mathbf{R}^d$ (as $\lambda \in \mathbf{R}^d \setminus \pi_d(V^d)$ is equivalent to $P^d(t, \lambda) > 0$ for all $t \in \mathbf{R}$). Let $\psi(t, \tilde{\lambda}) = \phi(\hat{\pi}_d(t, \tilde{\lambda}))$, $t \in \mathbf{R}$ and $\tilde{\lambda} \in R^{d-1}$.

By the chain rule $(d_\lambda J\phi)(\lambda) \cdot \dfrac{\partial(\lambda_1 \ldots \lambda_d)}{\partial(t, \lambda_1 \ldots \lambda_{d-1})} = d_{(t,\tilde{\lambda})}\psi$, where $\lambda_d = -t^d - \sum_{k=1}^{d-1} \lambda_k \cdot t^{d-k}$ and $\tilde{\lambda} = (\lambda_1, \lambda_2 \ldots \lambda_{d-1})$, or explicitly

$$(d_\lambda J\phi)(\pi_d(t, \lambda)) \cdot \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0\,1 & \cdots \\ -P_t'(t, \lambda) & -t^{d-1} & -t^{d-2} & \cdots & -t^2 & -t \end{pmatrix} = \begin{pmatrix} \psi_t'(t, \tilde{\lambda}) \\ \psi_{\lambda_1}'(t, \tilde{\lambda}) \\ \cdots \\ \cdots \\ \cdots \\ \psi_{\lambda_{d-1}}'(t, \tilde{\lambda}) \end{pmatrix}; \tag{19}$$

recall that $(t, \lambda) = (t, \lambda_1, \ldots \lambda_{d-1}, \lambda_d) = \hat{\pi}_d(t, \tilde{\lambda})$.

The main idea of the proof is

(a) *The function* $\phi^{\text{new}}(t, \lambda) = (d_\lambda J\phi)(\pi_d(t, \lambda))$ *as a function on* V^d *coincides with d-tuple of E-valued* C^∞ *function*;

(b) *The pair of functions* $(J\phi, J\phi^{\text{new}})$ *is a Whitney function of class* C^1 *on* $\pi_d^{-1}(V^d)$ (see p. 3 in [6]).

Repeating the procedure we obtain a linear continuous operator from $C_\pi^\infty(V^d, E)$ *into*

E-valued Whitney fields on $\pi_d(V^d) \subset \mathbf{R}^d$, *i.e.* $\phi(t, \tilde{\lambda}) \mapsto \{(d_\lambda{}^n J\phi)(\lambda)\}_{n \in \mathbf{Z}^+}$. *The Stein extension theorem allows to complete the proof* (see Ex. 2, p. 189[14]).

To prove (a) it is sufficient, due to (19), to prove that function $\psi_t'(t, \tilde{\lambda})$ is divisible by $P_t'(t, \lambda)$ (and continuity of this division in C^∞ topology). First of all, $P(t, \lambda) = P_t'(t, \lambda) = 0$ implies $\psi_t'(t, \tilde{\lambda}) = 0$, because $\psi(t, \tilde{\lambda})$ is constant on the fibres π_d and there exist sequences $\lambda_{(n)} \in \mathbf{R}^d$ and $S_{1n} \neq S_{2n}$ of real numbers with $P^d(S_{jn}, \lambda_{(n)}) = 0$, $\lim_{n\to\infty} \lambda_{(n)} = \lambda$ and $\lim_{n\to\infty} S_{jn} = t$, $j = 1, 2$. $\Big($Hence $\psi_t'(t, \tilde{\lambda}) = \lim_{n\to\infty} \dfrac{\psi(S_{1n}, \tilde{\lambda}_{(n)}) - \psi(S_{2n}, \tilde{\lambda}_{(n)})}{S_{1n} - S_{2n}} = 0.\Big)$ Now by algebraic change of coordinates $(t, \lambda_1 \dots \lambda_{d-1}) \mapsto (t, \lambda_1 \dots \lambda_{d-2}, p_1)$ where $p_1 = P_t'(t, \lambda)$ we obtain

$$\theta(t, \lambda_1 \dots \lambda_{d-2}, p_1) = \psi_t'(t, \lambda_1 \dots \lambda_{d-1}) \tag{20}$$

and $\theta(t, \lambda_1 \dots \lambda_{d-2}, 0) = 0$. Therefore $\theta(t, \lambda_1 \dots \lambda_{d-2}, p_1)$ is divisible by p_1 and consequently $\psi_t'(t, \tilde{\lambda})$ is divisible by $P_t'(t, \lambda)$. Therefore we have obtained the desired division of $\psi_t'(t, \tilde{\lambda})$ by $P_t'(t, \lambda)$ and the function $(d_\lambda J\phi)(\pi_d(t, \lambda))$ coincides with a C^∞ function $\phi^{\text{new}}(t, \lambda)$ on V^d (recall that $(t, \tilde{\lambda})$ are coordinates on V^d). The function $\phi^{\text{new}}(t, \lambda)$ is automatically constant on fibres of π_d.

Remark 9. It is clear that the function $\phi^{\text{new}}(t, \lambda)$ is a function of the same type as $\phi(t, \lambda)$, i.e. polynomial, analytic, or, as in our proof, C^∞. Also the mapping $\phi(t, \lambda) \mapsto \phi^{\text{new}}(t, \lambda)$ is a continuous linear operator for all the mentioned classes and also from $C^m(V^d, E)$ into $C^{m-2}(V^d, E) \otimes_{\mathbf{R}} \mathbf{R}^d$.

Let us prove (b). The function $(J\phi)(\lambda)$ is continuously differentiable on the set $\tilde{U} \subset \pi_d(V^d)$ and, as π_d is proper, the function $(d_\lambda J\phi)(\lambda)$ has a continuous extension $(J\phi^{\text{new}})(\lambda)$ to $\pi_d(V^d)$.

Let $\gamma = \{\lambda \in \pi_d(V^d) |$ such that $\exists\, a \in \mathbf{R}$, $P^d(z, \lambda) = (z-a)^d\}$. Let us prove, first of all, that $(J\phi)(\lambda)$ is continuously differentiable at all points $\lambda \in [\pi_d(V^d) \backslash \gamma]$. Consider (t^*, λ^*) such that $P^d(z, \lambda^*) = (z - t^*)^l \cdot P^{d-l}(z, \eta^*)$ where $l < d$, $t^* \in \mathbf{R}$, $\lambda^* \in \mathbf{R}^d$, $\eta^* \in \mathbf{R}^{d-l}$ and $P^{d-l}(t^*, \eta^*) \neq 0$. The following relation,

$$P^d(z, \lambda(\xi, \eta)) = P^l(z, \xi) \cdot P^{d-l}(z, \eta), \tag{21}$$

between polynomials in z, defines a mapping $(\xi, \eta) \mapsto \lambda(\xi, \eta)$ from $\mathbf{R}^l \times \mathbf{R}^{d-l}$ into $\mathbf{R}^d$, which is a local diffeomorphism at all points (ξ, η) where the resultant of $P^l(z, \xi)$ and $P^{d-l}(z, \eta)$ as polynomials in z is not zero $\left(\text{because this resultant is the jacobian } \dfrac{\partial \lambda(\xi, \eta)}{\partial(\xi, \eta)}\right)$. Using the change of variables $(\xi, \eta) \to \lambda(\xi, \eta)$ and $(t, \xi, \eta) \to (t, \lambda(\xi, \eta))$ in some neighborhoods of the points $\lambda(\xi^*, \eta^*) = \lambda^* \in \mathbf{R}^d$ and $(t^*, \lambda^*) \in V^d$ respectively, the projection π_d: $V^d \to \mathbf{R}^d$ will have the form $\pi_l \times \text{id}$: $V^l \times \mathbf{R}^{d-l} \to \mathbf{R}^l \times \mathbf{R}^{d-l}$, where $(\pi_l \times \text{id})(t, \xi, \eta) = (\xi, \eta)$ for every $(t, \xi) \in V^l$ (i.e. $t \in \mathbf{R}$, $\xi \in \mathbf{R}^l$ and $P^l(t, \xi) = 0$) and $\eta \in \mathbf{R}^{d-l}$. Indeed, $P(t^*, \eta^*) \neq 0$ implies $P^{d-l}(t, \eta) \neq 0$ near (t^*, η^*) and due to $0 = P^d(t, \lambda(\xi, \eta)) = P^l(t, \xi) \cdot P^{d-l}(t, \eta)$ we obtain $P^l(t, \xi) = 0$. By the induction hypothesis ($l < d$) we obtain continuous differentiability of $(J\phi)(\lambda)$ at every point $\lambda \in [\pi_d(V^d) \backslash \gamma]$.

It remains to prove, that $(J\phi)(\lambda)$ is continuously differentiable at the points $\lambda \in \gamma$; i.e., $P^d(z, \lambda) = (z-a)^d$ for some $a \in \mathbf{R}$. Due to diffeomorphism $\lambda \mapsto \lambda_a$ of $\mathbf{R}^d$ into $\mathbf{R}^d$, $a \in \mathbf{R}$, which is defined by equality of polynomials in z

$$P^d(z, \lambda_a) = P^d((z-a), \lambda) \tag{22}$$

it is enough to prove differentiability of $(J\phi)(\lambda)$ at the point $0 \in \mathbf{R}^d$.

Take an arbitrary $\lambda \in \pi_d(V^d)$; there exists a smooth path which connects points λ and 0 such that $\lambda(s) \in \pi_d(V^d) \backslash \gamma$ for all $s \in (0, 1)$ and of length $\leq$const. $\sqrt{(\sum_{k=1}^{d} \lambda_\kappa^2)}$ (for example a smoothing of the path going from λ to orthogonal projection $\mathcal{P}(\lambda)$ of the point λ onto hyperplane $\lambda_d = 0$ and from $\mathcal{P}(\lambda)$ to 0). By the same arguments as in [19] we get

$$(J\phi)(\lambda) - (J\phi)(0) = \int_0^1 (d_\lambda J\phi)(\lambda(\tau)) \cdot \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}(\tau)\, \mathrm{d}\tau \tag{23}$$

as the limit when s tends to 1 of the following equality

$$(J\phi)(\lambda(s)) - (J\phi)(\lambda(1-s)) = \int_{1-s}^{s} (d_\lambda J\phi)(\lambda(\tau)) \cdot \frac{d\lambda}{d\tau}(\tau)\, d\tau, \tag{24}$$

which together with estimate on the length of the path allows us to obtain the differentiability of $(J\phi)(\lambda)$ at $\lambda = 0$. This completes the proof of Theorem 1 if d is odd and reduces Theorem 1 to the extension of the Whitney field $\{(d_\lambda{}^k J\phi)(\lambda)\}_{k\in Z^+}$ from $\mathscr{E}(\pi_d(V^d))$ to the $C^\infty(\mathbf{R}^d; E)$ (see Ex. 2, p. 189[14].

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