Algebraic and Differential Operator Equations*

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ABSTRACT

Explicit expressions for solutions of boundary-value problems and Cauchy problems related to the operator differential equation $X^{(n)} + A_{n-1}X^{(n-1)} + \cdots + A_0 X = 0$ are given in terms of solutions of the algebraic operator equation $X^n + A_{n-1}X^{n-1} + \cdots + A_0 = 0$. A method for solving this algebraic equation is studied.

1. INTRODUCTION

The purpose of this paper is to show that in an analogous way to the scalar case, explicit expressions of solutions for operator differential Cauchy and boundary-value problems can be given in terms of solutions of algebraic operator ones. It is well known that the solutions of a scalar differential equation of the type

$$x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) + \cdots + a_0 x(t) = 0, \tag{1.1}$$

where a_i for $0 \leqslant i \leqslant n-1$ are complex numbers, are given in terms of solutions of the characteristic algebraic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0. \tag{1.2}$$

For the scalar case, the equation (1.2) is always solvable, but this does not occur for the operator case. For instance, if A_0 is an operator without square

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roots, then the equation $X^2 - A_0 = 0$ is not solvable. In this paper H will denote a complex separable Hilbert space, and $L(H)$ the algebra of all bounded linear operators on H with the operator norm. If B_i lies in $L(H)$ for $0 \leqslant i \leqslant m-1$, the existence of solutions of the algebraic operator equation

$$T^m + B_{m-1}T^{m-1} + \cdots + B_0 = 0 \tag{1.3}$$

is related to the existence of a linear factorization of the polynomial operator $P(\lambda) = \lambda^m + B_{m-1}\lambda^{m-1} + \cdots + B_0$; in fact, Z is a solution of (1.3) if and only if, $\lambda I - Z$ is a right divisor of $P(\lambda)$, i.e. $P(\lambda) = P_1(\lambda)(\lambda I - Z)$, for some polynomial operator of degree $m-1$. The problem of the linear factorization of a polynomial operator has been studied by several authors. The finite-dimensional case has been studied in [12], [23], [25], and the infinite-dimensional case in [28]. The existence of solutions of the operator equation (1.3) is closely related to the companion operator

$$C = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -B_0 & -B_1 & -B_2 & \cdots & -B_{m-1} \end{bmatrix}.$$

In [12] it is proved that $P(\lambda)$ admits a linear factorization of the type $P(\lambda) = (\lambda I - Z_1)(\lambda I - Z_2)\cdots(\lambda I - Z_n)$, if the operator C is diagonable and H is finite-dimensional.

We are interested in finding explicit expressions for solutions of boundary-value problems and Cauchy problems for the operator differential equation

$$X^{(n)} + B_{m-1}X^{(n-1)} + \cdots + B_0 = 0 \tag{1.4}$$

in terms of solutions of the equation (1.3).

This paper can be regarded as a continuation of the sequence [15], [17], [18], and [19]. In Section 2 we develop a method for solving the algebraic operator equation (1.3) by reduction of the degree of the equation. This reduction is based on the application of annihilating analytic functions of operators. If H is finite-dimensional, it is well known that any operator on H is annihilated by a polynomial. For the infinite-dimensional case this does not occur, and an operator with this property is called an algebraic operator. Examples and properties of algebraic operators can be found in [20]. The existence of annihilating analytic functions of operators has been successfully applied for the last years in the context of the invariant subspace problem [1, 2, 30]. In [13], P. R. Halmos proved that an operator which is annihilated by an entire function is algebraic. If D denotes the open unit disc in the complex plane and H^∞ denotes the algebra of all bounded analytic functions on D under the supremum norm, one has the Sz.-Nagy–Foias functional calculus [26], and for the operators T in the class C_0, there is a nonzero function f in H^∞ such that $f(T) = 0$.

In analogous way to the scalar case, in Section 3 explicit expressions of solutions for Cauchy problems and boundary-value problems related to the equation (1.4) are given. Operator differential equations with constant coefficient operators are important in the theory of damped oscillatory systems and vibrational systems [12, 16, 21]. For the infinite-dimensional case, these equations occur in denumerable Markov chains [21]. Infinite-dimensional systems of differential equations have been studied with several different techniques in [7], [8], [14], etc.

If T is an operator in $L(H)$, we represent by $\sigma_\pi(T)$ its approximate point spectrum, defined as the set of all complex numbers λ such that $\lambda I - T$ is not bounded below, and we represent by $\sigma_\delta(T)$ its approximate defect spectrum, defined as the set of all complex numbers λ such that $\lambda I - T$ is not onto [20, p. 42]. If B_{ij} lies in $L(H)$ for $1 \leqslant i, j \leqslant m$, and $B = (B_{ij})$ is the associated operator matrix in $L(H^m)$, we consider the following norm in $L(H^m)$: $\|B\| = \max_{1 \leqslant j \leqslant m} \sum_{i=1}^{m} \|B_{ij}\|$, under which this space is a Banach space.

2. ON THE ALGEBRAIC OPERATOR EQUATION $T^m + B_{m-1}T^{m-1} + \cdots + B_0 = 0$

The first result of this section is a theorem which permit us to reduce the degree of the algebraic operator equation (1.3) by application of annihilating analytic operator functions of operators. We recall that finite-dimensional operators and infinite-dimensional algebraic operators are annihilated by polynomials, and for the classes pointed out in the introduction, their operators are annihilated by different classes of analytic functions. Results of this section can be regarded as a nontrivial generalization of some results of [18] obtained for $n = 2$.

THEOREM 1. *Consider the operator equation*

$$T^m + B_{m-1}T^{m-1} + \cdots + B_0 = 0 \tag{2.1}$$

in $L(H)$, *and let* $c = \max_{0 \leqslant j \leqslant m-1} \|B_j\|$. *If* T *is a solution of* (2.1) *and* $f(z) = \sum_{n \geqslant 0} a_n z^n$, *is an analytic function in the disc* $|z| < 1 + \delta$, *with* $c < \delta$, *such that* $f(T) = 0$, *then* T *satisfies the equation*

$$C_{m-1} T^{m-1} + C_{m-2} T^{m-2} + \cdots + C_0 = 0, \tag{2.2}$$

where

$$C_j = \sum_{n \geqslant 0} a_n W_{n,j}, \qquad 0 \leqslant j \leqslant m-1, \tag{2.3}$$

and the operators $W_{n,j}$ *are recurrently defined by the expressions*

$$W_{n,j} = W_{n-1,j-1} - W_{n-1,m-1} B_j,$$

$$[W_{0,0}; \ldots ; W_{0,m-1}] = [0; \ldots ;0; I] \tag{2.4}$$

for $n \geqslant 1$, $0 \leqslant j \leqslant m-1$, *and with the agreement that* $W_{n-1,-1} = 0$ *for* $n \geqslant 1$.

Proof. Let T be a solution of (2.1) and let $W_0(T)$ be the operator T^{m-1}. Then it follows that

$$\begin{aligned}
T^{m-1} &= W_0(T),\\
T^m &= -B_0 - B_1 T - \cdots - B_{m-1} T^{m-1} = W_1(T),\\
T^{m+1} &= -B_0 T - B_1 T^2 - \cdots - B_{m-2} T^{m-1}\\
&\quad - B_{m-1}\left(-B_0 - B_1 T - \cdots - B_{m-1} T^{m-1}\right)\\
&= B_{m-1} B_0 + \cdots + \left(B_{m-1}^2 - B_{m-2}\right) T^{m-1} = W_2(T),\\
&\ \ \vdots\\
T^{m+n-1} &= W_n(T),\\
&\ \ \vdots
\end{aligned}$$

Every operator $W_n(T)$ is a polynomial in T of degree at most $m-1$. In the following we prove that

$$W_n(T) = W_{n,0} + W_{n,1}T + \cdots + W_{n,m-1}T^{m-1}, \qquad n \geqslant 0. \tag{2.5}$$

From the expression $W_n(T) = W_{n-1}(T)T$, it follows that

$$\begin{aligned} W_n(T) &= \left(W_{n-1,0} + \cdots + W_{n-1,m-1}T^{m-1}\right)T \\ &= W_{n-1,0}T + \cdots + W_{n-1,m-2}T^{m-1} \\ &\quad + W_{n-1,m-1}\left(-B_0 - \cdots - B_{m-1}T^{m-1}\right) \\ &= -W_{n-1,m-1}B_0 + (W_{n-1,0} - W_{n-1,m-1}B_1)T + \cdots \\ &\quad + (W_{n-1,m-2} - W_{n-1,m-1}B_{m-1})T^{m-1}. \end{aligned}$$

Considering the expression (2.5), one gets (2.4). Let C be the operator matrix

$$C = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -B_0 & -B_1 & -B_2 & \cdots & -B_{m-1} \end{bmatrix}. \tag{2.6}$$

Let $C^n = (C_{ij}^{(n)})$ for $1 \leqslant i,\ j \leqslant m,\ n \geqslant 1$, and $C_{ij}^{(n)}$ belonging to $L(H)$. It is easy to show that

$$\begin{aligned} C_{ij}^{(n)} &= C_{i+1,j}^{(n-1)}, \qquad 1 \leqslant i \leqslant m-1, \quad 1 \leqslant j \leqslant m, \\ C_{mj}^{(n)} &= \sum_{i=1}^{m} (-B_{i-1})C_{ij}^{(n-1)}, \qquad 1 \leqslant j \leqslant m. \end{aligned} \tag{2.7}$$

From (2.7) it follows that

$$\begin{aligned} \|C^n\| &= \max_{1 \leqslant j \leqslant m} \left(\sum_{i=1}^{m} \|C_{ij}^{(n)}\| \right) = \max_{1 \leqslant j \leqslant m} \left(\sum_{i=1}^{m-1} \|C_{ij}^{(n)}\| + \|C_{mj}^{(n)}\| \right) \\ &= \max_{1 \leqslant j \leqslant m} \left(\sum_{i=1}^{m-1} \|C_{i+j,j}^{(n-1)}\| + \left\| \sum_{i=1}^{m} (-B_{i-1})C_{ij}^{(n-1)} \right) \right\| \\ &\leqslant (1+c)\|C^{n-1}\|. \end{aligned}$$

Thus the operator series $\Sigma_{n\geqslant 0} a_n C^n$ is convergent in $L(H^m)$. Moreover, from the expression (2.6) it follows that

$$[W_{n,0};\dots;W_{n,m-1}] = [W_{n-1,0};\dots;W_{n-1,m-1}]C,$$

and recurrently one gets

$$[W_{n,0};\dots;W_{n,m-1}] = [0;\dots;0;I]C^n, \qquad n \geqslant 0. \tag{2.8}$$

From (2.8) and the convergence of the operator series $\Sigma_{n\geqslant 0} a_n C^n$, it follows that $\Sigma_{n\geqslant 0} a_n W_n(T)$ converges in $L(H)$. Postmultiplying $f(T) = \Sigma_{n\geqslant 0} a_n T^n = 0$; by T^{m-1}, it follows that

$$0 = \sum_{n\geqslant 0} a_n T^{m+n-1} = \sum_{n\geqslant 0} a_n W_n(T)$$

$$= \sum_{n\geqslant 0} a_n \left(\sum_{j=0}^{m-1} W_{n,j} T^j \right) = \sum_{j=0}^{m-1} \left(\sum_{n\geqslant 0} a_n W_{n,j} \right) T^j$$

Thus, T satisfies the equation (2.2), with C_j given by (2.3). ∎

The next result is a converse of Theorem 1 and shows under which conditions the solutions of the reduced equation of degree $m-1$ are solutions of the initial equation of degree m.

THEOREM 2. *Let C be the operator defined by* (2.6), *suppose that $\|B_i\| < c$ for $0 \leqslant i \leqslant m-1$, let δ be a real number such that $0 < c < \delta$ and let $f(z) = \Sigma_{n\geqslant 0} a_n z^n$, be an analytic function in the disc $|z| < 1 + \delta$. Let the operator matrix*

$$S = f(C) = \begin{bmatrix} C_{0,m-1} & \cdots & C_{m-1,m-1} \\ \vdots & & \vdots \\ C_{0,1} & \cdots & C_{m-1,1} \\ C_0 & \cdots & C_{m-1} \end{bmatrix}$$

be such that

$$C_{m-1} \textit{ is invertible} \tag{i}$$

and

$$\begin{bmatrix} C_{0,m-1} & C_{1,m-2} & \cdots & C_{m-2,m-1} \\ & \vdots & & \vdots \\ C_{0,1} & C_{1,1} & \cdots & C_{m-2,1} \end{bmatrix} = \begin{bmatrix} C_{m-1,m-1} \\ \vdots \\ C_{m-1,1} \end{bmatrix} C_{m-1}^{-1}\left[C_0, \ldots, C_{m-2} \right]. \qquad \text{(ii)}$$

Then any solution of the equation (2.2) *is a solution of the equation* (2.1).

Proof. From the hypothesis imposed on f it is clear that $S = f(C)$ is well defined. Let T be any solution of (2.2). From the hypothesis (i)–(ii) it follows that

$$C_{j,h} = C_{m-1,h}C_{m-1}^{-1}C_j, \qquad 0 \leqslant j \leqslant m-2, \quad 1 \leqslant h \leqslant m-1. \qquad (2.9)$$

Premultiplying the equation (2.2) by $C_{m-1,h}C_{m-1}^{-1}$ and substituting (2.9), it follows that

$$\sum_{j=0}^{m-1} C_{j,h}T^j = 0, \qquad 0 \leqslant h \leqslant m-1. \qquad (2.10)$$

Considering $f(C)C = Cf(C)$, equating the operators of the last row in the two members of this equality, one gets

$$-C_{m-1}B_0 = -B_0C_{0,m-1} - B_1C_{0,m-2} - \cdots - B_{m-2}C_{0,1} - B_{m-1}C_0,$$

$$C_0 - C_{m-1}B_1 = -B_0C_{1,m-1} - B_1C_{1,m-2} - \cdots - B_{m-2}C_{1,1} - B_{m-1}C_1,$$

$$\vdots$$

$$C_{m-2} - C_{m-1}B_{m-1} = -B_0C_{m-1,m-1} - B_1C_{m-1,m-2} - \cdots - B_{m-2}C_{m-1,1} - B_{m-1}C_{m-1}.$$

From this, with the agreement that $C_{-1} = 0$,

$$B_{m-1}C_j = C_{m-1}B_j - \sum_{h=1}^{m-1} B_{m-h-1}C_{j,h} - C_{j-1} \qquad (2.11)$$

for $0 \leqslant j \leqslant m-1$. Premultiplying (2.2) by B_{m-1}, from (2.11) it follows that

$$\sum_{j=0}^{m-1} B_{m-1} C_j T^j = 0,$$

$$\sum_{j=0}^{m-1} \left(C_{m-1} B_j - \sum_{h=1}^{m-1} B_{m-h-1} C_{j,h} - C_{j-1} \right) T^j = 0,$$

$$C_{m-1} B_0 - \sum_{h=1}^{m-1} B_{m-h-1} C_{0,h} + \sum_{j=1}^{m-1} \left(C_{m-1} B_j - \sum_{h=1}^{m-1} B_{m-h-1} C_{j,h} - C_{j-1} \right) T^j = 0. \tag{2.12}$$

From (2.10) one gets $C_{0,h} = -\sum_{j=1}^{m-1} C_{j,h} T^j$, $1 \leqslant h \leqslant m-1$, and substituting these expressions into (2.12) yields

$$C_{m-1} B_0 - \sum_{h=1}^{m-1} B_{m-h-1} \left(-\sum_{j=1}^{m-1} C_{j,h} T^j \right) + \sum_{j=1}^{m-1} C_{m-1} B_j T^j - \sum_{j=1}^{m-1} \left(\sum_{h=1}^{m-1} B_{m-h-1} C_{j,h} \right) T^j - \sum_{j=1}^{m-1} C_{j-1} T^j = 0,$$

that is

$$C_{m-1} B_0 + \sum_{j=1}^{m-1} C_{m-1} B_j T^j - \sum_{j=1}^{m-1} C_{j-1} T^j = 0. \tag{2.13}$$

Postmultiplying (2.2) by T and solving, it follows that $C_{m-1} T^m = -\sum_{j=1}^{m-1} C_{j-1} T^j$, and from (2.13) results

$$C_{m-1}\left(B_0 + B_1 T + \cdots + B_{m-1} T^{m-1} + T^m \right) = 0.$$

Premultiplying by C_{m-1}^{-1}, one concludes that T is a solution of (1.2). ∎

REMARK 1. For the scalar case it is well known that an equation of the type (1.2) has at most m different solutions in the complex plane. If we consider the equation (2.1) in $L(H)$, H being a separable complex Hilbert space, this does not occur. In fact, any nontrivial projection P on H satisfies the second-degree equation $T^2 - T = 0$. Moreover, notice that the coefficient operators of this equation are $B_0 = I$, $B_1 = -I$; thus from the expression (2.3), for any annihilating analytic function f, the reduced equation $C_1T + C_0 = 0$ has coefficient operators which are scalar multiples of the identity operator, but a nontrivial projection cannot be a scalar multiple of the identity operator. An easy computation shows that taking $f(z) = z^2 - z$, the coefficient operators are $C_0 = C_1 = 0$. So our method yields in this case a trivial equation, because the reduction is not possible.

In the finite-dimensional case, an effective reduction of the equation (2.1) is available even when the operator C_{m-1} is singular, by using generalized inverses [5, 27]. For the infinite-dimensional case, the generalized inverse technique presents serious problems [3, 4], and in order to yield an effective reduction of the degree of the equation we need the invertibility of the operator C_{m-1}.

Let H_i be a Hilbert space for $i = 1,2$, and let L be the operator matrix $L = (L_{ij})$ where $L_{ij}: H_j \to H_i$, for $i, j = 1,2$. If we assume that the operator L_{22} is invertible, we can decompose the operator L in the following way:

$$L = \begin{bmatrix} I & L_{12}L_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} L_{11} - L_{12}L_{22}^{-1}L_{21} & 0 \\ 0 & L_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ L_{22}^{-1}L_{21} & I \end{bmatrix}. \quad (2.14)$$

Thus, as the first and the third factor in the decomposition (2.14) are invertible operators, the invertibility of L is equivalent to the invertibility of the operator $K = L_{11} - L_{12}L_{22}^{-1}L_{21}$. Notice that we denote $L_{22} = C_{m-1}$, $L_{21} = [C_0, \ldots, C_{m-2}]$, and

$$S = f(C), \qquad L_{11} = \begin{bmatrix} C_{0,m-1} & \cdots & C_{m-2,m-1} \\ \vdots & & \vdots \\ C_{0,1} & \cdots & C_{m-2,1} \end{bmatrix},$$

$$L_{12} = \begin{bmatrix} C_{m-1,m-1} \\ \vdots \\ C_{m-1,1} \end{bmatrix}.$$

Then, hypotheses (i)–(ii) of Theorem 2 imply that $S = f(C)$ is not invertible and $K = 0$; in particular, if H is a n-dimensional complex Hilbert space, the hypotheses (i) and (ii) of Theorem 2 are equivalent to the conditions that C_{m-1} is invertible and S has rank n.

3. BOUNDARY-VALUE PROBLEMS AND CAUCHY PROBLEMS

We start this section with a definition. If $\{x_i;\ 0 \leqslant i \leqslant n-1\}$ is a set of n different complex numbers, then the Vandermonde determinant

$$\det V(x_0,\ldots,x_{n-1}) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_{n-1} \\ \vdots & \vdots & & \vdots \\ x_0^{n-1} & x_1^{n-1} & \cdots & x_{n-1}^{n-1} \end{vmatrix}$$

is nonzero, and thus the Vandermonde associated matrix is invertible. For the operator case this does not occur, and we are interested in finding sufficient conditions imposed on a set of operators $X_0, X_1,\ldots, X_{n-1}$, in order to ensure that the Vandermonde operator of $\{X_i;\ 0 \leqslant i \leqslant n-1\}$, defined by

$$V(X_0,\ldots,X_{n-1}) = \begin{bmatrix} I & I & \cdots & I \\ X_0 & X_1 & \cdots & X_{n-1} \\ \vdots & \vdots & & \vdots \\ X_0^{n-1} & X_1^{n-1} & \cdots & X_{n-1}^{n-1} \end{bmatrix}, \tag{3.1}$$

is invertible. The Vandermonde operator (3.1) has been studied by several authors in different contexts [10, 22]. Let us consider some examples.

EXAMPLE 1. Let $n = 2$, and let X_0, X_1 be two different operators in $L(H)$. Then it follows that

$$V(X_0, X_1) = \begin{bmatrix} I & 0 \\ X_0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X_1 - X_0 \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}.$$

From this, $V(X_0, X_1)$ is invertible if and only if the operator $X_1 - X_0$ is invertible.

Let $L = (L_{ij})$ for $i, j = 1,2$ be the operator matrix introduced in Remark 1 above. If we suppose that L_{11} is invertible, we may decompose L in the following way:

$$L = \begin{bmatrix} I & 0 \\ L_{21}L_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} - L_{21}L_{11}^{-1}L_{12} \end{bmatrix} \begin{bmatrix} I & L_{11}^{-1}L_{12} \\ 0 & I \end{bmatrix}. \tag{3.2}$$

From (3.2) it follows that L is invertible if and only if the operator $K = L_{22} - L_{21}L_{11}^{-1}L_{12}$ is invertible, because the first and the third factor of the decomposition (3.2) are invertible operators. Moreover it is a straightforward matter to show that in this case one has

$$L^{-1} = \begin{bmatrix} L_{11}^{-1}(I + L_{12}K^{-1}L_{21}L_{11}^{-1}) & -L_{11}^{-1}L_{12}K^{-1} \\ -K^{-1}L_{21}L_{11}^{-1} & K^{-1} \end{bmatrix}. \tag{3.3}$$

Let $\{X_i;\ 0 \leqslant i \leqslant 2\}$ be a set of different operators in $L(H)$ such that $X_1 - X_0$ is invertible. Then if we denote

$$L_{11} = V(X_0, X_1), \qquad L_{21} = [X_0^2, X_1^2], \qquad L_{22} = X_2^2, \quad \text{and} \quad L_{12} = \begin{bmatrix} I \\ X_2 \end{bmatrix},$$

taking into account (3.2), it follows that $V(X_0, X_1, X_2)$ is invertible if and only if the following operator is invertible:

$$K = X_2^2 - [X_0^2, X_1^2] \begin{bmatrix} I + (X_1 - X_0)^{-1}X_0 & -(X_1 - X_0)^{-1} \\ -(X_1 - X_0)^{-1}X_0 & (X_1 - X_0)^{-1} \end{bmatrix} \begin{bmatrix} I \\ X_2 \end{bmatrix}$$

$$= X_2^2 - X_0^2[I + (X_1 - X_0)^{-1}(X_0 - X_2)] + X_1^2(X_1 - X_0)^{-1}(X_2 - X_0)$$

$$= X_2^2 - X_0^2 - (X_0^2 - X_1^2)(X_1 - X_0)^{-1}(X_0 - X_2). \tag{3.4}$$

From (3.4) several different hypotheses can be imposed on X_0, X_1, and X_2 in order to obtain the invertibility of $V(X_0, X_1, X_2)$.

Example 2. If $X_0X_2 = X_2X_0$, $X_0X_1 = X_1X_0$, $X_1 - X_0$, $X_2 - X_1$, and $X_2 - X_0$ are invertible operators in $L(H)$, then the Vandermonde operator $V(X_0, X_1, X_2)$ is invertible, and its inverse operator is given by (3.3) with

$$L_{11} = V(X_0, X_1), \qquad L_{22} = X_2^2, \qquad L_{21} = [X_0^2, X_1^2], \qquad L_{12} = \begin{bmatrix} I \\ X_2 \end{bmatrix},$$

and $K = (X_2 - X_1)(X_2 - X_0)$.

The result is a consequence of (3.2), (3.3) and (3.4), because from (3.4) and the hypothesis it follows that $K = (X_2 + X_0)(X_2 - X_0) + (X_0 + X_1)(X_0 - X_2) = (X_2 - X_1)(X_2 - X_0)$.

Although the Vandermonde operator $V(X_0, X_1, X_2)$ may be invertible with $V(X_0, X_1)$ singular (an example is given in [10]), the following result gives a sufficient condition in order to ensure the invertibility of the Vandermonde operator $V(X_0,\dots, X_{n-1})$, $n \geqslant 3$, under the hypothesis of invertibility of $V(X_0,\dots, X_{n-2})$.

PROPOSITION 1. *Consider a set of n different operators in $L(H)$, $\{X_i;\ 0 \leqslant i \leqslant n-1\}$. Then $V(X_0,\dots, X_{n-1})$ is invertible if the following conditions are satisfied:*

(i) *$V(X_0,\dots, X_{n-2})$ is invertible.*
(ii) *The matrix*

$$X_{n-1}^{n-1} - \left[X_0^{n-1},\dots, X_{n-2}^{n-1}\right]\left[V(X_0,\dots, X_{n-2})\right]^{-1}\begin{bmatrix} I \\ X_{n-1} \\ \vdots \\ X_{n-1}^{n-2} \end{bmatrix} = K \quad (3.5)$$

is invertible.

In this case, $V(X_0,\dots, X_{n-1})^{-1}$ is given by (3.3), *taking as K the operator given in* (3.5), $L_{11} = V(X_0,\dots, X_{n-2})$, $L_{21} = [X_0^{n-1},\dots, X_{n-2}^{n-1}]$, $L_{22} = X_{n-1}^{n-1}$, *and*

$$L_{12} = \begin{bmatrix} I \\ X_{n-1} \\ \vdots \\ X_{n-1}^{n-2} \end{bmatrix}.$$

Proof. The result is an easy consequence of (3.2), (3.3), and the following decomposition of $V(X_0,\dots, X_{n-1})$:

$$V(X_0,\dots, X_{n-1}) = \left[\begin{array}{ccc|c} & & & I \\ & V(X_0,\dots, X_{n-2}) & & X_{n-1} \\ & & & \vdots \\ & & & X_{n-1}^{n-2} \\ \hline X_0^{n-1} & \cdots & X_{n-2}^{n-1} & X_{n-1}^{n-1} \end{array}\right]. \quad \blacksquare$$

Let us consider the Cauchy problem

$$X^{(n)} + A_{n-1}X^{(n-1)} + \cdots + A_0X = 0$$

$$X(0) = C_0, \qquad X^{(1)}(0) = C_1, \ldots, X^{(n-1)}(0) = C_n. \tag{3.6}$$

Considering $Y_1 = X$, $Y_2 = X^{(1)}, \ldots,$ $Y_n = X^{(n-1)}$, the problem (3.6) is equivalent to the Cauchy problem on $L(H^n)$

$$Y^{(1)} = \begin{bmatrix} Y_1^{(1)} \\ \vdots \\ Y_n^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{n-1} \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix},$$

$$\begin{bmatrix} C_0 \\ \vdots \\ C_{n-1} \end{bmatrix} = \begin{bmatrix} Y_1(0) \\ \vdots \\ Y_n(0) \end{bmatrix}. \tag{3.7}$$

Thus, if we denote by A the operator matrix of coefficients of (3.7), the operator $\exp(A(t-s))$ is a fundamental operator of (3.7) and the Cauchy problem has only solution [22].

Let us consider the algebraic operator equation

$$X^n + A_{n-1}X^{n-1} + \cdots + A_0 = 0 \tag{3.8}$$

We say that a set $\{X_i;\ 0 \leqslant i \leqslant n-1\}$ of n different solutions of (3.8) is a fundamental set of solutions of (3.8) if the Vandermonde operator $V(X_0,\ldots,X_{n-1})$ defined by (3.1) is invertible. Note that an operator function of the type $X(t)=\exp(X_0t)D_0+\cdots+\exp(X_{n-1}t)D_{n-1}$, for any set of operators D_i, $0\leqslant i\leqslant n-1$, is a solution of the differential equation arising in (3.6) if X_i satisfies (3.8) for $0\leqslant i\leqslant n-1$.

The following result proves that if $\{X_i;\ 0\leqslant i\leqslant n-1\}$ is a fundamental set of solutions of (3.8), then any solution of (3.6) may be expressed in this form.

THEOREM 3. *Let $\{X_i;\ 0\leqslant i\leqslant n-1\}$ be a fundamental set of solutions of the equation* (3.8), *and let X be a solution of the operator differential equation of* (3.6). *Then there are operators $D_0,\ldots,D_{n-1}$ in $L(H)$, uniquely determined by X, such that*

$$X(t)=\sum_{i=0}^{n-1}\exp(X_it)D_i.$$

These operators are defined by the expression

$$\begin{bmatrix} D_0\\ \vdots\\ D_{n-1}\end{bmatrix}=\left[V(X_0,\ldots,X_{n-1})\right]^{-1}\begin{bmatrix} X(0)\\ \vdots\\ X^{(n-1)}(0)\end{bmatrix}. \tag{3.9}$$

Proof. Given the solution X of the differential equation of (3.6), we use the uniqueness property for the Cauchy problem (3.6) taking $C_i=X^{(i)}(0)$, for $0\leqslant i\leqslant n-1$, and note that every expression $\sum_{i=0}^{n-1}\exp(X_it)D_i$ satisfies the corresponding differential equation for any operators $D_0,\ldots,D_{n-1}$ belonging to $L(H)$. Thus, in order to prove the result, we must find operators D_i in $L(H)$, for $0\leqslant i\leqslant n-1$, such that

$$\begin{aligned} D_0+D_1+\cdots+D_{n-1}&=C_0,\\ X_0D_0+X_1D_1+\cdots+X_{n-1}D_{n-1}&=C_1,\\ &\ \vdots\\ X_0^{n-1}D_0+X_1^{n-1}+\cdots+X_{n-1}^{n-1}D_{n-1}&=C_{n-1}. \end{aligned} \tag{3.10}$$

Note that the system (3.10) obtained by successive differentiations of the operator function $W(t)=\sum_{i=0}^{n-1}\exp(X_i t)D_i$, and imposing $W^{(i)}(0)=C_i$, for $0 \leqslant i \leqslant n-1$, is equivalent to the system

$$V(X_0,\ldots,X_{n-1})\begin{bmatrix} D_0 \\ \vdots \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} C_0 \\ \vdots \\ C_{n-1} \end{bmatrix} \tag{3.11}$$

From the hypothesis the system (3.11) has only one solution, given by (3.9). ∎

REMARK 2. Note that an explicit expression for the operators D_i, for $0 \leqslant i \leqslant n-1$, is available when $V(X_0,\ldots,X_{n-1})$ is invertible, by application of Proposition 1 and Examples 1 and 2. Note also, that an equation of the type (3.8) can be unsolvable, as we pointed out in the introduction; thus for certain equations it is not possible to find a fundamental set of solutions. Moreover, given a solvable equation, a set of n different solutions, n being the degree of the equation, is not necessarily a fundamental set. For instance, let us consider the operator differential equation $X^{(2)}-X^{(1)}=0$. Then the algebraic equation $X^2-X=0$, has the fundamental set of solutions $\{0, I\}$; but if we consider two different projections P_1 and P_2 such that their ranges satisfy $\{0\} \subsetneq \operatorname{range}(P_1) \subsetneq \operatorname{range}(P_2) \neq H$, then the solution set $\{P_1, P_2)$, is not a fundamental set of solutions because of Example 1 and the fact that $P_2 - P_1$ is not invertible in $L(H)$.

The following result is concerned with the study of the next boundary-value problem.

LEMMA 1. *Let* $\{X_i;\ 0 \leqslant i \leqslant n-1\}$ *be a fundamental set of solutions of the equation* (3.8), *and let* B_i *be an invertible operator such that* $B_iX_i = X_iB_i$, *for* $0 \leqslant i \leqslant n-1$. *Then the operator*

$$Z = \begin{bmatrix} B_0 & B_1 & \cdots & B_{n-1} \\ B_0X_0 & B_1X_1 & \cdots & B_{n-1}X_{n-1} \\ \vdots & \vdots & & \vdots \\ B_0X_0^{n-1} & B_1X_1^{n-1} & \cdots & B_{n-1}X_{n-1}^{n-1} \end{bmatrix}$$

is invertible in $L(H^n)$.

Proof. The result is a direct consequence of the commutativity hypothesis and the following decomposition:

$$Z = V(X_0, \dots, X_{n-1}) \begin{bmatrix} B_0 & 0 & \cdots & 0 \\ 0 & B_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_{n-1} \end{bmatrix}. \qquad \blacksquare$$

The following result is concerned with the boundary-value problem

$$X^{(n)} + A_{n-1}X^{(n-1)} + \cdots + A_0 X = 0,$$
$$X^{(i)}(b_i) - X^{(i)}(0) = E_i, \qquad 0 \leqslant i \leqslant n-1, \quad b_i > 0, \quad (3.12)$$

where E_i and A_i are operators in $L(H)$ and b_i are real numbers, for $0 \leqslant i \leqslant n-1$.

THEOREM 4. *Let* $\{X_i;\ 0 \leqslant i \leqslant n-1\}$ *be a fundamental set of solutions of the equation* (3.8) *such that*

$$z_j \neq \frac{2k\pi i}{b_j}, \qquad 0 \leqslant j \leqslant n-1, \qquad (3.13)$$

where z_j *belongs to the spectrum* $\sigma(X_j)$ *for* $0 \leqslant j \leqslant n-1$, *and* k *is any integer. Then the boundary-value problem* (3.12) *has only one solution given by* $X(t) = \sum_{i=0}^{n-1} \exp(X_i t) D_i$, *where the operators* D_i, *for* $0 \leqslant i \leqslant n-1$, *are determined by the expression*

$$\begin{bmatrix} \exp(X_0 b_0) - I & \cdots & \exp(X_{n-1} b_{n-1}) - I \\ [\exp(X_0 b_0) - I] X_0 & \cdots & [\exp(X_{n-1} b_{n-1}) - I] X_{n-1} \\ \vdots & & \vdots \\ [\exp(X_0 b_0) - I] X_0^{n-1} & \cdots & [\exp(X_{n-1} b_{n-1}) - I] X_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} D_0 \\ \vdots \\ D_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} E_0 \\ \vdots \\ E_{n-1} \end{bmatrix}. \qquad (3.14)$$

Proof. From Proposition 1, the general solution of the differential equation arising in (3.12) is given by $W(t)=\sum_{i=0}^{n-1}\exp(X_i t)D_i$, for arbitrary operators D_i in $L(H)$, for $0 \leqslant i \leqslant n-1$. Thus in order to obtain a solution of the boundary-value problem (3.12), it is sufficient to find operators D_i such that the boundary conditions of (3.12) are satisfied. Imposing these conditions on $W(t)$, it follows that the operators D_i must verify

$$\sum_{i=0}^{n-1} \exp(X_i b_i) X_i^j D_i - \sum_{i=0}^{n-1} X_i^j D_i = E_j, \qquad 0 \leqslant j \leqslant n-1. \tag{3.15}$$

The system (3.15) coincides with (3.14). From the hypothesis (3.13) and the spectral mapping theorem [9], the operators $\exp(X_j b_j) - I$ for $0 \leqslant j \leqslant n-1$ are invertible, and it is clear that $B_j = \exp(X_j b_j) - I$ satisfies $B_j X_j = X_j B_j$ for $0 \leqslant j \leqslant n-1$. From Lemma 1, the coefficient operator matrix of the system (3.14) is invertible in $L(H^n)$, and thus there are operators D_i, for $0 \leqslant i \leqslant n-1$, uniquely determined, such that $X(t)=\sum_{i=0}^{n-1}\exp(X_i t)D_i$ is the only solution of the problem (3.12). ■

REMARK 3. In order to compute the operators D_i, $0 = i = n-1$, it is necessary to compute the inverse of the coefficient operator matrix arising in the system (3.14). A method for obtaining it is suggested in (3.2) and (3.3). Thus an explicit expression for the operators D_i in terms of the data and the fundamental set of solutions is available.

In the following result we study a different boundary-value problem with only one boundary condition and where only one solution of the algebraic equation (3.8) is sufficient in order to obtain an explicit expression for the solutions.

THEOREM 5. *Let us consider the boundary-value problem*

$$X^{(n)} + A_{n-1}X^{(n-1)} + \cdots + A_0 X = 0$$

$$EX(b) - X(0)F = G, \qquad b > 0, \tag{3.16}$$

where E, F, and G are operators in $L(H)$ and b is a positive real number. Let X_0 be a solution of (3.8).

(i) *If $\sigma_\delta(E\exp(X_0 b)) \cap \sigma_\pi(F) = \varnothing$, then a solution of the problem* (3.16) *is given by the operator function*

$$X(t) = \exp(X_0 t)\, D_0, \tag{3.17}$$

where D_0 *is a solution of the algebraic equation*

$$E\exp(X_0 b)U - UF = G. \tag{3.18}$$

(ii) *If* $\sigma(E\exp(X_0 b)) \cap \sigma(F) = \varnothing$, *and* F *is an algebraic operator annihilated by the polynomial* $p(z) = \sum_{k=0}^{q} p_k z^k$, *then a solution of the problem* (3.16) *is given by* (3.17), *where* D_0 *is given by the expression*

$$\left(\sum_{k=0}^{q} p_k [E\exp(X_0 b)]^k\right)^{-1} \left(\sum_{k=1}^{q} \sum_{j=1}^{k} p_j [E\exp(bX_0)]^{j-1} G F^{k-j}\right). \tag{3.19}$$

Proof. For any operator D in $L(H)$, it is clear that $X(t) = \exp(X_0 t)D$ satisfies the differential equation arising in (3.16). This function X satisfies the boundary condition of (3.16) if and only if D satisfies the equation (3.18). From the hypothesis of (i) and Theorem 5 of [6], the equation (3.18) is solvable. Thus (i) is proved.

(ii): Under the Rosenblum condition imposed in the hypothesis, the equation (3.18) has only one solution. This solution is given by (3.19): see [1], [29]. ■

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