An infinite family of tetravalent half-arc-transitive graphs

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Abstract

A graph is half-arc-transitive if its automorphism group acts transitively on vertices and edges, but not on arcs. In this paper, a new infinite family of tetravalent half-arc-transitive graphs with girth 4 is constructed, each of which has order 16\(m\) such that \(m > 1\) is a divisor of \(2t^2 + 2t + 1\) for a positive integer \(t\) and is tightly attached with attachment number \(4m\). The smallest graph in the family has order 80.

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1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, unless specified otherwise, connected and undirected (but with an implicit orientation of the edges when appropriate). For a graph \(X\) we let \(V(X)\), \(E(X)\), \(A(X)\) and \(\text{Aut}(X)\) be the vertex set, the edge set, the arc set and the automorphism group of \(X\), respectively.

A graph \(X\) is said to be vertex-transitive, edge-transitive, or arc-transitive if \(\text{Aut}(X)\) acts transitively on \(V(X)\), \(E(X)\) or \(A(X)\), respectively. A graph is said to be \(1/2\)-arc-transitive or half-arc-transitive provided that it is vertex-transitive, edge-transitive, but not arc-transitive. More generally, a subgroup \(G\) of the automorphism group \(\text{Aut}(X)\) of a graph \(X\) is said to be half-arc-transitive if \(G\) is vertex-transitive and edge-transitive, but not arc-transitive on \(X\). In this case, we shall say that the graph \(X\) is \((G, 1/2)\)-arc-transitive.

Let \(X\) be a tetravalent \((G, 1/2)\)-arc-transitive graph. Then in the natural action of \(G\) on \(V(X) \times V(X)\), the arc set of \(X\) is the union of two \(G\)-orbits, say \(A_1\) and \(A_2\), which are paired with each other, that is, \(A_2 = \{(u, v) | (u, v) \in A_1\}\). Each of the two corresponding oriented graphs \((V(X), A_1)\) and \((V(X), A_2)\) has out-valency and in-valency equal to 2, and admits \(G\) as a vertex- and arc-transitive group of automorphisms. Moreover, each of them has \(X\) as its underlying graph. Let \(D_G(X)\) be one of these two oriented graphs, fixed from now on. For \(u, v \in V(X)\) such that \((u, v)\) is an arc in \(D_G(X)\), we say that \(u\) and \(v\) are the tail and the head of the arc \((u, v)\), respectively. An even length cycle \(C\) in \(X\) is called a \(G\)-alternating cycle if the vertices of \(C\) are alternately, the tail or the head (in \(D_G(X)\)) of their two

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incident edges in \( C \). It is proved in [16, Proposition 2.4(i)] that all \( G \)–alternating cycles in \( X \) have the same length and form a decomposition of the edge set of \( X \), half of this length is denoted by \( r_G(X) \) and is called the \( G \)-radius of \( X \). It is proved in [16, Proposition 2.6] that any two adjacent \( G \)–alternating cycles in \( X \) intersect in the same number of vertices, called the \( G \)-attachment number \( a_G(X) \) of \( X \). We say that \( X \) is tightly \( G \)-attached if \( r_G(X) = a_G(X) \). In [16], all tetravalent \(( G, \frac{1}{2})\)-arc-transitively \( G \)-attached graphs with odd \( G \)-radius have been classified and, moreover, it has also been specified which among them are \( \frac{1}{2} \)-arc-transitive and which are arc-transitive. A description of all tetravalent \(( G, \frac{1}{2})\)-arc-transitively \( G \)-attached graphs with even \( G \)-radius is given in [24]. It is not known, however, that which among them are \( \frac{1}{2} \)-arc-transitive. If \( X \) is \( \frac{1}{2} \)-arc-transitive, the terms an \( \text{Aut}(X) \)-alternating cycle, \( \text{Aut}(X) \)-radius, and \( \text{Aut}(X) \)-attachment number are referred to as an alternating cycle of \( X \), radius of \( X \) and attachment number of \( X \), respectively. Similarly, if \( X \) is tightly \( \text{Aut}(X) \)-attached, we say that \( X \) is tightly attached.

Let \( K \) be a group and \( X \) a graph. Let \( \phi: A(X) \mapsto K \) be the so-called voltage assignment, that is, a function from the set of arcs of \( X \) into the group \( K \), where reverse arcs carry inverse voltages. We, thus, have a labelling of the arcs of \( X \) by elements in \( K \) such that \( \phi(u, v)\phi(v, u) = id \) for all pairs of adjacent vertices \( u, v \) in \( X \), where \( \phi(u, v) \) denotes the element in \( K \) assigned to the arc \(( u, v) \). The voltage assignment \( \phi \) on arcs extends to a voltage assignment on walks in a natural way and for a walk \( W \) of \( X \), we let \( \phi(W) \) denote the voltage of \( W \). The covering graph \( X \times_\phi K \) of \( X \) with respect to \( \phi \) have vertex set \( V(X) \times K \) and edge set \( \{(u, g)(v, h) | uv \in E(X), h = \phi(u, v)g\} \). The graph \( X \) is said to be the base graph of \( X \times_\phi K \), and the latter is sometimes referred to as an regular covering (or \( K \)-covering) of \( X \). The set of vertices \( \{(u, k) | k \in K\} \) is called the fibre of \( u \). For \( k \in K \), by defining \((u, g)^k := (u, gk)\) for any \((u, g) \in V(X \times_\phi K) \), \( K \) becomes a subgroup of \( \text{Aut}(X \times_\phi K) \) which acts regularly on each fibre. If \( X \times_\phi K \) is connected \( K \) is called the covering transformation group. Clearly, a covering transformation maps each fibre onto itself. An automorphism of \( X \times_\phi K \) is said to be fibre-preserving if it maps a fibre to a fibre (not necessary itself). All such fibre-preserving automorphisms form a group called the fibre-preserving group. It is easy to see that each fibre is a block of the fibre-preserving group and if \( X \times_\phi K \) is connected, \( K \) is the kernel of the fibre-preserving group acting on the set of fibres.

The investigation of \(( G, \frac{1}{2})\)-arc-transitive graphs was initiated by Tutte [29] who proved that a vertex- and edge-transitive graph with odd valency must be arc-transitive. In 1970, Bouwer [3] constructed the first infinite family of half-arc-transitive graphs and later more such graphs were constructed (see [2,3,6,9,12,17,23,28,30]). There are now currently four main open areas of research in half-arc-transitive graphs. The first area of research is the study of half-arc-transitive graphs as addressed in [5,19]. And finally, the fourth area of research concerns the “attachment of alternating cycles” question. In that respect, odd radius tightly attached tetravalent half-arc-transitive graphs have been classified as mentioned before [16] and even radius tightly attached tetravalent half-arc-transitive graphs are definitely objects worth exploring. Wilson [31] gave a description of tetravalent graphs admitting half-arc-transitive group action with respect to which they are of even radius and tightly attached. However, no examples of even radius attached half-arc-transitive graphs were constructed in [31] and in this paper, an infinite family of even radius tightly attached tetravalent half-arc-transitive graphs are constructed. For each number \( m > 1 \) being a divisor of \( 2r^2 + 2t + 1 \) for a positive integer \( t \), we construct a tetravalent half-arc-transitive graph of order \( 16m \), which is a regular covering of the complete bipartite graph \( K_{4,4} \). Clearly, \( m \) is odd. These half-arc-transitive graphs have girth 4 and are tightly attached with attachment number \( 4m \). Since these half-arc-transitive graphs have solvable automorphism groups and order 16 times an odd integer, one may show that they do not belong to any family of half-arc-transitive graphs discussed above (note that the half-arc-transitive graphs constructed in [27, Theorem 3.1] have orders \( 2r^t \) times an odd integer with \( 2 \leq r \leq 3 \) (see [27, Lemma 3.5]) and the half-arc-transitive graphs constructed in [19, Proposition 5.1] have girth 3). Constructing and characterizing tetravalent half-arc-transitive graphs is currently an active topic in algebraic graph theory (see [1,4,7,11–14,16,22,20,19,24–27,32]).

To state the main result of this paper, we first introduce an infinite family of tetravalent half-arc-transitive graphs. Let \( \mathbb{Z}_n \) be the cyclic group of order \( n \), as well as the ring of integers modulo \( n \). Denote by \( \mathbb{Z}_n^* \) the multiplicative group of \( \mathbb{Z}_n \) consisting of numbers coprime to \( n \). Denote by \( V(K_{4,4}) = \{a, b, c, d, u, v, w, x\} \) the vertex set of the bipartite graph \( K_{4,4} \) as in Fig. 1. Let \( t \) be a positive integer and let \( m > 1 \) be a divisor of \( 2t^2 + 2t + 1 \). The graph \( \mathcal{C}_H(16m) \) is
Theorem 1.1. Let \( t \) be a positive integer and let \( \tilde{\ g \} \) be fibre-preserving and let \( \tilde{\ g \} \) be an automorphism of \( X \). Let \( \tau \) be a half-arc-transitive graph of order \( 4 \). It is easy to see that \( \mathcal{G}(X, K) \) is connected if and only if the voltages on the cotree arcs generate the voltage group \( K \). The notation \( \mathcal{G}(X, K) \) means cyclic covering of \( K, 4 \). It is easy to see that \( \mathcal{G}(X, K) \) is bipartite. The main result of this paper is the following theorem.

**Theorem 1.1.** Let \( t \) be a positive integer and let \( m > 1 \) be a divisor of \( 2t^2 + t + 1 \). Then, the graph \( \mathcal{G}(X, K) \) is a half-arc-transitive graph of order \( 16m \) with girth \( 4 \). Furthermore, \( \mathcal{G}(X, K) \) is tightly attached with attachment number \( 4m \).

2. Lifting automorphisms

Let \( X \) be a graph and \( K \) a group. Let \( X \times K \) be a regular covering of \( X \). Given a spanning tree \( T \) of the graph \( X \), the voltage assignment \( \phi \) is said to be \( T \)-reduced if the voltages on the tree arcs are the identity. Gross and Tucker [8] showed that every regular covering \( X \times K \) of a graph \( X \) can be derived from a \( T \)-reduced voltage assignment with respect to an arbitrary fixed spanning tree \( T \) of \( X \). It is clear that if \( \phi \) is \( T \)-reduced, the covering graph \( X \times K \) is connected if and only if the voltages on the cotree arcs generate the voltage group \( K \).

Let \( \tilde{\ g \} \in \text{Aut}(X \times K) \) be fibre-preserving and let \( \tau \in \text{Aut}(X) \). Clearly, the permutation induced by \( \tilde{\ g \} \) on the set of fibres is an automorphism of \( X \) (identifying the fibres with the vertices of \( X \)). If the automorphism is exactly \( \tau \) we call \( \tilde{\ g \} \) a lift of \( \tau \), and \( \tau \) a projection of \( \tilde{\ g \} \). Concepts such as a lift of a subgroup of \( \text{Aut}(X) \) and the projection of a subgroup of \( \text{Aut}(X \times K) \) are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in \( \text{Aut}(X \times K) \) and \( \text{Aut}(X) \), respectively. In particular, if the covering graph \( X \times K \) is connected, then the covering transformation group \( K \) is the lift of the identity subgroup. Clearly, if \( \tilde{\ g \} \) is a lift of \( \tau \), then \( K \tilde{\ g \} \) are all the lifts of \( \tau \).

The problem whether an automorphism \( \tau \) of \( X \) lifts or not can be grasped in terms of voltages as follows. Given \( \tau \in \text{Aut}(X) \), we define a function \( \mathcal{Z} \) from the set of voltages of fundamental closed walks based at a fixed vertex \( v \in V(X) \) to the voltage group \( K \) by

\[
(\phi(C))^{\mathcal{Z}} = \phi(C^2),
\]
where $C$ ranges over all fundamental closed walks at $v$, and $\phi(C)$ and $\phi(C^2)$ are the voltages of $C$ and $C^2$, respectively. Note that if $K$ is abelian, $\Xi$ does not depend on the choice of the base vertex, and the fundamental closed walks at $v$ can be substituted by the fundamental cycles corresponding to the cotree arcs of $X$. The following proposition is a special case of [13, Theorem 4.2].

**Proposition 2.1.** Let $X \times \phi K \mapsto X$ be a connected $K$-covering where $\phi$ is $T$-reduced. Then, an automorphism $x$ of $X$ lifts if and only if $\Xi$ extends to an automorphism of $K$.

### 3. Proof of Theorem 1.1

Let $t$ be a positive integer and let $m > 1$ be a divisor of $2t^2 + 2t + 1$. Thus, $4t^2 + 4t + 2 = 0$ in the ring $\mathbb{Z}_{2m}$. Let $\tilde{X} = K_{4,4} \times \phi \mathbb{Z}_{2m}$ be a regular covering of the graph $K_{4,4}$ such that $\phi = 0$ on the spanning tree $T$ which is illustrated by dark lines in Fig. 1 and we assign voltages $\tilde{z}_1 = 1, \tilde{z}_2 = t + 1, \tilde{z}_3 = -t, \tilde{z}_4 = 0, z_5 = 2t^2 + t, z_6 = 2t^2 + t, z_7 = 2t^2 + 2t + 1, z_8 = t$ and $z_9 = 2t^2 + 3t + 1$ to the cotree arcs of $K_{4,4}$. Note that $z_i \in \mathbb{Z}_{2m}$ for $1 \leq i \leq 9$ and the vertex set of $K_{4,4}$ is $\{a, b, c, d, u, v, w, x\}$. Clearly, $\tilde{X}$ is connected because $\langle \tilde{z}_1 \rangle = \mathbb{Z}_{2m}$. By the definition of the graph $\mathcal{C}(16m)$ in the paragraph preceding the statement of Theorem 1.1, $\tilde{X} = K_{4,4} \times \phi \mathbb{Z}_{2m} = \mathcal{C}(16m)$.

Let $x = (a \ u \ c \ w) (b \ v \ d \ x)$ and $\beta = (u \ v) (w \ x)$. Then $x$ and $\beta$ are automorphisms of $K_{4,4}$. Let $G = \langle x, \beta \rangle$. It is clear that $(\beta, \beta^2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $(\beta, \beta^2)$ is a normal subgroup of index 4 in $G$. Thus, one may show that $|G| = 16$ and that $K_{4,4}$ is $(G, \frac{1}{2})$-arc-transitive.

Denote by $i_1i_2 \ldots i_s$ the cycle with consecutive adjacent vertices $i_1, i_2, \ldots, i_s$. There are nine fundamental cycles $\text{ubva, wbva, xbva, ucwa, vcwa, xcwa, udxa, vdxax}$ and $\text{wdxa}$ in $K_{4,4}$, which are generated by the nine cotree arcs $(u, b), (w, b), (x, b), (u, c), (v, c), (x, c), (u, d), (v, d)$ and $(w, d)$, respectively. Each cycle maps to a cycle of the same length under the actions of $x$ and $\beta$. We list these cycles and their voltages in Table 1, where $C$ denotes a fundamental cycle of $K_{4,4}$ and $\phi(C)$ denotes the voltage on $C$.

Since $4t^2 + 4t + 2 = 0$, one has $(2t + 1)^2 = -1$ in $\mathbb{Z}_{2m}$. Clearly, $m > 1$ implies that $2 \neq 0$ in $\mathbb{Z}_{2m}$ and so $(2t + 1)^2 \neq 1$. Thus, $2t + 1 \in \mathbb{Z}_{2m}$ has order 4. Consider the mappings $\Xi$ and $\overline{\beta}$ from the set of voltages of the nine fundamental cycles of $K_{4,4}$ to the cyclic group $\mathbb{Z}_{2m}$, defined by $\phi(C)^2 = \phi(C^2)$ and $\phi(C^2) = \phi(C^2)$, respectively, where $C$ ranges over the nine fundamental cycles. With $4t^2 + 4t + 2 = 0$ and $(2t + 1)^2 = -1$, by Table 1 one may easily check that $\Xi$ and $\overline{\beta}$ can be extended to the automorphisms of $\mathbb{Z}_{2m}$ induced by $1 \mapsto 2t + 1$ and $1 \mapsto -1$, respectively. By Proposition 2.1, $x$ and $\beta$ lift, and so $G$ lifts. Let $\overline{x}^*$ and $\overline{\beta}^*$ be one of the lifts of $x$ and $\beta$, respectively. Then, $B := \langle \mathbb{Z}_{2m}, \overline{x}^*, \overline{\beta}^* \rangle$ is the subgroup of Aut($\tilde{X}$) lifted by $G$, where $\mathbb{Z}_{2m}$ is identified as a subgroup of Aut($\tilde{X}$) by right multiplication on the second coordinate of each vertex in $K_{4,4} \times \phi \mathbb{Z}_{2m}$ (see Section 1). Since $G$ is half-arc-transitive on $K_{4,4}$, $B$ is half-arc-transitive on the covering graph $\tilde{X} = K_{4,4} \times \phi \mathbb{Z}_{2m}$. Furthermore, $|G| = 16$ implies that $|B| = 32m$. Let $A := \text{Aut}(\tilde{X})$. Then, $B \leq A$.

To prove half-arc-transitivity of $\tilde{X}$, it suffices to show that $A = B$.

For the convenience of statement, we use $z_i$ to denote the vertex $(\overline{x}, x)$ of $\tilde{X}$ where $z \in V(K_{4,4})$ and $x \in \mathbb{Z}_{2m}$. One may easily show that $(a_0, u_0, c_0, w_0)$ and $(a_0, v_0, c_2t^2 + 2t, x_0)$ are precisely the two 4-cycles, that is, cycles of length 4, passing through $a_0$ in $\tilde{X}$. Since $B \leq A$, it follows that $X$ is vertex- and edge-transitive. Vertex-transitivity of $\tilde{X}$ implies

<table>
<thead>
<tr>
<th>Cycle</th>
<th>$\phi(C)$</th>
<th>$C^2$</th>
<th>$\phi(C^2)$</th>
<th>$C^\beta$</th>
<th>$\phi(C^\beta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ubva</td>
<td>$z_1 = 1$</td>
<td>cvdu</td>
<td>$-z_5 + z_6 - z_7 + z_4$</td>
<td>vbuva</td>
<td>$-z_1$</td>
</tr>
<tr>
<td>wbva</td>
<td>$z_2 = t + 1$</td>
<td>avdu</td>
<td>$z_8 - z_7$</td>
<td>xbuva</td>
<td>$z_3 - z_1$</td>
</tr>
<tr>
<td>xbva</td>
<td>$z_3 = -t$</td>
<td>bvdu</td>
<td>$z_8 - z_7 + z_1$</td>
<td>wbva</td>
<td>$z_2 - z_1$</td>
</tr>
<tr>
<td>ucwa</td>
<td>$z_4 = 0$</td>
<td>cwau</td>
<td>$z_4$</td>
<td>ucxaw</td>
<td>$z_5 - z_6$</td>
</tr>
<tr>
<td>vcwa</td>
<td>$z_5 = 2t^2 + t$</td>
<td>dwau</td>
<td>$-z_9 + z_7$</td>
<td>ucxaw</td>
<td>$z_4 - z_6$</td>
</tr>
<tr>
<td>xcwa</td>
<td>$z_6 = 2t^2 + t$</td>
<td>bwau</td>
<td>$-z_2 + z_1$</td>
<td>wcxaw</td>
<td>$-z_6$</td>
</tr>
<tr>
<td>udxa</td>
<td>$z_7 = 2t^2 + 2t + 1$</td>
<td>cxbu</td>
<td>$-z_6 + z_3 - z_1 + z_4$</td>
<td>vdxaw</td>
<td>$z_8 - z_9$</td>
</tr>
<tr>
<td>vdxax</td>
<td>$z_8 = t$</td>
<td>dxbu</td>
<td>$z_3 - z_1 + z_7$</td>
<td>udwa</td>
<td>$z_7 - z_9$</td>
</tr>
<tr>
<td>wdxa</td>
<td>$z_9 = 2t^2 + 3t + 1$</td>
<td>axbu</td>
<td>$z_3 - z_1$</td>
<td>xdxwa</td>
<td>$-z_9$</td>
</tr>
</tbody>
</table>
that there are exactly two 4-cycles passing through any given vertex in $V(X)$. Clearly, the two 4-cycles passing through $a_x$ for each $x \in \mathbb{Z}_{2m}$ are $(a_x, u_x, c_x, w_x)$ and $(a_x, v_x, c_x+2^2+1, x_x)$. We say that the two 4-cycles have types $\text{auce}$ and $\text{avce}$, respectively. Furthermore, every 4-cycle in $X$ has such a type. We identify the types with a cyclic order as the same type. Thus, $\text{auce}$, $\text{ucwa}$, $\text{caue}$ and $\text{auwc}$ are of the same type. Remember that $\alpha^u$ and $\beta^u$ are the lifts of the $x\rightarrow (a\ u\ e\ w)(b\ v\ d\ x)$ and $\beta= (u\ v)(w\ x)$, respectively. It is easy to see that the images of the 4-cycle $(a_0, u_0, c_0, w_0)$ under $(\alpha^u)^2\beta^u\alpha^u$ and $(\beta^u)^2\beta^u\alpha^u$ have types $\text{bvdx}$ and $\text{budw}$, respectively. Noting that every vertex in $V(K_{4,4})$ appears twice in the following four types, one may easily show that each of 4-cycles in $X$ has one of following types because there are exactly two 4-cycles passing through any given vertex in $V(X)$:

(1) $\text{auce}$,
(2) $\text{avce}$,
(3) $\text{budx}$,
(4) $\text{budw}$.

Thus, if $z \in V(K_{4,4})$ is one of the four letters in type $(i)$ for some $1 \leq i \leq 4$ then there is a unique 4-cycle passing through the vertex $z_i$ ($x \in \mathbb{Z}_{2m}$) which has type $(i)$. We denote by $z_i$ $(i)$ this 4-cycle. For example, the two 4-cycles passing through $v_x$ have types $(2)$ and $(3)$ because $v$ is a letter in $(2)$ and $(3)$. By Fig. 1 and by taking into account the expression for the two types $(2)$ and $(3)$ of 4-cycles above, these two 4-cycles are $v_x(2) = (v_x, c_x+2^2+1, x_x, a_x)$ and $v_x(3) = (v_x, d_x+1, x_x+1, b_x)$.

Let $A_{a_0}$ be the stabilizer of $a_0$ in $A$. We know that $a_{0}(1)$ and $a_{0}(2)$ are the only two 4-cycles passing through $a_0$. Let $A_{a_0}^\ast$ be the subgroup of $A_{a_0}$ fixing the two 4-cycles $a_{0}(1)$ and $a_{0}(2)$ setwise. We claim that $A_{a_0}^\ast = 1$.

By Fig. 1 and the expression for the four types of 4-cycles above, one may obtain the following 4-cycles:

$$
\begin{align*}
\text{a}_0(1) &= (a_0, u_0, c_0, w_0) = c_0(1), \\
\text{c}_0(2) &= (c_0, x_{-2^2-1}, a_{-2^2-1}, v_{-2^2-1}), \\
\text{u}_0(4) &= (u_0, d_{2^2+1}, w_{-1}, b_1) = u_2(4), \\
\text{w}_{-1}(1) &= (u_1, c_{-1}, u_{-1}, e_{-1}), \\
\text{a}_1(1) &= (a_1, c_1, w_1, a_{-1}).
\end{align*}
$$

Let $\gamma \in A_{a_0}^\ast$. Then, $\gamma$ fixes $a_0$ and the 4-cycle $a_0(1)$, implying that $\gamma$ fixes $\{u_0, w_0\}$ setwise. We now prove that $\gamma$ fixes $u_0$ and $w_0$. To do it, we shall show that $\gamma$ fixes $a_0$ by the first two 4-cycles above and that $\gamma$ cannot interchanges $u_0$ and $w_0$ by the last four 4-cycles above.

Since $\gamma$ fixes $a_0$ and the 4-cycle $a_0(1)$, it fixes $c_0$, the antipodal vertex of $a_0$ on the 4-cycle $a_0(1)$. Since the two 4-cycles passing through $c_0$ are $c_0(1)$ and $c_0(2)$, $\gamma$ fixes $c_0(2)$ and so the antipodal vertex $a_{-2^2-1}$ of $c_0$ on $c_0(2)$. By induction on $k$, one may easily show that $\gamma$ fixes $a_k(2)$ for each $k \in \mathbb{Z}_{2m}$. Since $2t+1 \in \mathbb{Z}_{2m}$, $\gamma$ fixes $a_{k+1}$ for each $k \in \mathbb{Z}_{2m}$. Specially, $\gamma$ fixes $a_1$. Suppose that $\gamma$ interchanges $u_0$ and $w_0$. Since the two 4-cycles passing through $u_0$ or $w_0$ are $u_0(1) = a_0(1)$ and $u_0(4)$ or $w_0(1) = a_0(1)$ and $w_0(4)$, respectively, $\gamma$ interchanges the 4-cycles $u_0(4)$ and $w_0(4)$. Since the antipodal vertices of $u_0$ and $w_0$ on $u_0(4)$ and $w_0(4)$ are $w_{-1}$ and $u_{-1}$, respectively, it follows that $\gamma$ interchanges $w_{-1}$ and $u_{-1}$. Note that the two 4-cycles passing through $w_{-1}$ or $u_{-1}$ are $u_{-1}(4) = u_0(4)$ and $w_{-1}(1) = u_1(1)$ or $u_{-1}(4) = w_0(4)$ and $u_{-1}(1)$, respectively. Since $\gamma$ interchanges $u_0(4)$ and $w_0(4)$, we have that $\gamma$ interchanges $w_{-1}(1)$ and $u_{-1}(1)$. Since $a_1$ is a vertex on $u_1$, we have that $(a_1)^+ = (a_1)$ is a vertex on $w_{-1}(1)$. Noting that $\gamma$ fixes $a_1$, $(a_1)^+ = (a_1)$ and $a_{-1}$ is a vertex on $w_{-1}(1)$, one has that $a_1 = a_{-1}$ and so $2t = 0$. However, this implies that $2 = 0$ because $4t^2 + 4t + 2 = 0$, which is impossible in $\mathbb{Z}_{2m}$ $(m > 1)$. Thus, $\gamma$ fixes $u_0$ and $w_0$.

Since $\gamma$ fixes $a_1$, one may show that $\gamma$ fixes $v_0$ and $x_0$ in a similar way by using the following 4-cycles:

$$
\begin{align*}
\text{a}_0(2) &= (a_0, v_0, c_{2^2+2}, x_0) = v_0(2) = x_0(2), \\
\text{v}_0(3) &= (v_0, d_1, x_1, b_0) = x_0(3), \\
\text{x}_0(3) &= (x_0, b_{-1}, v_{-1}, d_0) = v_{-1}(3), \\
\text{x}_{-1}(2) &= (x_{-1}, a_{-1}, v_{-1}, c_{2^2+2}), \\
\text{v}_{-1}(2) &= (v_{-1}, c_{2^2}, x_{-1}, a_{-1}).
\end{align*}
$$

In fact, it is easy to see that $\gamma$ fixes $\{v_0, x_0\}$ setwise. Suppose $\gamma$ interchanges $v_0$ and $x_0$. Then, $\gamma$ interchanges the 4-cycles $v_0(3)$ and $x_0(3)$, and the 4-cycles $x_0(2)$ and $v_{-1}(2)$. One may again get the same contradiction as above, that is, $a_1 = a_{-1}$.

We know that $A_{a_0}^\ast$ fixes each neighbour of $a_{-1}$ for any $y \in V(K_{4,4})$ and $x \in \mathbb{Z}_{2m}$, where $A_{a_0}^\ast$ is the subgroup of the stabilizer $A_{a_0}$ of $a_{-1}$ in $A$ that fixes the two 4-cycles passing through $a_{-1}$ setwise. In particular, $A_{a_0}^\ast$ fixes each vertex with distance 2 from $a_0$. By connectivity of $\tilde{X}$, one has $A_{a_0}^\ast = 1$, as claimed.
Consider the orbit $O$ of $u_0$ under $A_{a_0}$. Remember that the two 4-cycles passing through $a_0$ are $a_0(1) = (a_0, u_0, c_0, w_0)$ and $a_0(2) = (a_0, v_0, c_{2m+1}, x_0)$. Since $A_{a_0}^0 = 1$, one has $w_0 \not\in O$. Since $\beta = (u \ v \ w \ x)$ lift, there is an $\delta \in A_{a_0}$ such that $\overline{\delta}$ interchanges $u_0$ and $v_0$, and $w_0$ and $x_0$. Thus, $v_0 \in O$. Suppose $x_0 \in O$. Then there is an $\eta \in A_{a_0}$ such that $(u_0)^{\eta} = x_0$. It follows that $(u_0)^{\eta\delta} = w_0$. Thus, $\eta\delta$ fixes the 4-cycle $a_0(1)$ and $\eta\delta \neq 1$. The former implies that $\eta\delta \in A_{a_0}^*$. Since $A_{a_0}^* = 1$, one has $\eta\delta = 1$, a contradiction. Thus, $x_0 \not\in O$ and so $O$ has length 2. It follows that $|A_{a_0}| = 2|A_{(a_0, u_0)}|$, where $A_{(a_0, u_0)}$ is the subgroup of $A_{a_0}$ fixing $u_0$. Clearly, $A_{(a_0, u_0)} \leq A_{a_0}^* = 1$. It follows that $|A_{a_0}| = 2$ and $|A| = 16m |A_{a_0}| = 32m$. Thus, $A = B$ because $|A| = |B| = 32m$ and so $X$ is half-arc-transitive.

As mentioned in Section 1, we fix the oriented graph $D_A(X)$ having $(a_0, u_0)A$ as its arc set, where $A = \text{Aut}(\widehat{X})$. Since $\beta = (u \ v \ (w \ x))$ and $\beta^2 = (a \ b \ (c \ d))$ lift, one may show that all arcs from a vertex in the fibres of $a$ and $b$ to a vertex in the fibres of $u$ and $v$ are arcs in $D_A(X)$. Thus, there is an alternating cycle, say $O(a_0)$, containing $a_0, u_0, b_1, v_1, a_1, u_1$ as a subgraph of the consecutive vertices of the alternating cycle. Since there is a unique alternating cycle passing through any given arc in $\widehat{X}$ and $Z_{2m} \leq \text{Aut}(\widehat{X})$, it follows that $O(a_0)$ contains all vertices $a_0, b_0, u_0, v_0$ for $x \in Z_{2m}$. Similarly, one may show that there is another alternating cycle, say $I(a_0)$, which passes through $a_0$ and consists of all vertices $a_1, b_1, w_0, x_0$. The common vertices in $O(a_0)$ and $I(a_0)$ are $a_0, b_0, u_0, v_0$ for $x \in Z_{2m}$. Thus, $\widehat{X}$ is tightly attached with attachment number 4$m$. This complete the proof of Theorem 1.1.

**Remark.** Let $t$ be a positive integer and let $m > 1$ be a divisor of $2^t + t + 1$. Since $2^m + 1$ has order 4 in $Z_{2m}$, the smallest order in the half-arc-transitive graphs constructed in Theorem 1.1 is $16 \times 5 = 80$. Since $\mathcal{K}(16m)$ is tightly attached with attachment number 4$m$, the alternating cycles in $\mathcal{K}(16m)$ have length 8$m$ and by [22, Corollary 4.2], all 4-cycles in $\mathcal{K}(16m)$ are directed cycles in $D_A(\mathcal{K}(16m))$ where $A = \text{Aut}(\mathcal{K}(16m))$.

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**References**