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An infinite family of tetravalent half-arc-transitive graphs

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Abstract

A graph is half-arc-transitive if its automorphism group acts transitively on vertices and edges, but not on arcs. In this paper, a new infinite family of tetravalent half-arc-transitive graphs with girth 4 is constructed, each of which has order $16m$ such that $m > 1$ is a divisor of $2t^2 + 2t + 1$ for a positive integer t and is tightly attached with attachment number $4m$. The smallest graph in the family has order 80.

MSC: 05C25; 20B25

Keywords: Half-arc-transitive graph; Alternating cycle; Attachment number

1. Introduction

Throughout this paper, graphs are assumed to be finite, simple, unless specified otherwise, connected and undirected (but with an implicit orientation of the edges when appropriate). For a graph X we let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively.

A graph X is said to be *vertex-transitive*, *edge-transitive*, or *arc-transitive* if $\mathrm{Aut}(X)$ acts transitively on $V(X)$, $E(X)$ or $A(X)$, respectively. A graph is said to be $\frac{1}{2}$*-arc-transitive* or *half-arc-transitive* provided that it is vertex-transitive, edge-transitive, but not arc-transitive. More generally, a subgroup G of the automorphism group $\mathrm{Aut}(X)$ of a graph X is said to be *half-arc-transitive* if G is vertex-transitive and edge-transitive, but not arc-transitive on X. In this case, we shall say that the graph X is $(G, \frac{1}{2})$*-arc-transitive*.

Let X be a tetravalent $(G, \frac{1}{2})$-arc-transitive graph. Then in the natural action of G on $V(X) \times V(X)$, the arc set of X is the union of two G-orbits, say A_1 and A_2, which are paired with each other, that is, $A_2 = \{(v, u) | (u, v) \in A_1\}$. Each of the two corresponding oriented graphs $(V(X), A_1)$ and $(V(X), A_2)$ has out-valency and in-valency equal to 2, and admits G as a vertex- and arc-transitive group of automorphisms. Moreover, each of them has X as its underlying graph. Let $D_G(X)$ be one of these two oriented graphs, fixed from now on. For $u, v \in V(X)$ such that (u, v) is an arc in $D_G(X)$, we say that u and v are the *tail* and the *head* of the arc (u, v), respectively. An even length cycle C in X is called a G*-alternating* cycle if the vertices of C are alternately, the tail or the head (in $D_G(X)$) of their two

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incident edges in C. It is proved in [16, Proposition 2.4(i)] that all G-alternating cycles in X have the same length and form a decomposition of the edge set of X, half of this length is denoted by $r_G(X)$ and is called the *G-radius* of X. It is proved in [16, Proposition 2.6] that any two adjacent G-alternating cycles in X intersect in the same number of vertices, called the *G-attachment number* $a_G(X)$ of X. We say that X is *tightly G-attached* if $r_G(X)=a_G(X)$. In [16], all tetravalent $(G, \frac{1}{2})$-arc-transitive tightly G-attached graphs with odd G-radius have been classified and, moreover, it has also been specified which among them are $\frac{1}{2}$-arc-transitive and which are arc-transitive. A description of all tetravalent $(G, \frac{1}{2})$-arc-transitive tightly G-attached graphs with even G-radius is given in [24]. It is not known, however, that which among them are $\frac{1}{2}$-arc-transitive. If X is $\frac{1}{2}$-arc-transitive, the terms an $\mathrm{Aut}(X)$-alternating cycle, $\mathrm{Aut}(X)$-radius, and $\mathrm{Aut}(X)$-attachment number are referred to as an *alternating cycle* of X, *radius* of X and *attachment number* of X, respectively. Similarly, if X is tightly $\mathrm{Aut}(X)$-attached, we say that X is *tightly attached*.

Let K be a group and X a graph. Let ϕ: $A(X) \mapsto K$ be the so-called *voltage assignment*, that is, a function from the set of arcs of X into the group K, where reverse arcs carry inverse voltages. We, thus, have a labelling of the arcs of X by elements in K such that $\phi(u, v)\phi(v, u) = id$ for all pairs of adjacent vertices u, v in X, where $\phi(u, v)$ denotes the element in K assigned to the arc (u, v). The voltage assignment ϕ on arcs extends to a voltage assignment on walks in a natural way and for a walk W of X, we let $\phi(W)$ denote the voltage of W. The *covering graph* $X\times_\phi K$ of X with respect to ϕ have vertex set $V(X) \times K$ and edge set $\{(u, g)(v, h)|uv \in E(X),\ h = \phi(u, v)g\}$. The graph X is said to be the *base graph* of $X\times_\phi K$, and the latter is sometimes referred to as an *regular covering* (or *K-covering*) of X. The set of vertices $\{(u, k)|k \in K\}$ is called the *fibre* of u. For $k \in K$, by defining $(u, g)^k := (u, gk)$ for any $(u, g) \in V(X\times_\phi K)$, K becomes a subgroup of $\mathrm{Aut}(X\times_\phi K)$ which acts regularly on each fibre. If $X\times_\phi K$ is connected K is called the *covering transformation group*. Clearly, a covering transformation maps each fibre onto itself. An automorphism of $X\times_\phi K$ is said to be *fibre-preserving* if it maps a fibre to a fibre (not necessary itself). All such fibre-preserving automorphisms form a group called the *fibre-preserving group*. It is easy to see that each fibre is a block of the fibre-preserving group and if $X\times_\phi K$ is connected, K is the kernel of the fibre-preserving group acting on the set of fibres.

The investigation of half-arc-transitive graphs was initiated by Tutte [29] who proved that a vertex- and edge-transitive graph with odd valency must be arc-transitive. In 1970, Bouwer [3] constructed the first infinite family of half-arc-transitive graphs and later more such graphs were constructed (see [2,3,6,9,12,17,23,28,30]). There are now currently four main open areas of research in half-arc-transitive graphs. The first area of research is the study of half-arc-transitive graphs of valency 4 with large order vertex stabilizers, initiated in Marušič and Nedela [21], with some open problems partly answered in Marušič [18], and with a number of problems that remain open to this day. The second area of research concerns investigation of primitivity/imprimitivity of action for half-arc-transitive graphs, see for example [28] but also a more recent paper by Li et al. [10]. The third area of research encompasses geometry related question about half-arc-transitive graphs as addressed in [5,19]. And finally, the fourth area of research concerns the "attachment of alternating cycles" question. In that respect, odd radius tightly attached tetravalent half-arc-transitive graphs have been classified as mentioned before [16] and even radius tightly attached tetravalent half-arc-transitive graphs are definitely objects worth exploring. Wilson [31] gave a description of tetravalent graphs admitting half-arc-transitive group action with respect to which they are of even radius and tightly attached. However, no examples of even radius attached half-arc-transitive graphs were constructed in [31] and in this paper, an infinite family of even radius tightly attached tetravalent half-arc-transitive graphs are constructed. For each number $m > 1$ being a divisor of $2t^2+2t+1$ for a positive integer t, we construct a tetravalent half-arc-transitive graph of order $16m$, which is a regular covering of the complete bipartite graph $K_{4,4}$. Clearly, m is odd. These half-arc-transitive graphs have girth 4 and are tightly attached with attachment number $4m$. Since these half-arc-transitive graphs have solvable automorphism groups and order 16 times an odd integer, one may show that they do not belong to any family of half-arc-transitive graphs discussed above (note that the half-arc-transitive graphs constructed in [27, Theorem 3.1] have orders 2^r times an odd integer with $2\leqslant r\leqslant 3$ (see [27, Lemma 3.5]) and the half-arc-transitive graphs constructed in [19, Proposition 5.1] have girth 3). Constructing and characterizing tetravalent half-arc-transitive graphs is currently an active topic in algebraic graph theory (see [1,4,7,11,14–16,22,20,19,24–27,32]).

To state the main result of this paper, we first introduce an infinite family of tetravalent half-arc-transitive graphs. Let $\mathbb{Z}_n$ be the cyclic group of order n, as well as the ring of integers modulo n. Denote by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n. Denote by $V(K_{4,4}) = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$ the vertex set of the bipartite graph $K_{4,4}$ as in Fig. 1. Let t be a positive integer and let $m > 1$ be a divisor of $2t^2 + 2t + 1$. The graph $\mathcal{CK}(16m)$ is

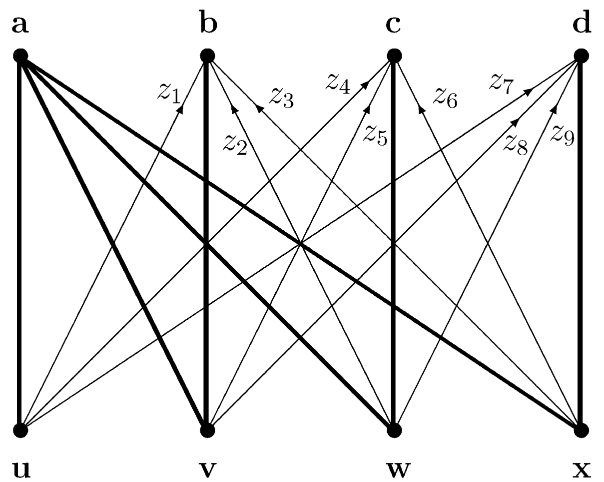

Fig. 1. The graph $K_{4,4}$ with voltage assignment ϕ.

defined to have the vertex set $V(\mathscr{CK}(16m)) = V(K_{4,4}) \times \mathbb{Z}_{2m}$ and edge set

$$\begin{aligned} E(\mathscr{CK}(16m)) = \{ & (\mathbf{u}, x)(\mathbf{a}, x), (\mathbf{v}, x)(\mathbf{a}, x), (\mathbf{v}, x)(\mathbf{b}, x), (\mathbf{w}, x)(\mathbf{a}, x), \\ & (\mathbf{w}, x)(\mathbf{c}, x), (\mathbf{x}, x)(\mathbf{a}, x), (\mathbf{x}, x)(\mathbf{d}, x), (\mathbf{u}, x)(\mathbf{b}, x+1), \\ & (\mathbf{u}, x)(\mathbf{c}, x), (\mathbf{u}, x)(\mathbf{d}, x+2t^2+2t+1), (\mathbf{v}, x)(\mathbf{c}, x+2t^2+t), \\ & (\mathbf{v}, x)(\mathbf{d}, x+t), (\mathbf{w}, x)(\mathbf{b}, x+t+1), (\mathbf{w}, x)(\mathbf{d}, x+2t^2+3t+1), \\ & (\mathbf{x}, x)(\mathbf{b}, x-t), (\mathbf{x}, x)(\mathbf{c}, x+2t^2+t) \mid x \in \mathbb{Z}_{2m}\}. \end{aligned}$$

Note that the second coordinate in each vertex of $V(K_{4,4}) \times \mathbb{Z}_{2m}$ is taken modulo $2m$. The notation $\mathscr{CK}$ means cyclic covering of $K_{4,4}$. It is easy to see that $\mathscr{CK}(16m)$ is bipartite. The main result of this paper is the following theorem.

Theorem 1.1. *Let t be a positive integer and let $m > 1$ be a divisor of $2t^2 + t + 1$. Then, the graph $\mathscr{CK}(16m)$ is a half-arc-transitive graph of order $16m$ with girth 4. Furthermore, $\mathscr{CK}(16m)$ is tightly attached with attachment number $4m$.*

2. Lifting automorphisms

Let X be a graph and K a group. Let $X \times_\phi K$ be a regular covering of X. Given a spanning tree T of the graph X, the voltage assignment ϕ is said to be *T-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [8] showed that every regular covering $X \times_\phi K$ of a graph X can be derived from a T-reduced voltage assignment with respect to an arbitrary fixed spanning tree T of X. It is clear that if ϕ is T-reduced, the covering graph $X \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group K.

Let $\widetilde{\alpha} \in \mathrm{Aut}(X \times_\phi K)$ be fibre-preserving and let $\alpha \in \mathrm{Aut}(X)$. Clearly, the permutation induced by $\widetilde{\alpha}$ on the set of fibres is an automorphism of X (identifying the fibres with the vertices of X). If the automorphism is exactly α we call $\widetilde{\alpha}$ a *lift* of α, and α a *projection* of $\widetilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ and the projection of a subgroup of $\mathrm{Aut}(X \times_\phi K)$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\mathrm{Aut}(X \times_\phi K)$ and $\mathrm{Aut}(X)$, respectively. In particular, if the covering graph $X \times_\phi K$ is connected, then the covering transformation group K is the lift of the identity subgroup. Clearly, if $\widetilde{\alpha}$ is a lift of α, then $K\widetilde{\alpha}$ are all the lifts of α.

The problem whether an automorphism α of X lifts or not can be grasped in terms of voltages as follows. Given $\alpha \in \mathrm{Aut}(X)$, we define a function $\overline{\alpha}$ from the set of voltages of fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by

$$(\phi(C))^{\overline{\alpha}} = \phi(C^\alpha),$$

where C ranges over all fundamental closed walks at v, and $\phi(C)$ and $\phi(C^{\alpha})$ are the voltages of C and C^{α}, respectively. Note that if K is abelian, $\overline{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles corresponding to the cotree arcs of X. The following proposition is a special case of [13, Theorem 4.2].

Proposition 2.1. *Let $X \times_{\phi} K \mapsto X$ be a connected K-covering where ϕ is T-reduced. Then, an automorphism α of X lifts if and only if $\overline{\alpha}$ extends to an automorphism of K.*

3. Proof of Theorem 1.1

Let t be a positive integer and let $m > 1$ be a divisor of $2t^2 + 2t + 1$. Thus, $4t^2 + 4t + 2 = 0$ in the ring $\mathbb{Z}_{2m}$. Let $\widetilde{X} = K_{4,4} \times_{\phi} \mathbb{Z}_{2m}$ be a regular covering of the graph $K_{4,4}$ such that $\phi = 0$ on the spanning tree T which is illustrated by dark lines in Fig. 1 and we assign voltages $z_1 = 1$, $z_2 = t + 1$, $z_3 = -t$, $z_4 = 0$, $z_5 = 2t^2 + t$, $z_6 = 2t^2 + t$, $z_7 = 2t^2 + 2t + 1$, $z_8 = t$ and $z_9 = 2t^2 + 3t + 1$ to the cotree arcs of $K_{4,4}$. Note that $z_i \in \mathbb{Z}_{2m}$ for $1 \leqslant i \leqslant 9$ and the vertex set of $K_{4,4}$ is $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}\}$. Clearly, $\widetilde{X}$ is connected because $\langle z_1 \rangle = \mathbb{Z}_{2m}$. By the definition of the graph $\mathscr{CK}(16m)$ in the paragraph preceding the statement of Theorem 1.1, $\widetilde{X} = K_{4,4} \times_{\phi} \mathbb{Z}_{2m} = \mathscr{CK}(16m)$.

Let $\alpha = (\mathbf{a\ u\ c\ w})(\mathbf{b\ v\ d\ x})$ and $\beta = (\mathbf{u\ v})(\mathbf{w\ x})$. Then α and β are automorphisms of $K_{4,4}$. Let $G = \langle \alpha, \beta \rangle$. It is clear that $\langle \beta, \beta^{\alpha} \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\langle \beta, \beta^{\alpha} \rangle$ is a normal subgroup of index 4 in G. Thus, one may show that $|G| = 16$ and that $K_{4,4}$ is $(G, \frac{1}{2})$-arc-transitive.

Denote by $i_1 i_2 \ldots i_s$ the cycle with consecutive adjacent vertices $i_1, i_2, \ldots, i_s$. There are nine fundamental cycles **ubva**, **wbva**, **xbva**, **ucwa**, **vcwa**, **xcwa**, **udxa**, **vdxa** and **wdxa** in $K_{4,4}$, which are generated by the nine cotree arcs $(\mathbf{u}, \mathbf{b})$, $(\mathbf{w}, \mathbf{b})$, $(\mathbf{x}, \mathbf{b})$, $(\mathbf{u}, \mathbf{c})$, $(\mathbf{v}, \mathbf{c})$, $(\mathbf{x}, \mathbf{c})$, $(\mathbf{u}, \mathbf{d})$, $(\mathbf{v}, \mathbf{d})$ and $(\mathbf{w}, \mathbf{d})$, respectively. Each cycle maps to a cycle of the same length under the actions of α and β. We list these cycles and their voltages in Table 1, where C denotes a fundamental cycle of $K_{4,4}$ and $\phi(C)$ denotes the voltage on C.

Since $4t^2 + 4t + 2 = 0$, one has $(2t + 1)^2 = -1$ in $\mathbb{Z}_{2m}$. Clearly, $m > 1$ implies that $2 \neq 0$ in $\mathbb{Z}_{2m}$ and so $(2t + 1)^2 \neq 1$. Thus, $2t + 1 \in \mathbb{Z}_{2m}^*$ has order 4. Consider the mappings $\overline{\alpha}$ and $\overline{\beta}$ from the set of voltages of the nine fundamental cycles of $K_{4,4}$ to the cyclic group $\mathbb{Z}_{2m}$, defined by $\phi(C)^{\overline{\alpha}} = \phi(C^{\alpha})$ and $\phi(C)^{\overline{\beta}} = \phi(C^{\beta})$, respectively, where C ranges over the nine fundamental cycles. With $4t^2 + 4t + 2 = 0$ and $(2t + 1)^2 = -1$, by Table 1 one may easily check that $\overline{\alpha}$ and $\overline{\beta}$ can be extended to the automorphisms of $\mathbb{Z}_{2m}$ induced by $1 \mapsto 2t + 1$ and $1 \mapsto -1$, respectively. By Proposition 2.1, α and β lift, and so G lifts. Let α^* and β^* be one of the lifts of α and β, respectively. Then, $B := \langle \mathbb{Z}_{2m}, \alpha^*, \beta^* \rangle$ is the subgroup of $\mathrm{Aut}(\widetilde{X})$ lifted by G, where $\mathbb{Z}_{2m}$ is identified as a subgroup of $\mathrm{Aut}(\widetilde{X})$ by right multiplication on the second coordinate of each vertex in $K_{4,4} \times_{\phi} \mathbb{Z}_{2m}$ (see Section 1). Since G is half-arc-transitive on $K_{4,4}$, B is half-arc-transitive on the covering graph $\widetilde{X} = K_{4,4} \times_{\phi} \mathbb{Z}_{2m}$. Furthermore, $|G| = 16$ implies that $|B| = 32m$. Let $A = \mathrm{Aut}(\widetilde{X})$. Then, $B \leqslant A$. To prove half-arc-transitivity of $\widetilde{X}$, it suffices to show that $A = B$.

For the convenience of statement, we use $\mathbf{z}_x$ to denote the vertex $(\mathbf{z}, x)$ of $\widetilde{X}$ where $\mathbf{z} \in V(K_{4,4})$ and $x \in \mathbb{Z}_{2m}$. One may easily show that $(\mathbf{a}_0, \mathbf{u}_0, \mathbf{c}_0, \mathbf{w}_0)$ and $(\mathbf{a}_0, \mathbf{v}_0, \mathbf{c}_{2t^2 \pm t}, \mathbf{x}_0)$ are precisely the two 4-cycles, that is, cycles of length 4, passing through $\mathbf{a}_0$ in $\widetilde{X}$. Since $B \leqslant A$, it follows that $\widetilde{X}$ is vertex- and edge-transitive. Vertex-transitivity of $\widetilde{X}$ implies

Table 1
Voltages on fundamental cycles and their images under α and β

C	$\phi(C)$	C^{α}	$\phi(C^{\alpha})$	C^{β}	$\phi(C^{\beta})$
ubva	$z_1 = 1$	**cvdu**	$-z_5 + z_8 - z_7 + z_4$	**vbua**	$-z_1$
wbva	$z_2 = t + 1$	**avdu**	$z_8 - z_7$	**xbua**	$z_3 - z_1$
xbva	$z_3 = -t$	**bvdu**	$z_8 - z_7 + z_1$	**wbua**	$z_2 - z_1$
ucwa	$z_4 = 0$	**cwau**	z_4	**vcxa**	$z_5 - z_6$
vcwa	$z_5 = 2t^2 + t$	**dwau**	$-z_9 + z_7$	**ucxa**	$z_4 - z_6$
xcwa	$z_6 = 2t^2 + t$	**bwau**	$-z_2 + z_1$	**wcxa**	$-z_6$
udxa	$z_7 = 2t^2 + 2t + 1$	**cxbu**	$-z_6 + z_3 - z_1 + z_4$	**vdwa**	$z_8 - z_9$
vdxa	$z_8 = t$	**dxbu**	$z_3 - z_1 + z_7$	**udwa**	$z_7 - z_9$
wdxa	$z_9 = 2t^2 + 3t + 1$	**axbu**	$z_3 - z_1$	**xdwa**	$-z_9$

that there are exactly two 4-cycles passing through any given vertex in $V(\widetilde{X})$. Clearly, the two 4-cycles passing through $\mathbf{a}_x$ for each $x \in \mathbb{Z}_{2m}$ are $(\mathbf{a}_x, \mathbf{u}_x, \mathbf{c}_x, \mathbf{w}_x)$ and $(\mathbf{a}_x, \mathbf{v}_x, \mathbf{c}_{x+2t^2+t}, \mathbf{x}_x)$. We say that the two 4-cycles have *types* **aucw** and **avcx**, respectively. Furthermore, every 4-cycle in $\widetilde{X}$ has such a type. We identify the types with a cyclic order as the same type. Thus, **aucw**, **ucwa**, **cwau** and **wauc** are of the same type. Remember that α^* and β^* are the lifts of $\alpha = (\mathbf{a}\ \mathbf{u}\ \mathbf{c}\ \mathbf{w})(\mathbf{b}\ \mathbf{v}\ \mathbf{d}\ \mathbf{x})$ and $\beta = (\mathbf{u}\ \mathbf{v})(\mathbf{w}\ \mathbf{x})$, respectively. It is easy to see that the images of the 4-cycle $(\mathbf{a}_0, \mathbf{u}_0, \mathbf{c}_0, \mathbf{w}_0)$ under $(\alpha^*)^3\beta^*\alpha^*\beta^*$ and $(\alpha^*)^3\beta^*\alpha^*$ have types **bvdx** and **budw**, respectively. Noting that every vertex in $V(K_{4,4})$ appears twice in the following four types, one may easily show that each of 4-cycles in $\widetilde{X}$ has one of following types because there are exactly two 4-cycles passing through any given vertex in $V(\widetilde{X})$:

(1) **aucw**,
(2) **avcx**,
(3) **bvdx**,
(4) **budw**.

Thus, if $\mathbf{z} \in V(K_{4,4})$ is one of the four letters in type (i) for some $1 \leqslant i \leqslant 4$ then there is a unique 4-cycle passing through the vertex $\mathbf{z}_x$ $(x \in \mathbb{Z}_{2m})$ which has type (i). We denote by $\mathbf{z}_x(i)$ this 4-cycle. For example, the two 4-cycles passing through $\mathbf{v}_x$ have types (2) and (3) because $\mathbf{v}$ is a letter in (2) and (3). By Fig. 1 and by taking into account the expression for the two types (2) and (3) of 4-cycles above, these two 4-cycles are $\mathbf{v}_x(2) = (\mathbf{v}_x, \mathbf{c}_{x+2t^2+t}, \mathbf{x}_x, \mathbf{a}_x)$ and $\mathbf{v}_x(3) = (\mathbf{v}_x, \mathbf{d}_{x+t}, \mathbf{x}_{x+t}, \mathbf{b}_x)$.

Let $A_{\mathbf{a}_0}$ be the stabilizer of $\mathbf{a}_0$ in A. We know that $\mathbf{a}_0(1)$ and $\mathbf{a}_0(2)$ are the only two 4-cycles passing through $\mathbf{a}_0$. Let $A^*_{\mathbf{a}_0}$ be the subgroup of $A_{\mathbf{a}_0}$ fixing the two 4-cycles $\mathbf{a}_0(1)$ and $\mathbf{a}_0(2)$ setwise. We claim that $A^*_{\mathbf{a}_0} = 1$.

By Fig. 1 and the expression for the four types of 4-cycles above, one may obtain the following 4-cycles:

$$\mathbf{a}_0(1) = (\mathbf{a}_0, \mathbf{u}_0, \mathbf{c}_0, \mathbf{w}_0) = \mathbf{c}_0(1), \quad \mathbf{c}_0(2) = (\mathbf{c}_0, \mathbf{x}_{-2t^2-t}, \mathbf{a}_{-2t^2-t}, \mathbf{v}_{-2t^2-t}),$$
$$\mathbf{u}_0(4) = (\mathbf{u}_0, \mathbf{d}_{2t^2+2t+1}, \mathbf{w}_{-t}, \mathbf{b}_1) = \mathbf{w}_{-t}(4), \quad \mathbf{w}_0(4) = (\mathbf{w}_0, \mathbf{b}_{t+1}, \mathbf{u}_t, \mathbf{d}_{2t^2+3t+1}) = \mathbf{u}_t(4),$$
$$\mathbf{w}_{-t}(1) = (\mathbf{w}_{-t}, \mathbf{a}_{-t}, \mathbf{u}_{-t}, \mathbf{c}_{-t}), \quad \mathbf{u}_t(1) = (\mathbf{u}_t, \mathbf{c}_t, \mathbf{w}_t, \mathbf{a}_t).$$

Let $\gamma \in A^*_{\mathbf{a}_0}$. Then, γ fixes $\mathbf{a}_0$ and the 4-cycle $\mathbf{a}_0(1)$, implying that γ fixes $\{\mathbf{u}_0, \mathbf{w}_0\}$ setwise. We now prove that γ fixes $\mathbf{u}_0$ and $\mathbf{w}_0$. To do it, we shall show that γ fixes $\mathbf{a}_t$ by the first two 4-cycles above and that γ cannot interchanges $\mathbf{u}_0$ and $\mathbf{w}_0$ by the last four 4-cycles above.

Since γ fixes $\mathbf{a}_0$ and the 4-cycle $\mathbf{a}_0(1)$, it fixes $\mathbf{c}_0$, the antipodal vertex of $\mathbf{a}_0$ on the 4-cycle $\mathbf{a}_0(1)$. Since the two 4-cycles passing through $\mathbf{c}_0$ are $\mathbf{c}_0(1)$ and $\mathbf{c}_0(2)$, γ fixes $\mathbf{c}_0(2)$ and so the antipodal vertex $\mathbf{a}_{-2t^2-t}$ of $\mathbf{c}_0$ on $\mathbf{c}_0(2)$. By induction on k, one may easily show that γ fixes $\mathbf{a}_{k(-2t^2-t)}$ for each $k \in \mathbb{Z}_{2m}$. Since $2t+1 \in \mathbb{Z}^*_{2m}$, γ fixes $\mathbf{a}_{kt}$ for each $k \in \mathbb{Z}_{2m}$. Specially, γ fixes $\mathbf{a}_t$. Suppose that γ interchanges $\mathbf{u}_0$ and $\mathbf{w}_0$. Since the two 4-cycles passing through $\mathbf{u}_0$ or $\mathbf{w}_0$ are $\mathbf{u}_0(1) = \mathbf{a}_0(1)$ and $\mathbf{u}_0(4)$ or $\mathbf{w}_0(1) = \mathbf{a}_0(1)$ and $\mathbf{w}_0(4)$, respectively, γ interchanges the 4-cycles $\mathbf{u}_0(4)$ and $\mathbf{w}_0(4)$. Since the antipodal vertices of $\mathbf{u}_0$ and $\mathbf{w}_0$ on $\mathbf{u}_0(4)$ and $\mathbf{w}_0(4)$ are $\mathbf{w}_{-t}$ and $\mathbf{u}_t$, respectively, it follows that γ interchanges $\mathbf{w}_{-t}$ and $\mathbf{u}_t$. Note that the two 4-cycles passing through $\mathbf{w}_{-t}$ or $\mathbf{u}_t$ are $\mathbf{w}_{-t}(4) = \mathbf{u}_0(4)$and $\mathbf{w}_{-t}(1)$ or $\mathbf{u}_t(4) = \mathbf{w}_0(4)$ and $\mathbf{u}_t(1)$, respectively. Since γ interchanges $\mathbf{u}_0(4)$ and $\mathbf{w}_0(4)$, we have that γ interchanges $\mathbf{w}_{-t}(1)$ and $\mathbf{u}_t(1)$. Since $\mathbf{a}_t$ is a vertex on $\mathbf{u}_t(1)$, we have that $(\mathbf{a}_t)^\gamma$ is a vertex on $\mathbf{w}_{-t}(1)$. Noting that γ fixes $\mathbf{a}_t$ $(\mathbf{a}_t^\gamma = \mathbf{a}_t)$ and $\mathbf{a}_{-t}$ is a vertex on $\mathbf{w}_{-t}(1)$, one has that $\mathbf{a}_t = \mathbf{a}_{-t}$ and so $2t = 0$. However, this implies that $2 = 0$ because $4t^2 + 4t + 2 = 0$, which is impossible in $\mathbb{Z}_{2m}$ $(m > 1)$. Thus, γ fixes $\mathbf{u}_0$ and $\mathbf{w}_0$.

Since γ fixes $\mathbf{a}_t$, one may show that γ fixes $\mathbf{v}_0$ and $\mathbf{x}_0$ in a similar way by using the following 4-cycles:

$$\mathbf{a}_0(2) = (\mathbf{a}_0, \mathbf{v}_0, \mathbf{c}_{2t^2+t}, \mathbf{x}_0) = \mathbf{v}_0(2) = \mathbf{x}_0(2),$$
$$\mathbf{v}_0(3) = (\mathbf{v}_0, \mathbf{d}_t, \mathbf{x}_t, \mathbf{b}_0) = \mathbf{x}_t(3), \quad \mathbf{x}_0(3) = (\mathbf{x}_0, \mathbf{b}_{-t}, \mathbf{v}_{-t}, \mathbf{d}_0) = \mathbf{v}_{-t}(3),$$
$$\mathbf{x}_t(2) = (\mathbf{x}_t, \mathbf{a}_t, \mathbf{v}_t, \mathbf{c}_{2t^2+2t}), \quad \mathbf{v}_{-t}(2) = (\mathbf{v}_{-t}, \mathbf{c}_{2t^2}, \mathbf{x}_{-t}, \mathbf{a}_{-t}).$$

In fact, it is easy to see that γ fixes $\{\mathbf{v}_0, \mathbf{x}_0\}$ setwise. Suppose γ interchanges $\mathbf{v}_0$ and $\mathbf{x}_0$. Then, γ interchanges the 4-cycles $\mathbf{v}_0(3)$ and $\mathbf{x}_0(3)$, and the 4-cycles $\mathbf{x}_t(2)$ and $\mathbf{v}_{-t}(2)$. One may again get the same contradiction as above, that is, $\mathbf{a}_t = \mathbf{a}_{-t}$.

We know that $A^*_{\mathbf{a}_0}$ fixes each neighbour of $\mathbf{a}_0$. By vertex-transitivity of $\widetilde{X}$, we have that $A^*_{\mathbf{z}_x}$ fixes each neighbor of $\mathbf{z}_x$ for any $\mathbf{z} \in V(K_{4,4})$ and $x \in \mathbb{Z}_{2m}$, where $A^*_{\mathbf{z}_x}$ is the subgroup of the stabilizer $A_{\mathbf{z}_x}$ of $\mathbf{z}_x$ in A that fixes the two 4-cycles passing through $\mathbf{z}_x$ setwise. In particular, $A^*_{\mathbf{a}_0}$ fixes each vertex with distance 2 from $\mathbf{a}_0$. By connectivity of $\widetilde{X}$, one has $A^*_{\mathbf{a}_0} = 1$, as claimed.

Consider the orbit O of $\mathbf{u}_0$ under $A_{\mathbf{a}_0}$. Remember that the two 4-cycles passing through $\mathbf{a}_0$ are $\mathbf{a}_0(1)=(\mathbf{a}_0, \mathbf{u}_0, \mathbf{c}_0, \mathbf{w}_0)$ and $\mathbf{a}_0(2)=(\mathbf{a}_0, \mathbf{v}_0, \mathbf{c}_{2t^2+t}, \mathbf{x}_0)$. Since $A^*_{\mathbf{a}_0}=1$, one has $\mathbf{w}_0 \notin O$. Since $\beta=(\mathbf{u}\ \mathbf{v})(\mathbf{w}\ \mathbf{x})$ lifts, there is an $\delta \in A_{\mathbf{a}_0}$ such that δ interchanges $\mathbf{u}_0$ and $\mathbf{v}_0$, and $\mathbf{w}_0$ and $\mathbf{x}_0$. Thus, $\mathbf{v}_0 \in O$. Suppose $\mathbf{x}_0 \in O$. Then there is an $\eta \in A_{\mathbf{a}_0}$ such that $(\mathbf{u}_0)^\eta=\mathbf{x}_0$. It follows that $(\mathbf{u}_0)^{\eta\delta} = \mathbf{w}_0$. Thus, $\eta\delta$ fixes the 4-cycle $\mathbf{a}_0(1)$ and $\eta\delta \neq 1$. The former implies that $\eta\delta \in A^*_{\mathbf{a}_0}$. Since $A^*_{\mathbf{a}_0}=1$, one has $\eta\delta=1$, a contradiction. Thus, $\mathbf{x}_0 \notin O$ and so O has length 2. It follows that $|A_{\mathbf{a}_0}|=2|A_{(\mathbf{a}_0,\mathbf{u}_0)}|$, where $A_{(\mathbf{a}_0,\mathbf{u}_0)}$ is the subgroup of $A_{\mathbf{a}_0}$ fixing $\mathbf{u}_0$. Clearly, $A_{(\mathbf{a}_0,\mathbf{u}_0)} \leqslant A^*_{\mathbf{a}_0}=1$. It follows that $|A_{\mathbf{a}_0}|=2$ and $|A|=16m|A_{\mathbf{a}_0}|=32m$. Thus, $A = B$ because $|A| = |B| = 32m$ and so $\widetilde{X}$ is half-arc-transitive.

As mentioned in Section 1, we fix the oriented graph $D_A(\widetilde{X})$ having $(\mathbf{a}_0, \mathbf{u}_0)^A$ as its arc set, where $A = \mathrm{Aut}(\widetilde{X})$. Since $\beta = (\mathbf{u}\ \mathbf{v})(\mathbf{w}\ \mathbf{x})$ and $\beta^\alpha = (\mathbf{a}\ \mathbf{b})(\mathbf{c}\ \mathbf{d})$ lift, one may show that all arcs from a vertex in the fibres of $\mathbf{a}$ and $\mathbf{b}$ to a vertex in the fibres of $\mathbf{u}$ and $\mathbf{v}$ are arcs in $D_A(\widetilde{X})$. Thus, there is an alternating cycle, say $O(\mathbf{a}_0)$, containing $\mathbf{a}_0$, $\mathbf{u}_0$, $\mathbf{b}_1$, $\mathbf{v}_1$, $\mathbf{a}_1$, $\mathbf{u}_1$ as a subgraph of consecutive vertices of the alternating cycle. Since there is a unique alternating cycle passing through any given arc in $\widetilde{X}$ and $\mathbb{Z}_{2m} \leqslant \mathrm{Aut}(\widetilde{X})$, it follows that $O(\mathbf{a}_0)$ contains all vertices $\mathbf{a}_x$, $\mathbf{b}_x$, $\mathbf{u}_x$ and $\mathbf{v}_x$ for $x \in \mathbb{Z}_{2m}$. Similarly, one may show that there is another alternating cycle, say $I(\mathbf{a}_0)$, which passes through $\mathbf{a}_0$ and consists of all vertices $\mathbf{a}_x$, $\mathbf{b}_x$, $\mathbf{w}_x$ and $\mathbf{x}_x$ for $x \in \mathbb{Z}_{2m}$. The common vertices in $O(\mathbf{a}_0)$ and $I(\mathbf{a}_0)$ are $\mathbf{a}_x$ and $\mathbf{b}_x$ for $x \in \mathbb{Z}_{2m}$. Thus, $\widetilde{X}$ is tightly attached with attachment number $4m$. This complete the proof of Theorem 1.1.

Remark. Let t be a positive integer and let $m > 1$ be a divisor of $2t^2 + t + 1$. Since $2m + 1$ has order 4 in $\mathbb{Z}^*_{2m}$, the smallest order in the half-arc-transitive graphs constructed in Theorem 1.1 is $16 \times 5 = 80$. Since $\mathscr{CK}(16m)$ is tightly attached with attachment number $4m$, the alternating cycles in $\mathscr{CK}(16m)$ have length $8m$ and by [22, Corllary 4.2], all 4-cycles in $\mathscr{CK}(16m)$ are directed cycles in $D_A(\mathscr{CK}(16m))$ where $A = \mathrm{Aut}(\mathscr{CK}(16m))$.

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