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On the Numerical Integration of a Symmetric System of Difference-Differential Equations of Neutral Type*

R. K. BRAYTON AND R. A. WILLOUGHBY

IBM Watson Research Center, Yorktown Heights, New York

Submitted by Richard Bellman

1. INTRODUCTION

For a large class of electrical networks containing lossless transmission lines, the describing equations can be reduced to a system of difference-differential equations. For examples of this see Miranker [1] and Brayton [2]. The simple reason for this reduction is that each transmission line can be described in terms of a time delay τ and a characteristic impedance z. In general, the equations are of the neutral type (see [3] p. 166 for a definition). Such networks arise in high speed computers where nearly lossless transmission lines are used to interconnect switching circuits. In general, in circuits where the propagation time in a length of wire is significant compared to the characteristic frequencies of the circuit, it is necessary to use transmission lines to have an accurate model.

It is not the purpose of this paper to describe the circuits or the reduction of the equations to difference-differential equations, but to give some results which are useful in choosing the step-size for the numerical integration of systems of difference-differential equations. The system that will be investigated is given by

$$\dot{x}(t) + A\dot{x}(t-\tau) + Bx(t) + Cx(t-\tau) = 0, \tag{1}$$

where A, B, C are symmetric $n \times n$ matrices and x is an n-vector. Also certain conditions on A, B, C which imply the asymptotic stability of (1) are assumed and these are given in Theorem 1. The case where A, B, C are non-symmetric is much more difficult and is not treated here.

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A one-parameter family of difference schemes is investigated. This is obtained by making the replacements

$$\dot{x}(t) \rightarrow \frac{x(t+h) - x(t)}{h},$$

$$\dot{x}(t-\tau) \rightarrow \frac{x(t+h-\tau) - x(t-\tau)}{h},$$

$$x(t) \rightarrow \mu x(t+h) + (1-\mu)\, x(t),$$

$$x(t-\tau) \rightarrow \mu x(t+h-\tau) + (1-\mu)\, x(t-\tau),$$

where $0 \leqslant \mu \leqslant 1$. Sufficient conditions in terms of the size of h for the asymptotic stability of the resulting family of difference equations are given in Theorem 2. When $\mu \geqslant \frac{1}{2}$, the result is that the equations are unconditionally asymptotically stable. For the scalar case and when $\mu = 0$, sufficient conditions for instability are given in Theorem 3 and this result shows that the delay terms cannot be neglected in determining the integration step-size h.

2. Stability of Equation (1)

THEOREM 1. *If A, B, and C are real symmetric $n \times n$ matrices, $\tau > 0$ and $I \pm A$, $B \pm C$ are positive definite then*

$$\dot{x}(t) + A\dot{x}(t-\tau) + Bx(t) + Cx(t-\tau) = 0$$

is asymptotically stable.

PROOF. According to a theorem by Miranker [4], (1) is stable if the values of s for which there exist a unit vector u satisfying

$$su + Ase^{-s\tau}u + Bu + Ce^{-s\tau}u = 0 \tag{2}$$

have the property

$$\operatorname{Re}(s) \leqslant -\delta < 0.$$

The reason for requiring that the roots be uniformly bounded away from the imaginary axis is because (1) is of the neutral type and there exists the possibility that the characteristic roots of such an equation could accumulate at $\pm i\infty$. Indeed it has been shown by Snow [6] that even though the characteristic roots may satisfy $\operatorname{Re}(s) < 0$, it is possible for solutions to be unbounded as $t \rightarrow \infty$.

The proof is by contradiction. First suppose s satisfies (2) and Re $(s) \geqslant 0$ for some unit vector u. Then taking inner products in (2) we have

$$s(u, u) + se^{-s\tau}(u, Au) + (u, Bu) + e^{-s\tau}(u, Cu) = 0.$$

Let $a = (u, Au)$, $b = (u, Bu)$ and $c = (u, Cu)$. Then a, b, and c are real, $|a| < 1$, $b > |c|$ by assumption and

$$s + ase^{-s\tau} + b + ce^{-s\tau} = 0. \tag{4}$$

Solving for s we have

$$s = -\frac{be^{s\tau} + c}{e^{s\tau} + a}, \tag{5}$$

or with

$$v = e^{s\tau} \tag{6}$$

we have

$$v = e^{-\tau[(bv+c)/(v+a)]}. \tag{7}$$

If $v = \alpha + \imath\beta$, then

$$\operatorname{Re}\left(\frac{bv + c}{v + a}\right) = \frac{b\left\{\left[\alpha + \frac{1}{2}\left(a + \frac{c}{b}\right)\right]^2 + \beta^2 - \frac{(a - c/b)^2}{4}\right\}}{(\alpha + a)^2 + \beta^2}.$$

The circle

$$\left[\alpha + \frac{1}{2}\left(a + \frac{c}{b}\right)\right]^2 + \beta^2 = \frac{(a - c/b)^2}{4}$$

lies inside the unit circle $|v| = 1$ since

$$\left|-\frac{1}{2}\left(a + \frac{c}{b}\right) - \frac{1}{2}\left(a - \frac{c}{b}\right)\right| = |-a| < 1,$$

and

$$\left|-\frac{1}{2}\left(a + \frac{c}{b}\right) + \frac{1}{2}\left(a - \frac{c}{b}\right)\right| = \left|-\frac{c}{b}\right| < 1.$$

Thus for $|v| \geqslant 1$, $\operatorname{Re}[(bv + c)/(v + a)] > 0$ which by (7) implies $|v| < 1$ and hence we have a contradiction. Therefore the roots s of the characteristic equation (2) are in the left half plane.

To show that the roots are uniformly bounded away from the imaginary axis we note that the function

$$g(s) = s + ase^{-s\tau} + b + ce^{-s\tau}$$

is analytic and therefore its zeros cannot have an accumulation point in the finite plane. Thus if we assume there exists a set $\{w_n\}$, $|w_n| \to \infty$, such that $g(iw_n) \to 0$, we can write

$$g(s) = s(1 + ae^{-s\tau})(1 + 0(|s|) \qquad |s| \to \infty$$

and hence for n sufficiently large,

$$|g(iw_n)| \geqslant (1 - |a|) > 0,$$

a contradiction. This completes the proof of Theorem 1.

3. Stability of the Difference Equations

From

$$\dot{x}(t) + A\dot{x}(t-\tau) + Bx(t) + Cx(t-\tau) = 0$$

we obtain the one-parameter family of difference equations

$$\frac{x(t+h) - x(t)}{h} + A\,\frac{x(t+h-\tau) - x(t-\tau)}{h} + B[\mu x(t+h) + (1-\mu)\,x(t)]$$
$$+ C[\mu x(t+h-\tau) + (1-\mu)\,x(t-\tau)] = 0, \qquad (8)$$

where $0 \leqslant \mu \leqslant 1$.

Theorem. *If $I \pm A$, $B \pm C$ and $(I \pm A) - (\frac{1}{2} - \mu)\,h(B \pm C)$ are positive definite, then (8) is asymptotically stable.*

Remark 1. We consider (8) as a continuous difference equation applying for all $t \geqslant 0$. If h is chosen so that $mh = \tau$ where m is an integer, then (8) becomes a discrete difference relation

$$x_{n+1} = (I + \mu hB)^{-1}\,[(I - (1-\mu)\,hB)\,x_n - (A + \mu hC)\,x_{n+1-m} + (A - (1-\mu)\,hC)\,x_{n-m}].$$

In practice one would probably choose h so that m is an integer. However no such restriction needs to be made here.

Remark 2. If $\mu \geqslant \frac{1}{2}$, then the last two matrices are positive definite if the first two are, and hence we have unconditional asymptotic stability which is analogous to A-stability as defined by Dahlquist [5].

Proof. It is enough to show that the characteristic equation

$$\det M \equiv \det\Big(\frac{e^{sh}-1}{h} + Ae^{-s\tau}\frac{e^{sh}-1}{h} + B(\mu e^{sh}+1-\mu) + Ce^{-s\tau}(\mu e^{sh}+1-\mu)\Big) = 0 \tag{9}$$

has no roots in the half plane $\operatorname{Re}(s) \geqslant -\delta < 0$. Let $v = e^{s\tau}$ and $m = \tau/h$. If (9) holds for some s, then there exists a unit vector u such that $Mu = 0$. Therefore

$$(u, Mu) = \frac{v^{1/m}-1}{h}(1 + av^{-1}) + (b + cv^{-1})(\mu v^{1/m} + 1 - \mu) = 0, \tag{10}$$

where $a = (u, Au)$, $b = (u, Bu)$, and $c = (u, Cu)$. The quantities a, b, c are real and satisfy $|a| < 1$ and $b > |c|$ because of the assumptions made on A, B, and C. Solving (10) for $v^{1/m}$ we have

$$v = \Big(\frac{v(1-(1-\mu)hb) + a - (1-\mu)hc}{v(1+\mu hb) + a + \mu hc}\Big)^m. \tag{11}$$

Let

$$v^{1/m} \equiv z = \Big(\frac{v(1-(1-\mu)hb) + a - (1-\mu)hc}{v(1+\mu hb) + a + \mu hc}\Big) \tag{12}$$

and note that (12) defines a map from the complex v-plane to the complex z plane. Except for the special case $c = ab$ (in which case the entire v-plane is mapped into the point

$$z_0 = \frac{1-(1-\mu)hb}{1+\mu hb}\Big),$$

the inverse map of (12) exists and takes the circle $|z| = \rho$ into a circle in the v-plane with real center. (When $c = ab$, it will be shown later that $|z_0| < 1$ and hence the only solution to (11) is $v_0 = z_0^m = \exp(s_0\tau)$ implying that $\operatorname{Re}(s_0) < 0$.)[1] If for $\rho \geqslant 1$, we can show that the v-plane circle satisfies $|v| < 1$, then it follows, since this circle is compact, that (11) can have no solution $v^* = e^{s^*\tau}$ where $\operatorname{Re}(s^*) \geqslant -\delta$ for some $\delta > 0$. (Accumulation on the imaginary axis is impossible since this would imply the existence of roots s_r such that $|\exp(s_r\tau)| \to 1$ as $r \to \infty$.)

Since the v-plane circle has real center, it is enough to show that for $\rho \geqslant 1$

[1] In addition, if $c = ab$, Eq. (10) has the solution $v = -a$ which is clearly less than unity in modulus.

the real values of this circle are inside the unit circle. Denote these values by $v_{\pm}$. Then

$$\pm\rho = \frac{v_{\pm}(1-(1-\mu)\,hb)+a-(1-\mu)\,hc}{v_{\pm}(1+\mu hb)+a+\mu hc}$$

or

$$v_{\pm} = -\frac{a(\pm\rho-1)+hc(1-\mu\pm\rho\mu)}{\pm\rho-1+hb(1-\mu\pm\rho\mu)}.$$

Clearly since $\rho \geqslant 1$, $|a| < 1$, and $b > |c|$, then $|v_{+}| < 1$. Therefore it remains to prove that

$$0 \leqslant 1 \pm v_{-} = \frac{(\rho+1)(1\mp a)+(\mu(\rho+1)-1)\,h(b\mp c)}{(\rho+1)(1+\mu hb)-hb}. \tag{13}$$

Adding the two positive (by assumption) quantities

$$\begin{aligned} 0 &< [(1-a)-(\tfrac{1}{2}-\mu)\,h(b-c)]+[(1+a)-(\tfrac{1}{2}-\mu)\,h(b+c)] \\ &= 2(1+\mu hb)-hb \leqslant (\rho+1)(1+\mu hb)-hb, \end{aligned} \tag{14}$$

we see that the denominator in (13) is positive. The numerator is positive since

$$\begin{aligned} &(\rho+1)(1\mp a)+(\mu(\rho+1)-1)\,h(b\mp c) \\ &\quad\geqslant 2[(1\mp a)-(\tfrac{1}{2}-\mu)\,h(b\mp c)] > 0. \end{aligned}$$

Hence $1 \pm v_{-} > 0$.

It remains to be shown that for $c = ab$, $|z_0| < 1$. For this case

$$z_0 = \frac{1-(1-\mu)\,hb}{1+\mu hb}$$

and clearly if $z_0 > 0$ then $z_0 < 1$. However

$$1+z_0 = \frac{2(1+\mu hb)-hb}{1+\mu hb} > 0$$

by (14) and hence $|z_0| < 1$.

If we take $n = 1$ and $\mu = 0$ so that (1) and (8) are scalar equations, then we have

Theorem 3. *If $\tau > 0$, $|A| < 1$, $B > |C|$ and $h = \tau/m$ where m is a positive integer, then Eq. (8) is unstable provided*

$$h > \frac{2(1+A)}{B+C} \qquad \textit{and} \qquad m \textit{ is even}$$

or

$$h > \frac{2(1-A)}{B-C} \qquad \textit{and} \qquad m \textit{ is odd.}$$

Before proving these results we compare them with the stability bound

$$0 < h < \frac{2}{B} \tag{15}$$

for the Euler integration of the differential equation

$$\dot{x}(t) = -Bx(t) + f(t).$$

This is the equation obtained from (1) if we treat the delay terms as a forcing function. Here we are dealing with the difference equation

$$\frac{x(t+h) - x(t)}{h} = -Bx(t) + f(t).$$

The conclusions of Theorem 3 show that it is incorrect to treat the terms $A\dot{x}(t-\tau)$ and $Cx(t-\tau)$ as forcing terms in the stability analysis. For example, suppose $A = \frac{1}{2}$, $B = 3$, $C = 1$, $\tau = 3$ and $m = 5$. Then $h = \tau/m = \frac{3}{5}$, $2(1-A)/(B-C) = \frac{1}{2}$, $2/B = \frac{2}{3}$ and hence

$$\frac{2(1-A)}{B-C} < h < \frac{2}{B}.$$

However, since m is odd, (8) is unstable even though we are satisfying the Euler stability condition (15).

Proof of Theorem 3. It suffices to show that (9) with $u = 1$ has a solution s where $\operatorname{Re}(s) > 0$. With $w = e^{sh}$, (9) becomes

$$p(w) \equiv w^{m+1} + (Bh - 1)\, w^m + Aw + (Ch - A) = 0. \tag{16}$$

We have instability if (16) holds for some w with $|w| > 1$. If m is even and $m < (\tau/2)\,[(B + C)/(1 + A)]$, then

$$p(-1) = (B + C)\, h - 2(1 + A) > 0$$

while $p(w) < 0$ for sufficiently large negative w. Hence $p(w) = 0$ has a real root $w < -1$. If m is odd and $m \leqslant (\tau/2)\,[(B - C)/(1 - A)]$, then

$$p(-1) = 2(1 - A) - (B - C)\, h \leqslant 0$$

while $p(w) > 0$ for sufficiently large negative w so again there is a root of

$p(w) = 0$ with $w < -1$. Thus the instability statements in the theorem have been established.

We now show that the condition

$$I \pm A - (\tfrac{1}{2} - \mu)\, h(B \pm C) > 0 \tag{17}$$

is not necessary for the stability of (8) by considering the scalar case with $\mu = 0$ and $0 \leqslant \tau = h$. Since $m = 1$, Eq. (15) can be written as

$$w^2 + (A + B\tau - 1)\, w + (C\tau - A) = 0. \tag{18}$$

The quadratic equation $w^2 + \alpha w + \beta = 0$, where α and β are real, has roots satisfying $|w| < 1$ if and only if $1 + \alpha + \beta > 0$, $1 - \alpha + \beta \geqslant 0$ and $\beta < 1$. In Eq. (18), $1 + \alpha + \beta = (B + C)\,\tau > 0$, so the necessary and sufficient conditions for stability of (8) when $0 < \tau = h$, $|A| < 1$, $B > |C|$ are

$$2(1 - A) - (B - C)\,\tau > 0$$

and

$$C\tau - A < 1.$$

If $A = \frac{1}{2}$, $B = 1$, $C = \frac{3}{5}$, $\tau = h = 2$, then $2(1 - A) - (B - C)\,\tau = \frac{1}{5} > 0$ and $C\tau - A = \frac{7}{10} < 1$ so (8) is stable. However (17) is not satisfied since

$$1 + A - \tfrac{1}{2}\, h(B + C) = -.1 < 0.$$

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