On nonconvex equilibrium problems

Muhammad Aslam Noor\textsuperscript{a}, Themistocles M. Rassias\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a} Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan
\textsuperscript{b} Mathematics Department, National Technical University of Athens, Zografou Campus, 15773 Athens, Greece

Received 16 March 2005
Available online 13 May 2005

Abstract

In this paper, we introduce a new class of equilibrium problems, known as mixed quasi nonconvex equilibrium problems. We suggest some iterative schemes for solving nonconvex equilibrium problems by using the auxiliary principle technique. The convergence of the proposed methods either requires partially relaxed strongly monotonicity or pseudomonotonicity. As special cases, we obtain a number of known and new results for solving various classes of equilibrium and variational inequality problems.

\textcopyright 2005 Elsevier Inc. All rights reserved.

Keywords: Equilibrium problems; Auxiliary principle; Iterative methods; Convergence; Dual problems

1. Introduction

Equilibrium problems theory provides us a natural, novel and unified framework to study a wide class of problems arising in economics, finance, transportation, network and structural analysis, elasticity and optimization. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative; see [1–12]. Equilibrium problems also include variational inequalities and related optimization problems as special cases. Almost all the results obtained so far in this area are in the setting of convexity. It has been noted that these results may not hold in the nonconvex

\textsuperscript{*} Corresponding author.
E-mail addresses: noormaslam@hotmail.com (M.A. Noor), trassias@math.ntua.gr (T.M. Rassias).
setting. In recent years, the concept of convexity has been generalized in many directions, which has potential and important applications in various fields. A significant generalization of the convex functions is the introduction of $g$-convex functions. It is well known that the $g$-functions and $g$-convex sets may not be convex functions and convex sets [13,14]. However, it has been shown that the class of $g$-convex function have some nice properties, which the convex functions have. In particular, it has been shown [15] that the minimum of the $g$-functions over the $g$-convex sets can be characterized by a class of variational inequalities, which is called the nonconvex ($g$-convex) variational inequality. Inspired and motivated by the recent research work going in this field, we consider a new class of equilibrium problems, which is called mixed quasi nonconvex equilibrium problems, where the convex set is replaced by the so-called $g$-convex set. There are several methods including projection and its variant forms, Wiener–Hopf equations and auxiliary principle for solving variational inequalities. On the other hand, there are only few iterative methods for solving (nonconvex) equilibrium problems. It is known that projection methods and variant forms including Wiener–Hopf equations cannot be extended for equilibrium, since it is not possible to evaluate the projection of the trifunction. This fact has motivated to use the auxiliary principle technique. Glowinski, Lions and Tremolieres [16] used this technique to study the existence of a solution of the mixed variational inequalities, whereas Noor, Noor and Rassias [11] used this technique to suggest and analyze an iterative method for solving mixed quasi variational inequalities. It is well known that a substantial number of numerical methods can be obtained as special cases from this technique; see [9–12,17]. We again use the auxiliary principle technique to suggest a class of iterative methods for solving nonconvex equilibrium problems. The convergence of these methods requires only that the trifunction is partially relaxed strongly jointly monotone, which is weaker than monotonicity. We also use the auxiliary principle technique to suggest and analyze a proximal method for solving equilibrium problem, which was introduced as a regularization of convex optimization in Hilbert space. We prove that the convergence of proximal method requires only jointly pseudomonotonicity, which is a weaker condition than jointly monotonicity. This clearly improves the known results. Since mixed quasi nonconvex equilibrium problems include equilibrium, nonconvex variational inequalities and complementarity problems as special cases, results obtained in this paper continue to hold for these problems. Our results can be considered an important and significant extension of the known results for solving equilibrium, variational inequalities and complementarity problems.

2. Preliminaries

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Let $K$ be a nonempty and closed set in $H$. First of all, we recall the following concepts and results.

**Definition 2.1.** Let $K$ be any set in $H$. The set $K$ is said to be $g$-convex, if there exists a function $g : K \rightarrow K$ such that

$$g(u) + t(g(v) - g(u)) \in K, \quad \forall u, v \in K, \quad t \in [0, 1].$$
Note that every convex set is $g$-convex, but the converse is not true, see [13,14]. In passing, we remark that the notion of the $g$-convex set was introduced by Noor [18] implicitly in 1988.

*From now onward, we assume that $K$ is a $g$-convex set, unless otherwise specified.*

**Definition 2.2.** The function $f : K \to H$ is said to be $g$-convex, if

$$ f\left(g(u) + t(g(v) - g(u))\right) \leq (1-t)f\left(g(u)\right) + tf\left(g(v)\right), \quad \forall u, v \in K, \ t \in [0,1]. $$

Clearly every convex function is $g$-convex, but the converse is not true; see [13,14].

**Definition 2.3.** A function $f$ is said to be strongly $g$-convex on the $g$-convex set $K$ with modulus $\mu > 0$, if,

$$ f\left(g(u) + t(g(v) - g(u))\right) \leq (1-t)f\left(g(u)\right) + tf\left(g(v)\right) - t(1-t)\mu \|g(v) - g(u)\|^2, $$

or

$$ \|f'(g(u)) - f'(g(v)), g(u) - g(v)\| \geq 2\mu \|g(v) - g(u)\|^2, $$

that is, $f'(g(u))$ is a strongly monotone operator.

It is well known [14] that the $g$-convex functions are not convex function, but they have some nice properties which the convex functions have. Note that for $g = I$, the $g$-convex functions are convex functions and Definition 2.3 is a well-known result in convex analysis.

For a given nonlinear continuous trifuction $F(.,.,.) : K \times K \times K \to H$ and a continuous bifunction $\varphi(.,.) : H \times H \to \mathbb{R} \cup \infty$, consider the problem of finding $u \in K$ such that

$$ F\left(g(u), T\left(g(u)\right), g(v)\right) + \varphi\left(g(v), g(u)\right) - \varphi\left(g(u), g(u)\right) \geq 0, \quad \forall v \in K, $$

which is called is called the *mixed quasi nonconvex equilibrium problem with trifuction*. For $g \equiv I$, where $I$ is the identity operator, the $g$-convex set $K$ becomes the convex set $K$ and consequently, problem (1) is equivalent to finding $u \in K$ such that

$$ F\left(u, T\left(u\right), v\right) + \varphi\left(v, u\right) - \varphi\left(u, u\right) \geq 0, \quad \forall v \in K, $$

which is the mixed quasi equilibrium problem with trifuction, introduced and studied by Noor [19].

We note that for $F(g(u), T(g(u)), g(v)) = \langle T(g(u)), g(v) - g(u)\rangle$, where $T : H \to H$ is a nonlinear continuous operator, problem (1) is equivalent to finding $u \in K$ such that

$$ \langle T\left(g(u)\right), g(v) - g(u)\rangle + \varphi\left(g(v), g(u)\right) - \varphi\left(g(u), g(u)\right) \geq 0, \quad \forall v \in K. $$

(3)
Inequality (3) is known as the \textit{mixed quasi nonconvex variational inequality}, which was introduced by Noor [19].

If \( \varphi(.,.) = \varphi(.) \) is the indicator function of a closed and \( g \)-convex set \( K \), then problem (1) reduces to finding \( u \in K \) such that

\[
F\left( g(u), T\left( g(u) \right), g(v) \right) \geq 0, \quad \forall v \in K,
\]

which is called the \( g \)-convex equilibrium problem and appears to be a new one.

If \( F(g(u), T(g(u)), g(v)) = \langle T(g(u)), g(v) - g(u) \rangle \), then problem (4) is equivalent to finding \( u \in K \) such that

\[
\langle T\left( g(u) \right), g(v) - g(u) \rangle \geq 0, \quad \forall v \in K,
\]

which is known as the nonconvex variational inequality introduced by Noor [15]. It is worth mentioning that nonconvex variational inequalities (5) are quite different from the so-called general variational inequalities, introduced and studied by Noor [18] in 1988. For the applications and numerical methods of general variational inequalities see [11,12,19,22] and the references therein.

If \( g = I \), the identity operator, then the \( g \)-convex set \( K \) becomes the convex set \( K \), and consequently the nonconvex variational inequalities (3) are equivalent to finding \( u \in K \) such that

\[
\langle Tu, v - u \rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in K,
\]

which are known as the mixed quasi variational inequalities; see [6–8,20,21].

It is clear that problems (2)–(6) are special cases of the nonconvex equilibrium problems (1). In brief, for a suitable and appropriate choice of the operators \( T, g \), and the space \( H \), one can obtain a wide class of equilibrium, variational inequalities and complementarity problems. This clearly shows that problem (1) is quite general and unifying one. Furthermore, problem (1) has important applications in various branches of pure and applied sciences; see [1–22].

We also need the following concepts.

**Definition 2.4.** The trifunction \( F(.,.,.) : K \times K \times K \rightarrow H \) with respect to the operators \( T, g \), is said to be:

(i) \textit{partially relaxed jointly strongly monotone}, if there exists a constant \( \alpha > 0 \) such that

\[
F\left( g(u), T\left( g(u) \right), g(v) \right) + F\left( g(v), T\left( g(v) \right), g(z) \right) \\
\leq \alpha \| g(z) - g(u) \|^2, \quad \forall u, v, z \in K;
\]

(ii) \textit{jointly monotone}, if

\[
F\left( g(u), T\left( g(u) \right), g(v) \right) + F\left( g(v), T\left( g(v) \right), g(u) \right) \leq 0, \quad \forall u, v \in K;
\]

(iii) \textit{jointly pseudomonotone}, if

\[
F\left( g(u), T\left( g(u) \right), g(v) \right) + \varphi(g(v) - g(u)) - \varphi(g(u), g(u)) \geq 0 \\
\Rightarrow -F\left( g(v), T\left( g(v) \right), g(u) \right) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \\
\forall u, v \in K;
\]
(iv) jointly hemicontinuous, \( \forall u, v \in K, t \in [0, 1], \) if the mapping \( F(g(u) + t(g(v) - g(u)), T(g(u) + t(g(v) - g(u)), g(v)) \) is continuous.

We remark that if \( z = u \), then partially relaxed jointly strongly monotonicity is exactly jointly monotonicity of the operator \( F(.,.,.) \). For \( g \equiv I \), the identity operator, then Definition 2.1 reduces to the standard definition of partially relaxed jointly strongly monotonicity, jointly monotonicity, and jointly pseudomonotonicity. It is known that monotonicity implies pseudomonotonicity, but not conversely. This implies that the concepts of partially relaxed strongly monotonicity and pseudomonotonicity are weaker than monotonicity.

**Lemma 2.1.** Let \( F(.,.,.) \) be jointly pseudomonotone, jointly hemicontinuous and \( g \)-convex with respect to third argument. If the bifunction \( \varphi(.,.) \) is \( g \)-convex with respect to first argument, then the nonconvex equilibrium problem (1) is equivalent to finding \( u \in K \) such that

\[
-F(g(v), T(g(v)), g(u)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in K.
\]

**Proof.** Let \( u \in K \) be a solution of (1). Then

\[
F(g(u), T(g(u)), g(v)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in K,
\]

which implies

\[
-F(g(v), T(g(v)), g(u)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in K,
\]

since \( F(.,.,.) \) is jointly pseudomonotone.

Conversely, let \( u \in K \) satisfy (7). Since \( K \) is a \( g \)-convex set, \( \forall u, v \in K, t \in [0, 1], g(v_t) = g(u) + t(g(v) - g(u)) \equiv (1 - t)g(u) + tg(v) \in K \).

Taking \( g(v) = g(v_t) \) in (8), we have

\[
F(g(v_t), T(g(v_t)), g(u)) \leq \varphi(g(v_t), g(u)) - \varphi(g(u), g(u))
\]

\[
\leq t\{\varphi(g(v), g(u)) - \varphi(g(u), g(u))\}.
\]

(8)

Now using (8) and \( g \)-convexity of \( F(.,.) \) with respect to third argument, we have

\[
0 \leq F(g(v_t), T(g(v_t)), g(v_t))
\]

\[
= F(g(v_t), T(g(v_t)), (1 - t)g(u) + tg(v))
\]

\[
\leq tF(g(v_t), T(g(v_t)), g(v)) + (1 - t)F(g(v_t), T(g(v_t)), g(u))
\]

\[
\leq tF(g(v_t), T(g(v_t)), g(v)) + t(1 - t)\{\varphi(g(v), g(u)) - \varphi(g(u), g(u))\}.
\]

(9)

Dividing (9) by \( t \) and letting \( t \to 0 \), we have

\[
F(g(u), T(g(u)), g(v)) + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in K,
\]

the required (1). \( \square \)

**Remark 2.1.** Problem (7) is known as the dual mixed quasi nonconvex equilibrium problem. One can easily show that the solution set of problem (7) is closed and \( g \)-convex set. From Lemma 2.1, it follows that the solution set of problems (1) and (7) are the same. This
inter relationship has played an important role in the study of well-posedness of equilibrium problems and variational inequalities. In fact, Lemma 2.1 can be viewed as a natural generalization and extension of a well-known Minty’s lemma in variational inequalities theory; see [12,21].

**Definition 2.5.** The bifunction \( \varphi(.,.) : H \times H \to R \cup \{+\infty\} \) is called skew-symmetric, if and only if,
\[
\varphi(u,u) - \varphi(u,v) - \varphi(v,u) - \varphi(v,v) \geq 0, \quad \forall u, v \in H.
\]
Clearly if the skew-symmetric bifunction \( \varphi(.,.) \) is bilinear, then
\[
\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) = \varphi(u - v, u - v) \geq 0, \quad \forall u, v \in H.
\]

3. Iterative schemes and convergence analysis

In this section, we suggest and analyze some new iterative methods for solving the problem (1) by using the auxiliary principle technique as developed by Noor [9,10,12] in recent years.

For a given \( u \in K \), consider the problem of finding a unique \( w \in K \) satisfying the auxiliary nonconvex equilibrium problem
\[
\rho F(g(u), T(g(u)), g(v)) + \langle E'(g(w)) - E'(g(u)), g(v) - g(w) \rangle \\
\geq \rho \{ \varphi(g(w), g(w)) - \varphi(g(v), g(w)) \}, \quad \forall v \in K,
\]
(10)
where \( \rho > 0 \) is a constant and \( E' \) is the differential of a strongly \( g \)-convex function \( E \).

**Remark 3.1.** The function \( B(w, u) = E(g(w)) - E(g(u)) - \langle E'(g(u)), g(w) - g(u) \rangle \) associated with the \( g \)-convex function \( E(u) \) is called the generalized Bregman function. We note that if \( g = I \), then \( B(w, u) = E(w) - E(u) - \langle E'(u), w - u \rangle \) is the well known Bregman function.

We note that if \( w = u \), then clearly \( w \) is a solution of the nonconvex equilibrium problems (1). This observation enables us to suggest the following method for solving (1).

**Algorithm 3.1.** For a given \( u_0 \in H \), compute the approximate solution \( u_{n+1} \) by the iterative schemes
\[
\rho F(g(u_n), T(g(u_n)), g(v)) + \langle E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1}) \rangle \\
\geq \rho \{ \varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(v), g(u_{n+1})) \}, \quad \forall v \in K,
\]
(11)
where \( \rho > 0 \) is a constant.

Note that if \( g \equiv I \), the identity operator, the \( g \)-convex set \( K \) becomes a convex set \( K \), then Algorithm 3.1 reduces to a method for solving the equilibrium problems with trifunction (2).
Algorithm 3.2. For a given $u_0 \in H$, compute $u_{n+1}$ by the iterative scheme

$$\rho F(u_n, T(u_n), v) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1} \rangle$$

$$\geq \rho \{ \varphi(u_{n+1}, u_n) - \varphi(v, u_n) \}, \quad \forall v \in K,$$

which appears to be a new one.

For $F(g(u), T(g(u)), (v)) = \langle T(g(u)), g(v) - g(u) \rangle$, where $T : H \rightarrow H$ is a nonlinear continuous operator, Algorithm 3.1 reduces to:

Algorithm 3.3. For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative scheme

$$\rho T(g(u)) + E'(g(u_{n+1})) - E'(g(u)), g(v) - g(u_{n+1})$$

$$\geq \rho \{ \varphi(g(u_{n+1}), g(u)) - \varphi(g(v), g(u_{n+1})) \}, \quad \forall v \in K,$$

for solving mixed quasi nonconvex variational inequalities [19].

For suitable and appropriate choice of the operators and the space $H$, one can obtain various new and known methods for solving equilibrium, variational inequalities and complementarity problems.

For the convergence analysis of Algorithm 3.1, we need the following result.

Theorem 3.1. Let $E(u)$ be a strongly $g$-convex with modulus $\beta > 0$ and the trifunction $F(.,.,.)$ is partially relaxed jointly strongly monotone with constant $\alpha > 0$. If $0 < \rho < \frac{\beta}{\alpha}$ and the bifunction $\varphi(.,.)$ is skew-symmetric, then the approximate solution $u_{n+1}$ obtained from Algorithm 3.1 converges to a exact solution of (1).

Proof. Let $u \in K$ be a solution of (1). Then

$$F(g(u), T(g(u)), g(v)) \geq \varphi(g(u), g(u)) - \varphi(g(v), g(u)), \quad \forall v \in K,$$  \hspace{1cm} (12)

where $\rho > 0$ is a constant.

Now taking $v = u_{n+1}$ in (12) and $v = u$ in (11), we have

$$F(g(u), T(g(u)), g(u_{n+1})) \geq \varphi(g(u), g(u)) - \varphi(g(u_{n+1}), g(u))$$  \hspace{1cm} (13)

and

$$\rho F(g(u), T(g(u)), g(u)) + \langle E'(g(u_{n+1})) - E'(g(u)), g(u) - g(u_{n+1}) \rangle$$

$$\geq \rho \{ \varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(u), g(u_{n+1})) \}.$$  \hspace{1cm} (14)

We consider the Bregman function

$$B(u, w) = E(g(u)) - E(g(w)) - \langle E'(g(w)), g(u) - g(w) \rangle$$

$$\geq \beta \| g(u) - g(w) \|^2,$$  \hspace{1cm} (15)

using strongly $g$-convexity of $E$. Now combining (13)–(15), we have
Algorithm 3.4. For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative scheme
\[
\rho F(g(u_{n+1}), T(g(u_{n+1})), g(v)) + \{E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1})\} \\
\geq \rho \{\varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(v), g(u_{n+1}))\}, \quad \forall v \in K.
\] (17)

Algorithm 3.4 is known as the proximal method for solving nonconvex equilibrium problem (1). For $g \equiv I$, where $I$ is the identity operator, the $g$-convex set $K$ becomes the convex set $K$ and we obtain a proximal method for equilibrium problems (2), that is,
Algorithm 3.5. For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative scheme

$$
\rho F(u_{n+1}, T(u_{n+1}), v) + \left\{ E'(u_{n+1}) - E'(u_n), v - u_{n+1} \right\} \\
\geq \rho \left\{ \varphi(u_{n+1}, u_n) - \varphi(v, u_n) \right\}, \quad \forall v \in K.
$$

Note that $E'(u)$ is the differential of a differentiable strongly convex function $E$ at $u \in K$.

If $F(g(u), g(v)) = \langle T(g(u)), g(v) - g(u) \rangle$, then Algorithm 3.4 reduces to:

Algorithm 3.6. For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative scheme

$$
\left\{ \rho T(g(u_{n+1})) + E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1}) \right\} \\
\geq \rho \left\{ \varphi(g(u_{n+1}), g(u_n)) - \varphi(g(v), g(u_{n+1})) \right\}, \quad \forall v \in K.
$$

In a similar way, one can obtain a variant form of proximal methods for solving variational inequalities and equilibrium problems as special cases.

We now study the convergence analysis of Algorithm 3.4 using the technique of Theorem 3.1. For the sake of completeness and to convey an idea of the techniques involved, we sketch the main points only.

Theorem 3.2. Let $E(u)$ be a strongly $g$-convex with modulus $\beta > 0$ and the trifunction $F(.,.,.)$ be jointly pseudomonotone. If the bifunction $\varphi(.,.)$ is skew-symmetric, then the approximate solution $u_{n+1}$ obtained from Algorithm 3.4 converges to a exact solution of (1).

Proof. Let $u \in K$ be a solution of (1). Then

$$
F(g(u), T(g(u)), g(v)) \geq \varphi(g(u), g(u)) - \varphi(g(v), g(u)), \quad \forall v \in K,
$$

which implies that

$$
-F(g(v), T(g(v)), g(u)) \geq \varphi(g(u), g(u)) - \varphi(g(v), g(u)), \quad \forall v \in K, \quad (18)
$$

since $F(.,.)$ is jointly pseudomonotone.

Taking $v = u_{n+1}$ in (18), we have

$$
-F(g(u_{n+1}), T(g(u_{n+1})), g(u)) \geq \varphi(g(u), g(u)) - \varphi(g(u_{n+1}), g(u)). \quad (19)
$$

Now as in Theorem 3.1, from (15), (17) and (18), we have

$$
B(u, u_n) - B(u, u_{n+1}) \\
= E(g(u_{n+1})) - E(g(u_n)) + \left\{ E'(g(u_{n+1})), g(u_{n+1}) - g(u_n) \right\} \\
+ \left\{ E'(g(u_{n+1})), g(u) - g(u_{n+1}) \right\} \\
\geq \beta \| g(u_{n+1}) - g(u_n) \|^2 + \left\{ E'(g(u_{n+1})), g(u) - g(u_{n+1}) \right\} \\
\geq \beta \| g(u_{n+1}) - g(u_n) \|^2 - \rho F(g(u_{n+1}), T(g(u_{n+1})), g(u)) \\
+ \rho \left\{ \varphi(g(u_{n+1}), g(u_{n+1})) - \varphi(g(u), g(u_{n+1})) \right\}
$$
\[
\begin{align*}
&\geq \beta \| g(u_{n+1}) - g(u_n) \|^2 + \rho \{ \phi(g(u_{n+1}), g(u_{n+1})) - \phi(g(u), g(u_{n+1})) \\
&\quad - \phi(g(u_{n+1}), g(u)) + \phi(g(u), g(u)) \} \\
&\geq \beta \| g(u_{n+1}) - g(u_n) \|^2,
\end{align*}
\]
where we have used the fact that the bifunction $\phi(\cdot, \cdot)$ is skew symmetric.

If $u_{n+1} = u_n$, then clearly $u_n$ is a solution of the nonconvex equilibrium problems (1).

Otherwise, it follows that $B(u, u_n) - B(u, u_{n+1})$ is nonnegative, and we must have

\[
\lim_{n \to \infty} \| u_{n+1} - u_n \| = 0.
\]

Now using the technique of Zhu and Marcotte [17], it can be shown that the entire sequence \{u_n\} converges to the cluster point $u$ satisfying the nonconvex equilibrium problem (1). \[ \square \]

References
