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Factorization of injective ideals by interpolation

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Abstract

We construct a factorization of certain multilinear mappings through linear operators belonging to closed, injective operator ideals using interpolation technique. An extension of the duality theorem for interpolation spaces is also obtained.

Keywords: Factorization; Operator ideals; Multilinear mappings; Infinite holomorphy

1. Introduction

The by now classical paper [2] states as a main result the equivalence of the following three statements for multilinear mappings $M \in \mathcal{L}(E_1, \dots, E_m; F)$:

(1) M is weak-to-norm continuous on bounded sets,
(2) the linearizations

$$M^{(j)} : E_j \to \mathcal{L}(E_1, \dots, E_{j-1}, E_{j+1}, \dots, E_m; F)$$

given by $M^{(j)}(x_j)(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m) = M(x_1, \dots, x_m)$ are linear compact operators,
(3) there is a factorization $M = L(T_1, \dots, T_m)$ with compact linear operators $T_j \in \mathcal{L}(E_j; G_j)$ and a multilinear bounded mapping $L \in \mathcal{L}(G_1, \dots, G_m; F)$.

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The importance of this theorem is evident: It reduces topological properties of multilinear mappings (and finally of holomorphic functions) to associated linear operators and makes the linear theory applicable. Therefore, several authors have tried to extend this result to multilinear mappings with other topological properties instead of compactness [1,4,5,8]. S. Geiss [4] and H. Junek [8] were the first who used operator ideals and interpolation technique. Recall that a subclass $\mathcal{A} \subseteq \mathcal{L}$ of linear bounded operators with the components $\mathcal{A}(E;F) = \mathcal{A} \cap \mathcal{L}(E;F)$ is called to be an *operator ideal*, if all components $\mathcal{A}(E;F)$ are linear subspaces of $\mathcal{L}(E;F)$ containing the space $\mathcal{F}(E;F)$ of all operators of finite rank and if the components satisfy the typical *ideal condition*:

For all operators $T \in \mathcal{A}(E;F)$ and all linear bounded operators $S \in \mathcal{L}(E_1;E)$ and $R \in \mathcal{L}(F;F_1)$ the product satisfies $RTS \in \mathcal{A}(E_1;F_1)$.

The ideals $\mathcal{K}$ and $\mathcal{W}$ of all compact or weakly compact operators, respectively, are important examples. For more details on operator ideals we refer to [9]. Ideals of linear operators were used by A. Pietsch in [10] to generate ideals of multilinear mappings in two different ways:

Definition 1.1. For ideals $\mathcal{A}_1, \dots, \mathcal{A}_m$ of linear operators we define the classes $[\mathcal{A}_1, \dots, \mathcal{A}_m]$ and $\mathcal{L}(\mathcal{A}_1, \dots, \mathcal{A}_m)$ as follows: For $M \in \mathcal{L}(E_1, \dots, E_m; F)$ we put

(1) $M \in [\mathcal{A}_1, \dots, \mathcal{A}_m](E_1, \dots, E_m; F)$ if all linearizations

$$M^{(j)} : E_j \to \mathcal{L}(E_1, \dots, E_{j-1}, E_{j+1}, \dots, E_m; F)$$

belong to $\mathcal{A}_j(E_j; \mathcal{L}(E_1, \dots, E_{j-1}, E_{j+1}, \dots, E_m; F))$.

(2) $M \in \mathcal{L}(\mathcal{A}_1, \dots, \mathcal{A}_m)(E_1, \dots, E_m; F)$ if M admits a factorization

$$M = L(T_1, \dots, T_m)$$

with some operators $T_j \in \mathcal{A}_j(E_j; G_j)$ and some bounded multilinear mapping $L \in \mathcal{L}(G_1, \dots, G_m; F)$.

Obviously, we always have an inclusion $\mathcal{L}(\mathcal{A}_1, \dots, \mathcal{A}_m) \subseteq [\mathcal{A}_1, \dots, \mathcal{A}_m]$. For $\mathcal{A}_j = \mathcal{K}$ the Aron–Hervés–Valdivia result mentioned above states

$$\mathcal{L}(\mathcal{K}, \dots, \mathcal{K}) = [\mathcal{K}, \dots, \mathcal{K}],$$

while

$$\mathcal{L}(\mathcal{W}, \dots, \mathcal{W}) = [\mathcal{W}, \dots, \mathcal{W}]$$

was shown in [1]. Here, we will investigate for what further ideals a factorization formula

$$\mathcal{L}(\mathcal{A}_1, \dots, \mathcal{A}_m) = [\mathcal{A}_1, \dots, \mathcal{A}_m]$$

holds true. This cannot be expected for all operator ideals. Indeed, if $\mathcal{S}_2(H;H)$ denotes the ideal of the Hilbert–Schmidt operators on an infinite dimensional Hilbert space, then we obviously have

$$\mathcal{L}(\mathcal{S}_2, \mathcal{S}_2)(H, H; \mathbb{C}) \neq [\mathcal{S}_2, \mathcal{S}_2](H, H; \mathbb{C}).$$

Therefore, we need additional conditions on the ideals $\mathcal{A}_j$ to get factorization theorems. The following definition is important for this purpose.

Definition 1.2. Let $\mathcal{A}$ be any operator ideal.

(1) $\mathcal{A}$ is called to be closed, if the components $\mathcal{A}(E; F)$ are closed subspaces of $\mathcal{L}(E; F)$ for all pairs E, F of Banach spaces.
(2) $\mathcal{A}$ is called to be injective, if any $T \in \mathcal{L}(E; F)$ belongs to $\mathcal{A}(E; F)$ provided that there is an isomorphic embedding $J : F \to F_1$ such that $JT \in \mathcal{A}(E; F_1)$.
(3) $\mathcal{A}$ is called to be surjective, if $T \in \mathcal{L}(E; F)$ belongs to $\mathcal{A}(E; F)$ provided that there is some quotient map $Q : E_0 \to E$ such that $TQ \in \mathcal{A}(E_0; F)$.

The ideals $\mathcal{W}$ and $\mathcal{K}$ are both injective and surjective, the ideal $\mathcal{G}$ of all approximable operators is neither injective nor surjective, and the ideals $\mathcal{U}$ of all unconditionally summing operators (weakly summable sequences are mapped into norm summable sequences) and $\mathcal{V}$ of all completely continuous operators (weakly convergent sequences are mapped into norm convergent sequences) are closed and obviously injective, but not surjective.

It was shown in [4] that a factorization theorem holds true, if the closed ideals $\mathcal{A}_j$ are both injective and surjective, and it was stated in [5] that a factorization theorem holds also true for closed, injective ideals $\mathcal{A}_1 = \cdots = \mathcal{A}_m = \mathcal{A}$. The wrong proof given in [5] was improved later on in [6].

In the present paper we give an alternative proof of the factorization formula for closed, injective ideals using interpolation spaces. This method works well also for the case of different ideals $\mathcal{A}_1, \ldots, \mathcal{A}_m$ and additionally it provides us with numerous possible factorization spaces. The application of the interpolation technique to our situation requires a vector valued version of the duality theorem for interpolation spaces given in this paper. This could also be useful in other applications.

2. Operators on interpolation spaces

Let us recall some notions of the interpolation theory. For details we refer to the textbook [3].

Definition 2.1. A couple $\bar{X} = (X_0, X_1)$ of Banach spaces is called to be compatible, if X_0 and X_1 are continuously embedded into some Hausdorff topological vector space Z. With respect to this embedding we define as usually

$$\Delta\bar{X} = X_0 \cap X_1 \quad \text{equipped with } \|x\|_{\Delta\bar{X}} = \max\big\{\|x\|_{X_0}, \|x\|_{X_1}\big\},$$
$$\Sigma\bar{X} = X_0 + X_1 \quad \text{equipped with } \|x\|_{\Sigma\bar{X}} = \inf_{x=x_0+x_1} \|x_0\|_{X_0} + \|x_1\|_{X_1},$$

and for all $t > 0$ we define the both functionals

$$J(t,x) = J(t,x,\bar{X}) = \max\big\{\|x\|_{X_0}, t\|x\|_{X_1}\big\} \quad \text{for } x \in \Delta\bar{X},$$
$$K(t,x) = K(t,x,\bar{X}) = \inf\big\{\|x_0\|_{X_0} + t\|x_1\|_{X_1}\colon x = x_0 + x_1\big\} \quad \text{for } x \in \Sigma\bar{X}.$$

In particular, we have $\|x\|_{\Delta\bar{X}} = J(1,x)$ and $\|x\|_{\Sigma\bar{X}} = K(1,x)$.

Using these functionals the K- and the J-interpolation methods can be established:

Definition 2.2. For $0 < \Theta < 1$ and $1 \leqslant q < \infty$ we define the interpolation spaces

$$(X_0, X_1)_{K,\Theta,q} = \left\{ x \in \Sigma\bar{X}\colon\ \|x\|^q_{K,\Theta,q} = \sum_{\nu=-\infty}^{\infty} \bigl(2^{-\Theta\nu} K(2^\nu, x)\bigr)^q < \infty \right\},$$

$$(X_0, X_1)_{J,\Theta,q} = \left\{ x \in \Sigma\bar{X}\colon\ \|x\|^q_{J,\Theta,q} = \inf_{x=\sum_\nu x_\nu} \sum_{\nu=-\infty}^{\infty} \bigl(2^{-\Theta\nu} J(2^\nu, x_\nu)\bigr)^q < \infty \right\},$$

where the convergence $x = \sum_\nu x_\nu$ with $x_\nu \in \Delta\bar{X}$ is taken in $\Sigma\bar{X}$. For $q = \infty$ we put

$$\|x\|_{K,\Theta,\infty} = \sup_{\nu\in\mathbb{Z}} 2^{-\Theta\nu} K(2^\nu, x) \quad \text{and} \quad \|x\|_{J,\Theta,\infty} = \inf_{x=\sum_\nu x_\nu} \sup_{\nu\in\mathbb{Z}} 2^{-\Theta\nu} J(2^\nu, x_\nu),$$

respectively.

Because of $K(t, x + y) \leqslant K(t, x) + K(t, y)$ and $J(t, x + y) \leqslant J(t, x) + J(t, y)$ both functionals $\|\cdot\|_{K,\Theta,q}$ and $\|\cdot\|_{J,\Theta,q}$ are norms. The central result of the interpolation theory is the equivalence of both methods:

Theorem 2.3 [3, Theorem 3.3.1]. *For $0 < \Theta < 1$ and $1 \leqslant q \leqslant \infty$ we have $(X_0, X_1)_{K,\Theta,q} = (X_0, X_1)_{J,\Theta,q}$ with equivalent norms.*

Now, we are going to extend the duality theorem on interpolation spaces from the case of linear functionals to linear operators.

Definition 2.4. A compatible pair $\bar{X} = (X_0, X_1)$ is called to be fully compatible, if $\Delta\bar{X}$ is dense in both, X_0 and X_1.

Proposition 2.5. *The pair $\bar{X} = (X_0, X_1)$ is fully compatible if and only if for every Banach space Y the restriction mappings*

$$\rho_j : \mathcal{L}(X_j; Y) \to \mathcal{L}(\Delta\bar{X}; Y), \quad j = 0, 1,$$

are injective. In this case, the pair

$$\mathcal{L}(\bar{X}; Y) = \bigl(\mathcal{L}(X_0; Y), \mathcal{L}(X_1; Y)\bigr)$$

is compatible with respect to the embeddings $\mathcal{L}(X_j; Y) \subseteq \mathcal{L}(\Delta\bar{X}; Y)$ given by ρ_j. In particular, we have $\Sigma\mathcal{L}(\bar{X}; Y) \subseteq \mathcal{L}(\Delta\bar{X}; Y)$.

Proof. If $\Delta\bar{X}$ is dense in X_0 then $T|_{\Delta\bar{X}} = 0$ implies $T|_{X_0} = 0$ for all $T \in \mathcal{L}(X_0; Y)$, and the same holds true for X_1. The "only if" part is a consequence of the Hahn–Banach theorem. □

Proposition 2.6. *Let $\bar{X} = (X_0, X_1)$ be any fully compatible pair and let Y be any Banach space. Then we have*

$$\Delta\mathcal{L}(\bar{X}; Y) = \mathcal{L}(\Sigma\bar{X}; Y) \quad \textit{and} \quad \|T : \Sigma\bar{X} \to Y\| = \|T\|_{\Delta\mathcal{L}(\bar{X};Y)}.$$

Proof. Each $T \in \Delta\mathcal{L}(\bar{X};Y)$ admits a unique linear extension $T:\Sigma\bar{X} \to Y$ defined by $T(x_0+x_1) = Tx_0 + Tx_1$ for $x_0 \in X_0$ and $x_1 \in X_1$. We are going to check the equality of the associated norms. Let $x \in \Sigma\bar{X}$ with $\|x\|_{\Sigma\bar{X}} < 1$ be given. We choose a representation $x = x_0 + x_1$ with $\|x_0\|_{X_0} + \|x_1\|_{X_1} < 1$. Then we get

$$\begin{aligned}\|Tx\| &= \|Tx_0 + Tx_1\| \leqslant \|T:X_0 \to Y\| \cdot \|x_0\|_{X_0} + \|T:X_1 \to Y\| \cdot \|x_1\|_{X_1} \\ &\leqslant \max\bigl\{\|T:X_0 \to Y\|, \|T:X_1 \to Y\|\bigr\} \cdot \bigl(\|x_0\|_{X_0} + \|x_1\|_{X_1}\bigr),\end{aligned}$$

and this implies $\|T:\Sigma\bar{X} \to Y\| \leqslant \|T\|_{\Delta\mathcal{L}(\bar{X};Y)}$. To prove the converse we may suppose $\|T\|_{\Delta\mathcal{L}(\bar{X};Y)} = \|T:X_0 \to Y\|$. For $\varepsilon > 0$ there is some $x_0 \in X_0$ with $\|x_0\|_{X_0} \leqslant 1$ and $\|T:X_0 \to Y\| \leqslant \|Tx_0\| + \varepsilon \leqslant \|T:\Sigma\bar{X} \to Y\| \cdot \|x_0\|_{\Sigma\bar{X}} + \varepsilon \leqslant \|T:\Sigma\bar{X} \to Y\| + \varepsilon$. □

Corollary 2.7. *Let $\bar{X} = (X_0, X_1)$ be any fully compatible pair and let Y be any Banach space. Then we have*

$$J\bigl(t, T, \mathcal{L}(\bar{X};Y)\bigr) = \sup_{x \in \Sigma\bar{X}} \frac{\|Tx\|}{K(t^{-1}, x, \bar{X})}$$

for all $T \in \mathcal{L}(\Sigma\bar{X};Y)$ and all $t > 0$.

Proof. For $X_t = (X_1, \|\cdot\|_t)$ with $\|x\|_t = t^{-1}\|x\|_{X_1}$ and $T \in \mathcal{L}(\Sigma\bar{X};Y)$ we have

$$\begin{aligned}J\bigl(t, T, \mathcal{L}(\bar{X};Y)\bigr) &= \max\bigl\{\|T:X_0 \to Y\|, t\|T:X_1 \to Y\|\bigr\} \\ &= \max\bigl\{\|T:X_0 \to Y\|, \|T:X_t \to Y\|\bigr\} \\ &= \|T\|_{\Delta\mathcal{L}((X_0,X_t);Y)},\end{aligned}$$

and for $x \in \Sigma\bar{X}$ we get

$$K(t^{-1}, x, \bar{X}) = \inf_{x=x_0+x_1} \bigl(\|x_0\|_{X_0} + t^{-1}\|x_1\|_{X_1}\bigr) = \|x\|_{\Sigma(X_0,X_t)}.$$

Now, the statement follows from Proposition 2.6. □

A Banach space Y is said to have the metric extension property, if for any Banach space X and any subspace $X_0 \subseteq X$ each operator $T \in \mathcal{L}(X_0;Y)$ admits a norm preserving extension $\tilde{T} \in \mathcal{L}(X;Y)$. For example, all L_∞-spaces have the metric extension property.

Proposition 2.8. *Let $\bar{X} = (X_0, X_1)$ be any fully compatible pair. Let Y be any Banach space with the metric extension property. Then we have*

$$\Sigma\mathcal{L}(\bar{X};Y) = \mathcal{L}(\Delta\bar{X};Y) \quad \textit{and} \quad \|T:\Delta\bar{X} \to Y\| \leqslant \|T\|_{\Sigma\mathcal{L}(\bar{X};Y)} \leqslant 2\|T:\Delta\bar{X} \to Y\|.$$

Proof. To prove the first inequality, let $T = T_0 + T_1$ be any representation of $T \in \mathcal{L}(X_0;Y) + \mathcal{L}(X_1;Y) \subseteq \mathcal{L}(\Delta\bar{X};Y)$. Let $x \in \Delta\bar{X}$ be any point. Then we get

$$\begin{aligned}\|Tx\| &= \|T_0x + T_1x\| \leqslant \|T_0x\| + \|T_1x\| \\ &\leqslant \|T_0:X_0 \to Y\| \cdot \|x\|_{X_0} + \|T_1:X_1 \to Y\| \cdot \|x\|_{X_1} \\ &\leqslant \bigl(\|T_0\| + \|T_1\|\bigr) \cdot \max\bigl(\|x\|_{X_0}, \|x\|_{X_1}\bigr) = \bigl(\|T_0\| + \|T_1\|\bigr) \cdot \|x\|_{\Delta\bar{X}}.\end{aligned}$$

This implies $\|T:\Delta\bar{X}\to Y\|\leqslant\|T\|_{\Sigma\mathcal{L}(\bar{X};Y)}$. Conversely, let any $T:\Delta\bar{X}\to Y$ be given. We consider the normed subspace $E=\{x\oplus x\colon x\in\Delta\bar{X}\}$ of $X_0\oplus_\infty X_1$ and define a linear operator $S:E\to Y$ by $S(x\oplus x)=Tx$. Then we get

$$\big\|S(x\oplus x)\big\|=\|Tx\|\leqslant\|T:\Delta\bar{X}\to Y\|\cdot\|x\|_{\Delta\bar{X}}\leqslant\|T:\Delta\bar{X}\to Y\|\cdot\|x\oplus x\|_E.$$

This implies $\|S\|\leqslant\|T:\Delta\bar{X}\to Y\|$. Since Y has the metric extension property, there is an extension

$$\tilde{S}:X_0\oplus_\infty X_1\to Y$$

of S satisfying $\|\tilde{S}\|=\|S\|$. Let $\iota_j:X_j\to X_0\oplus_\infty X_1$ for $j=0,1$ denote the canonical embeddings. Putting

$$T_j=\tilde{S}\cdot\iota_j$$

we obtain $T_0x+T_1x=\tilde{S}(x\oplus 0)+\tilde{S}(0\oplus x)=\tilde{S}(x\oplus x)=Tx$ for $x\in\Delta\bar{X}$. This shows $\Sigma\mathcal{L}(\bar{X};Y)=\mathcal{L}(\Delta\bar{X};Y)$. Moreover, we have

$$\|T\|_{\Sigma\mathcal{L}(\bar{X};Y)}\leqslant\|T_0:X_0\to Y\|+\|T_1:X_1\to Y\|\leqslant 2\|\tilde{S}\|\leqslant 2\|T:\Delta\bar{X}\to Y\|,$$

and the proof is finished. □

Corollary 2.9. *Let $\bar{X}=(X_0,X_1)$ be any fully compatible pair and let Y be any Banach space with the metric extension property. Then we have*

$$K\big(t,T,\mathcal{L}(\bar{X};Y)\big)\leqslant 2\sup_{x\in\Delta\bar{X}}\frac{\|Tx\|}{J(t^{-1},x,\bar{X})}\leqslant 2K\big(t,T,\mathcal{L}(\bar{X};Y)\big)$$

for all $T\in\Sigma\mathcal{L}(\bar{X};Y)$ and all $t>0$.

Proof. For $t>0$ we define the Banach space $X_t=(X_1,\|\cdot\|_t)$ with $\|x\|_t=t^{-1}\|x\|_{X_1}$. For $T\in\Sigma\mathcal{L}(\bar{X};Y)$ we get

$$\begin{aligned}K\big(t,T,\mathcal{L}(\bar{X};Y)\big)&=\inf_{T=T_0+T_1}\big(\|T_0:X_0\to Y\|+t\|T_1:X_1\to Y\|\big)\\&=\inf_{T=T_0+T_1}\big(\|T_0:X_0\to Y\|+\|T_1:X_t\to Y\|\big)\\&=\|T\|_{\Sigma\mathcal{L}((X_0,X_t);Y)}.\end{aligned}$$

On the other hand, we have $\|x\|_{\Delta(X_0,X_t)}=\max(\|x\|_{X_0},t^{-1}\|x\|_{X_1})=J(t^{-1},x,\bar{X})$ for $x\in\Delta\bar{X}$. The statement of the corollary follows now from Proposition 2.8. □

Proposition 2.10. *Let $\bar{X}=(X_0,X_1)$ be any fully compatible pair and let Y be any Banach space. For any $0<\Theta<1$ and any $1\leqslant q<\infty$ we have*

$$\big(\mathcal{L}(X_0;Y),\mathcal{L}(X_1;Y)\big)_{J,\Theta,q'}\subseteq\mathcal{L}\big((X_0;X_1)_{K,\Theta,q};Y\big)$$

with

$$\big\|T:(X_0,X_1)_{K,\Theta,q}\to Y\big\|\leqslant\|T\|_{J,\Theta,q'}.$$

Proof. For $1 \leqslant p \leqslant \infty$ we denote the norm of the sequence space $\ell_p(\mathbb{Z})$ by $\|\cdot\|_p$. Let $T \in \Sigma\mathcal{L}(\bar{X}; Y)$ with $0 \neq \|T\|_{J,\Theta,q'} < \infty$ be given. For $\varepsilon > 0$ we choose a representation $T = \sum_{\nu\in\mathbb{Z}} T_\nu$ with $T_\nu \in \mathcal{L}(\Sigma\bar{X}; Y)$ and

$$\left\|\left(2^{-\nu\Theta} J\left(2^\nu, T_\nu, \mathcal{L}(\bar{X}; Y)\right)\right)\right\|_{q'} \leqslant (1+\varepsilon)\|T\|_{J,\Theta,q'}.$$

Let $x \in \Delta\bar{X}$ be fixed. Using Corollary 2.7 and Hölder's inequality we get

$$\begin{aligned}\|Tx\|_Y &\leqslant \sum_{\nu\in\mathbb{Z}} \|T_\nu x\| \leqslant \sum_{\nu\in\mathbb{Z}} K(2^{-\nu}, x, \bar{X}) \cdot J\left(2^\nu, T_\nu, \mathcal{L}(\bar{X}; Y)\right) \\ &= \sum_{\nu\in\mathbb{Z}} 2^{\Theta\nu} K(2^{-\nu}, x, \bar{X}) \cdot 2^{-\Theta\nu} J\left(2^\nu, T_\nu, \mathcal{L}(\bar{X}; Y)\right) \\ &\leqslant \left\|\left(2^{-\Theta\nu} K(2^\nu, x, \bar{X})\right)\right\|_q \cdot \left\|\left(2^{-\Theta\nu} J\left(2^\nu, T_\nu, \mathcal{L}(\bar{X}; Y)\right)\right)\right\|_{q'} \\ &\leqslant \|x\|_{K,\Theta,q} \cdot (1+\varepsilon)\|T\|_{J,\Theta,q'}.\end{aligned}$$

Since $\Delta\bar{X}$ is dense in $(X_0, X_1)_{K,\Theta,q}$ (cf. [3, Theorem 3.4.2]), the inequality

$$\|Tx\|_Y \leqslant \|x\|_{K,\Theta,q} \cdot (1+\varepsilon)\|T\|_{J,\Theta,q'}$$

holds true for all $x \in (X_0, X_1)_{K,\Theta,q}$, and this implies the claimed estimation. □

An inverse inclusion holds true at least for the case $q = 1$:

Proposition 2.11. *Let $\bar{X} = (X_0, X_1)$ be any fully compatible pair and let Y be any Banach space with the metric extension property. For any $0 < \Theta < 1$ we have*

$$\mathcal{L}\left((X_0, X_1)_{J,\Theta,1}; Y\right) \subseteq \left(\mathcal{L}(X_0; Y), \mathcal{L}(X_1; Y)\right)_{K,\Theta,\infty}$$

with

$$\|T\|_{K,\Theta,\infty} \leqslant 2\left\|T : (X_0, X_1)_{J,\Theta,1} \to Y\right\|.$$

Proof. We put $X = (X_0, X_1)_{J,\Theta,1}$. Let $T \in \mathcal{L}(X; Y)$ be given. We fix $\varepsilon > 0$. Proposition 2.8 implies $T \in \Sigma\mathcal{L}(\bar{X}; Y)$. By Corollary 2.9, for each $\nu \in \mathbb{Z}$ there is some $x \in \Delta\bar{X}$ such that

$$K\left(2^{-\nu}, T, \mathcal{L}(\bar{X}; Y)\right) \leqslant (2+\varepsilon)\frac{\|Tx\|}{J(2^\nu, x, \bar{X})} \leqslant (2+\varepsilon)\|T : X \to Y\|\frac{\|x\|_{J,\Theta,1}}{J(2^\nu, x, \bar{X})}.$$

Since $\|x\|_{J,\Theta,1} \leqslant 2^{-\Theta\nu} J(2^\nu, x)$ for $x \in \Delta\bar{X}$ and all $\nu \in \mathbb{Z}$, we get

$$2^{\Theta\nu} K\left(2^{-\nu}, T, \mathcal{L}(\bar{X}; Y)\right) \leqslant (2+\varepsilon)\|T : X \to Y\|$$

for all $\nu \in \mathbb{Z}$, and this implies $\|T\|_{K,\Theta,\infty} \leqslant (2+\varepsilon)\|T : X \to Y\|$. □

As a consequence we obtain from Theorem 2.3, Propositions 2.10 and 2.11 the following generalization of the duality theorem:

Theorem 2.12. *Let $\bar{X} = (X_0, X_1)$ be any fully compatible pair and let Y be any Banach space with the metric extension property. Then we have*

$$\mathcal{L}\big((X_0, X_1)_{\Theta,1}; Y\big) = \big(\mathcal{L}(X_0; Y), \mathcal{L}(X_1; Y)\big)_{\Theta,\infty}$$

with equivalent norms.

3. Ideals of operators on interpolation spaces

In this section we are going to study injective ideals on interpolation spaces. Let (Y_0, Y_1) be any compatible pair. Recall that a Banach space Y with $\Delta\bar{Y} \subseteq Y \subseteq \Sigma\bar{Y}$ is of the class $J(\Theta, Y_0, Y_1)$ for $0 < \Theta < 1$, if there is some constant C such that $\|y\|_Y \leqslant Ct^{-\Theta} J(t, y)$ holds true for all $t > 0$ and all $y \in \Delta\bar{Y}$. Obviously, the interpolation spaces $(Y_0, Y_1)_{J,\Theta,q}$ are of class $J(\Theta, Y_0, Y_1)$ for all $1 \leqslant q \leqslant \infty$ (see [3, Theorem 3.2.2]).

The starting point is now the following result due to Stephan Heinrich proved in [7] as Proposition 1.6:

Proposition 3.1. *Let $\mathcal{A}$ be any closed, injective operator ideal, let (Y_0, Y_1) be a compatible pair and let Y be of class $J(\Theta, Y_0, Y_1)$ for some $0 < \Theta < 1$. If $T \in \mathcal{A}(X; Y_0)$ and $T \in \mathcal{L}(X; Y_1)$ then $T \in \mathcal{A}(X; Y)$.*

In order to extend at least some version of this result to the multilinear case we will use the canonical embedding of a Banach space Y into a space with the metric extension property:

Definition 3.2. For any Banach space Y we define

$$Y^{\text{inj}} = \ell_\infty(B_{Y'}) \quad \text{and} \quad J_Y : Y \to Y^{\text{inj}} \quad \text{by } J_Y y = \langle y, \cdot\rangle.$$

The mapping J_Y is obviously an isometric embedding and Y^{inj} shares the metric extension property with all L_∞-spaces. It is easy to see that for all Banach spaces G and Y an isometric embedding $\mathcal{L}(G; Y) \to \mathcal{L}(G; Y^{\text{inj}})$ is given by $T \mapsto J_Y \cdot T$.

Proposition 3.3. *Let $\bar{X} = (X_0, X_1)$ be any fully compatible pair, let $E_2, \ldots, E_m$ and F be any Banach spaces, and let $X = (X_0, X_1)_{\Theta,1}$ for some $0 < \Theta < 1$ be given. If both,*

$$M \in [\mathcal{L}, \mathcal{A}_2, \ldots, \mathcal{A}_m](X_0, E_2, \ldots, E_m; F),$$
$$M \in [\mathcal{L}, \mathcal{L}, \ldots, \mathcal{L}](X_1, E_2, \ldots, E_m; F),$$

then

$$M \in [\mathcal{L}, \mathcal{A}_2, \ldots, \mathcal{A}_m](X, E_2, \ldots, E_m; F).$$

Proof. It is sufficient to prove

$$M^{(m)} : E_m \to \mathcal{L}(X, E_2, \ldots, E_{m-1}; F) \in \mathcal{A}_m.$$

For each Banach space Z we have a canonical isometry

$$\mathcal{L}(Z, E_2, \ldots, E_{m-1}; F) = \mathcal{L}\big(Z; \mathcal{L}(E_2, \ldots, E_{m-1}; F)\big).$$

For abbreviation we put $Y = \mathcal{L}(E_2, \ldots, E_{m-1}; F)$. Then the above formula reads as $\mathcal{L}(Z, E_2, \ldots, E_{m-1}; F) = \mathcal{L}(Z; Y)$. With this identification we get from the assumption

$$M^{(m)} \in \mathcal{L}\big(E_m; \mathcal{L}(X_1; Y)\big) \quad \text{and} \quad M^{(m)} \in \mathcal{A}_m\big(E_m; \mathcal{L}(X_0; Y)\big).$$

Using the isometric embedding $\mathcal{L}(G; Y) \subseteq \mathcal{L}(G; Y^{\mathrm{inj}})$ constructed above, we get

$$M^{(m)} \in \mathcal{L}\big(E_m; \mathcal{L}(X_1; Y^{\mathrm{inj}})\big) \quad \text{and} \quad M^{(m)} \in \mathcal{A}_m\big(E_m; \mathcal{L}(X_0; Y^{\mathrm{inj}})\big).$$

Now, the interpolation formula (Theorem 2.12) together with Proposition 3.1 gives us

$$M^{(m)} \in \mathcal{A}_m\big(E_m; \mathcal{L}(X; Y^{\mathrm{inj}})\big).$$

Since $\mathcal{A}_m$ is injective, this yields

$$M^{(m)} \in \mathcal{A}_m\big(E_m; \mathcal{L}(X; Y)\big) = \mathcal{A}_m\big(E_m; \mathcal{L}(X, E_2, \ldots, E_{m-1}; F)\big). \qquad \square$$

Theorem 3.4. *For all closed, injective operator ideals $\mathcal{A}_1, \ldots, \mathcal{A}_m$ the following factorization formula holds true*:

$$[\mathcal{A}_1, \ldots, \mathcal{A}_m] = \mathcal{L}(\mathcal{A}_1, \ldots, \mathcal{A}_m).$$

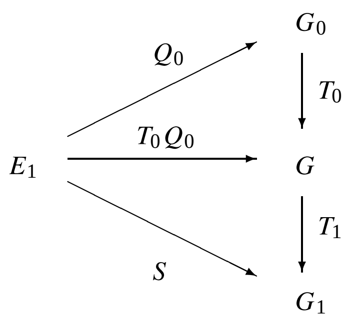
Proof. Let $M \in [\mathcal{A}_1, \ldots, \mathcal{A}_m](E_1, \ldots, E_m; F)$ be given. It is sufficient to construct a Banach space G, an operator $R \in \mathcal{A}_1(E_1; G)$ and a multilinear mapping

$$M_G \in [\mathcal{L}, \mathcal{A}_2, \ldots, \mathcal{A}_m](G, E_2, \ldots, E_m; F) \quad \text{satisfying } M = M_G(R, \mathrm{Id}, \ldots, \mathrm{Id}).$$

By assumption, we have

$$M^{(1)} \in \mathcal{A}_1\big(E_1; \mathcal{L}(E_2, \ldots, E_m; F)\big).$$

We put $Y = \mathcal{L}(E_2, \ldots, E_m; F)$. Let G_1 be the closure of $M^{(1)}(E_1)$ in Y and let $J: G_1 \to Y$ be the canonical injection. Then we have $M^{(1)} = J \cdot S$ for some $S: E_1 \to G_1$. Since $\mathcal{A}_1$ is supposed to be injective, we have $S \in \mathcal{A}_1(E_1; G_1)$. Let $G_0 = E_1/\ker S$ and let $Q_0: E_1 \to G_0$ be the canonical quotient map. Then we have a canonical linear continuous dense embedding $T: G_0 \to G_1$ such that $S = T \cdot Q_0$. With respect to this embedding, the pair (G_0, G_1) is fully compatible. Let $0 < \Theta < 1$ be fixed and put $G = (G_0, G_1)_{\Theta,1}$. Then we get the following commutative diagram with $T_1 T_0 = T$:



From Proposition 3.1 we conclude $T_0 Q_0 \in \mathcal{A}_1(E_1; G)$. Next, we define multilinear mappings M_0, M_G, and M_1 to make the following diagram commutative:

$$\begin{array}{ccc}
E_1 \times E_2 \times \cdots \times E_m & \xrightarrow{M} & F \\
Q_0 \downarrow \quad \mathrm{Id} \downarrow \quad \mathrm{Id} \downarrow & & \downarrow \mathrm{Id} \\
G_0 \times E_2 \times \cdots \times E_m & \xrightarrow{M_0} & F \\
T_0 \downarrow \quad \mathrm{Id} \downarrow \quad \mathrm{Id} \downarrow & & \downarrow \mathrm{Id} \\
G \times E_2 \times \cdots \times E_m & \xrightarrow{M_G} & F \\
T_1 \downarrow \quad \mathrm{Id} \downarrow \quad \mathrm{Id} \downarrow & & \downarrow \mathrm{Id} \\
G_1 \times E_2 \times \cdots \times E_m & \xrightarrow{M_1} & F
\end{array}$$

We start with the definition of M_1. To ensure the commutativity we should have

$$\begin{aligned} M_1(T_1 T_0 Q_0 x_1, \ldots, x_m) &= M(x_1, \ldots, x_m) = M^{(1)}(x_1)(x_2, \ldots, x_m) \\ &= S(x_1)(x_2, \ldots, x_m) = (T_1 T_0 Q_0 x_1)(x_2, \ldots, x_m), \end{aligned}$$

i.e., the first component applies to the remaining ones. Therefore, we define

$$M_1(L, x_2, \ldots, x_m) = L(x_2, \ldots, x_m) \quad \text{for } L \in G_1 \subseteq \mathcal{L}(E_2, \ldots, E_m; F).$$

Then M_1 is obviously a multilinear bounded mapping. Next, we define

$$M_G = M_1(T_1, \mathrm{Id}, \ldots, \mathrm{Id}) \quad \text{and} \quad M_0 = M_1(T_1 T_0, \mathrm{Id}, \ldots, \mathrm{Id}).$$

This gives us the factorization

$$M = M_G(T_0 Q_0, \mathrm{Id}, \ldots, \mathrm{Id}) \quad \text{with } T_0 Q_0 \in \mathcal{A}_1(E_1; G).$$

Finally, we show $M_G \in [\mathcal{L}, \mathcal{A}_2, \ldots, \mathcal{A}_m](G, E_2, \ldots, E_m; F)$. First, we define

$$J_0 : \mathcal{L}(G_0, E_2, \ldots, E_{m-1}; F) \to \mathcal{L}(E_1, E_2, \ldots, E_{m-1}; F)$$

by $J_0(U) = U(Q_0, \mathrm{Id}, \ldots, \mathrm{Id})$. Since Q_0 is a quotient map, J_0 comes out to be an isometric embedding. Now we have

$$J_0 \cdot M_0^{(m)} = M^{(m)} \in \mathcal{A}_m\bigl(E_m; \mathcal{L}(E_1, \ldots, E_{m-1}; F)\bigr),$$

and this implies $M_0^{(m)} \in \mathcal{A}_m(E_m; \mathcal{L}(G_0, E_2, \ldots, E_{m-1}; F))$ by the injectivity of $\mathcal{A}_m$. The same holds true for the other coordinates. This shows

$$M_0 \in [\mathcal{L}, \mathcal{A}_2, \ldots, \mathcal{A}_m](G_0, E_2, \ldots, E_m; F).$$

Since (G_0, G_1) is fully compatible by construction, we can apply Proposition 3.3. This finally shows $M_G \in [\mathcal{L}, \mathcal{A}_2, \dots, \mathcal{A}_m](G, E_2, \dots, E_m; F)$. $\square$

A large variety of closed and injective ideals can be found in [9] and in [7], and this shows that the theorem has a wide field of applications.

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