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Interpolation by Piecewise-Linear Radial Basis Functions, I

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In the two-dimensional plane, a set of points $x_1, x_2, ..., x_n$ (called "nodes") is given. It is desired to interpolate arbitrary data given on the nodes by continuous functions having piecewise-linear ("$\mathscr{PL}$") structure. For this purpose, one can employ the space of all $\mathscr{PL}$-functions on a rectangular grid generated by the nodes. We study this space first. Next, we investigate the special $\mathscr{PL}$-functions that are linear combinations of functions $h_i(x) = \|x - x_i\|_1$, in which the l_1-norm on $\mathbb{R}^2$ is employed. The "dual" case, involving the two-dimensional l_∞-norm, is included in our results, as are certain general interpolating functions of the form

$$(s, t) \mapsto F(s - s_i) + G(t - t_i).$$

1. INTRODUCTION

Throughout the paper, $\mathscr{N}$ denotes a set of n distinct points in $\mathbb{R}^2$ designated by $x_1, x_2, ..., x_n$. These points are called *nodes*. (In Section 9, we consider nodes in $\mathbb{R}^d$ for $d \geqslant 2$.) The basic problem of two-dimensional

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interpolation addressed here is as follows. A "data-function" $d\colon \mathcal{N} \to \mathbb{R}$ is given, and we seek a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f \mid \mathcal{N} = d$; i.e., $f(x_i) = d_i$ for $i = 1, 2, ..., n$. Such a function f is said to *interpolate* d. Usually the search for f is restricted to a class of functions that (a) are easily computed and (b) have some prescribed smoothness.

If the set of nodes has no special structure capable of being exploited, then this problem is called *scattered data interpolation*. Many methods proposed for this problem are discussed in the surveys of Schumaker [13] and Franke [4, 5]. One method that has been used successfully employs *radial basis functions*. In the simplest case of this, one seeks an interpolant in the linear space generated by the n functions $h_j(x) = \|x - x_j\|$ $(1 \leqslant j \leqslant n)$, where the norm can be any convenient one on $\mathbb{R}^2$. The existence of an interpolant $f = \sum_{j=1}^{n} c_j h_j$ for arbitrary data depends upon the invertibility of the *interpolation matrix* A, whose elements are $A_{ij} = h_j(x_i)$. For the Euclidean norm, a result of Schoënberg [11] asserts that the matrix A is *always* nonsingular. (Schoenberg's result holds in any inner-product space.) Micchelli [9] proved a striking generalization of this result in which h_j can be replaced by $h_j(x) = F(\|x - x_j\|^2)$, where F comes from a certain class of functions. The papers of Powell [10], Jackson [6], Madych and Nelson [7, 8], Dyn [1], and Dyn *et al.* [2] contain further important contributions to this field. Practical experience with this type of interpolation is reported by Franke [3, 4].

We consider radial basis functions that are generated by the l_1-norm. Thus, if $x = (s, t)$ and $x_i = (s_i, t_i)$, then

$$h_i(x) = \|x - x_i\|_1 = |s - s_i| + |t - t_i| \qquad (1 \leqslant i \leqslant n).$$

Since these functions are piecewise linear on a rectangular grid, we devote several sections (2–5) to a study of piecewise linear functions in general. Sections 6, 7, 9 concern the space of radial basis functions, and emphasize its role as a linear subspace in the space of piecewise linear functions. Section 8 is devoted to radial basis functions employing the l_∞-norm. Our results provide a geometric property of $\mathcal{N}$ that is necessary and sufficient for the invertibility of the interpolation matrix.

The following notation is adopted. Orthogonal *projections* onto the coordinate axes are denoted by P and Q. Explicitly,

$$Px = s, \qquad Qx = t, \qquad x = (s, t) \in \mathbb{R}^2.$$

The projections of $\mathcal{N}$ are denoted by

$$P(\mathcal{N}) = \{\sigma_1, \sigma_2, ..., \sigma_m\}, \qquad \sigma_1 < \sigma_2 < \cdots < \sigma_m$$
$$Q(\mathcal{N}) = \{\tau_1, \tau_2, ..., \tau_k\}, \qquad \tau_1 < \tau_2 < \cdots < \tau_k.$$

The *rectangular grid* and the *rectangular hull* determined by the node set $\mathcal{N}$ are the sets

$$\mathrm{RG} = \{\sigma_1, \sigma_2, ..., \sigma_m\} \times \{\tau_1, \tau_2, ..., \tau_k\}$$

$$\mathrm{RH} = \{(s, t) : \sigma_1 \leqslant s \leqslant \sigma_m \quad \text{and} \quad \tau_1 \leqslant t \leqslant \tau_k\}.$$

It is assumed always that $\#\mathcal{N} = n$ (i.e., the nodes $x_1, ..., x_n$ are distinct) and that $m \geqslant 2$, $k \geqslant 2$.

2. THE SPACE $\mathcal{PL}$ OF PIECEWISE LINEAR FUNCTIONS

The horizontal and vertical lines through the points of $\mathcal{N}$ divide the plane into rectangles, some of which are unbounded. These rectangles are expressible as Cartesian products of intervals. There are $m+1$ such intervals on the s-axis and there are $k+1$ intervals on the t-axis. The space $\mathcal{PL}(\mathcal{N})$, or simply $\mathcal{PL}$, is defined to be the space of all continuous functions $f: \mathbb{R}^2 \to \mathbb{R}$ such that the restriction of f to each of these rectangles is a linear function of (s, t).

The dimension of the space of all piecewise linear functions is obviously $3(m+1)(k+1)$ since there are $(m+1)(k+1)$ rectangles, and a linear function has three coefficients. On the other hand, if continuity *is* imposed, there will be three conditions required at each grid point to ensure that the linear functions in four adjacent rectangles are equal there. There are $3mk$ conditions of this type. Also, on each of the semi-infinite lines which emanate from RH one continuity condition must be imposed. This provides $2(m+k)$ further conditions. The number of parameters minus the number of conditions is $m+k+3$, and one can prove that this is indeed the dimension of $\mathcal{PL}$. To this end, we now define σ_0, τ_0, σ_{m+1}, and τ_{k+1} to be any real numbers satisfying

$$\sigma_0 < \sigma_1, \qquad \sigma_m < \sigma_{m+1}, \qquad \tau_0 < \tau_1, \qquad \tau_k < \tau_{k+1}.$$

2.1. THEOREM. *A $\mathcal{PL}$-function is uniquely determined by assigning to it arbitrary values at the points (σ_0, τ_j) and (σ_i, τ_0), where $0 \leqslant j \leqslant k+1$ and $1 \leqslant i \leqslant m+1$. Consequently,* $\dim(\mathcal{PL}) = m+k+3$.

Proof. Let $f \in \mathcal{PL}$. By 5.1, f can be written

$$f(s, t) = u(s) + v(t),$$

where u is a piecewise linear function in $C(\mathbb{R})$ having knots $\sigma_1, \sigma_2, ..., \sigma_m$, and v is a piecewise linear function in $C(\mathbb{R})$ having knots $\tau_1, \tau_2, ..., \tau_k$. By adding a constant to u and subtracting it from v we can arrange that $u(\sigma_0)=0$. Then v is uniquely determined by the equation

$$v(\tau_j)=f(\sigma_0,\tau_j), \qquad 0\leqslant j\leqslant k+1.$$

After that, u is uniquely specified by the equation

$$u(\sigma_i)=f(\sigma_i,\tau_0)-v(\tau_0),\ 1\leqslant i\leqslant m+1. \quad \blacksquare$$

Theorem 2.1 is already known but is included here for completeness. See Schumaker [15].

2.2. COROLLARY. *Let* $S=\{\sigma_0,\sigma_1,...,\sigma_{m+1}\}$ *and* $T=\{\tau_0,\tau_1,...,\tau_{k+1}\}$. *Let* $\Pi_0(S)$ *and* $\Pi_0(T)$ *denote the spaces of constant functions on* S *and* T, *respectively. The space* $\mathscr{PL}$, *when interpreted as a subspace of* $C([\sigma_0,\sigma_{m+1}]\times[\tau_0,\tau_{k+1}])$, *is isometrically isomorphic to* $l_\infty(S)\otimes\Pi_0(T)+\Pi_0(S)\otimes l_\infty(T)$ *as a subspace of* $l_\infty(S\times T)$.

2.3. THEOREM. *When* $\mathscr{PL}$ *is restricted to the rectangular hull,* $[\sigma_1,\sigma_m]\times[\tau_1,\tau_k]$, *its dimension is* $m+k-1$.

Proof. This follows at once from 2.1 upon changing k to $k-2$ and m to $m-2$. $\blacksquare$

2.4. THEOREM. *Every* $\mathscr{PL}$*-function can be written uniquely in the form*

$$f(s,t)=\sum_{i=1}^{m}\alpha_i\,|s-\sigma_i|+\sum_{i=1}^{k}\beta_i\,|t-\tau_i|+as+bt+c. \tag{1}$$

Proof. The function on the right in this equation is a linear combination of $m+k+3$ functions. By 2.1, dim $\mathscr{PL}=m+k+3$. Thus it suffices to prove that every $\mathscr{PL}$-function has a representation as claimed. Let f be any $\mathscr{PL}$-function. Then f can be written in the form

$$f(s,t)=u(s)+v(t) \tag{2}$$

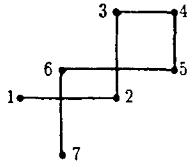
with u and v piecewise linear functions having knots $\sigma_1\cdots\sigma_m$ and $\tau_1\cdots\tau_k$, respectively. By a well-known theorem in spline theory, Schumaker [14], we can write

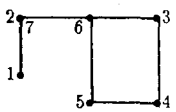
$$u(s)=\sum_{i=1}^{m}\alpha_i|s-\sigma_i|+as+c', \qquad v(t)=\sum_{i=1}^{k}\beta_i\,|t-\tau_i|+bt+c''. \quad \blacksquare$$

3. PATHS AND PATH FUNCTIONALS

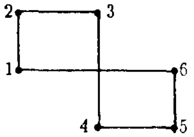
A *path* is a finite ordered set in RG, $[y_1, y_2, ..., y_r]$, such that the line segments joining consecutive points are of positive length and are alternately horizontal and vertical. (Repetitions of points are permitted.) The number r is the *length* of the path.

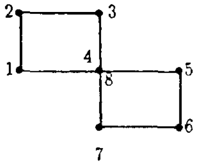
Pictures of paths are shown below.





A path is said to be *closed* if r is even, if $y_r \neq y_1$, and if the line segment joining y_1 with y_2 is perpendicular to the line segment joining y_r with y_1. Typical closed paths are shown below.





An equivalent formulation of the definition is this: the path $[y_1, y_2, ..., y_r]$ is closed if r is even and if $[y_1, y_2, ..., y_r, y_1]$ is also a path.

If $[y_1, y_2, ..., y_r]$ is a closed path, then a linear functional, called a *path functional*, is associated with it as follows:

$$\phi = \sum_{i=1}^{r} (-1)^i \hat{y}_i.$$

Here $\hat{y}$ denotes the point-evaluation functional associated with the point y; i.e., for any function f whose domain includes y,

$$\hat{y}(f) = f(y).$$

If r is 4, the path functional is called "the 4-point rule."

3.1. LEMMA. *Any path of length $n+1$ in $\mathscr{N}$ contains a closed path.*

Proof. Select a path of length $n+1$ in $\mathscr{N}$: $[z_0, z_1, ..., z_n]$. Since

$\#\mathcal{N}=n$, there is a first index i such that $z_i \in \{z_0, z_1, ..., z_{i-1}\}$. Select $j<i$ such that $z_i = z_j$. Now there are several cases.

If $t_j = t_{j+1}$ and $t_i = t_{i-1}$ then $[z_{j+1}, z_{j+2}, ..., z_{i-1}]$ is a closed path. To verify this, notice that the line segment from z_j to z_{j+1} is horizontal and the line segment from z_{j+1} to z_{j+2} must therefore be vertical. Next, since $t_{i-1} = t_i = t_j = t_{j+1}$, we see that the line segment from z_{i-1} to z_{j+1} is horizontal. The number of entries in the ordered set is even, by the following reasoning. The segments joining z_ν to $z_{\nu+1}$ in the path have vertical and horizontal orientations in the pattern

$$V\,H\,V\,H\,V\,H \cdots V\,H\,V.$$

Hence there is an odd number of segments and an even number of points $z_{j+1}, ..., z_{i-1}$.

If $t_j = t_{j+1}$ and $s_i = s_{i-1}$, a closed path is $[z_j, z_{j+1}, ..., z_{i-1}]$. Indeed the segment from z_j to z_{j+1} is horizontal, and the segment from z_{i-1} to $z_i = z_j$ is vertical. The number of entries in the ordered set is again even.

There are two remaining cases described by the conditions ($s_i = s_{j+1}$ and $s_i = s_{i-1}$) and ($s_j = s_{j+1}$ and $t_i = t_{i-1}$). These require no further proof since they follow from the first two cases upon interchanging s and t. ∎

3.2. Lemma. *Let A be a nonvoid subset of $\mathcal{N}$ such that $\#(A \cap L) \neq 1$ for any horizontal or vertical line, L. Then A contains a closed path.*

Proof. Assume the hypotheses, and select any point z_0 in A. The vertical line L through z_0 must satisfy $\#(A \cap L) \geqslant 2$, and one can select $z_1 \in A \cap L$ with $z_1 \neq z_0$. Similarly, one can select z_2 on the horizontal line through z_1, with $z_2 \neq z_1$. We continue in this way until we have a list of $n+1$ points $[z_0, z_1, ..., z_n]$. An application of 3.1 completes the proof. ∎

3.3. Lemma. *For a subset Z of the rectangular grid the following properties are equivalent*:

(1) *Z contains a closed path,*

(2) *There is a nontrivial functional supported on Z that annihilates $\mathscr{PL}$.*

Proof. If (2) is true, let $\phi \equiv \sum_{i=1}^{r} a_i \hat{z}_i$ annihilate $\mathscr{PL}$, where $z_i \in Z$ and all coefficients a_i are nonzero. Put $Z' = \{z_1, ..., z_r\}$. We observe that every horizontal and vertical line that intersects Z' does so in at least two points. Indeed, if (for example) the vertical line through z_j contains no other z_i, then an element f in $\mathscr{PL}$ can be constructed that is a function of s only and satisfies $f(z_i) = \delta_{ij}$. Then we have the contradiction $0 = \phi(f) = a_j$. An application of 3.2 establishes (1).

Now suppose that Z contains a closed path, $[z_1, ..., z_r]$. Then the functional $\phi \equiv \sum_{i=1}^{r} (-1)^i \hat{z}_i$ annihilates $C(S)+C(T)$, as is easily verified by considered functions of s and functions of t separately. By 2.4 $\mathscr{PL}$ is a subspace of $C(S)+C(T)$, and so is annihilated by ϕ. ▌

3.4. LEMMA. *Any subset of the grid that contains* $m+k$ *points must contain a closed path.*

Proof. Let A be such a set. Let S' be the set of all σ_i such that the vertical line through σ_i contains at least two points of A. If $\# S' = 0$, then each vertical grid line contains at most one point of A, and thus $\# A \leqslant m$, contrary to hypotheses. If $\# S' = 1$, then one vertical line contains v points of A, with $v \geqslant 2$. Each of the remaining $m-1$ vertical grid lines contains at most one point of A. Since $\# T = k$ we have the contradiction

$$m+k \leqslant \# A \leqslant m-1+v \leqslant m-1+k.$$

Thus we conclude that $\# S' \geqslant 2$. Similarly $\# T' \geqslant 2$, where T' is the set of τ_j such that the horizontal line through τ_j contains at least two points of A. By 3.2, the set $A \cap (S' \times T')$ contains a closed path. ▌

3.5. LEMMA. *Let* g *and* h *belong to* $C(\mathbb{R})$. *Let* $\mathscr{N}$ *be a set of nodes,* $x_i = (s_i, t_i)$, *in* $\mathbb{R}^2$. *Let*

$$H_i(s, t) = g(s-s_i) + h(t-t_i) \qquad (1 \leqslant i \leqslant n).$$

If $\mathscr{N}$ *contains a closed path, then the functions* H_i *form a dependent (indexed) set.*

Proof. Renumber the nodes, if necessary, so that $[x_1, ..., x_q]$ is a closed path. It is easily seen that the functional $\phi = \sum_{i=1}^{q} (-1)^i \hat{x}_i$ annihilates $C(S)+C(T)$. Hence it annihilates the function given by $(s, t) \mapsto g(\sigma - s) + h(\tau - t)$ where (σ, τ) is any fixed point in $\mathbb{R}^2$. Thus

$$\sum_{i=1}^{q} (-1)^i \{g(\sigma - s_i) + h(\tau - t_i)\} = 0$$

and therefore

$$\sum_{i=1}^{q} (-1)^i H_i(\sigma, \tau) = 0, \qquad (\sigma, \tau) \in \mathbb{R}^2. \quad ▌$$

3.6. LEMMA. *If a path* $[z_1, z_2, ..., z_r]$ *satisfies* $r \geqslant 4$ *and if either* $P(z_1) = P(z_r)$ *or* $Q(z_1) = Q(z_r)$, *then it contains a closed subpath.*

Proof. Select integers p and q in $\{1, 2, ..., r\}$ to minimize $q-p$ under the constraints

(1) $q-p \geqslant 3$,

(2) $[P(z_p)-P(z_q)][Q(z_p)-Q(z_q)]=0$.

We shall prove that $z_p \neq z_q$. Suppose that $z_p = z_q$. Then $q-p \geqslant 4$. Since (p, q) is minimal, $(p+1, q)$ does not satisfy the constraints. Since it satisfies (1), it does not satisfy (2). Hence $P(z_{p+1}) \neq P(z_q)$ and $Q(z_{p+1}) \neq Q(z_q)$. These inequalities violate the definition of a path, and we conclude that $z_p \neq z_q$.

Next we prove that $[z_p, z_{p+1}, ..., z_q]$ is a closed path. Note that $z_p \neq z_q$ by the preceding paragraph. With no loss of generality, we assume that the first factor in (2) is 0. By the minimality of (p, q), the segment joining z_p to z_{p+1} and the segment joining z_{q-1} to z_q are both horizontal. Hence the number of elements in the ordered set $[z_p, ..., z_q]$ is even. Finally, the segent from z_p to z_q is vertical and hence perpendicular to the segment from z_p to z_{p+1}. ∎

4. INTERPOLATION PROPERTIES OF $\mathscr{PL}$

The dimension of $\mathscr{PL}$ when restricted to the rectangular hull of $\mathscr{N}$ is $m+k-1$, as shown in 2.3. It is of interest to know what sets of $m+k-1$ grid points are suitable as nodes for interpolation. In this section we answer this question by exploiting the intuitive geometric idea of a path. Although one would normally expect to compute the coefficients in an interpolating function by inverting a linear system of order $m+k-1$, a much more economical algorithm is available.

4.1. THEOREM. *The space $\mathscr{PL}$ can interpolate arbitrary data on a set of $m+k-1$ grid points if and only if that set does not contain a closed path.*

Proof. Let Y be a set of $m+k-1$ grid points, $\{y_1, ..., y_{m+k-1}\}$. Let $\{f_j\colon 1 \leqslant j \leqslant m+k-1\}$ be a basis for $\mathscr{PL}$ restricted to RG. Interpolation is possible if and only if the matrix $(f_j(y_i))$ is nonsingular. If Y contains a closed path, then by 3.3, a nontrivial linear combination of the point functionals $\hat{y}_i$ annihilates $\mathscr{PL}$, and the matrix in question is singular. Conversely, if the matrix is singular then a linear combination of its rows is 0, and this gives an annihilating functional supported on Y. By 3.3, Y contains a closed path. ∎

Now let us assume that $\mathscr{N}$ contains no closed path and that $n \leqslant m+k-1$. We shall give an algorithm which produces a continuous $\mathscr{PL}$ interpolant for arbitrary data given on the node set $\mathscr{N} =$

$\{x_1, x_2, ..., x_n\}$. As in classical polynomial interpolation by the Lagrange method, it suffices to construct functions $l_1, ..., l_n$ in $\mathcal{PL}(\mathcal{N})$ with the "cardinal property"

$$l_i(x_j) = \delta_{ij} \qquad (1 \leqslant i, j \leqslant n).$$

For $i = 1, 2, ..., n$, define the set $\mathcal{H}_i$ to consist of x_i and all other nodes that can be connected to x_i by a path in $\mathcal{N}$ starting at x_i with a horizontal segment. Similarly, $\mathcal{V}_i$ contains x_i and all nodes that can be reached along a path in $\mathcal{N}$ starting with a vertical segment at x_i. Figure 4.1 shows a typical situation; the nodes labeled v belong to $\mathcal{V}_i$ and the nodes labeled h belong to $\mathcal{H}_i$,

Now define a_i on $P(\mathcal{N})$ and b_i on $Q(\mathcal{N})$ as follows:

$$a_i(s) = \begin{cases} +1 & \text{if } s \in P(\mathcal{V}_i) \\ -1 & \text{if } s \in P(\mathcal{H}_i \backslash x_i), \\ 0 & \text{otherwise} \end{cases} \qquad b_i(t) = \begin{cases} +1 & \text{if } t \in Q(\mathcal{H}_i) \\ -1 & \text{if } t \in Q(\mathcal{V}_i \backslash x_i). \\ 0 & \text{otherwise.} \end{cases}$$

Finally, define $l_i(s, t) = \frac{1}{2}[a_i(s) + b_i(t)]$.

4.2. THEOREM. *If $\mathcal{N}$ contains no closed path, then a_i and b_i are well defined, and $l_i(x_j) = \delta_{ij}$.*

Proof. Let $\sigma \in P(\mathcal{N})$, and suppose that $a_i(\sigma)$ is not well defined. Then $\sigma \in P(\mathcal{V}_i) \cap P(\mathcal{H}_i \backslash x_i)$. Hence there exist nodes x_j and x_v such that $x_v \in \mathcal{V}_i$, $x_j \in \mathcal{H}_i \backslash x_i$, and $P(x_v) = P(x_j) = \sigma$. (Under some circumstances, x_v can be x_i.) There exists a path, starting at x_v, progressing to x_i and then to x_j. By 3.6, such a path contains a closed subpath, contrary to hypotheses.

Now observe that $s_i = P(x_i) \in P(\mathcal{V}_i)$ and $t_i = Q(x_i) \in Q(\mathcal{H}_i)$. Consequently

$$l_i(x_i) = \tfrac{1}{2}[a_i(s_i) + b_i(t_i)] = \tfrac{1}{2}(1 + 1) = 1.$$

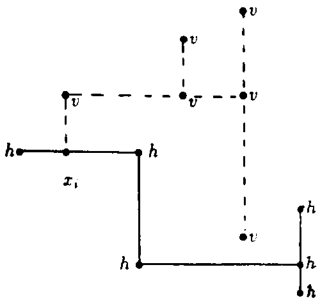


FIGURE 4.1

On the other hand, let $x = (s, t)$ be any node other than x_i. If there is a path from x_i to x, then either $x \in \mathscr{H}_i$ or $x \in \mathscr{V}_i$. In these two cases we have

$$\begin{aligned} 2l_i(x) &= a_i(s) + b_i(t) = -1 + 1 = 0, && x \in \mathscr{H}_i \\ 2l_i(x) &= a_i(s) + b_i(t) = 1 - 1 = 0, && x \in \mathscr{V}_i. \end{aligned}$$

If x is not connected by any path to x_i then $s \notin P(\mathscr{V}_i)$, $s \notin P(\mathscr{H}_i)$, $t \notin Q(\mathscr{H}_i)$, and $t \notin Q(\mathscr{V}_i)$. Consequently $l_i(x) = 0$. ▮

5. Degree of Approximation by $\mathscr{P}\mathscr{L}$-Interpolants

In the preceding section, a necessary and sufficient condition was given on a set of grid points in order that $\mathscr{P}\mathscr{L}$-interpolation at those points would be possible. The natural question arises of what happens when the grid is refined and the set of interpolation points is chosen to fill out the rectangular hull. It turns out that in general a continuous function on RH *cannot* be approximated to arbitrary precision by $\mathscr{P}\mathscr{L}$-interpolants. This is a corollary of a more general result to which we now turn.

In the next theorem we consider functions defined in a piecewise manner on Cartesian products. The setting will be as follows. There are two topological spaces given, S and T. Each is expressed as a union of nonempty sets

$$S = S_1 \cup \cdots \cup S_m, \qquad T = T_1 \cup \cdots \cup T_n.$$

We assume that $S_i \cap S_{i+1}$ and $T_j \cap T_{j+1}$ are nonempty for $1 \leqslant i \leqslant m-1$, $1 \leqslant j \leqslant n-1$. The notation $C(S)$ denotes the space of continuous real-valued functins on a topological space S. Subspaces are prescribed as follows

$$G_i \subset C(S_i)\ (1 \leqslant i \leqslant m), \qquad H_j \subset C(T_j)\ (1 \leqslant j \leqslant n).$$

It is assumed that these subspaces contain the constant functions. Three spaces of continuous piecewise-defined functions (or "generalized splines") are given as follows, with $|$ signifying restriction of a function to a subset of its domain:

$$\begin{aligned} G &= \{ g \in C(S) : g \mid S_i \in G_i, \text{ all } i \} \\ H &= \{ h \in C(T) : h \mid T_j \in H_j, \text{ all } j \} \\ K &= \{ f \in C(S \times T) : f \mid (S_i \times T_j) \in G_i + H_j, \text{ all } i \text{ and } j \}. \end{aligned}$$

5.1. Theorem. *The spaces G, H, and K defined above satisfy the equation $K = G + H$.*

Proof. It is clear that if $f(s, t) = g(s) + h(t)$, with $g \in G$ and $h \in H$, then $f \in K$, since

$$f \mid (S_i \times T_j) = g \mid S_i + h \mid T_j \in G_i + H_j.$$

Thus $G + H \subset K$.

Now let $f \in K$. We can find $u_{ij} \in G_i$ and $v_{ij} \in H_j$ so that

$$f \mid (S_i \times T_j) = u_{ij} + v_{ij} \qquad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n).$$

Suppose $j > 1$. Select $t_j \in T_j \cap T_{j-1}$. Then for all $s \in S_i$ we have $(s, t_j) \in (S_i \times T_j) \cap (S_i \times T_{j-1})$. Hence, for such a point,

$$u_{ij}(s) + v_{ij}(t_j) = u_{i,j-1}(s) + v_{i,j-1}(t_j).$$

This proves that $u_{ij}(s) - u_{i,j-1}(s)$ is a constant, c_{ij}, on S_i. Using this equation repeatedly, we obtain

$$u_{ij} = u_{i,j-1} + c_{ij} = u_{i,j-2} + c_{i,j-1} + c_{ij} = \cdots = u_{i1} + \alpha_{ij},$$

where α_{ij} is a constant. Because of the symmetry in the situation, we obtain a similar equaton for v_{ij}:

$$v_{ij} = v_{1j} + \beta_{ij}.$$

From a previous equation we have

$$u_{i1}(s) + \alpha_{ij} + v_{1j}(t_j) + \beta_{ij} = u_{i1}(s) + \alpha_{i,j-1} + v_{1,j-1}(t_j) + \beta_{i,j-1}.$$

Putting $\gamma_{ij} = \alpha_{ij} + \beta_{ij}$, we have

$$\gamma_{ij} - \gamma_{i,j-1} = v_{1,j-1}(t_j) - v_{1j}(t_j) = \delta_j.$$

Iterating this equation produces

$$\gamma_{ij} = \gamma_{i1} + \delta_j + \delta_{j-1} + \cdots + \delta_2 = \gamma_{i1} + d_j.$$

Thus on $S_i \times T_j$ we have

$$\begin{aligned} f(s, t) &= u_{i1}(s) + \alpha_{ij} + v_{1j}(t) + \beta_{ij} \\ &= u_{i1}(s) + v_{1j}(t) + \gamma_{ij} \\ &= [u_{i1}(s) + \gamma_{i1}] + [v_{1j}(t) + d_j]. \end{aligned}$$

The first bracketed expression can be denoted by $g_i(s)$, where $g_i \in G_i$. The second can be denoted by $h_j(t)$, where $h_j \in H_j$. Our analysis shows that

$$f(s, t) = g_i(s) + h_j(t), \qquad (s, t) \in S_i \times T_j.$$

If $s \in S_i \cap S_k$ then $g_i(s) = g_k(s)$ because for any $t \in T_j$

$$f(s, t) = g_i(s) + h_j(t) = g_k(s) + h_j(t).$$

This shows that there are unique functions g and h such that $g \mid S_i = g_i$ and $h \mid T_j = h_j$ for all i and j. Since $f(s, t) = g(s) + h(t)$, the functions g and h are continuous. ▌

5.2. COROLLARY. *Assume the hypotheses made at the beginning of this section. Then*

$$\{f \in C(S \times T) : f \mid (S_i \times T_j) \in C(S_i) + C(T_j), \textit{ all } i \textit{ and } j\} = C(S) + C(T).$$

Proof. One inclusion is trivial, and the other is a consequence of the preceding theorem, letting $G_i = C(S_i)$ and $H_j = C(T_j)$. ▌

5.3. THEOREM. *It is not possible to approximate with arbitrary precision all functions in $C(S \times T)$ by use of functions which are piecewise of the form $g(s) + h(t)$, no matter how finely we partition $S \times T$ into Cartesian-product subsets.*

Proof. The result follows from 5.2 and the fact that the subspace $C(S) + C(T)$ is not dense in $C(S \times T)$. Indeed, it is annihilated by every path functional. ▌

5.4. COROLLARY. *There exists a function f in $C(S \times T)$ such that* $\operatorname{dist}(f, \mathscr{PL}) \geqslant 1$ *for all grids.*

6. THE SPACE $\mathscr{RB}$

As in Section 1, a set of nodes

$$\mathscr{N} = \{x_1, x_2, ..., x_n\}$$

is given in $\mathbb{R}^2$, and we define *radial basis functions*

$$h_i(x) = \|x - x_i\|_1 = |s - s_i| + |t - t_i|.$$

The norm symbol is henceforth reserved for the l_1-norm on $\mathbb{R}^2$.

We denote by $\mathscr{RB}$ the linear space generated by the radial basis functions h_i $(1 \leqslant i \leqslant n)$. This section investigates the structure of $\mathscr{RB}$. Since interpolation by $h_1, ..., h_n$ at a set of n nodes certainly requires the linear independence of these functions, the first step is to characterize the sets $\mathscr{N}$ for which $\mathscr{RB}$ is of dimension n.

6.1. LEMMA. *Let Φ be a function in $C^1(\mathbb{R})$ that satisfies $\Phi'(s)>0$ for all s. Let $\mathcal{N}=\{x_1, ..., x_n\} \subset \mathbb{R}^2$, with $\#\mathcal{N}=n$ and $0 \notin \mathcal{N}$. If the equation*

$$\Phi(\|x\|_1)=\sum_{i=1}^{n} a_i \Phi(\|x-x_i\|_1) \tag{1}$$

is valid in a neighborhood of 0, *then* $0 \in P(\mathcal{N})$ *and* $0 \in Q(\mathcal{N})$.

Proof. Assume the hypotheses and deny the conclusion. With no loss of generality we suppose that $0 \notin P(\mathcal{N})$. Select $\varepsilon>0$ so that Eq. (1) is valid for $\|x\|<\varepsilon$. If necessary, reduce ε so that $(-\varepsilon, \varepsilon)$ contains no element of $P(\mathcal{N})$. If $|s|<\varepsilon$ then Eq. (1) is valid for $x=(s, 0)$, and thus

$$\Phi(|s|)=\sum_{i=1}^{n} a_i \Phi(|s-s_i|+|t_i|), \qquad |s|<\varepsilon.$$

The function on the right in this equation is differentiable at $s=0$, but the function on the left is not. This contradiction completes the proof. ∎

6.2. LEMMA. *Let Φ be a function in $C^1(\mathbb{R})$ that satisfies $\Phi'(s)>0$ and $\Phi(0)=0$. Then any set of three functions $H_i(s)=\Phi(\|x-x_i\|)$ is linearly independent on the corresponding set of three nodes. (In this result, any norm can be used.)*

Proof. The value of the 3×3 determinant $\det(H_i(x_j))$ is

$$2\Phi(\|x_1-x_2\|)\,\Phi(\|x_1-x_3\|)\,\Phi(\|x_2-x_3\|) \neq 0. \quad ∎$$

6.3. THEOREM. *Let $\mathcal{N}=\{x_1, ..., x_n\} \subset \mathbb{R}^2$, with $\#\mathcal{N}=n$. Let $\Phi \in C^1(\mathbb{R})$ and satisfy $\Phi'(s)>0$ and $\Phi(0)=0$. Put $H_i(x)=\Phi(\|x-x_i\|_1)$. In order that the indexed set $[H_1, ..., H_n]$ be linearly independent it is sufficient that $\mathcal{N}$ contain no closed path. If Φ is a linear function the condition is also necessary.*

Proof. Assume that the set is dependent. With no loss of generality we suppose that it has no proper linearly dependent subset. By 6.2, $n \geqslant 4$. Let $\sum_1^n c_i H_i=0$ with $\sum_1^n |c_i|>0$. Then $c_i \neq 0$ for all i, and each H_i is a linear combination of the others. By 6.1, each node x_i has the property that the vertical line and the horizontal line through x_i each contains another node. It follows from 3.2 that $\mathcal{N}$ contains a closed path.

To complete the proof, assume that Φ is linear and that $\mathcal{N}$ contains a closed path. If $\Phi(r)=ar+b$, then $H_i(x)=a|s-s_i|+a|t-t_i|+b$. The result now follows from 3.5. ∎

6.4. LEMMA. *Let $x_1, ..., x_n$ be n distinct points in a normed space. Let $f(x) = \sum_{i=1}^n a_i \|x - x_i\|$. In order that f be bounded, it is necessary and sufficient that $\sum_{i=1}^n a_i = 0$.*

Proof. Put $I = \{i : a_i > 0\}$, $J = \{i : a_i < 0\}$. For all x,

$$f(x) = \sum_I a_i \|x - x_i\| + \sum_J a_i \|x - x_i\|$$

$$\leqslant \sum_I a_i(\|x\| + \|x_i\|) + \sum_J a_i(\|x\| - \|x_i\|) = \|x\| \sum a_i + c,$$

where $c = \sum |a_i| \, \|x_i\|$. Similarly,

$$f(x) \geqslant \|x\| \sum a_i - c. \quad \blacksquare$$

7. INTERPOLATION BY $\mathscr{R}\mathscr{B}$

This section contains the central result of the paper. It is shown that the interpolation problem

$$\sum_{j=1}^n a_j \|x_i - x_j\|_1 = d_i \qquad (1 \leqslant i \leqslant n)$$

has a unique solution for every data function d if and only if the set of nodes $\{x_1, x_2, ..., x_n\}$ contains no closed path. The work of Micchelli [9] enables us to generalize this problem. We assume throughout this section that F is a function fulfilling five requirements:

(M1) $F: [0, \infty) \to [0, \infty)$,

(M2) F is C^∞ on $(0, \infty)$ and continuous at 0,

(M3) $F(t) > 0$ when $t > 0$,

(M4) F' is not constant,

(M5) $(-1)^\nu F^{(\nu+1)}(t) \geqslant 0$ for $\nu = 0, 1, 2, ...$, and $t > 0$.

Suppose that G is a function having the same properties. We consider interpolation at the nodes by a linear combination of these functions:

$$H_j(x) = F((s - s_j)^2) + G((t - t_j)^2) \qquad (1 \leqslant j \leqslant n).$$

In this equation, $x = (s, t)$ and $x_j = (s_j, t_j)$. The coefficient matrix A that arises in this more general interpolation problem is given by

$$A = B + C \qquad \text{where} \quad B_{ij} = F((s_i - s_j)^2) \text{ and } C_{ij} = G((t_i - t_j)^2).$$

If F and G are chosen to be the square-root function, then we recover the original problem. Micchelli [9] establishes the following important theorem.

7.1. THEOREM (Micchelli). *If F satisfies the five conditions* (M1–M5) *given above and if $r_1, ..., r_p$ are distinct reals, then the $p \times p$ matrix $D_{ij} = F((r_i - r_j)^2)$ is nonsingular. Also, $c^T Dc < 0$ for every nonzero vector c such that $\sum_{j=1}^{p} c_j = 0$.*

From Micchelli's theorem, we see immediately that the rank of B is m. Indeed, we can remove from B rows and columns which are duplicates of other rows and columns, arriving at an $m \times m$ matrix B' whose elements are $F((\sigma_i - \sigma_j)^2)$. This matrix is nonsingular, by Micchelli's theorem.

Our next task is to describe a basis for $\ker(B)$. If $1 \leqslant j < i \leqslant n$, then f^{ij} will denote a vector in $\mathbb{R}^n$ having 1 as its ith component, -1 as its jth component, and 0 components elsewhere. Thus, $f^{ij}_\mu = \delta_{i\mu} - \delta_{j\mu}$. Define also

$$J = \{(i, j): 1 \leqslant j < i \leqslant n, s_i = s_j, s_\mu \neq s_j \text{ if } \mu < j\}.$$

Notice that J is a function: different elements of J cannot have the same first component.

7.2. LEMMA. *A basis for* $\ker(B)$ *is* $\{f^{ij}: (i, j) \in J\}$.

Proof. First we establish that the purported basis is a subset of $\ker(B)$. If $(i, j) \in J$, then $Bf^{ij} = 0$ because

$$(Bf^{ij})_\nu = \sum_{\mu=1}^{n} B_{\nu\mu} f^{ij}_\mu = B_{\nu i} - B_{\nu j} = F((s_\nu - s_i)^2) - F((s_\nu - s_j)^2) = 0.$$

Next we prove that the purported basis is linearly independent. Suppose that $\sum \alpha_{ij} f^{ij} = 0$, the sum being over $(i, j) \in J$. Select any $(\nu, \mu) \in J$. We shall show that $\alpha_{\nu\mu} = 0$. This follows from the calculation

$$0 = \sum \alpha_{ij} f^{ij}_\nu = \alpha_{\nu\mu} f^{\nu\mu}_\nu = \alpha_{\nu\mu}.$$

To justify this, we only have to prove that if $f^{ij}_\nu \neq 0$ then $(i, j) = (\nu, \mu)$. If $f^{ij}_\nu \neq 0$, then either $\nu = i$ or $\nu = j$. If $\nu = i$, then $(i, j) \in J$ and $(i, \mu) \in J$. Since J is a function, $j = \mu$. If $\nu = j$ we have $(i, j) \in J$, $(j, \mu) \in J$, $\mu < j < i$, and $s_i = s_j = s_\mu$, which contradicts $(i, j) \in J$. (This case can therefore not arise.)

Lastly, we observe that J has the correct cardinality, namely $n - m$, which is the dimension of $\ker(B)$. This assertion follows from the equation

$$n = \#\mathcal{N} = \#P(\mathcal{N}) + \sum_{i=1}^{m} [\#P^{-1}(\sigma_i) - 1] = m + \#J. \quad \blacksquare$$

7.3. Lemma. *If* $u \in \ker(B)$ *then* $\sum_{i=1}^{n} u_i = 0$ *and*

$$\sum \{u_i : s_i = \sigma_j\} = 0 \qquad (1 \leqslant j \leqslant m).$$

Proof. The first equation follows from the second by summing for $1 \leqslant j \leqslant m$ (or it can be proved directly for the basis vectors in 7.2). In proving the second equation, it suffices to verify it for any one of the basis vectors in $\ker(B)$ as described above. To this end, fix $(\mu, \nu) \in J$ and $j \in \{1, ..., m\}$. By the definitions of J and $f^{\mu\nu}$,

$$\sum \{f_i^{\mu\nu} : s_i = \sigma_j\} = \sum \{\delta_{i\mu} - \delta_{i\nu} : s_i = \sigma_j\}.$$

This is obviously zero unless $s_\mu = \sigma_j$ or $s_\nu = \sigma_j$. But these equations imply each other, and if $s_\mu = \sigma_j = s_\nu$, the sum in question reduces to $1 - 1 = 0$. ▌

7.4. Lemma. *If* $v^T B v = \sum_{i=1}^{n} v_i = 0$, *then* $Bv = 0$.

Proof. For $\nu = 1, 2, ..., m$ put $I_\nu = \{i : 1 \leqslant i \leqslant n, s_i = \sigma_\nu\}$. The sets $I_1, ..., I_m$ form a partition of $\{1, 2, ..., n\}$. Hence any sum of the form $\sum_{i=1}^{n}$ can be expressed as a double sum $\sum_{\nu=1}^{m} \sum_{i \in I_\nu}$. We observe also that if $i \in I_\nu$ and $j \in I_\mu$ then

$$B_{ij} = F((s_i - s_j)^2) = F((\sigma_\nu - \sigma_\mu)^2) \equiv B'_{\nu\mu}.$$

Now assume the hypotheses and put $v'_\nu = \sum_{i \in I_\nu} v_i$. Applying the above principles we have

$$\begin{aligned} 0 = v^T B v &= \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} v_i v_j = \sum_{\nu=1}^{m} \sum_{i \in I_\nu} \sum_{\mu=1}^{m} \sum_{j \in I_\mu} B'_{\nu\mu} v_i v_j \\ &= \sum_{\nu=1}^{m} \sum_{\mu=1}^{m} B'_{\nu\mu} v'_\nu v'_\mu = (v')^T B' v'. \end{aligned}$$

Notice also that $\sum_{\nu=1}^{m} v'_\nu = \sum_{i=1}^{n} v_i = 0$. Since the points σ_ν are distinct, the matrix B' has the properties in Micchelli's theorem (7.1). Hence by 7.1, $v' = 0$. It follows that $Bv = 0$ by the calculation

$$\begin{aligned} (Bv)_i &= \sum_{j=1}^{n} B_{ij} v_j = \sum_{\mu=1}^{m} \sum_{j \in I_\mu} B_{ij} v_j = \sum_{\mu=1}^{m} \sum_{j \in I_\mu} F((s_i - \sigma_\mu)^2 v_j \\ &= \sum_{\mu=1}^{m} F((s_i - \sigma_\mu)^2) v'_\mu = 0. \end{aligned}$$ ▌

7.5. Lemma. *If* $u \in \ker(B)$, *then every vertical line that intersects the set* $\Gamma(u) = \{x_i \in \mathcal{N} : u_i \neq 0\}$ *contains at least two points of* $\Gamma(u)$.

Proof. Let $x_i \in \Gamma(u)$ so that $u_i \neq 0$. By 7.3

$$\sum \{u_j : s_j = s_i\} = 0.$$

Thus there must exist at least one index j, different from i, for which $u_j \neq 0$ and $s_j = s_i$. Then x_j is an element of $\Gamma(u)$ on the vertical line through x_i. ▌

7.6. LEMMA. *If* $\ker(B) \cap \ker(C) \neq 0$ *then* $\mathcal{N}$ *contains a closed path.*

Proof. Suppose that u is a nonzero vector in $\ker(B) \cap \ker(C)$. Then $\Gamma(u)$ is nonvoid. By 7.5, every vertical line that intersects $\Gamma(u)$ contains at least two points of $\Gamma(u)$. Applying the same lemmas to C shows that every horizontal line that intersects $\Gamma(u)$ contains two points of $\Gamma(u)$. By 3.2, $\Gamma(u)$ contains a path and, a fortiori, so does $\mathcal{N}$. ▌

7.7. THEOREM. *Let* $\mathcal{N}$ *be a set of* n *distinct points* $x_i = (s_i, t_i) \in \mathbb{R}^2$. *Let* F *and* G *be functions satisfying hypotheses* (M1)–(M5) *above. The* $n \times n$ *matrix* A *defined by*

$$A_{ij} = F((s_i - s_j)^2) + G((t_i - t_j)^2),$$

is singular if and only if $\mathcal{N}$ *contains a closed path.*

Proof. If $\mathcal{N}$ contains a closed path, then the functions

$$g_i(s, t) = F((s - s_i)^2) + G((t - t_i)^2),$$

form a dependent set (by 3.5), and thus A is singular.

Now let v be any vector such that $v \neq 0$ and $v^T e = 0$, where $e = (1, 1, ..., 1)^T$. If the points $s_1, s_2, ..., s_n$ were distinct, then 7.1 would imply $v^T B v < 0$. Since the points s_i are not necessarily distinct, a limit argument yields $v^T B v \leqslant 0$. Similarly $v^T C v \leqslant 0$. Hence

$$v^T A v = v^T B v + v^T C v \leqslant 0.$$

Since A is symmetric, its eigenvalues are real and can be ordered $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. Since $e^T A e > 0$, it follows from the Courant–Fischer ("minimax") theorem that $\lambda_n > 0$. Using this result again, we have

$$\lambda_{n-1} = \min_{\dim V = n-1} \max_{\substack{v \in V \\ \|v\| = 1}} v^T A v \leqslant \max_{\substack{v^T e = 0 \\ \|v\| = 1}} v^T A v \leqslant 0.$$

Now assume that A is singular. Then $\lambda_{n-1} = 0$, and hence there is a vector v satisfying $\|v\| = 1$, $v^T e = 0$, $v^T A v = 0$. It follows that $v^T B v = v^T C v = 0$. By 7.4, $v \in \ker(B) \cap \ker(C)$, and by 7.6, $\mathcal{N}$ contains a closed path. ▌

7.8. COROLLARY. *Let $\mathcal{N}$ be a set of n distinct points $x_i = (s_i, t_i)$ in $\mathbb{R}^2$. Let $0 < \alpha < 2$. The $n \times n$ matrix A given by*

$$A_{ij} = |s_i - s_j|^\alpha + |t_i - t_j|^\alpha$$

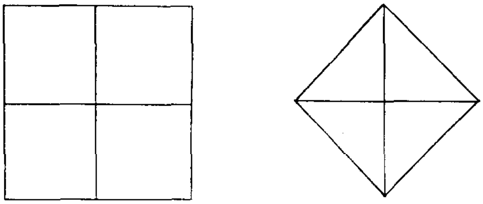
is singular if and only if $\mathcal{N}$ contains a closed path.

Proof. In 7.7, let $F(u) = G(u) = u^{\alpha/2}$. ∎

The results of this section are extended in [3] *to interpolation by sums of radial functions.*

8. RADIAL BASIS FUNCTIONS WITH THE MAXIMUM NORM

All of what has been proved for radial basis functions with the l_1-norm can be proved, *mutatis mutandis*, for the l_∞-norm. This assertion depends upon the isometry between $l_1^{(2)}$ and $l_\infty^{(2)}$ that must exist because of the similarity in the unit spheres in these two spaces. (See the figure.)

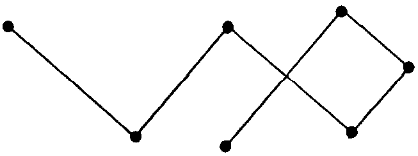


The isometry from $l_1^{(2)}$ to $l_\infty^{(2)}$ is given by $(s, t) \rightarrow (s+t, s-t)$.

Given $\mathcal{N} = \{x_1, x_2, ..., x_n\} \subset \mathbb{R}^2$, our basis functions are now

$$h_j(x) = \|x - x_j\|_\infty = \max[|s - s_j|, |t - t_j|].$$

The notion of a path must now be modified; we refer to the new concept as an l_∞-path. It is an ordered set of points $[z_1, z_2, ..., z_l]$ such that the line segments joining successive points are of positive length and have inclinations alternately 45° and 135°.



The grid lines generated by $\mathcal{N}$ consist now of lines with inclinations 45° and 135° through points of $\mathcal{N}$.

8.1. THEOREM. *Let $\mathcal{N}$ be a set of n points, x_i, in $\mathbb{R}^2$. The functions*

$x \mapsto \|x - x_i\|_\infty$ $(1 \leqslant i \leqslant n)$ are capable of interpolating arbitrary data on $\mathcal{N}$ if and only if $\mathcal{N}$ contains no closed l_∞-path.

The preceding considerations provide an example of the following general principle. If functions $f_1, ..., f_n$ are capable of interpolating arbitrary data at nodes $x_1, ..., x_m$, and if L is a nonsingular linear transformation, then the functions $f_1 \circ L^{-1}, ..., f_n \circ L^{-1}$ are capable of interpolating arbitrary data at nodes $Lx_1, ..., Lx_m$.

9. GENERALIZATIONS TO HIGHER-DIMENSIONAL SPACES

The basic interpolation results of Section 7 can be generalized to the space $\mathbb{R}^d$, $d \geqslant 2$. To describe the results, a somewhat different formalism from that used in the previous sections is needed.

If x is a point in $\mathbb{R}^d$, we write $x = (\xi_1, \xi_2, ..., \xi_d)$. Coordinate functionals p_i are defined by setting $p_i(x) = \xi_i$. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is said to be *degenerate* if it has the form $f = \sum_i^d g_i \circ p_i$ for suitable $g_i\colon \mathbb{R} \to \mathbb{R}$. Such a function f is a sum of univariate functions. The space of all degenerate functions on $\mathbb{R}^d$ is denoted by $\mathcal{D}$.

If f is a function on a finite set $S = \{s_1, s_2, ..., s_n\}$, we write

$$\sum [f(s) : s \in S] = f(s_1) + f(s_2) + \cdots + f(s_n).$$

Note that if f is not injective, the sum will contain repeated terms. That is why we eschew the notation $\sum \{f(s) : s \in S\}$, which—if strictly interpreted—means a sum without repetitions.

9.1. LEMMA. *In order that a functional of the form $\phi = \sum_1^N c_i \hat{y}_i$ annihilate $\mathcal{D}$ it is necessary and sufficient that for all $t \in \mathbb{R}$ and for all $v \in \{1, 2, ..., d\}$,*

$$\sum [c_i : p_v(y_i) = t] = 0.$$

Proof. Let $g\colon \mathbb{R} \to \mathbb{R}$, and fix v. Let

$$A = \{p_v(y_i) : 1 \leqslant i \leqslant N\}.$$

We have then

$$\phi(g \circ p_v) = \sum_{i=1}^{N} c_i g(p_v(y_i)) = \sum_{t \in A} \sum [c_i g(p_v(y_i)) : p_v(y_i) = t]$$

$$= \sum_{t \in A} g(t) \sum [c_i : p_v(y_i) = t].$$

The sufficiency of the given condition is now clear. For the necessity, select $t \in A$, and construct g so that $g(t) = 1$ and $g(s) = 0$ for all $s \in A \setminus \{t\}$. The preceding calculation gives us

$$0 = \phi(g \circ p_\nu) = \sum [c_i : p_\nu(y_i) = t]. \quad \blacksquare$$

If $v_1, v_2, ..., v_n$ are elements in a vector space, we adopt the usual meaning for linear dependence of the *indexed* set $[v_1, v_2, ..., v_n]$. It can happen that the *unindexed* set $\{v_1, v_2, ..., v_n\}$ is linearly independent while $[v_1, v_2, ..., v_n]$ is linearly dependent.

9.2. LEMMA. *The following properties of a set* $\mathcal{N} = \{x_1, x_2, ..., x_n\}$ *in* $\mathbb{R}^d$ *are equivalent*:

(a) *There is a nonzero functional in* $\mathcal{D}^\perp$ *that is supported on* $\mathcal{N}$.

(b) *For every* $f \in \mathcal{D}$, *the indexed set of translates* $x \mapsto f(x - x_i)$ *is linearly dependent.*

Proof. Let E_u denote the translation operator, defined by $(E_u F)(x) = F(x-u)$. Define an operator B by putting $(BF)(x) = F(-x)$. Let $\phi = \sum_{i=1}^n c_i \hat{x}_i$, and assume that $\phi \in \mathcal{D}^\perp$. If $f \in \mathcal{D}$, then $E_u B f \in \mathcal{D}$, and consequently

$$0 = \phi(E_u Bf) = \sum c_i (E_u Bf)(x_i) = \sum c_i f(u - x_i) = \sum c_i (E_{x_i} f)(u).$$

This proves that (a) implies (b). Observe that the proof requires of $\mathcal{D}$ only its invariance under the operators B and E_u. For the other half of the proof, assume (b). Let $f \in \mathcal{D}$. Then $Bf \in \mathcal{D}$, and by (b) there exist coefficients c_i, not all zero, such that $\sum c_i E_{x_i} Bf = 0$. Evaluating at 0, we have

$$0 = \sum c_i (E_{x_i} Bf)(0) = \sum c_i f(x_i) = \sum c_i \hat{x}_i(f). \quad \blacksquare$$

Now select functions $F_1, F_2, ..., F_d$ satisfying the five axioms (M1)–(M5) of Section 7. Define

$$H(x) = \sum_{\nu=1}^{d} F_\nu((p_\nu(x)^2), \qquad x \in \mathbb{R}^d.$$

As before a set of nodes is given: $\mathcal{N} = \{x_1, x_2, ..., x_n\}$, with $x_i \in \mathbb{R}^d$. Interpolation at the nodes by the x_i-translates of H requires the nonsingularity of the interpolation matrix A given by

$$A_{ij} = H(x_i - x_j) \qquad (1 \leqslant i, j \leqslant n).$$

It is clear from the definition of H that A is the sum of matrices $A^{(\nu)}$ given by

$$A_{ij}^{(\nu)} = F_\nu[(p_\nu(x_i) - p_\nu(x_j))^2 \qquad (1 \leqslant \nu \leqslant d).$$

9.3. Lemma. *If the matrix A is singular, then there exists a vector $u \in \mathbb{R}^n$ such that $u \neq 0$, $u^T e = 0$, and $A^{(\nu)} u = 0$ for $1 \leqslant \nu \leqslant d$.*

Proof. Proceed exactly as in the proof of 7.7, obtaining thereby a vector u having the desired properties. ▌

9.4. Theorem. *The following are equivalent properties of the node set $\mathscr{N}$:*

(a) *The interpolation matrix A is singular;*

(b) *There is a linear dependence among the n basis functions $x \mapsto H(x - x_i)$.*

Proof. That (b) implies (a) is obvious. Assume that (a) is true. By the preceding lemma, there is a nonzero vector u such that $A^{(\nu)} u = 0$ for $1 \leqslant \nu \leqslant d$. By 7.3, we have

$$\sum [u_i : p_\nu(x_i) = t] = 0 \qquad (t \in \mathbb{R},\ 1 \leqslant \nu \leqslant d).$$

By Lemma 9.1, the functional $\sum u_i \hat{x}_i$ annihilates $\mathscr{D}$. By Lemma 9.2, the set of functions $E_{x_i} H$ is linearly dependent. ▌

The geometrical characteristics of $\mathscr{N}$ that are equivalent to Properties (a) and (b) in 9.4 will be explored in the second half of this paper.

Notice that if $d > 2$, the theory of radial basis functions using the l_∞-norm is not a simple consequence of the theory in the case of the l_1-norm. This is because there is no isometric isomorphism between the spaces $\mathbb{R}^d$ when these two norms are used. For example, the unit balls in $\mathbb{R}^3$ are a cube and an octahedron for these two norms.

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