The extremal zeros of a perturbed orthogonal polynomials system

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Abstract

The purpose of this paper is to study how the extremal zeros of a family of orthogonal polynomials evolve when we perturb the coefficient of the recurrence relation defining the family. To this end we shall compare the extremal zeros with the corresponding zeros in the perturbed case. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let \( \{P_n\}_{n \geq 0} \) a family of orthogonal polynomials be defined by

\[
\begin{align*}
P_{-1} & \equiv 0, \quad P_0 \equiv 1, \\
0 \leq n \leq N - 1, \quad P_{n+1}(x) = (x - b_n)P_n(x) - a_nP_{n-1}(x),
\end{align*}
\]

where

\[
b_n \in \mathbb{R}; \quad a_0 \equiv 0, \quad 0 < n < N, \quad a_n > 0.
\]

This case is not restrictive because in the general case if the family is defined by

\[
\begin{align*}
Q_{-1} & \equiv 0, \quad Q_0 \equiv q_0 \neq 0, \\
0 \leq n \leq N - 1, \quad Q_{n+1}(x) = (A_n x + B_n)Q_n(x) - D_nQ_{n-1}(x).
\end{align*}
\]

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Then we can compute $a_n$ and $b_n$ in order to obtain the following relation:

$$0 \leq n \leq N - 1 \quad Q_{n+1}(x) = \left( q_0 \prod_{i=0}^{n} A_i \right) P_{n+1}(x).$$

**Definition 1.1.** We call the extremal zeros of the family (1.1) the smallest and the greatest zeros of this family.

It is well known that these zeros are the extremal zeros of the polynomial $P_N$.

**Definition 1.2.** We call the perturbed family of the family (1.1) the new system of orthogonal polynomials $\{P_n^*\}_{n \geq 0}$ defined by

$$P_{-1}^* = 0, \quad P_0^* = 1,$$

$$0 \leq n \leq N - 1: \quad P_{n+1}^*(x) = (x - b_n^*)P_n^*(x) - a_n^*P_{n-1}^*(x), \quad (1.2)$$

where

$$b_n^* = (b_n + \beta_n) \in \mathbb{R}; \quad a_n^* \equiv 0, \quad 0 < n < N, \quad \alpha_n \in \mathbb{R}; \quad a_n^* = (a_n + \alpha_n) > 0. \quad (1.3)$$

In this case, we call $a_n^*$ and $b_n^*$ $0 \leq n \leq N - 1$ the perturbed coefficients and $\alpha_n$ and $\beta_n$ the perturbations.

Let

$$E, F, \quad (1.4)$$

be the extremal zeros of the family (1.1) (respectively the smallest and the greatest zeros of the polynomial $P_N$) and

$$E^*, F^* \quad (1.5)$$

those of the family (1.2).

Our problem is to compare $E$ with $E^*$ and $F$ with $F^*$ according to the perturbations $(\alpha_n)_{n \geq 0}$ and $(\beta_n)_{n \geq 0}$.

We shall also give the bounds on $E - E^*$ and $F - F^*$ in the general case.

Many authors have studied the problem of perturbations of (1.1) in other contexts, see for example [4–6].

In the case of zeros, see [1–3] for the location and for the perturbation of zeros, some other results can be found in [7, 8].

2. Some tools

**Definition 2.1.** Formally put, for $N \geq 2$

$$0 \leq n \leq N - 2: \quad c_n = c_n(d_n, d_{n+1}) = d_n + \frac{a_{n+1}}{d_{n+1} - b_{n+1}}, \quad (2.1)$$
\[ f_0(x) \equiv 1, \ 1 \leq n \leq N - 1: \quad f_n(x) = x - b_{n-1} - \frac{a_{n-1}}{f_{n-1}(x)}, \quad (2.2) \]
\[ g_N(x) \equiv 0, \ N - 1 \geq n \geq 1: \quad g_n(x) = -\frac{a_n}{x - b_n + g_{n+1}(x)}, \quad (2.3) \]

In order to avoid repeating cumbersome notations, we shall adopt the following definition.

**Definition 2.2.** Let

\[ J_N(\text{symb}_1, \text{symb}_2, t) = \{ d_n \in R \mid d_0 = b_0, \ 0 < n < N \ \text{symb}_1 b_n \]

and \((\beta_{n-1} - t)d_n \ \text{symb}_2 (\beta_{n-1} - t)b_n - \alpha_n\}, \quad (2.4)\]

\[ J_N^*(\text{symb}_1, \text{symb}_2, t) = \{ d_n \in R \mid d_0 = b_0^*, \ 0 < n < N \ \text{symb}_1 b_n^* \]

and \((\beta_{n-1} - t)d_n \ \text{symb}_2 (\beta_{n-1} - t)b_n^* - \alpha_n\}, \quad (2.5)\]

\[ K_N(\text{symb}_1, \text{symb}_2) = \{ d_n \in R \mid d_0 = b_0, \ 0 < n < N \ \text{symb}_1 b_n \]

and \(\alpha_n d_n \ \text{symb}_2 \alpha_n b_n - a_n \beta_n\}, \quad (2.6)\]

\[ K_N^*(\text{symb}_1, \text{symb}_2) = \{ d_n \in R \mid d_0 = b_0^*, \ 0 < n < N \ \text{symb}_1 b_n^* \]

and \(\alpha_n d_n \ \text{symb}_2 \alpha_n b_n^* - a_n^* \beta_n\}, \quad (2.7)\]

\[ L_N(t) = \left\{d_n \in R \mid d_0 = b_0, \ 0 < n < N \ b_n + \frac{a_n}{f_n(t)} \leq d_n \leq b_n - \frac{a_n}{g_n(t)} \right\} \quad (2.8)\]

and

\[ l_N(t) = \left\{d_n \in R \mid d_0 = b_0, \ 0 < n < N \ b_n - \frac{a_n}{g_n(t)} \leq d_n \leq b_n + \frac{a_n}{f_n(t)} \right\}. \quad (2.9)\]

For example, if in the set \(J_N(\text{symb}_1, \text{symb}_2, t)\), we replace the \(\text{symb}_1\) by \(>\), the \(\text{symb}_2\) by \(\leq\) and \(t\) by 0 then we obtain the corresponding set

\[ J_N(>, \leq, 0) = \{d_n \in R \mid d_0 = b_0, \ 0 < n < N \ b_n \\

\beta_{n-1}d_n \leq \beta_{n-1}b_n - \alpha_n\}. \]

**Remark.** According to (1.3), we can also consider that the family \(\{P_n\}_{n \geq 0}\) is the perturbed family of the family \(\{P_{n+1}\}_{n \geq 0}\). In this case we put

\[ 0 \leq n < N: \quad a_n^{**} = a_n = a_n + (-\alpha_n) \quad \text{and} \quad b_n^{**} = b_n = b_n + (-\beta_n). \quad (2.10)\]

**Lemma 2.1** (Gilewicz and Leopold [1]). The smallest zero \(E\) and the greatest zero \(F\) of the family \(\{P_n\}_{n \geq 0}\) defined by (1.1) are given by

\[ E = \max_{d_0 \leq b_0, \ 0 < n < N: \ d_n \leq b_n} \min_{0 \leq n \leq N - 2} (c_n, d_{N-1}), \quad (2.11)\]
We recall that the functions $c_n$ are defined by (2.1).

We shall adopt the simplified notation $c_n$ instead of $c_n(d_n, d_{n+1})$.

**Lemma 2.2** (Leopold [2]). The smallest zero $E$ and the greatest zero $F$ of the family $\{P_n\}_{n \geq 0}$ are also given by

\[
E = \max_{\gamma \in \gamma_N} \min_{0 \leq n \leq N-2} (c_n, d_{N-1}),
\]  
(2.13)

\[
F = \min_{\Gamma \in \Gamma_N} \max_{0 \leq n \leq N-2} (c_n, d_{N-1}).
\]
(2.14)

For all $\gamma$, $\Gamma$ satisfying

\[
g \leq E, \quad \Gamma \geq F.
\]  
(2.15)

The sets $\gamma_N$ and $\Gamma_N$ are given by (2.8) and (2.9).

**Remarks.** A well-known theorem of Hadamard–Gershgorin applied to Jacobie’s representation of family (1.1) allows one to obtain the estimates given by (2.16). Also, we can take for example:

\[
(a_0 = a_N \equiv 0), \quad \gamma = \min_{0 \leq n < N} (b_n - \sqrt{a_n} - \sqrt{a_{n+1}})
\]  
(2.16)

and

\[
\Gamma = \max_{0 \leq n < N} (b_n + \sqrt{a_n} + \sqrt{a_{n+1}}).
\]

Lemma 2.2 is more precise than Lemma 2.1. But, we shall use the two versions in the following.

**Lemma 2.3** (Leopold [2]). The functions $f_n$ and $g_n$ defined by (2.2) and (2.3) satisfy the following for $0 < n < N$:

(i) $\forall x \leq E, \forall y \leq E$ and $x < y$,

\[
x - b_{n-1} \leq f_n(x) < f_n(y) \leq f_n(E) = -g_n(E) \leq -g_n(y) < -g_n(x) \leq \frac{a_n}{x - b_n} < 0
\]  
(2.17)

(ii) $\forall x \geq F, \forall y \geq F$ and $y < x$,

\[
0 < \frac{a_n}{x - b_n} \leq -g_n(x) < -g_n(y) \leq -g_n(F) = f_n(F) \leq f_n(y) < f_n(x) \leq x - b_{n-1}.
\]  
(2.18)

**Lemma 2.4.** The bounds $E^*$ and $F^*$ defined by (1.5) associated to the family $\{P_n^*\}_{n \geq 0}$ given by (1.2) satisfy

\[
E^* = \max_{d_0 \leq d_0^*, 0 < s < N; d_s^* < d_s^*} \min_{0 \leq n \leq N-2} (c_n^*, d_{N-1}^*),
\]  
(2.19)
\[ E^* = \max_{d_0 \leq b_0, \ 0 < n < N: d_n < b_n, \ 0 \leq n \leq N-2} \min (c_n + \theta_{n+1}, d_{N-1} + \beta_{N-1}) \]

\[ E^* = \max_{d_0 \leq b_0, \ 0 < n < N: d_n < b_n, \ 0 \leq n \leq N-2} \min (c_n + \theta_{n+1} - \beta_{N-1}, d_{N-1}) + \beta_{N-1} \]

and

\[ F^* = \min_{d_0 \geq b^*_0, \ 0 < n < N: d_n > b_n, \ 0 \leq n \leq N-2} \max (c^*_n, d_{N-1}) \]

\[ F^* = \min_{d_0 \geq b_0, \ 0 < n < N: d_n > b_n, \ 0 \leq n \leq N-2} \max (c_n + \theta_{n+1}, d_{N-1} + \beta_{N-1}) \]

\[ F^* = \min_{d_0 \geq b_0, \ 0 < n < N: d_n > b_n, \ 0 \leq n \leq N-2} \max (c_n + \theta_{n+1} - \beta_{N-1}, d_{N-1}) + \beta_{N-1} \]

For \( 0 \leq n \leq N - 2 \), \( c_n \) are defined by (2.1), \( c^*_n \) the corresponding function of \( c_n \) with \( a^*_n, b^*_n \) and the functions \( \theta_{n+1} \) by

\[ \theta_{n+1} = \frac{\alpha_{n+1}}{d_{n+1} - b_{n+1}} + \beta_n. \]

**Proof.** Formulae (2.19) and (2.22) correspond to cases (2.11) and (2.12) where we have replaced the coefficients \( a_n \) and \( b_n \) by \( a^*_n \) and \( b^*_n \) according to (1.3). In (2.19) according to the remark on the function \( c^*_n \) we have

\[ d_0 \leq b^*_0 = b_0 + \beta_0, \ 0 < n < N: \ d_n < b^*_n = b_n + \beta_n, \]

\[ 0 \leq n \leq N - 2: \ c^*_n = d_n + \frac{a^*_{n+1}}{d_{n+1} - b^*_{n+1}} = d_n - \beta_n + \beta_n + \frac{a_{n+1}}{d_{n+1} - \beta_{n+1} - \beta_n} + \frac{\alpha_{n+1}}{d_{n+1} - \beta_{n+1} - \beta_n}. \]

We can also rewrite \( d_{N-1} = d_{N-1} - \beta_{N-1} + \beta_{N-1} \).

Now, if we replace \( (d_m - \beta_m) \) by \( d_m \) and

\[ d_0 \leq b_0 + \beta_0, \ 0 < n < N: \ d_n \leq b_n + \beta_n \]

by \( d_0 \leq b_0, \ 0 < n < N: \ d_n < b_n \), we obtain (2.20). The statement (2.21) is an immediate consequence of the representation (2.20). Using the same proof for the bound \( F^* \), we obtain (2.22)–(2.24) \( \square \)

3. The results

3.1. Some bounds in the general case

In this section we shall give two theorems which allow to surround \( E, F \) and \( E^* - E, F^* - F \) in the general case. For this, it is not necessary to know any of the quantities \( E, E^*, F \) or \( F^* \).
Theorem 3.1. Let $E$ and $F$ be defined by (1.4). Then

\begin{align*}
\forall \gamma_N \leq E, \quad &\forall \Gamma_N \geq F, \\
\gamma_N \leq E \leq \tilde{\gamma}_N & \tag{3.1} \\
\text{and} & \\
\tilde{\Gamma}_N \leq F \leq \Gamma_N. & \tag{3.3}
\end{align*}

The quantities $\tilde{\gamma}_N$ and $\tilde{\Gamma}_N$ are defined by

\begin{equation}
\tilde{\gamma}_N = \min_{0 \leq n \leq N-1} (h_n(\gamma_N)), \quad \tilde{\Gamma}_N = \max_{0 \leq n \leq N-1} (h_n(\Gamma_N)), \tag{3.4}
\end{equation}

the functions $h_n$ are given by

\begin{equation}
0 \leq n \leq N-1: \quad h_n(x) = b_n + \frac{a_n}{f_n(x)} - g_{n+1}(x) \tag{3.2}
\end{equation}

and the functions $f_n, g_{n+1}$ by (2.2) and (2.3).

**Remarks.** According to the property (cf. Lemma 2.3) of functions $f_n$ and $g_{n+1}$, the functions $h_n$ are decreasing on the intervals $(-\infty, E]$ and $[F, +\infty)$. Moreover, it is easy to see that we have for $0 \leq n \leq N-1$:

\begin{align*}
h_n(E) &= E, \quad h_n(F) = F \quad \text{and} \quad h_n(-\infty) = h_n(+\infty) = b_n.
\end{align*}

Also, we can improve the inequalities (3.2) and (3.3) if we replace $\gamma_N$ and $\Gamma_N$ by better estimations of $E$ and $F$. For initial values of $\gamma_N$ and $\Gamma_N$, we can take those given by (2.16).

**Proof of the Theorem 3.1.** The inequalities

\begin{align*}
\gamma_N \leq E, \quad &\Gamma_N \geq F,
\end{align*}

are self-evident consequences of their constructions (3.1). We can take for $\gamma_N$ and $\Gamma_N$ the values given by (2.16); with the definition (2.1) of $c_n$ and the expression (2.13) of $E$, we have

\begin{equation}
0 \leq n \leq N-2: \quad c_n \leq b_n + \frac{a_n}{f_n(\gamma_N)} - g_{n+1}(\gamma_N) = h_n(\gamma_N) \tag{3.5}
\end{equation}

and

\begin{equation}
d_{N-1} \leq b_{N-1} + \frac{a_{N-1}}{f_{N-1}(\gamma_N)} = h_{N-1}(\gamma_N). \tag{3.6}
\end{equation}

But the optimal parameters $\{d_n\}_{n=0}^{N-1}$ verify (cf. [1] and according to the definition (2.1), $c_n = c_n(d_n, d_{n+1}))$

\begin{equation}
c_0 = c_1 = \cdots = c_{N-2} = d_{N-1} = E. \tag{3.7}
\end{equation}

Therefore, the quantity $E$ satisfies $E \leq \min_{0 \leq n \leq N-1} (h_n(\gamma_N))$. Hence, we obtain with (3.4) the right member of the inequality (3.2). For $F$, a same technique yields to the left member of the inequality (3.3). This completes the proof of the theorem. \(\square\)
Remark. By the construction (2.2), (2.3) and the property (2.17) of functions \( f_n \) and \( g_n \), it is easy to see that for \( \forall \gamma_n \leq E, \ 0 \leq n \leq N - 2 \) we have
\[
c_n \geq b_n - \frac{a_n}{g_n(\gamma_n)} + f_{n+1}(\gamma_n) = \gamma_n - a_n \left( \frac{f_n(\gamma_n) + g_n(\gamma_n)}{f_n(\gamma_n)g_n(\gamma_n)} \right) \leq \gamma_n
\]
and
\[
d_{N-1} \geq b_{N-1} - \frac{a_{N-1}}{g_{N-1}(\gamma_n)} = \gamma_n.
\]
The above inequalities mean that we have \( \gamma_n \geq \max_{0 \leq n \leq N-2} (b_n - a_n/g_n(\gamma_n) + f_{n+1}(\gamma_n)) \).

Theorem 3.2. Let \{\alpha_n\}_{n \geq 0} \{\beta_n\}_{n \geq 0} \) be the perturbations defined by (1.3), then the extremal zeros \( E, E^*, F \) and \( F^* \) given by (1.4) and (1.5) satisfy
\[
\forall \gamma_n \leq E, \ \forall \gamma_n^* \leq E^* \quad \phi_1(\gamma_n, \gamma_n^*) \leq E^* - E \leq \phi_2(\gamma_n, \gamma_n^*) \tag{3.5}
\]
and
\[
\forall \Gamma_n \geq F, \ \forall \Gamma_n^* \geq F^* \quad \phi_1(\Gamma_n, \Gamma_n^*) \leq F^* - F \leq \phi_2(\Gamma_n, \Gamma_n^*). \tag{3.6}
\]
The bounds are defined by
\[
\phi_1(\gamma_n, \gamma_n^*) = \max \left[ \min_{0 < n < N} (S_n(\gamma_n), \beta_{N-1}), \gamma_n^* - \gamma_n \right],
\]
\[
\phi_2(\gamma_n, \gamma_n^*) = \min \left[ \max_{0 < n < N} (R_n(\gamma_n^*), \beta_{N-1}), \gamma_n^* - \gamma_n \right], \tag{3.7}
\]
\[
\phi_1(\Gamma_n, \Gamma_n^*) = \max \left[ \min_{0 < n < N} (R_n(\Gamma_n^*), \beta_{N-1}), \Gamma_n^* - \Gamma_n \right],
\]
\[
\phi_2(\Gamma_n, \Gamma_n^*) = \min \left[ \max_{0 < n < N} (S_n(\Gamma_n), \beta_{N-1}), \Gamma_n^* - \Gamma_n \right]. \tag{3.8}
\]
and for \( 0 < n < N \),
\[
R_n(x) = \frac{\alpha_n}{2a_n^*} \left[ (1 - \text{Sgn}(\alpha_n))f_n^*(x) - (1 + \text{Sgn}(\alpha_n))g_n^*(x) \right] + \beta_{n-1},
\]
\[
S_n(x) = \frac{\alpha_n}{2a_n} \left[ (1 + \text{Sgn}(\alpha_n))f_n(x) - (1 - \text{Sgn}(\alpha_n))g_n(x) \right] + \beta_{n-1} \tag{3.9}
\]
and
\[
\text{Sgn}(\alpha_n) = \begin{cases} 1 & \text{if } \alpha_n \geq 0, \\ -1 & \text{otherwise}. \end{cases} \tag{3.10}
\]
The quantities \( \tilde{\gamma}_n \) and \( \tilde{\Gamma}_n \) are defined by (3.4). We can take for \( \gamma_n \) and \( \Gamma_n \) the values given, for example, by (2.16). The corresponding values with star notations can be obtained with the same definition but by replacing the coefficients \( a_n \) and \( b_n \) by \( a_n^* \) and \( b_n^* \) following (1.3).
Remarks. According to the definition (3.7) and (3.8) of the bounds $\phi_1, \phi_2, \phi_1$ and $\phi_2$, these estimates will be better if the estimates for $\gamma_N, \gamma_N^*, \Gamma_N$ and $\Gamma_N^*$ are better. In the optimal case: if $\gamma_N = E$ and $\gamma_N^* = E^*$, we have $\phi_1 = \phi_2$ or if $\Gamma_N = F$ and $\Gamma_N^* = F^*$ then we have $\phi_1 = \phi_2$. The Theorem 3.2 allows one to obtain the estimates (3.5) and (3.6) even if none of the quantities $E, E^*, F$ and $F^*$ are known.

Proof of the Theorem 3.2. With the Theorem 3.1 and (3.2) and (3.3), it is easy to see that the bounds $E, E^*, F$ and $F^*$ defined by (1.4) and (1.5) satisfy,

$$\gamma_N^* - \gamma_N \leq E^* - E \leq \gamma_N^* - \gamma_N$$

and

$$\Gamma_N^* - \Gamma_N \leq F^* - F \leq \Gamma_N^* - \Gamma_N.$$  

The bounds $\gamma_N^*$ and $\Gamma_N^*$ are obtained by replacing in the Theorem 3.1, the functions $h_n$ by

$$0 \leq n \leq N - 1, \quad h_n^*(x) = b_n^* + \frac{a_n^*}{f_n^*(x)} - g_{n+1}(x).$$

and the variable $x$ by $\gamma_N^*$ or by $\Gamma_N^*$. 

From the definition (2.9) of the set $I_N(\gamma_N)$ and the result (2.20) we can write

$$E^* \geq \max_{I_N(\gamma_N)} \min_{0 \leq n \leq N - 2} (c_n + \theta_{n+1}, d_{N-1} + \beta_{N-1}).$$

Because the set $I_N(\gamma_N)$ verifies, following the properties (2.17):

$$I_N(\gamma_N) \subset \{d_0 \leq b_0; \quad 0 < n < N \quad d_n < b_n\}.$$ 

Hence, we obtain with (2.13) $E^* \geq E + \min_{I_N(\gamma_N)} \min_{0 \leq n \leq N - 2} (\theta_{n+1}, \beta_{N-1})$. But, with definition (2.25) of $\theta_{n+1}$ and definition (3.9) of $S_n(\gamma_N)$ we can see that

$$\min_{I_N(\gamma_N)} \min_{0 \leq n \leq N - 2} (\theta_{n+1}, \beta_{N-1}) \geq \min_{0 \leq n \leq N - 2} (S_{n+1}(\gamma_N), \beta_{N-1}).$$

Hence, with (3.14), we have $\varphi_1$ of (3.7). If we apply (2.13) to $E^*$, we obtain

$$E^* = \max_{I_N^*(\gamma_N)} \min_{0 \leq n \leq N - 2} (c_n^*, d_{N-1}).$$

The set $I_N^*(\gamma_N)$ is the corresponding set to $I_N(\gamma_N)$ with the star notation. Now, by putting:

$$G = \left\{ d_n \in R \mid d_0 = b_0; \quad 0 < n < N: \quad b_n - \frac{a_n^*}{g_n^*(\gamma_N^*)} \leq d_n \leq b_n + \frac{a_n^*}{f_n^*(\gamma_N^*)} \right\},$$

we obtain

$$E^* = \max_{G} \min_{0 \leq n \leq N - 2} (c_n + \theta_{n+1}, d_{N-1} + \beta_{N-1})$$

$$\leq \max_{G} \min_{0 \leq n \leq N - 2} (c_n, d_{N-1}) + \max_{0 < n < N} (R_N(\gamma_N^*), \beta_{N-1})$$
where $R_\gamma(y^n)$ is defined by (3.9). But according to the sign of functions $f_n^*$ on $(-\infty, E^*]$ (cf. (2.17)), we have: $0 < n < N$, \( b_n + a_n^*/f_n^*(\gamma^n) \). Therefore, with (2.11), we obtain $E^* \leq E + \max_{0 < n < N} (R_\gamma(y^n), \beta_{N-1})$, which, with (3.11), yields the bound $\phi_2$ of (3.7). Hence (3.5) follows. A same proof for $F$ gives $\phi_1 \leq F^* - F \leq \phi_2$. Thus, we get the relation (3.6) and we complete the proof of the Theorem 3.2.

In the next sections, we shall give some inequalities between $E$ and $E^*$, between $F$ and $F^*$ in terms of the perturbations $\{\alpha_n\}_{n=0}^{N-1}$ and $\{\beta_n\}_{n=0}^{N-1}$ following (1.3). First, we shall treat the general case with:

### 3.2. Some perturbations with arbitrary signs

**Theorem 3.3.** Let $\{\alpha_n\}_{n=0}^{N-1}$ and $\{\beta_n\}_{n=0}^{N-1}$ be the perturbations defined by (1.3). Then the following properties hold:

\[
\begin{align*}
(F = \min \max_{J_N(>\leq, \leq, \beta_{N-1})} (c_n, d_{N-1}) \Rightarrow F^* &\leq F + \beta_{N-1}, & (3.16) \\
(\beta_{N-1} \leq 0 \text{ and } F = \min \max_{J_N(>\leq, \leq, 0)} (c_n, d_{N-1}) \Rightarrow F^* &\leq F, & (3.17) \\
(F^* = \min \max_{J_N^+(>\geq, >, \beta_{N-1})} (c_n^*, d_{N-1}) \Rightarrow F \leq F^* - \beta_{N-1}, & (3.18) \\
(\beta_{N-1} \geq 0 \text{ and } F^* = \min \max_{J_N^+(>\geq, >, 0)} (c_n^*, d_{N-1}) \Rightarrow F \leq F^*, & (3.19) \\
(E^* = \max \min_{J_N^+(<\geq, \leq, \beta_{N-1})} (c_n^*, d_{N-1}) \Rightarrow E^* \leq E + \beta_{N-1}, & (3.20) \\
(\beta_{N-1} \leq 0 \text{ and } E^* = \max \min_{J_N^+(<\geq, \leq, 0)} (c_n^*, d_{N-1}) \Rightarrow E^* \leq E, & (3.21) \\
(E = \max \min_{J_N(<\leq, \leq, \beta_{N-1})} (c_n, d_{N-1}) \Rightarrow E \leq E^* - \beta_{N-1} & (3.22) \\
and \beta_{N-1} \geq 0 \text{ and } E = \max \min_{J_N(<\leq, \leq, 0)} (c_n, d_{N-1}) \Rightarrow E \leq E^*. & (3.23)
\end{align*}
\]

The sets $J_N$ and $J_N^+$ are defined by (2.4) and (2.5), and the function $c_n$ by (2.1), $c_n^*$ is its equivalent with the star notation (i.e. the coefficients $a_n$ and $b_n$ replaced by $a_n^*$ and $b_n^*$ following (1.3)).

**Remarks.** When $\beta_{N-1} \leq 0$ the case (3.17) is less restrictive than the case (3.16). Because in (3.16) the set $J_N(>\leq, \leq, \beta_{N-1})$ satisfies in this case

\[ J_N(>\leq, \leq, \beta_{N-1}) \subset J_N(>\leq, \leq, 0). \]
If \( d_n \) belongs to \( J_N(\succ, \preceq, \beta_{N-1}) \) then it verifies

\[
d_n > b_n \quad (\beta_{n-1} - \beta_{N-1})d_n \leq (\beta_{n-1} - \beta_{N-1})b_n - \alpha_n.
\]

Hence, we can write

\[
d_n > b_n \quad \beta_{n-1}d_n - \beta_{N-1}d_n \leq \beta_{n-1}b_n - \beta_{N-1}d_n - \alpha_n,
\]

since \(-\beta_{N-1}b_n \leq -\beta_{N-1}d_n\). One has \( d_n \in J_N(\succ, \preceq, 0)\).

However, inequality (3.16) is more precise than (3.17). If \( \beta_{N-1} \) is negative then \( F^* \leq F + \beta_{N-1} \leq F \).

It is the same for the cases (3.18)-(3.19), (3.20)-(3.21) and (3.22)-(3.23).

Proof of the Theorem 3.3. First, we see that if case (3.16) is true, then case (3.18) is also true.

From remarks (2.10), we have \( F^{**} = F \).

Now by applying (3.16) to \( F^* \), if \( F^* = \min_{J_N^*}(\succ, \preceq, -\beta_{N-1}) \max_{0 \leq n \leq N-2} (c_n^*, d_{N-1}) \) then \( F = F^{**} \leq F^* - \beta_{N-1} \). But, from the definition (2.5) of \( J_N^* \) we have

\[
J_N^*(\succ, \preceq, -\beta_{N-1}) = \{d_n \in R \mid d_0 = b_0^*, \quad 0 < n < N \quad d_n > b_n^* \quad \text{and} \quad (-\beta_{n-1} + \beta_{N-1})d_n \leq (-\beta_{n-1} + \beta_{N-1})b_n + \alpha_n = J_N^*(\succ, \preceq, \beta_{N-1}).
\]

Hence, (3.18) holds. Using this remark we see that case (3.17) gives (3.19), case (3.20) gives (3.22) and case (3.21) leads to (3.23). Also, we shall only give the proofs of (3.16), (3.17), (3.20), and of (3.21).

According to (2.24):

\[
F^* = \min_{d_0 \geq b_0, 0 < n < N : d_n \geq b_n, 0 \leq n \leq N-2} \max (c_n + \theta_{n+1} - \beta_{N-1}, d_{N-1}) + \beta_{N-1},
\]

and following definition (2.25) of \( \theta_{n+1} \) we get

\[
\theta_{n+1} - \beta_{N-1} = \frac{\alpha_{n+1}}{d_{n+1} - b_{n+1}} + \beta_n - \beta_{N-1}.
\]

It is easy to see that if \( d_{n+1} \) belongs to \( J_N(\succ, \preceq, \beta_{N-1}) \) then the quantity \( (\theta_{n+1} - \beta_{N-1}) \) is negative. Thus, if the quantity \( F \) is defined by (3.16), then we obtain

\[
F \geq \min_{J_N(\succ, \preceq, -\beta_{N-1})} \max_{0 \leq n \leq N-2} (c_n + \theta_{n+1} - \beta_{N-1}, d_{N-1})
\]

\[
\geq \min_{d_0 \geq b_0, 0 < n < N : d_n \geq b_n, 0 \leq n \leq N-2} \max (c_n + \theta_{n+1} - \beta_{N-1}, d_{N-1}) = F^* - \beta_{N-1},
\]

because we have: \( J_N(\succ, \preceq, -\beta_{N-1}) \subset \{d_0 \geq b_0, 0 < n < N : d_n > b_n \} \).

Hence, property (3.16) follows. Now, using (2.23), if \( F \) verifies the assumption of (3.17) then we have \( 0 \leq n \leq N - 2, \quad \theta_{n+1} \leq 0 \) and with \( \beta_{N-1} \leq 0 \), we can conclude that

\[
F \geq \min_{J_N(\succ, \preceq, 0)} \max_{0 \leq n \leq N-2} (c_n + \theta_{n+1}, d_{N-1} + \beta_{N-1}) \geq F^*.
\]

Thus, we obtain (3.17). In (3.20), suppose that the hypothesis is satisfied. However, following (2.5) the set \( J_N^*(\prec, \succ, \beta_{N-1}) \) is defined by

\[
0 < n < N \quad d_n < b_n^* \quad \text{and} \quad (\beta_{n-1} - \beta_{N-1})d_n \geq (\beta_{n-1} - \beta_{N-1})b_n^* - \alpha_n.
\]
which is equivalent to

$$0 < n < N \quad d_n - \beta_n < b_n \quad \text{and} \quad (\beta_{n-1} - \beta_{N-1})(d_n - \beta_n) \geq (\beta_{n-1} - \beta_{N-1})b_n - \alpha_n.$$ 

Now, if we replace \((d_n - \beta_n)\) by \(d_n\) and \(d_0 - \beta_0\) by \(d_0\), we obtain with definition (2.25) of \(\theta_n\),

$$d_0 \leq b_0, \quad 0 < n < N \quad d_n < b_n \quad \text{and} \quad \theta_n - \beta_{N-1} \leq 0.$$ 

Hence, with (2.21) we have \(E^* \leq \max_{\beta_0(\leq, \geq, \beta_{N-1})} \min_{0 \leq n \leq N-2} (c_n, d_{N-1}) + \beta_{N-1}^{-1}\), because, with the above, we can write

$$d_n \in J_n^*(\leq, \geq, \beta_{N-1}) \Leftrightarrow (d_n - \beta_n) \in J_n^*(\leq, \geq, \beta_{N-1}).$$

Obviously, the right member of this inequality is bounded by \(E + \beta_{N-1}\), which yields the result (3.20).

In property (3.21), if the assumption is verified, then the conclusion follows. It is the same proof as the case (3.20). In this situation if \(d_{n+1}\) belongs to \(J_n^*(\leq, \geq, 0)\) then the quantity \(\theta_{n+1}\) in (2.20) is negative and with the sign of \(\beta_{N-1}\) we have

$$E^* \leq \max_{\beta_0(\leq, \geq, 0)} \min_{0 \leq n \leq N-2} (c_n, d_{N-1}) \leq E.$$ 

Thus, the proof of the Theorem is completed. \(\square\)

Now, we give some sufficient conditions in order to satisfy the assumptions of properties (3.16)-(3.23).

**Corollary 3.1.** Let \(\gamma_N \leq E, \gamma_N^* \leq E^*, \Gamma_N \geq F, \Gamma_N^* \geq F^*\) and \(\{\alpha_n\}_{n=0}^{N-1}, \{\beta_n\}_{n=0}^{N-1}\) be the perturbations defined by (1.3). Suppose \(0 < n < N, \alpha_n > -\alpha_n\), then the following properties hold:

\[
\begin{cases}
\alpha_n \leq \delta_n a_n / g_n(\Gamma_N) \quad (< 0) & \text{if } \delta_n > 0,
\alpha_n \leq -\delta_n a_n / f_n(\Gamma_N) \quad (\geq 0) & \text{if } \delta_n \leq 0,
\end{cases}
\]

implies (3.16): \(F^* \leq F + \beta_{N-1}\) and if \(\beta_{N-1} \leq 0\) then, we can replace \(\beta_{N-1}\) by 0.

\[
\begin{cases}
\alpha_n \geq -\delta_n a_n^* / f_n^*(\Gamma_N^*) \quad (< 0) & \text{if } \delta_n > 0,
\alpha_n \geq \delta_n a_n^* / g_n^*(\Gamma_N^*) \quad (\geq 0) & \text{if } \delta_n \leq 0,
\end{cases}
\]

implies (3.18): \(F \leq F^* - \beta_{N-1}\) and if \(\beta_{N-1} \geq 0\) then, we can replace \(\beta_{N-1}\) by 0.

\[
\begin{cases}
\alpha_n \geq \delta_n a_n^* / g_n^*(\gamma_N^*) \quad (> 0) & \text{if } \delta_n > 0,
\alpha_n \geq -\delta_n a_n^* / f_n^*(\gamma_N^*) \quad (\leq 0) & \text{if } \delta_n \leq 0,
\end{cases}
\]

implies (3.22): \(E^* \leq E + \beta_{N-1}\) and if \(\beta_{N-1} \leq 0\) then, we can replace \(\beta_{N-1}\) by 0.

\[
\begin{cases}
\alpha_n \leq -\delta_n a_n / f_n(\gamma_N) \quad (> 0) & \text{if } \delta_n > 0,
\alpha_n \leq \delta_n a_n / g_n(\gamma_N) \quad (\leq 0) & \text{if } \delta_n \leq 0,
\end{cases}
\]

implies (3.22): \(E \leq E^* - \beta_{N-1}\) and if \(\beta_{N-1} \geq 0\) then, we can replace \(\beta_{N-1}\) by 0.

The quantity \(\delta_n\) is defined by

$$0 < n < N, \quad \delta_n = \beta_{n-1} - \beta_{N-1},$$

(3.28)
the functions $f_n$, $g_n$ by (2.2), (2.3) and the functions $f_n^*$, $g_n^*$ are the corresponding functions with star notation (where the coefficients $a_n$ and $b_n$ are replaced by $a_n^*$ and $b_n^*$ according to (1.3)).

Remarks. For example the quantities $\gamma_N$, $\Gamma_N$ can be given by (2.16). Corollary 3.1 allows one to obtain the inequalities between the quantities $E$, $E^*$ and $F$, $F^*$ even if none is known.

In certain inequalities we also give the sign of the members. For example, in (3.24):

$$\delta_n \frac{a_n}{g_n(\Gamma_N)} \ ( < 0)$$

means that this quantity is negative. We know the signs, because we can use Lemma 2.3 which gives the sign of $g_n(\Gamma_N)$.

As the case was with Theorem 3.3, for example, in (3.24) if $\beta_{N-1} < 0$ then we have

$$\delta_n \frac{a_n}{g_n(\Gamma_N)} < \beta_{n-1} \frac{a_n}{g_n(\Gamma_N)},$$

which means, that if we replace $\beta_{N-1}$ by 0 in (3.24), we obtain a condition which is less restrictive. But in this case inequality (3.16) is more precise than the one obtained with $\beta_{N-1}$ replaced by 0. Because, we have

$$F^* \leq F + \beta_{N-1} < F.$$  

Corollary 3.1 allows one to obtain, by sufficient conditions, simple conditions satisfying the assumptions of Theorem 3.3.

Proof of the Corollary 3.1. In order to avoid the same proofs, we shall only give the proof of (3.24). For (2.26) we can use the same technique as in (3.24). With (2.10), (3.25) and (3.27) are direct consequences of (3.24) and (3.26), respectively.

Conditions (3.24) is a sufficient condition to obtain relation (3.16). According to (2.14), the quantity $F$ can also be defined by

$$\forall \Gamma_N \geq F, \ F = \min_{L_N(I_N)} \max_{0 \leq n \leq N-2} (c_n, d_{N-1}),$$

with $L_N(I_N)$ given by (2.8). A sufficient condition to satisfy (3.16) is therefore: $L_N(I_N) \subseteq J_N(>, \leq, \beta_{N-1})$, where the set $J_N(>, \leq, \beta_{N-1})$ is defined by (2.4). From definition (2.12) of the quantity $F$, we have in this case

$$F = \min_{L_N(I_N)} \max_{0 \leq n \leq N-2} (c_n, d_{N-1})$$

$$\geq \min_{J_N(>, \leq, \beta_{N-1})} \max_{0 \leq n \leq N-2} (c_n, d_{N-1})$$

$$\geq F = \min_{d_0 \geq b_0, 0 < n < N} \max_{0 \leq n \leq N-2} (c_n, d_{N-1}).$$

We shall verify whether (3.24) ensures the above inclusion. It is clear that if $0 < n < N, \ d_n \in L_N(I_N)$, then $0 < n < N, \ d_n > b_n$. Because the functions $f_n$ (cf. (2.17)) are positive on $[F, +\infty)$. Moreover, with the definition (3.28) of $\delta_n$, if the first assumption of (3.24) is true:

$$\delta_n > 0 \ \text{and} \ -a_n < a_n \leq \delta_n \frac{a_n}{g_n(\Gamma_N)}$$
and \( d_n \) belongs to \( L_N(\Gamma_N) \), then we have \( \delta_n d_n \leq \delta_n b_n - \delta_n a_n / g_n(\Gamma_N) \). Hence, with the above inequality, we get \( \delta_n d_n \leq \delta_n b_n - \alpha_n \).

Thus, \( d_n \) belongs to \( J_N(\gamma, \leq, \beta_{N-1}) \). If the second assumption is true:

\[
\delta_n \leq 0 \quad \text{and} \quad -a_n < \alpha_n \leq -\frac{a_n}{f_n(\Gamma_N)}
\]

and if \( d_n \) belongs to \( L_N(\Gamma_N) \) we obtain: \( \delta_n d_n \leq \delta_n b_n + \delta_n a_n / f_n(\Gamma_N) \). This implies that \( d_n \) belongs to \( J_N(\gamma, \leq, \beta_{N-1}) \) and we have (3.16).

In the case \( \beta_{N-1} \leq 0 \), if we use the same technique, we obtain

\[
L_N(\Gamma_N) \subset J_N(\gamma, \leq, 0).
\]

Hence, with the corresponding hypothesis, we have (3.17), which is (3.16) where we have replaced \( \beta_{N-1} \) by 0. This completes the proof of the corollary. \( \square \)

**Remarks.** As we have already stated, we can enlarge all the inequalities which give the constraints on the perturbations \( \{a_n\}_{n=0}^{N-1} \) and \( \{b_n\}_{n=0}^{N-1} \) if we can choose \( \gamma_N, \gamma_N^*, \Gamma_N \) and \( \Gamma_N^* \) near \( E, E^*, F \) and \( F^* \). This is a consequence of Lemma 2.3.

If we want to obtain even simpler constraints on the perturbations than those given in the preceding corollary, we have:

**Corollary 3.2.** Let \( \gamma_N, \gamma_N^*, \Gamma_N \) and \( \Gamma_N^* \) be as in Corollary 3.1, then, conditions (3.24)-(3.27) can be replaced, respectively by

\[
\begin{align*}
\alpha_n &\leq \delta_n (b_n - \Gamma_N) \quad (<0) \quad \text{if} \ \delta_n > 0, \\
\alpha_n &\leq \delta_n a_n / (b_{n-1} - \Gamma_N) \quad (\geq 0) \quad \text{if} \ \delta_n \leq 0,
\end{align*}
\]

(3.29)

\[
\begin{align*}
\alpha_n &\geq \delta_n a_n^* / (b_{n-1}^* - \Gamma_N^*) \quad (<0) \quad \text{if} \ \delta_n > 0, \\
\alpha_n &\geq \delta_n (b_n^* - \Gamma_N^*) \quad (\geq 0) \quad \text{if} \ \delta_n \leq 0,
\end{align*}
\]

(3.30)

\[
\begin{align*}
\alpha_n &\geq \delta_n (b_n^* - \gamma_N) \quad (>0) \quad \text{if} \ \delta_n > 0, \\
\alpha_n &\geq \delta_n a_n^* / (b_{n-1}^* - \gamma_N) \quad (\leq 0) \quad \text{if} \ \delta_n \leq 0,
\end{align*}
\]

(3.31)

\[
\begin{align*}
\alpha_n &\leq \delta_n a_n / (b_{n-1} - \gamma_N) \quad (>0) \quad \text{if} \ \delta_n > 0, \\
\alpha_n &\leq \delta_n (b_n - \gamma_N) \quad (\leq 0) \quad \text{if} \ \delta_n \leq 0.
\end{align*}
\]

(3.32)

The quantity \( \delta_n \) is defined by (3.28).

**Proof.** It is a direct consequence of Corollary 3.1. From Lemma 2.3, the functions \( f_n \) and \( g_n \) satisfy (2.17) and (2.18). Particularly, for \( 0 < n < N \),

\[
\forall x \leq E \quad x - b_{n-1} \leq f_n(x) < 0 \quad \text{and} \quad 0 < \frac{a_n}{b_n - x} \leq g_n(x),
\]

(3.33)

\[
\forall x \geq F \quad 0 < f_n(x) \leq x - b_{n-1} \quad \text{and} \quad g_n(x) \leq \frac{a_n}{b_n - x} < 0.
\]

(3.34)
In order to avoid the repetitions, we shall give only the proof corresponding to the first assumption of (3.29). This is a simplification of the first condition of (3.24) of Corollary 3.1. Indeed with (3.34) we can write
\[ \forall \gamma_n \geq F, \ 0 < n < N \quad b_n - \Gamma_N \leq \frac{a_n}{g_n(\Gamma_N)}. \]

Thus, if \( \delta_n \) is positive we obtain \( \delta_n(b_n - \Gamma_N) \leq \delta_n a_n / g_n(\Gamma_N) \).

Therefore, if the first assumption of (3.29) is true then, the first condition of (3.24) holds. It is the same for the second assumption which is a sufficient condition to verify the second hypothesis of (3.24). Hence, we have (3.16).

**Remarks.** If we want to obtain finer constraints on the perturbations than those given by this corollary we can improve inequalities (3.33) and (3.34). Here we want to obtain very simple constraints on the perturbations \( \{a_n\}_{n=0}^{N-1} \) and \( \{b_n\}_{n=0}^{N-1} \).

In order to give some global inequalities between the extremal zeros \( E, E^*, F \) and \( F^* \) we have, from (3.29), (3.32); (3.30), (3.31); (3.29), (3.31); (3.30), (3.32), the following properties for \( 0 < n < N \), \( \alpha_n > a_n \):

\[
\begin{align*}
\alpha_n &\leq \delta_n(b_n - \Gamma_N)(<0) \quad \text{if } \delta_n > 0, \\
\alpha_n &\geq \delta_n(b_n - \gamma_n)(\leq 0) \quad \text{if } \delta_n \leq 0,
\end{align*}
\]

then
\[
E \leq E^* - \beta_{N-1} < F^* - \beta_{N-1} \leq F, \quad (3.35)
\]

\[
\begin{align*}
\alpha_n &\geq \delta_n(b_n^* - \gamma_n^*)(>0) \quad \text{if } \delta_n > 0, \\
\alpha_n &\geq \delta_n(b_n^* - \Gamma_n^*)(\geq 0) \quad \text{if } \delta_n \leq 0,
\end{align*}
\]

then
\[
E^* - \beta_{N-1} \leq E < F \leq F^* - \beta_{N-1}, \quad (3.36)
\]

\( \alpha_n > -a_n, \ \delta_n \leq 0 \)

and
\[
(0 \geq \delta_n) \frac{a_n^*}{b_{n-1}^* - \gamma_n^*} \leq \alpha_n \leq \delta_n \frac{a_n}{b_{n-1} - \Gamma_n} (\geq 0),
\]

then
\[
E^* \leq E + \beta_{N-1}, \quad F^* \leq F + \beta_{N-1}, \quad (3.37)
\]

\( \alpha_n > -a_n, \ \delta_n \geq 0 \)

and
\[
(0 \geq \delta_n) \frac{a_n^*}{b_{n-1}^* - \Gamma_n^*} \leq \alpha_n \leq \delta_n \frac{a_n}{b_{n-1} - \gamma_n} (\geq 0),
\]

then
\[
E + \beta_{N-1} \leq E^*, \quad F + \beta_{N-1} \leq F^*. \quad (3.38)
\]
3.3. The perturbations \( \{\beta_n\}_{n=0}^{N-1} \) with same signs

In this section we shall study the effects of the perturbations \( \beta_n \) on the extremal zeros \( E \) and \( F \) when

\[
0 \leq n \leq N - 1, \quad \beta_n \geq 0,
\]

or,

\[
0 \leq n \leq N - 1, \quad \beta_n \leq 0.
\]

In this particular case, we can obtain new conditions for the inequalities between \( E \) and \( E^* \), \( F \) and \( F^* \).

We shall treat the positive case first.

**Theorem 3.4.** Under condition (3.39), we can write

\[
\left( F^* = \min_{K_N^{*}(>,>)} \max_{0 \leq n \leq N-2} (c^*_n, d_{N-1}) \right) \Rightarrow F^* \geq F.
\]

\[
\beta_0 = 0; \; 0 < n < N: \; -a_n < \alpha_n \leq 0
\]

and

\[
F = \min_{K_N(>)} \max_{0 \leq n \leq N-2} (c_n, d_{N-1}) \Rightarrow F \geq F^*;
\]

\[
\left( E = \max_{K_N(<,>)} \min_{0 \leq n \leq N-2} (c_n, d_{N-1}) \right) \Rightarrow E \leq E^*,
\]

\[
(\beta_0 = 0; \; 0 < n < N: \; \alpha_n \geq 0)
\]

and

\[
E^* = \max_{K_N^{*}(<,>)} \min_{0 \leq n \leq N-2} (c^*_n, d_{N-1}) \Rightarrow E^* \leq E.
\]

The sets \( K_N \) and \( K_N^{*} \) are defined by (2.6), (2.7) and the functions \( c_n \) by (2.1) (\( c_n^* \) are the corresponding functions with the star notation)

**Proof.** If the assumption of (3.40) holds, then we can write

\[
F^* \geq \min_{K_N^{*}(>,>)} \max_{0 \leq n \leq N-2} (c_n, d_{N-1}).
\]

If \( 0 \leq n \leq N - 1, \; d_n \in K_N^{*}(>,>), \) then we obtain, from definition (2.1) of \( c_n \),

\[
0 \leq n \leq N - 2 \quad c_n^* \geq c_n.
\]

Hence, with (2.12), we obtain property (3.40). Now, suppose \( 0 < n < N, -a_n < \alpha_n \leq 0 \), then we can show that the constraint of (3.41) implies:

\[
0 < n < N, \quad d_n > b_n^*.
\]
If \( \alpha_n = 0 \) and \( d_n \) belongs to \( K_N(>, \leq) \) then we have \(-a_n \beta_n \geq 0\). Thus, with (3.39) we obtain \( \beta_n = 0 \), which yields \( d_n > b_n = b^*_n \).

In the case \(-a_n < \alpha_n < 0\) and \( d_n \in K_N(>, \leq) \), with (1.3), we have

\[
d_n > b_n - \frac{a_n \beta_n}{\alpha_n} = b^*_n - \frac{a_n \beta_n}{\alpha_n}
\]

and with (3.39), the inequality (3.44) follows. Moreover, if \( \beta_0 = 0 \), \( 0 \leq n \leq N - 1 \), \( d_n \in K_N(>, \leq) \), we also have, \( 0 \leq n \leq N - 2 \), \( c_n^* \leq c_n \). Hence, with (2.12) applied to \( F^* \), we get

\[
F \geq \min_{d_0 = b_0 = b^*_0, 0 \leq n < N; d_n < b^*_n, \alpha_n \beta_n \leq \alpha_n \beta_n - a_n \beta_n, 0 \leq n < N - 2} \max_{c_n^*, d_{N-1}} \min_{0 < n < N} \left( c_n^*, d_{N-1} \right) \geq F^*
\]

and (3.41) follows. From (2.11) applied to \( E^* \), we can write with (3.39)

\[
E^* = \max_{0 \leq n < N; 0 < n < N; d_n < b^*_n, 0 < n < N - 2} \min_{c_n^*, d_{N-1}} \left( c_n^*, d_{N-1} \right) = \min_{d_0 = b_0 = b^*_0, 0 < n < N; d_n < b^*_n, 0 < n < N - 2} \max_{c_n^*, d_{N-1}} \min_{0 < n < N} \left( c_n^*, d_{N-1} \right)
\]

However, for \( 0 \leq n \leq N - 1 \), \( d_n \in K_N(>, \geq) \), then \( 0 \leq n \leq N - 2 \), \( c_n^* \geq c_n \). Thus, with the hypothesis of (3.42) and of (3.45), we obtain the inequality

\[
E^* \geq \max_{K_N(>, \geq)} \min_{0 < n < N - 2} \left( c_n, d_{N-1} \right) = E,
\]

which yields (3.42). The same way as the case (3.41), shows that

\[
\beta_0 = 0, \quad 0 < n < N; \quad \alpha_n \geq 0 \quad \text{and} \quad d_n \in K_N^*(<, \leq)
\]

which implies that

\[
0 \leq n \leq N - 1: \quad d_n \in K_N(<, \leq), \quad 0 \leq n \leq N - 2: \quad c_n \geq c_n^*,
\]

which leads with (2.11) to

\[
E^* \leq \max_{K_N(<, \leq)} \min_{0 < n < N - 2} \left( c_n, d_{N-1} \right) \leq E.
\]

This gives (3.43) and completes the proof of the theorem. \( \square \)

As we have done for the Theorem 3.3 by means of Corollaries 3.1 and 3.2, we shall see how to obtain some sufficient conditions to satisfy the assumptions of Theorem 3.4.

**Corollary 3.3.** The properties of Theorem 3.4 can be modified as follows:

\[
(0 < n < N, \quad -a_n < \alpha_n \quad \text{and} \quad (0 \geq) \beta_n g^*(I^*_N) \leq \alpha_n)
\]

implies (3.40): \( F^* \geq F \).

\[
(\beta_0 = 0; \quad 0 < n < N; \quad -a_n < \alpha_n \leq -\beta_n f_n(I_N)(\leq 0))
\]
implies (3.41): \( F \geq F^* \).

\[(0 < n < N, \ -a_n < x_n \leq \beta_n g_n(y_N)(\geq 0))\]

implies (3.42): \( E \leq E^* \).

\[(\beta_0 = 0; \ 0 < n < N: \ x_n \geq -\beta_n f_n^*(y_N^*)(\geq 0))\]

implies (3.43): \( E^* \leq E \).

Some other simplified conditions to satisfy (3.40)-(3.43) are, respectively,

\[
\left\{ \begin{array}{l}
0 < n < N, \ -a_n < x_n \leq \beta_n(b_n - \gamma_N)(\geq 0)
\end{array} \right.
\]

and

\[
\left\{ \begin{array}{l}
0 < n < N: \ x_n \geq -\beta_n(b_n - \gamma_N^*)(\geq 0).
\end{array} \right.
\]

The functions \( f_n \) and \( g_n \) are given by (2.2) and (2.3) (\( f_n^* \) and \( g_n^* \) are the corresponding functions where we replace the coefficients \( a_n \) and \( b_n \) by \( a_n^* \) and \( b_n^* \) following (1.3)). The quantities \( y_N \) and \( \gamma_N \) may be given by (2.16). The quantities \( y_N^* \) and \( \gamma_N^* \) may be also given by (2.16) where we have replaced the coefficients with the star coefficients.

**Proof.** From Theorem 3.4, in (3.40) if the assumption holds, then we have the corresponding implication. But a sufficient condition to have this, when \( -a_n < x_n < 0 \) (the case \( x_n \geq 0 \) involves obviously, with (2.12) applied to \( F^* \), (3.40)) and with (2.14) applied to \( F^* \), is

\[
b_n = \frac{a_n^*}{g_n^*(\gamma_N^*)} \leq b_n^* - \frac{a_n^* \beta_n}{x_n}.
\]

Hence, in Corollary 3.3, the hypothesis is a sufficient condition to obtain (3.40). In the set \( K_N(> <) \) defined by (2.6), if \( d_n \) belongs to \( K_N(> <) \) with \( x_n = 0 \) then \( \beta_n = 0 \). If: \( -a_n < x_n < 0 \), then a sufficient condition to satisfy (3.41), according to (2.14), is

\[
b_n - \frac{a_n \beta_n}{x_n} \leq b_n + \frac{a_n}{f_n(\gamma_N)}
\]

which yields the sufficient condition in corollary. By a same proof, the last assumptions are sufficient conditions to ensure the conditions of properties (3.42) and (3.43). Hence with them, we obtain ours last conclusions.

For the four simplified conditions given at the end, it is a direct consequence of (3.33) and (3.34). This completes the proof of corollary. \( \square \)

**Remarks.** In order to complete the results in the case of perturbations \( \{\beta_n\}_n \) with same signs, one can also easily obtain the similar results in the case:

\[0 \leq n \leq N - 1, \ \beta_n \leq 0.\]
For that, by means of (2.10), we can interchange the role of polynomials \( P_n \) and \( P_n^* \) and use the same results as in the case (3.39): \( \beta_n \geq 0 \). For example, the property (3.40) in the Theorem 3.4, becomes

\[
\left( F = \min_{\mathcal{K}_n(\cdot, \cdot)} \max_{0 \leq n \leq N-2} (c_n, d_{N-1}) \right) \Rightarrow F \geq F^*
\]

and in Corollary 3.3, the first property can be rewritten as

\[(0 < n < N, -a_n < \alpha_n \leq \beta_n g_n(\Gamma_n) (\geq 0)) \Rightarrow F \geq F^* . \]

4. Some numerical examples

In this section, we shall give some numerical examples. We have limited material to do that. Also we compute the extremal zeros \( E \) and \( E^* \) with \( N = 5 \) only. The star notation follows (1.3).

We have chosen for the family \( \{P_n\}_{n=0}^5 \), the Chebyshev polynomials of the second kind as the nonperturbed family. This family can be defined, according to (1.1), with the following coefficients

\[
b_0 = \cdots = b_4 = 0; \quad a_1 = \cdots = a_4 = 0.25. \quad (4.1)
\]

We shall give the sizes of \( E \) and \( E^* \) only. The smallest zero of this family is given by

\[
E = \cos \left( \frac{5\pi}{6} \right) \approx -0.866025404. \quad (4.2)
\]

4.1. An example to illustrate Theorems 3.1 and 3.2 in the general case

We shall take for \( E \) and for \( E^* \), the values given by (4.1) and by the zero corresponding to the last Example 4.3.2: \( E^* \approx -1.283418 \). If we choose for the initial values, the ones defined by (2.16): \( \gamma_5 = -1 \) and \( \gamma_5^* \approx -1.5939 \), then we obtain, from (3.4)

\[-1 \leq E \leq -0.667 \quad \text{and} \quad -1.5939 \leq E^* \leq -0.881,\]

which agrees with (3.2). This, also leads to

\[
\psi_1(\gamma_5, \gamma_5^*) \approx -0.9279 \leq E^* - E \leq -0.4174 \leq \psi_2(\gamma_5, \gamma_5^*) \approx 0.119,
\]

and implies (3.5). Following the remarks on the properties of functions \( h_n \), given after Theorem 3.1, if we take the initial values of \( \gamma_5 \) and \( \gamma_5^* \) as

\[
\gamma_5 = -0.88 \quad \text{and} \quad \gamma_5^* = -1.3,
\]

we get better estimates:

\[-0.88 \leq E \approx -0.866025404 \leq -0.839 \quad \text{and} \quad -1.3 \leq E^* \approx -1.283418 \leq -1.252.\]

This, yields \(-0.461 \leq (E^* - E) \approx -0.4174 \leq -0.372.\)
4.2. Examples to illustrate some results of the section "some perturbations with arbitrary signs"

We recall (3.28): $0 < n < N$, $\delta_n = \beta_{n-1} - \beta_{N-1}$. In the examples

$$0 < n < 5, \quad \delta_n = \beta_{n-1} - \beta_4. \tag{4.3}$$

We shall give some inequalities of Corollary 3.2.

4.2.1. If assumption (3.31) is satisfied, for example

$$\alpha_n \geq \delta_n (b_n^* - \gamma_5^*) (\geq 0), \quad \text{if } \delta_n > 0$$

then, we have (3.20): $E^* \leq E + \beta_4$. We have chosen the numerical example:

$$\alpha_1 = 0.6, \quad \alpha_2 = 0.3, \quad \alpha_3 = 0.3, \quad \alpha_4 = 0.5,$$

$$\beta_0 = 1.25, \quad \beta_1 = 1.125, \quad \beta_2 = 1.125, \quad \beta_3 = 1.25 \quad \text{and} \quad \beta_4 = 1.$$

From (1.3) and (4.1), we get

$$a_1^* = 0.85, \quad a_2^* = 0.55, \quad a_3^* = 0.55, \quad a_4^* = 0.75,$$

$$b_0^* = 1.25, \quad b_1^* = 1.125, \quad b_2^* = 1.125, \quad b_3^* = 1.25 \quad \text{and} \quad b_4^* = 1.$$

Following (2.16), $\gamma_5^* \approx -0.5385$ and we obtain $E^* \approx -0.2255$. Now, with (4.2): $E \approx -0.866025$, which yields

$$E + \beta_4 \approx -0.134 \geq E^*.$$ 

Hence, (3.20) holds.

4.2.2. In order to illustrate only the second assumption of (3.31):

$$\alpha_n \geq \delta_n \frac{a_n^*}{b_{n-1}^* - \gamma_5^*} (\leq 0), \quad \text{if } \delta_n \leq 0,$$

we have taken the following example:

$$\alpha_1 = 0, \quad \alpha_2 = 0.5, \quad \alpha_3 = 0, \quad \alpha_4 = 0.5,$$

$$\beta_0 = -0.5, \quad \beta_1 = -0.5, \quad \beta_2 = 0, \quad \beta_3 = 0.5 \quad \text{and} \quad \beta_4 = 0.5.$$

With (1.3) and (4.1), we obtain the perturbed coefficients

$$a_1^* = 0.25, \quad a_2^* = 0.75, \quad a_3^* = 0.25, \quad a_4^* = 0.75,$$

$$b_0^* = -0.5, \quad b_1^* = -0.5, \quad b_2^* = 0, \quad b_3^* = 0.5 \quad \text{and} \quad b_4^* = 0.5.$$

For $\gamma_5^*$ we have taken the value given by (2.16): $\gamma_5^* \approx -1.8660$. The size of the extremal zero $E^*$ is: $E^* \approx -1.3940$, which leads, with (4.2)

$$E^* \leq E + \beta_4 \approx -0.3660.$$ 

Thus, (3.20) is verified.
Now, we give an example where two constraints are satisfied together.

4.2.3. We have taken case (3.32):

\[
\begin{cases}
\alpha_n \leq \delta_n a_n / (b_n - \gamma_5) (>0) & \text{if } \delta_n > 0, \\
\alpha_n \leq \delta_n (b_n - \gamma_5) (\leq 0)) & \text{if } \delta_n \leq 0,
\end{cases}
\]

then, we have (3.22): \( E \leq E^* - \beta_4 \). We have chosen the following example:

\[ \beta_0 = 0.1, \quad \beta_1 = -0.1, \quad \beta_2 = -0.2, \quad \beta_3 = 2 \quad \text{and} \quad \beta_4 = 0, \]

which yields, according to (4.3)

\[ \delta_1 = 0.1, \quad \delta_2 = -0.1, \quad \delta_3 = -0.2 \quad \text{and} \quad \delta_4 = 2. \]

We have taken \( \gamma_5 = -1 \) and for the first condition of (3.22)

\[ -0.25 < \alpha_1 = 0 \leq \delta_1 \frac{a_1}{b_0 - \gamma_5} = 0.025, \]

\[ -0.25 < \alpha_4 = 0.5 \leq \delta_4 \frac{a_4}{b_3 - \gamma_5} = 0.5 \]

and for the second, we have chosen

\[ -0.25 < \alpha_2 = -0.1 \leq \delta_2 (b_2 - \gamma_5) = -0.1, \]

\[ -0.25 < \alpha_3 = -0.2 \leq \delta_3 (b_3 - \gamma_5) = -0.2. \]

This gives

\[ a_1^* = 0.25, \quad a_2^* = 0.15, \quad a_3^* = 0.05, \quad a_4^* = 0.75, \]

\[ b_0^* = 0.1, \quad b_1^* = -0.1, \quad b_2^* = -0.2, \quad b_3^* = 2 \quad \text{and} \quad b_4^* = 0. \]

We obtain with (4.2)

\[ -0.866025 \leq E \leq E^* - \beta_4 = E^* \leq -0.71551, \]

which implies (3.22).

4.3. Examples to illustrate some results of the section “the perturbations \( \{\beta_n\} \) with same signs”

Here, we treat the case (3.39): \( 0 \leq n \leq N - 1, \beta_n \leq 0 \) and we shall illustrate those of Corollary 3.3.

4.3.1. We have chosen the third property, with the simplified condition

\[ \left( 0 < n < N, \quad -a_n < \alpha_n \leq \frac{a_n \beta_n}{b_n - \gamma_5} (\geq 0) \right) \Rightarrow E \leq E^*. \]
The numerical example is (in other cases $\beta_0$ may be strictly positive)

$$\beta_0 = 0, \quad \beta_1 = 1, \quad \beta_2 = 0.2, \quad \beta_3 = 2 \quad \text{and} \quad \beta_4 = 0.4.$$ 

This yields, with $\gamma_5 = -1$,

$$\frac{a_1 \beta_1}{b_1 - \gamma_5} = 0.25, \quad \frac{a_2 \beta_2}{b_2 - \gamma_5} = 0.05,$$

$$\frac{a_3 \beta_3}{b_3 - \gamma_5} = 0.5 \quad \text{and} \quad \frac{a_4 \beta_4}{b_4 - \gamma_5} = 0.1,$$

then we have taken

$$\alpha_1 = 0, \quad \alpha_2 = 0.05, \quad \alpha_3 = -0.2 \quad \text{and} \quad \alpha_4 = -0.1.$$

Hence, we obtain for the perturbed coefficients

$$a_1^* = 0.25, \quad a_2^* = 0.3, \quad a_3^* = 0.05, \quad a_4^* = 0.15,$$

$$b_0^* = 0, \quad b_1^* = 1, \quad b_2^* = 0.2, \quad b_3^* = 2 \quad \text{and} \quad b_4^* = 0.4.$$ 

In this case, we have

$$E \equiv -0.866025 \leq E^* \equiv -0.3974.$$ 

Thus, inequality (3.42) is satisfied.

4.3.2. For the last property of this corollary, namely

$$(\beta_0 = 0; \quad \alpha_n \geq \beta_n (b_{n-1}^* - \gamma_5^*) (\geq 0)) \Rightarrow E^* \leq E,$$

we have taken

$$\alpha_1 = 0.25, \quad \alpha_2 = 0.3, \quad \alpha_3 = 0.6 \quad \text{and} \quad \alpha_4 = 0.6.$$ 

Because, with

$$\beta_0 = 0, \quad \beta_1 = 0.125, \quad \beta_2 = 0.125, \quad \beta_3 = 0.25 \quad \text{and} \quad \beta_4 = 0.25,$$

we have chosen the above perturbations $\alpha_n$ such that $\gamma_5^*$ verifies

$$\gamma_5^* \geq -2.$$ 

Thus, we obtain

$$\beta_1 (b_0^* - \gamma_5^*) \leq 0.25, \quad \beta_2 (b_1^* - \gamma_5^*) \leq 0.265625,$$

$$\beta_3 (b_2^* - \gamma_5^*) \leq 0.5625 \quad \text{and} \quad \beta_4 (b_3^* - \gamma_5^*) \leq 0.5625.$$ 

Hence, we have ($\gamma_5^* \equiv -1.5939$)

$$a_1^* = 0.5, \quad a_2^* = 0.55, \quad a_3^* = 0.85, \quad a_4^* = 0.85.$$
We get: \( E^* \approx -1.283418 \) and with (4.1), we obtain the inequality

\[ E^* \leq E \approx -0.866025. \]

This is property (3.43).

References


