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Weighted Lorentz Spaces and the Hardy Operator

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We find a new expression for the norm of a function in the weighted Lorentz space, with respect to the distribution function, and obtain as a simple consequence a generalization of the classical embeddings $L^{p,1} \subset \cdots \subset L^p \subset \cdots \subset L^{p,r}$ and a new definition of the weak space $\Lambda_u^{p,r}(w)$. We also give some applications to the boundedness of the Hardy operator $Sf = \int_0^x f$ from $\Lambda_{u_0}^{p_0}(w_0)$ into $\Lambda_{u_1}^{p_1}(w_1)$ with $0 < p_0 \leq p_1$. © 1993 Academic Press, Inc.

1. INTRODUCTION

We say that a nonnegative function is a weight, if it is locally integrable. The weighted Lorentz space $\Lambda_u^p(w)$ is defined by those measurable functions in $\mathbf{R}^n$ or $\mathbf{R}^+$ such that $\|f\|_{\Lambda_u^p(w)} = (\int_0^\infty (f_u^*(x))^p\, w(x)\, dx)^{1/p} < +\infty$, where f_u^* denotes the decreasing rearrangement function with respect to the weight u (*see* [2]). These spaces have been widely studied in multiple contexts. In general, they are not Banach spaces; however, they are quasi-Banach spaces if and only if $W(x) = \int_0^x w(t)\, dt$ satisfies the Δ_2-condition (see Corollary 2.2).

The paper is organized as follows. In Section 2, we prove a distribution formula for the "norm" in $\Lambda_u^p(w)$ and study the natural generalization of the well known embeddings for the classical Lorentz spaces,

$$L^{p,1} \subset \cdots \subset L^p \subset \cdots \subset L^{p,q} \subset \cdots \subset L^{p,\infty},$$

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for $p \leqslant q$. We also study conditions on the weights u_j and w_j for the embedding $\Lambda_{u_0}^{p_0}(w_0) \subset \Lambda_{u_1}^{p_1}(w_1)$ to hold. The characterization of the weights w_j for this embedding, when $u_0 = u_1 = 1$, was done by E. Sawyer in [10]. His proof can be applied to the case $u_0 = u_1$. Section 3 is devoted to the boundedness of the Hardy operator $Sf(x) = \int_0^x f(t)\,dt$ from $\Lambda_{u_0}^{p_0}(w_0)$ into $\Lambda_{u_1}^{p_1}(w_1)$. The idea is to obtain a unified version of the boundedness of S in both $L^p(u)$ (as in [7, 3, 6]) and $\Lambda^p(w)$ (as in [1, 10]). We need to impose a condition on the weight w_0 and examples of weights satisfying this extra condition are given in Section 4.

Throughout this paper, $f \approx g$ will denote the existence of two positive constants a and b such that $af(x) \leqslant g(x) \leqslant bf(x)$ for almost every x, and constants such as C may change from one occurrence to the next.

2. Weighted Lorentz Spaces with Two Parameters

If u is a weight in $\mathbf{R}^n$, we call $\lambda_f^u(y) = \int_{\{x : |f(x)| > y\}} u(x)\,dx$, the distribution function of f with respect to the measure $u(x)\,dx$. Also, if A is a measurable subset of $\mathbf{R}^n$, we write $u(A) = \int_A u(x)\,dx$.

Theorem 2.1. *Suppose u is a weight in $\mathbf{R}^n$, w is a weight in $\mathbf{R}^+$, and $0 < p < \infty$. Then,*

$$\int_0^\infty (f_u^*(t))^p\, w(t)\,dt = p \int_0^\infty y^{p-1} \left(\int_0^{\lambda_f^u(y)} w(t)\,dt \right) dy.$$

Proof. It suffices to consider simple functions, since, given a function f, we can always find a sequence of simple functions $\{s_k\}_k$ so that $|s_k| \leqslant |f|$, $\lambda_{s_k}^u \uparrow \lambda_f^u$, and $s_k^* \uparrow f_u^*$, and the monotone convergence theorem would imply the result for f. So, consider $f(x) = \sum_{i=1}^N a_i \chi_{A_i}(x)$, where the sets A_i are disjoint, $0 < u(A_i) < \infty$, and $|a_1| > |a_2| > \cdots > |a_N| > 0$. Set $a_{N+1} = 0$, $\alpha_0 = 0$, and $\alpha_k = \sum_{j=1}^k u(A_j)$. Since,

$$\lambda_f^u(y) = \sum_{k=1}^N \alpha_k \chi_{\{|a_{k+1}| \leqslant y < |a_k|\}}(y), \quad \text{and} \quad f_u^*(t) = \sum_{k=1}^N |a_k|\, \chi_{\{\alpha_{k-1} < t \leqslant \alpha_k\}}(t),$$

then,

$$\begin{aligned}\int_0^\infty (f_u^*(t))^p\, w(t)\,dt &= \sum_{k=1}^N |a_k|^p \int_{\alpha_{k-1}}^{\alpha_k} w(t)\,dt \\ &= \sum_{k=1}^N |a_k|^p \left(\int_0^{\alpha_k} w(t)\,dt - \int_0^{\alpha_{k-1}} w(t)\,dt \right)\end{aligned}$$

$$= \sum_{k=1}^{N} (|a_k|^p - |a_{k+1}|^p) \int_0^{\alpha_k} w(t)\,dt$$

$$= p \sum_{k=1}^{N} \int_{|a_{k+1}|}^{|a_k|} y^{p-1} \left(\int_0^{\alpha_k} w(t)\,dt \right) dy$$

$$= p \int_0^{\infty} y^{p-1} \left(\int_0^{\lambda_f^u(y)} w(t)\,dt \right) dy. \quad \blacksquare$$

A weaker version of this result, for a particular class of weights w, can be found in [11]. A first application gives us a characterization of the Lorentz spaces for which $\|\cdot\|_{\Lambda_u^p(w)}$ is a quasi-norm. We say that a function F satisfies the Δ_2-condition if there exists a constant $C>0$ such that $F(2t) \leqslant CF(t)$ for all $t>0$.

Corollary 2.2. *If $0<p<\infty$, then $\|\cdot\|_{\Lambda_u^p(w)}$ is a quasi-norm if and only if $W(x)=\int_0^x w(t)\,dt$ satisfies the Δ_2-condition.*

Proof. Let A, B be two measurable subsets of $\mathbf{R}^n$ such that $A \cap B = \varnothing$ and $u(A)=u(B)=x$. If $\|\cdot\|_{\Lambda_u^p(w)}$ is a quasi-norm and $f=\chi_A$, $g=\chi_B$, then

$$W(2x) = \|f+g\|_{\Lambda_u^p(w)}^p \leqslant C(\|f\|_{\Lambda_u^p(w)}^p + \|g\|_{\Lambda_u^p(w)}^p) = 2CW(x).$$

Conversely, if W satisfies the Δ_2-condition, since W is a nondecreasing function one has that $W(x+y) \leqslant C(W(x)+W(y))$ and hence,

$$\|f+g\|_{\Lambda_u^p(w)}^p = p \int_0^{\infty} y^{p-1} \left(\int_0^{\lambda_{f+g}^u(y)} w(t)\,dt \right) dy$$

$$\leqslant C \int_0^{\infty} y^{p-1} \left(\int_0^{\lambda_f^u(y/2)} w(t)\,dt + \int_0^{\lambda_g^u(y/2)} w(t)\,dt \right) dy$$

$$\leqslant C(\|f\|_{\Lambda_u^p(w)}^p + \|g\|_{\Lambda_u^p(w)}^p). \quad \blacksquare$$

A second application is the following. We have that

$$\|f\|_{\Lambda_u^p(w)} = \left\| y \left(\int_0^{\lambda_f^u(y)} w(t)\,dt \right)^{1/p} \right\|_{L^p(dy/y)},$$

and hence, the space $\Lambda_u^p(w)$ can be seen as a two parameters function space $\Lambda_u^{p,p}(w)$, where for $0<p$, $q<+\infty$, $\Lambda_u^{p,q}(w)$ is the space of all measurable functions in $\mathbf{R}^n$ such that

$$\|f\|_{\Lambda_u^p(w)} = \left\| y \left(\int_0^{\lambda_f^u(y)} w(t)\,dt \right)^{1/p} \right\|_{L^q(dy/y)} < +\infty.$$

This definition also makes sense for $q = +\infty$ and we can define the weak Lorentz space $\Lambda_u^{p,\infty}(w)$ to be the class of all measurable functions f such that

$$\|f\|_{\Lambda_u^{p,\infty}(w)} = \sup_{y>0} y\left(\int_0^{\lambda_f^u(y)} w(t)\,dt\right)^{1/p} < +\infty.$$

Observe that $\|f\|_{\Lambda_u^{p,\infty}(w)} = \sup_{x>0} f_u^*(x)(\int_0^x w(t)\,dt)^{1/p}$.

In particular, if $w(t) = t^{q/p-1}$, then $\Lambda_u^q(w) = L_u^{p,q}$ and $\Lambda_u^{q,\infty}(w)$ coincides with $L_u^{p,\infty}$. One can also check that the classical Lorentz space $L_u^{p,q}$ is the space $\Lambda_u^{p,q}(w)$ when $w = 1$.

Since the function $\int_0^{\lambda_f^u(y)} w(t)\,dt$ is decreasing one can easily prove, by standard arguments using a dyadic decomposition, the following discretization formula.

Proposition 2.3. *For every measurable function f in $\Lambda_u^{p,q}(w)$, and $0 < p$, $q < \infty$*

$$\|f\|_{\Lambda_u^{p,q}(w)} \approx \left(\sum_{k=-\infty}^{+\infty} 2^{kq}\left(\int_0^{\lambda_f^u(2^k)} w(t)\,dt\right)^{q/p}\right)^{1/q},$$

and, if $0 < p < \infty$,

$$\|f\|_{\Lambda_u^{p,\infty}(w)} \approx \sup_{k\in\mathbf{Z}} 2^k\left(\int_0^{\lambda_f^u(2^k)} w(t)\,dt\right)^{1/p}.$$

Proposition 2.4. *If $0 < q_0 \leqslant q_1$, $\Lambda_u^{p,q_0}(w)$ is embedded* in $\Lambda_u^{p,q_1}(w)$.

Proof. We just have to use Proposition 2.3 and the embedding properties of the sequence spaces. ∎

That is, we have the following chain of embeddings

$$\cdots \subset \Lambda_u^{p,1}(w) \subset \cdots \subset \Lambda_u^p(w) \subset \cdots \subset \Lambda_u^{p,q}(w) \subset \cdots \subset \Lambda_u^{p,\infty}(w),$$

for $p \leqslant q$. In the case $w = 1$ these are the classical embeddings of the Lorentz spaces $L_u^{p,q}$ we mentioned in the Introduction (see [12]).

Remark 2.5. Observe that the spaces $\Lambda_u^{p,q}(w)$ are, in fact, weighted Lorentz spaces if $q < \infty$, and coincide with the spaces $\Lambda_u^q(w_p)$, where

$$w_p(x) = \left(\int_0^x w(t)\,dt\right)^{q/p-1} w(x).$$

PROPOSITION 2.6. *If $0<q\leqslant+\infty$ and for $j=0$, 1, u_j (w_j) is a weight in* $\mathbf{R}^n$ ($\mathbf{R}^+$), *then $\Lambda_{u_0}^{p_0,q}(w_0)$ is embedded in $\Lambda_{u_1}^{p_1,q}(w_1)$ if and only if there exists a constant $C>0$ such that, for every measurable set A in* $\mathbf{R}^n$,

$$\left(\int_0^{u_1(A)} w_1(t)\,dt\right)^{1/p_1}\leqslant C\left(\int_0^{u_0(A)} w_0(t)\,dt\right)^{1/p_0}. \tag{1}$$

Proof. To prove the necessary condition, we just have to apply the hypothesis to the function $f=\chi_A$. Conversely, by Theorem 2.1,

$$\begin{aligned}\|f\|_{\Lambda_{u_1}^{p_1,q}(w_1)}&=\left(\int_0^{\infty} y^{q-1}\left(\int_0^{\lambda_f^{u_1}(y)} w_1(t)\,dt\right)^{q/p_1}dy\right)^{1/q}\\&\leqslant C\left(\int_0^{\infty} y^{q-1}\left(\int_0^{\lambda_f^{u_0}(y)} w_0(t)\,dt\right)^{q/p_0}dy\right)^{1/q}=C\|f\|_{\Lambda_{u_0}^{p_0,q}(w_0)}. \qquad ∎\end{aligned}$$

COROLLARY 2.7. *If $p_0\leqslant p_1$, then $\Lambda_{u_0}^{p_0}(w_0)$ is embedded in $\Lambda_{u_1}^{p_1}(w_1)$ if and only if (1) holds.*

Proof. Again, the necessary condition is trivial. Conversely, by Propositions 2.4 and 2.6,

$$\Lambda_{u_0}^{p_0}(w_0)=\Lambda_{u_0}^{p_0,p_0}(w_0)\subset\Lambda_{u_0}^{p_0,p_1}(w_0)\subset\Lambda_{u_1}^{p_1,p_1}(w_1)=\Lambda_{u_1}^{p_1}(w_1). \qquad ∎$$

Condition (1) is necessary for the corresponding embedding, independently of the relation between the parameters p_0 and p_1. However, if $p_1<p_0$, (1) is not a sufficient condition.

When $u_0=u_1=1$, the embedding $\Lambda^{p_0}(w_0)\subset\Lambda^{p_1}(w_1)$ was proved by E. Sawyer (see [10]) in the case $1<p_0$, p_1, as a consequence of the following nice and important result:

THEOREM 2.8. *If $1<p<+\infty$, $v(x)$ and $g(x)$ are nonnegative measurable functions on* $\mathbf{R}^+$, *with v locally integrable, then*

$$\begin{aligned}&\sup\left(\int_0^{\infty} f(x)\,g(x)\,dx\right)\left(\int_0^{\infty}(f(x))^p\,v(x)\,dx\right)^{-1/p}\\&\quad\approx\left(\int_0^{\infty}\left(\int_0^x g\right)^{p'} v(x)\left(\int_0^x v\right)^{-p'}dx\right)^{1/p'}+\left(\int_0^{\infty} g\right)\left(\int_0^{\infty} v\right)^{-1/p}\end{aligned} \tag{2}$$

where the supremum is taken over all nonnegative and nonincreasing functions f.

Moreover, the right hand side of (2) *can be replaced with the integral,*

$$\left(\int_0^{\infty}\left(\int_0^{\infty} g\right)^{p'-1}\left(\int_0^x v\right)^{1-p'} g(x)\,dx\right)^{1/p'}.$$

Using this result, E. Sawyer shows (see [10]) that $\Lambda^{p_0}(w_0)$ is embedded in $\Lambda^{p_1}(w_1)$ when $p_1 < p_0$, if and only if

$$\int_0^\infty \left(\left(\int_0^x w_1(t)\,dt \right)^{1/p_0} \left(\int_0^x v(t)\,dt \right)^{1/p_0} \right)^r w_1(x)\,dx < +\infty, \tag{3}$$

where $1/r = 1/p_1 - 1/p_0$. Now, to study the embedding $\Lambda^{p_0}_{u_0}(w_0) \subset \Lambda^{p_1}_{u_1}(w_1)$, with $p_1 < p_0$ we need the following lemmas:

LEMMA 2.9. *If w_1 is a decreasing function, and $p > 1$, then*

$$\sup_{f \in \Lambda^p_{u_0}(w_0)} \left(\int_0^\infty f^*_{u_1}(s)\, w_1(s)\,ds \right) \|f\|^{-1}_{\Lambda^p_{u_0}(w_0)}$$
$$\approx \sup_{w:w^*_{u_1} = w_1} \left(\|w u_0^{-1} u_1\|_{\Gamma^{p'}_{u_0}(\tilde{w}_0)} + \left(\int_{\mathbf{R}^n} w(x)\, u_1(x)\,dx \right) \left(\int_0^\infty w_0(s)\,ds \right)^{-1/p} \right),$$

where $\tilde{w}_0(x) = ((1/x) \int_0^x w_0(t)\,dt)^{-p'} w_0(x)$ *and* $\Gamma^p_u(w)$ *is defined by the norm*

$$\|f\|_{\Gamma^p_u(w)} = \left(\int_0^\infty \left(\frac{1}{x} \int_0^x f^*_u(s)\,ds \right)^p w(x)\,dx \right)^{1/p} < +\infty.$$

Proof. We observe that (see [2])

$$\int_0^\infty f^*_{u_1}(s)\, w_1(s)\,ds = \sup_{w:w^*_{u_1} = w_1} \int_{\mathbf{R}^n} f(x)\, w(x)\, u_1(x)\,dx,$$

and hence,

$$\sup_{f \in \Lambda^p_{u_0}(w_0)} \left(\int_0^\infty f^*_{u_1}(s)\, w_1(s)\,ds \right) \|f\|^{-1}_{\Lambda^p_{u_0}(w_0)}$$
$$= \sup_{w:w^*_{u_1} = w_1} \left(\sup_{f \in \Lambda^p_{u_0}(w_0)} \left(\int_{\mathbf{R}^n} f(x)(w(x)\, u_1(x)\, u_0^{-1}(x))\, u_0(x)\,dx \right) \|f\|^{-1}_{\Lambda^p_{u_0}(w_0)} \right)$$
$$= \sup_{w:w^*_{u_1} = w_1} \left(\sup_{f \in \Lambda^p_{u_0}(w_0)} \left(\int_0^\infty f^*_{u_0}(s)(w u_1 u_0^{-1})^*_{u_0}(s)\,ds \right) \|f\|^{-1}_{\Lambda^p_{u_0}(w_0)} \right),$$

and thus, we just have to apply Theorem 2.8. ∎

LEMMA 2.10. *If w_1 is a weight in the class B_1 (see [1]), $p > 1$, and we write $w_1^1(t) = \int_t^\infty (w_1(s)/s)\,ds$, then*

$$\sup_{f\in \Lambda^p_{u_0}(w_0)} \left(\int_0^\infty f^*_{u_1}(s)\, w_1(s)\, ds\right) \|f\|^{-1}_{\Lambda^p_{u_0}(w_0)}$$
$$\approx \sup_{w:w^*_{u_1}=w_1^1} \left(\|wu_0^{-1}u_1\|_{L^{p'}_{u_0}(\tilde{w}_0)} + \left(\int_{\mathbf{R}^n} w(x)\, u_1(x)\, dx\right)\left(\int_0^\infty w_0(s)\, ds\right)^{-1/p}\right).$$

Proof. It is enough to observe that if $w_1 \in B_1$ then an integration by parts shows that $\|f\|_{\Lambda^1_{u_1}(w_1)} \approx \|f\|_{\Lambda^1_{u_1}(w_1^1)}$ and since w_1^1 is decreasing, we can apply Lemma 2.9. ∎

COROLLARY 2.11. *Let* $p_1 < p_0$.

(a) *If* w_1 *is nonincreasing, then* $\Lambda^{p_0}_{u_0}(w_0) \subset \Lambda^{p_1}_{u_1}(w_1)$, *if and only if*

$$\sup\left(\|wu_0^{-1}u_1\|_{L^{p_0/(p_0-p_1)}_{u_0}(\tilde{w}_0)} + \left(\int_{\mathbf{R}^n} w(x)\, u_1(x)\, dx\right)\left(\int_0^\infty w_0(s)\, ds\right)^{-p_1/p_0}\right) < \infty,$$

where the supremum extends over all functions w *such that* $w^*_{u_1} = w_1$.

(b) *If* $w_1 \in B_1$ *then* $\Lambda^{p_0}_{u_0}(w_0) \subset \Lambda^{p_1}_{u_1}(w_1)$, *if and only if*

$$\sup\left(\|wu_0^{-1}u_1\|_{L^{p_0/(p_0-p_1)}_{u_0}(\tilde{w}_0)} + \left(\int_{\mathbf{R}^n} w(x)\, u_1(x)\, dx\right)\left(\int_0^\infty w_0(s)\, ds\right)^{-p_1/p_0}\right) < \infty,$$

where the supremum extends over all functions w *such that* $w^*_{u_1} = w_1^1$ *and* w_1^1 *is defined as above.*

Proof. It suffices to study the embedding $\Lambda^p_{u_0}(w_0) \subset \Lambda^1_{u_1}(w_1)$, *with* $p = p_0/p_1$ and apply the previous lemmas. ∎

In general, let $\Phi(t) = \int_0^t (u_1 u_0^{-1})^*_{u_0}(s)\, ds$. Then, for every $y > 0$, we have that $\lambda_f^{u_1}(y) \leqslant \Phi(\lambda_f^{u_0}(y))$ and, hence, $f^*_{u_1}(x) \leqslant f^*_{u_0}(\Phi^{-1}(x))$ for every $x \leqslant \int_{\mathbf{R}^n} u_1(y)\, dy = \Phi(\infty)$. Then, since

$$\int_0^\infty f^*_{u_1}(x)\, w_1(x)\, dx = \int_0^{\Phi(\infty)} f^*_{u_1}(x)\, w_1(x)\, dx \leqslant \int_0^\infty f^*_{u_0}(\Phi^{-1}(x))\, w_1(x)\, dx$$
$$= \int_0^\infty f^*_{u_0}(t)\, w_1(\Phi(t))\, \Phi'(t)\, dt = A,$$

using (3) we obtain that $A \leqslant (\int_0^\infty f^*_{u_0}(x))^p\, w_0(x)\, dx)^{1/p}$ if and only if

$$\int_0^\infty \left(\int_0^{\Phi(x)} w_1(s)\, ds\right)^{p'-1} \left(\int_0^x w_0(s)\, ds\right)^{1-p'} w_1(x)\, dx < +\infty. \tag{4}$$

Therefore, (4) is a sufficient condition for the desired embedding. Similarly, if we set $\varphi(t)=\int_0^t (u_0 u_1^{-1})^*_{u_1}(s)\,ds$, we can show that a necessary condition for the embedding to hold, is that

$$\int_0^\infty \left(\int_0^{\varphi^{-1}(y)} w_1(s)\,ds\right)^{p'-1} \left(\int_0^y w_0(s)\,ds\right)^{1-p'} w_1(y)\,dy < +\infty.$$

By Theorem 2.8 one can identify the dual space of $\Lambda^p_u(w)$, if $1<p<\infty$ and $\int_0^\infty w(x)\,dx=+\infty$, with the space $\Gamma^{p'}_u(\tilde{w})$. By Remark 2.5, the dual space of $\Lambda^{p,q}_u(w)$ can be identified (if $\int_0^\infty w=\infty$) with the space $\Gamma^{q'}_u(\hat{w})$ where $\hat{w}(x)=x^{q'}(\int_0^x w(s)\,ds)^{-q'/p-1}\,w(x)$.

If $p\leqslant 1$, the analogy to Theorem 2.8 is much easier.

THEOREM 2.12. *Suppose $p\leqslant 1$, $v(x)$ and $g(x)$ are nonnegative measurable functions on $\mathbf{R}^+$, with v locally integrable, and u_j $(j=0,1)$ are two weights in $\mathbf{R}^n$. Then,*

$$\sup_{f\in \Lambda^p_{u_0}(w_0)} \left(\int_0^\infty f^*_{u_1}(x)\,g(x)\,dx\right) \|f\|^{-1}_{\Lambda^p_{u_0}(w_0)} \approx \sup \left(\int_0^{u_1(A)} g(x)\,dx\right)\left(\int_0^{u_0(A)} v(x)\,dx\right)^{-1/p},$$

where the supremum extends over all measurable sets A in $\mathbf{R}^n$.

In particular, if $u_j=1$, then

$$\sup \left(\int_0^\infty f(x)\,g(x)\,dx\right)\left(\int_0^\infty f(x)^p\,v(x)\,dx\right)^{-1/p} \approx \sup_{r>0}\left(\int_0^r g(x)\,dx\right)\left(\int_0^r v(x)\,dx\right)^{-1/p},$$

where the first supremum extends over all nonincreasing functions f.

Proof. To prove the inequality $\geqslant$, it is enough to consider $f=\chi_A$, and for the other inequality, we proceed as in Proposition 2.6. ∎

3. THE HARDY OPERATOR

Let $Sf(x)=\int_0^x f(s)\,ds$ be the Hardy operator. The classic Hardy inequality (see [5]) gives the boundedness of $S\colon L^p\to L^p(x^{-p}\,dx)$ for $p>1$. Muckenhoupt (see [7]) gave necessary and sufficient conditions on the weights u and v such that the operator $S\colon L^p(u)\to L^p(v)$ is bounded for $1\leqslant p\leqslant\infty$. Later, Bradley [3] generalized this result by considering different exponents $S\colon L^p(u)\to L^q(v)$, for $1\leqslant p\leqslant q\leqslant\infty$. The complete characterization of the boundedness of the operator $S\colon L^p(u)\to L^q(v)$ for $1\leqslant p$, $q\leqslant\infty$, can be found in [6].

In [9], Sawyer characterizes, under certain restrictions on the exponents p, q, r, and s, the weights u and v such that $S\colon L_u^{p,q}\to L_v^{r,s}$. In this section we study the boundedness of $S\colon \Lambda_{u_0}^{p_0}(w_0)\to\Lambda_{u_1}^{p_1}(w_1)$ with $p_0\leqslant p_1$, under certain restrictions in the weight w_0.

THEOREM 3.1. *Let* $0<p_0, p_1<\infty$.

(a) *If* $p_0>1$, *then* $S\colon \Lambda_{u_0}^{p_0}(w_0)\to\Lambda_{u_1}^{p_1,\infty}(w_1)$ *is bounded if and only if*

$$\sup_{x>0}\left(\|\chi_{(0,x)}u_0^{-1}\|_{\Gamma_{u_0}^{p_0'}(\tilde{w}_0)}+x\left(\int_0^\infty w_0(s)\,ds\right)^{-1/p_0}\right)\left(\int_0^{\int_x^\infty u_1(t)\,dt} w_1(s)\,ds\right)^{1/p_1}<+\infty. \tag{5}$$

(b) *If* $p_0\leqslant 1$, *then* $S\colon \Lambda_{u_0}^{p_0}(w_0)\to\Lambda_{u_1}^{p_1,\infty}(w_1)$ *is bounded if and only if*

$$\sup_{x>0}\ \sup_{0<r\leqslant u_0(\mathbf{R}^n)}\frac{\int_0^r(\chi_{(0,x)}u_0^{-1})_{u_0}^*(s)\,ds}{(\int_0^r w_0(s)\,ds)^{1/p_0}}\left(\int_0^{\int_x^\infty u_1(t)\,dt} w_1(s)\,ds\right)^{1/p_1}<+\infty. \tag{6}$$

Observe that if $w_0=1$ and $p_0<1$, then the supremum in r on (6) is infinite and hence, that quantity is finite only for trivial weights u_i in the sense that $u_0=+\infty$ in $(0,x)$ whenever $\int_x^\infty u_1(s)\,ds>0$. For the case of Lorentz spaces $L_u^{p,q}$, this was proved by E. Sawyer in [9].

Proof. (a) Let $f\in\Lambda_{u_0}^{p_0}(w_0)$ and assume that $f\geqslant 0$. Then, for every $t\in[x,\infty)$,

$$\int_0^x f(s)\,ds\leqslant S(\chi_{(0,x)}f)(t),$$

and hence, if we write $\xi<\int_0^x f(s)\,ds$, we get

$$\int_x^\infty u_1(s)\,ds\leqslant\int_{\{t:\,S(\chi_{(0,x)}f)(t)>\xi\}}u_1(s)\,ds=\lambda_{S(\chi_{(0,x)}f)}^{u_1}(\xi).$$

Then,

$$\begin{aligned}\xi\left(\int_0^{\int_x^\infty u_1(t)\,dt}w_1(s)\,ds\right)^{1/p_1}&\leqslant\xi\left(\int_0^{\lambda_{S(\chi_{(0,x)}f)}^{u_1}(\xi)}w_1(s)\,ds\right)^{1/p_1}\\&\leqslant\sup_{y>0}y\left(\int_0^{\lambda_{S(\chi_{(0,x)}f)}^{u_1}(y)}w_1(s)\,ds\right)^{1/p_1}\\&=\|S(f\chi_{(0,x)})\|_{\Lambda_{u_1}^{p_1,\infty}(w_1)}\leqslant C\,\|f\chi_{(0,x)}\|_{\Lambda_{u_0}^{p_0}(w_0)}\leqslant C\,\|f\|_{\Lambda_{u_0}^{p_0}(w_0)}.\end{aligned}$$

Therefore, taking the supremum over all $\xi < \int_0^x f(s)\,ds$, we get

$$\sup_{x>0}\left(\sup_f \frac{\int_0^x f(t)\,dt}{\|f\|_{\Lambda^{p_0}_{u_0}(w_0)}}\right)\left(\int_0^{\int_x^\infty u_1(t)\,dt} w_1(s)\,ds\right)^{1/p_1} < \infty,$$

and the first factor equals $\|\chi_{(0,x)}u_0^{-1}\|_{\Gamma^{p_0'}_{u_0}(\tilde{w}_0)} + x\left(\int_0^\infty w_0(s)\,ds\right)^{-1/p_0}$, which shows (5).

Conversely, let $f \geqslant 0$ in $\Lambda^{p_0}_{u_0}(w_0)$ and set x_k such that if $\int_0^\infty f(t)\,dt > 2^k$ then $Tf(x_k) = 2^k$. Then, by Proposition 2.3,

$$\|Sf\|^{p_1}_{\Lambda^{p_1,\infty}_{u_1}(w_1)} \leqslant C \sup_{k\in\mathbf{Z}} 2^{kp_1}\int_0^{\lambda^{u_1}_{Sf}(2^k)} w_1(s)\,ds.$$

Since,

$$\lambda^{u_1}_{Sf}(2^k) = \int_{\{x:\,(Sf)(x)>2^k\}} u_1(x)\,dx \leqslant \int_{x_k}^\infty u_1(x)\,dx,$$

we get

$$\begin{aligned}
\|Sf\|^{p_1}_{\Lambda^{p_1,\infty}_{u_1}(w_1)} &\leqslant C\sup_{k\in\mathbf{Z}}\left(\int_0^{x_k} f(t)\,dt\right)^{p_1}\left(\int_0^{\int_{x_k}^\infty u_1(t)\,dt} w_1(s)\,ds\right)\\
&\leqslant C\sup_{k\in\mathbf{Z}}\|f\chi_{(0,x_k)}\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)}\Bigg(\|\chi_{(0,x_k)}u_0^{-1}\|_{\Gamma^{p_0'}_{u_0}(\tilde{w}_0)}\\
&\qquad + x_k\left(\int_0^\infty w_0(s)\,ds\right)^{-1/p_0}\Bigg)\left(\int_0^{\int_{x_k}^\infty u_1(t)\,dt} w_1(s)\,ds\right)\\
&\leqslant C\sup_{k\in\mathbf{Z}}\|f\chi_{(0,x_k)}\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)} \leqslant C\|f\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)}.
\end{aligned}$$

(b) We just need to make the obvious changes applying Theorem 2.12. ▌

To study the strong boundedness we need the following lemma.

LEMMA 3.2. *Let $p_0 \leqslant p_1$. Then w_0 satisfies that for every $\{t_k\}_k \subset \mathbf{R}^+$,*

$$\left(\sum_k\left(\int_0^{t_k} w_0(s)\,ds\right)^{p_1/p_0}\right)^{p_0/p_1} \leqslant C\int_0^{\sum_k t_k} w_0(s)\,ds, \tag{7}$$

if and only if, for every collection of functions $\{f_k\}_k$ in $\Lambda^{p_0}_{u_0}(w_0)$ with pairwise disjoint support, there exists a constant $C>0$ such that

$$\sum_k \|f_k\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)} \leqslant C\left\|\sum_k f_k\right\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)}.$$

Proof. To prove the necessary condition, we use Theorem 2.1 and the Minkowski integral inequality to obtain

$$\begin{aligned}
\sum_k \|f_k\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)} &= \sum_k \left(\int_0^\infty y^{p_0-1}\left(\int_0^{\lambda^{u_0}_{f_k}(y)} w_0(t)\,dt\right)dy\right)^{p_1/p_0} \\
&\leqslant \left(\int_0^\infty y^{p_0-1}\left(\sum_k\left(\int_0^{\lambda^{u_0}_{f_k}(y)} w_0(t)\,dt\right)^{p_1/p_0}\right)^{p_0/p_1} dy\right)^{p_1/p_0} \\
&\leqslant C\left(\int_0^\infty y^{p_0-1}\left(\int_0^{\sum_k \lambda^{u_0}_{f_k}(y)} w_0(t)\,dt\right)dy\right)^{p_1/p_0} \\
&= C\left(\int_0^\infty y^{p_0-1}\left(\int_0^{\lambda^{u_0}_{\sum_k f_k}(y)} w_0(t)\,dt\right)dy\right)^{p_1/p_0} \\
&= C\left\|\sum_k f_k\right\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)}.
\end{aligned}$$

Conversely, $(\int_0^{t_k} w_0(s)\,ds)^{1/p_0} = \|f_k\|_{\Lambda^{p_0}_{u_0}(w_0)}$, where $(f_k)^*_{u_0} = \chi_{(0,t_k)}$. If $F_k(x) = \chi_{(t_k-1, t_{k-1}+t_k)}(u_0(B(0,|x|)))$ for $x \in \mathbf{R}^n$, then one can easily check that $(F_k)^*_{u_0}(s) = (f_k)^*_{u_0}$ and F_k have pairwise disjoint support. Therefore,

$$\begin{aligned}
\sum_k \left(\int_0^{t_k} w_0(s)\,ds\right)^{p_1/p_0} &= \sum_k \|f_k\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)} = \sum_k \|F_k\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)} \\
&\leqslant C\left\|\sum_k F_k\right\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)} = C\left(\int_0^{\sum_k t_k} w_0(s)\,ds\right)^{p_1/p_0}. \quad \blacksquare
\end{aligned}$$

THEOREM 3.3. *Let $0 < p_0 \leqslant p_1$, and let us assume that condition (7) holds. Then $S\colon \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1}_{u_1}(w_1)$ is bounded, if and only if condition (5) holds for $p_0 > 1$ and condition (6) holds for $p_0 \leqslant 1$.*

Proof. The necessary condition is trivial by Theorem 3.1.

Conversely, as in Theorem 3.1 and using the fact that $2^{kp_1} \approx (\int_{x_{k-1}}^{x_k} f(t)\,dt)^{p_1}$, we get, by Lemma 3.2,

$$\begin{aligned}
\|Sf\|^{p_1}_{\Lambda^{p_1}_{u_1}(w_1)} &\leqslant C \sum_{k\in\mathbf{Z}} \left(\int_{x_{k-1}}^{x_k} f(t)\,dt\right)^{p_1}\left(\int_0^{\int_{x_k}^\infty u_1(t)\,dt} w_1(s)\,ds\right) \\
&\leqslant C \sum_{k\in\mathbf{Z}} \|f\chi_{(x_{k-1},x_k)}\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)}\Bigg(\|\chi_{(0,x_k)}u_0^{-1}\|_{\Gamma^{p_0'}_{u_0}(\tilde{w}_0)} \\
&\qquad + x_k\left(\int_0^\infty w_0(s)\,ds\right)^{-1/p_0}\Bigg)\left(\int_0^{\int_{x_k}^\infty u_1(t)\,dt} w_1(s)\,ds\right) \\
&\leqslant C \sum_{k\in\mathbf{Z}} \|f\chi_{(x_{k-1},x_k)}\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)} \leqslant C\|f\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)}.
\end{aligned}$$

Analogously the case $p_0 \leqslant 1$ is proved. $\blacksquare$

Set $\tilde{S}f(x) = \int_x^\infty f(t)\,dt$. Then we have the following:

THEOREM 3.4. *Let* $0 < p_0, p_1 < \infty$.

(a) *If* $p_0 > 1$, *then* $\tilde{S}\colon \Lambda_{u_0}^{p_0}(w_0) \to \Lambda_{u_1}^{p_1,\infty}(w_1)$ *is bounded, if and only if*

$$\sup_{x>0}\left(\|\chi_{(0,x)}u_0^{-1}\|_{\Gamma_{u_0}^{p_0'}(\tilde{w}_0)} + x\left(\int_0^\infty w_0(s)\,ds\right)^{-1/p_0}\right)\left(\int_0^{\int_0^x u_1(t)\,dt} w_1(s)\,ds\right)^{1/p_1} < +\infty. \tag{8}$$

(b) *If* $p_0 \leqslant 1$, *then* $\tilde{S}\colon \Lambda_{u_0}^{p_0}(w_0) \to \Lambda_{u_1}^{p_1,\infty}(w_1)$ *is bounded, if and only if*

$$\sup_{x>0}\ \sup_{0<r\leqslant u_0(\mathbf{R}^n)} \frac{\int_0^r (\chi_{(0,x)}u_0^{-1})_{u_0}^*(s)\,ds}{\left(\int_0^r w_0(s)\,ds\right)^{1/p_0}}\left(\int_0^{\int_0^x u_1(t)\,dt} w_1(s)\,ds\right)^{1/p_1} < +\infty. \tag{9}$$

(c) *If condition* (7) *holds, then* $\tilde{S}\colon \Lambda_{u_0}^{p_0}(w_0) \to \Lambda_{u_1}^{p_1}(w_1)$ *is bounded, if and only if condition* (8) *holds for* $p_0 > 1$ *and condition* (9) *holds for* $p_0 \leqslant 1$.

Proof. Using that $\int_x^\infty f(s)\,ds \leqslant \tilde{S}(\chi_{(x,\infty)}f)(t)$ for every $t \leqslant x$, we conclude the result as in the previous theorems. ∎

4. WEIGHTS SATISFYING CONDITION (7)

In this section we study examples of weights which satisfy condition (7).

(I) If w_0 is a nondecreasing weight, then (7) holds for every $p_0 \leqslant p_1$ since

$$\left(\sum_k \left(\int_0^{t_k} w_0(s)\,ds\right)^{p_1/p_0}\right)^{p_0/p_1} \leqslant \sum_k \int_0^{t_k} w_0(s)\,ds = \sum_k t_k \int_0^1 w_0(st_k)\,ds$$
$$\leqslant \sum_k t_k \int_0^1 w_0\left(s\sum_k t_k\right)ds = \int_0^{\sum_k t_k} w_0(s)\,ds.$$

(II) Weights obtained by interpolation of Lorentz spaces by the real method with a function parameter:

DEFINITION 4.1. (see [4]). *Let* $\varphi\colon \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $\varphi(1) = 1$ *and* $\varphi \in \mathscr{C}^1$. *Then,* φ *is a function parameter,* $\varphi \in B_\Psi$, *if*

$$0 < \alpha_\varphi = \inf_{t>0} \frac{t\varphi'(t)}{\varphi(t)} \leqslant \sup_{t>0} \frac{t\varphi'(t)}{\varphi(t)} = \beta_\varphi < 1.$$

Proposition 4.2. *Let $\varphi \in B_\psi$ and let $\{a_n\}_n$ be a sequence of positive numbers such that $\sum_n a_n < +\infty$. Then, for every $1/\alpha_\varphi \leqslant q < +\infty$, $\sum_n \varphi^q(a_n) \leqslant \varphi^q(\sum_n a_n)$.*

Proof. It is known (see [11]) that if $h(t) = t\varphi'(t)/\varphi(t)$, then

$$\varphi(t) = \exp\left(\int_1^t h(s)\frac{ds}{s}\right),$$

with $0 < \alpha_\varphi \leqslant h(t) \leqslant \beta_\varphi < 1$ for all $t > 0$. Clearly, we only have to prove that if $0 < x \leqslant y < +\infty$, then $\varphi^q(x) + \varphi^q(y) \leqslant \varphi^q(x+y)$, or equivalently

$$\exp\left(\int_1^x qh(s)\frac{ds}{s}\right) + \exp\left(\int_1^y qh(s)\frac{ds}{s}\right) \leqslant \exp\left(\int_1^{x+y} qh(s)\frac{ds}{s}\right).$$

That is, dividing both sides by the right hand side term,

$$\exp\left(-\int_x^{x+y} qh(s)\frac{ds}{s}\right) + \exp\left(-\int_y^{x+y} h(s)\frac{ds}{s}\right) \leqslant 1,$$

and, since $q\alpha_\varphi \geqslant 1$ this is an immediate consequence of the equality $x/(x+y) + y/(x+y) = 1$. ▮

Given $\varphi \in B_\psi$, we consider $w_0(t) = (t^{1/p_0'}/\varphi(t))^{p_0}$.

Proposition 4.3. *If $1 < p_0 \leqslant p_1$ and $1 \leqslant p_1(1-\beta_\varphi)$, then $w_0(t) = (t^{1/p_0'}/\varphi(t))^{p_0}$ satisfies condition* (7).

Proof. We have that

$$\int_0^{t_k} w_0(s)\,ds = \int_0^{t_k} \left(\frac{s^{1/p_0'}}{\varphi(s)}\right)^{p_0} ds = \int_0^{t_k} \left(\frac{s}{\varphi(s)}\right)^{p_0} \frac{ds}{s}.$$

Now, since $t/\varphi(t) \in B_\psi$, we get (see [11]) that $\int_0^{t_k} w_0(s)\,ds \approx (t_k/\varphi(t_k))^{p_0}$ and hence,

$$\left(\sum_k \left(\int_0^{t_k} w_0(s)\,ds\right)^{p_1/p_0}\right)^{p_0/p_1} \approx \left(\sum_k \left(\frac{t_k}{\varphi(t_k)}\right)^{p_1}\right)^{p_0/p_1}.$$

Simple computations show that if $\phi(t) = t/\varphi(t)$ then $\alpha_\phi = 1 - \beta_\varphi$ and, since by hypothesis $p_1(1-\beta_\varphi) = p_1\alpha_\Phi \geqslant 1$, we can apply Proposition 4.2 to get

$$\left(\sum_k \left(\int_0^{t_k} w_0(s)\,ds\right)^{p_1/p_0}\right)^{p_0/p_1} \leqslant C\left(\frac{\sum_k t_k}{\varphi(\sum_k t_k)}\right)^{p_0}$$
$$\approx \int_0^{\sum_k t_k} \left(\frac{s}{\varphi(s)}\right)^{p_0} \frac{ds}{s} \approx \int_0^{\sum_k t_k} w_0(s)\,ds. \quad ▮$$

COROLLARY 4.4. (see[9]). *Suppose* $1 \leqslant q_0 \leqslant q_1 < \infty$ *and* $1 < p_0, p_1 < \infty$, *with* $p_0 \leqslant q_1$. *Then, the operator* $S\colon L_{u_0}^{p_0,q_0} \to L_{u_1}^{p_1,q_1}$ *is bounded, if and only if*

$$\sup_{x>0} \left(\int_x^\infty u_1(s)\, ds \right)^{1/q_1} \|\chi_{(0,x)} u_0^{-1}\|_{L_{u_0}^{p_0',q_0'}} < +\infty.$$

Proof. It is enough to take $\varphi_i(t) = t^{1/p_i'}$ for $i = 0, 1$ and apply Theorem 3.3. ∎

In particular, taking $p_i = q_i$ we get the condition of Bradley and Muckenhoupt [3, 7]. (See also [6].)

PROPOSITION 4.5. *Given a weight* w, *let us consider the function*

$$\varphi_w(r) = \left(\int_0^r \left(\frac{1}{s} \int_0^s w(x)\, dx \right)^{1-p'} ds \right)^{1/p'}.$$

Then, the dual space of $\Lambda_u^p(w)$, $\Gamma_u^{p'}(\tilde{w})$ *coincides with the Lorentz space* $\Lambda_u^{p'}(\tilde{w})$, *if and only if* $\beta_{\varphi_w} < 1$.

Proof. We know (see [1, 8]) that $\Gamma_u^{p'}(\tilde{w}) \equiv \Lambda_u^{p'}(\tilde{w})$, if and only if $\tilde{w} \in B_{p'}$. Hence

$$\int_r^\infty \frac{\tilde{w}(x)}{x^{p'}}\, dx \leqslant C \frac{1}{r^{p'}} \int_0^r \tilde{w}(x)\, dx,$$

and therefore,

$$\frac{1}{p'-1} \left(\int_0^r w(t)\, dt \right)^{1-p'} = \int_r^\infty \left(\int_0^x w(t)\, dt \right)^{-p'} w(x)\, dx = \int_r^\infty \frac{\tilde{w}(x)}{x^{p'}}\, dx$$

$$\leqslant C \frac{1}{r^{p'}} \int_0^r \tilde{w}(x)\, dx = C \frac{1}{r^{p'}} \int_0^r s^{p'} \left(\int_0^s w(x)\, dx \right)^{-p'} w(s)\, ds$$

$$= \frac{C}{r^{p'}} \left(\frac{r^{p'}}{1-p'} \left(\int_0^r w(x)\, dx \right)^{-p'} + \frac{p'}{p'-1} \int_0^r \left(\frac{1}{s} \int_0^s w(x)\, dx \right)^{1-p'} ds \right).$$

Then, $(1+C) \left(\int_0^r w(s)\, ds\right)^{1-p'} \leqslant Cp' \, (1/r^{p'}) \, \varphi_w^{p'}(r)$, that is,

$$\frac{r(\varphi_w)'(r)}{\varphi_w(r)} \leqslant \frac{C}{C+1},$$

or equivalently $\beta_{\varphi_w} < 1$. ∎

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