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Two phase transitions for the contact process on small worlds

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Abstract

In our version of Watts and Strogatz's small world model, space is a d-dimensional torus in which each individual has in addition exactly one long-range neighbor chosen at random from the grid. This modification is natural if one thinks of a town where an individual's interactions at school, at work, or in social situations introduce long-range connections. However, this change dramatically alters the behavior of the contact process, producing two phase transitions. We establish this by relating the small world to an infinite "big world" graph where the contact process behavior is similar to the contact process on a tree. We then consider the contact process on a slightly modified small world model in order to show that its behavior is decidedly different from that of the contact process on a tree.
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1. Introduction

Small world graphs were first introduced by Watts and Strogatz [23]. In their model, they take a one-dimensional ring lattice and connect all pairs of vertices that are distance m or less. They then "rewire" each edge with probability p by moving one of the ends at random, where the new end is chosen uniformly. This leads to a graph that has small diameter but, in contrast to the Erdös–Renyi model, has a nontrivial density of triangles. These are both properties that they

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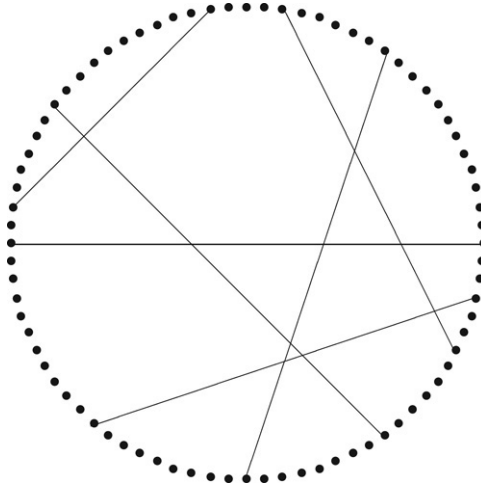 *E-mail address:* pauljung@gmail.com (P. Jung).

Fig. 1. The Newman and Watts version of a small world graph. For simplicity we have not drawn the short-range connections.

observed in the collaboration graph of film actors, the power grid, and the neural network of the nematode, *C. elegans.*

The small world model has been extensively studied, although most investigators have found it more convenient to study the Newman and Watts [15] version in which all pairs of vertices that are distance m or less are connected (we call these short-range connections), but in addition there is a density p of short-cuts that connect vertices to long-range neighbors chosen at random from the graph. (See Fig. 1). Newman and Watts aren't very precise about what this means but one can, for example if there are n vertices, flip a coin with probability $2p/(n-1)$ of heads to determine if each edge is present.

The graph theoretic properties (e.g., the average distance between two points and the clustering coefficient) of small world graphs are well understood (see [2] for the physicist's view point or [4] for rigorous results). Our focus here will be on the behavior of processes taking place on these networks. Chapter six of Watts [22] discusses the SIR (susceptible–infected–removed) disease model on small world graphs in which individuals that are *susceptible* (state 0) become *infected* (state 1) at a rate proportional to the number of infected neighbors. Infected individuals, after a random amount of time of fixed distribution, become *removed* (state 2), i.e., forever immune to further infection.

The SIR model on the small world graph has a detailed theory due to its connection to percolation: we draw an oriented edge from x to y if x will succeed in infecting y during the time it is infected and the persistence of the epidemic is equivalent to percolation. See Section 8.2 of Newman [14] for what is known about the SIR models on small world graphs. Here we will investigate the more difficult SIS (susceptible–infected–susceptible) model, known to probabilists as the contact process, where recovered individuals are immediately susceptible. Moreno, Pastor-Satorras and Vespignani [11] have studied this model by simulation, but we know of no rigorous results for the contact process on the small world. Berger, Borgs, Chayes and Saberi [5] have proved rigorous results for the contact process on the preferential attachment graph of Barabási and Albert [3]. For more on these results and on random graphs in general see [6].
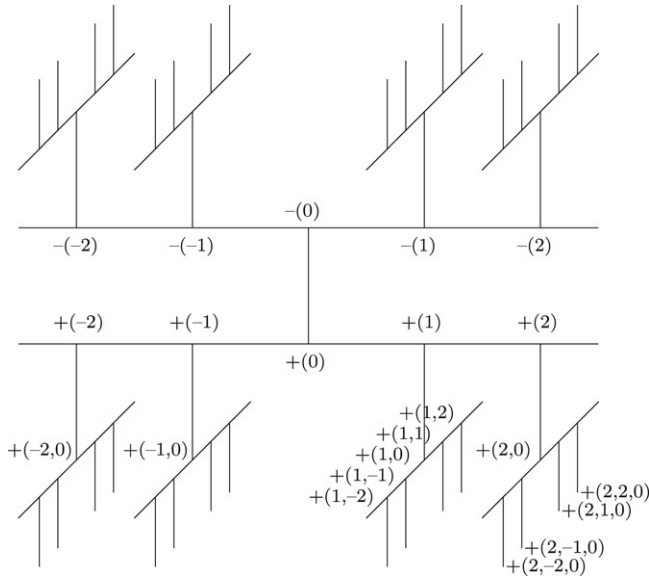
Fig. 2. The big world graph, $\mathcal{B}_1$.

Our version of the small world will be as follows. We start with a d-dimensional torus $(\mathbf{Z} \bmod R)^d$ and connect all pairs of vertices within distance m of each other using the $\| \cdot \|_\infty$ norm. We require R to be even so that we can partition the R^d vertices into $R^d/2$ pairs. Consider all such partitions and then pick one uniformly at random. A new edge is then drawn between each pair of vertices in the chosen partition. When $m \ll R$, we think of these new edges as long-range connections. We will call this graph $\mathcal{S}_m^R$, keeping in mind that this is a random graph.

The reason for insisting that all individuals have exactly one long-range neighbor is that we can define an associated "big world" graph $\mathcal{B}_m$ that is nonrandom. We define $\mathcal{B}_m$ to consist of all vectors $\pm(z_1, \ldots, z_n)$ with $n \geq 1$ components with $z_j \in \mathbf{Z}^d$ and $z_j \neq 0$ for $j < n$. Neighbors in the positive half-space are defined as follows: a point $+(z_1, \ldots, z_n)$ is adjacent to $+(z_1, \ldots, z_n + y)$ for all y with $0 < \|y\|_\infty \leq m$ (these are the short-range neighbors of $+(z_1, \ldots, z_n)$). The long-range neighbor is

$$
\begin{array}{ll}
+(z_1, \ldots, z_n, 0) & \text{if } z_n \neq 0 \\
+(z_1, \ldots, z_{n-1}) & \text{if } z_n = 0, n > 1 \\
-(0) & \text{if } z_n = 0, n = 1.
\end{array}
$$

See Fig. 2 for a picture of the one dimensional case with $m = 1$. Of course in this case the graph is a tree.

The drawing of the big world in Fig. 2 is more convenient for our proof, but the graph can be more succinctly described if all the long-range edges point down. In this case it is the free product $\mathbf{Z}^d * \{0, 1\}$ where the second factor is $\mathbf{Z} \bmod 2$. Elements of this group have the form $z_0 1 z_1 1 \ldots 1 z_n$ where $z_i \in \mathbf{Z}^d - \{0\}$ for $0 < i < n$. In words, this is the point you reach by moving z_0 in the first copy, going down a long-range edge, moving sideways by z_1, going down a long-range edge, etc.

We will consider the discrete-time contact process. On either the small world or the big world, an infected individual lives for one unit of time. During its infected period it will infect some of its neighbors. All infection events are independent, and each site that receives at least one infection is occupied with an infected individual at the next time step. A site infects itself or its short-range neighbors, each with probability $\alpha/(2m + 1)^d$. It infects its long-range neighbor with probability β. Let $\lambda = \alpha + \beta$ and $r = \alpha/\beta$. We are interested in the phase diagram as a function of (α, β) but in some cases we fix the ratio $0 < r < \infty$ and vary λ. We will use B_t to denote the contact process on the big world and ξ_t for the contact process on the small world.

The number of sites within distance n of a given site in the big world grows exponentially fast, so it is natural to guess that its contact process will behave like the contact process on a tree. Consider a tree in which each vertex has the same degree, let 0 be a distinguished vertex (the origin) of the tree, and let A_t^0 be the set of infected sites at time t on the tree starting from 0 occupied. For the contact process on the tree or on the big world, we can define two critical values:

$$\lambda_1 = \inf\{\lambda : \mathbf{P}(|A_t^0| = 0 \text{ eventually}) < 1\} \tag{1}$$
$$\lambda_2 = \inf\left\{\lambda : \liminf_{t\to\infty} \mathbf{P}(0 \in A_t^0) > 0\right\}.$$

We call λ_1 the weak survival critical value and λ_2 the strong survival critical value. Pemantle [16] showed that for homogeneous trees where every vertex has at least four neighbors, $\lambda_1 < \lambda_2$. Liggett [9] then extended Pemantle's result to trees with degree 3 by finding numerical bounds on the two critical values and thus showing that they are different. Later Stacey [21] found a proof that did not rely on numerical bounds. For another approach to separating the critical values see [8]. Other references concerning the contact process on trees include [24] and [18–20].

Theorem 1.1. *For each $0 < r < \infty$ there exists $\bar{m}$ such that for all $m \geq \bar{m}$, $\lambda_1 < \lambda_2$ for the contact process on $\mathcal{B}_m$.*

The proof of Theorem 1.1 will follow from Propositions 1.1 and 1.2 below. Our approach to proving Theorem 1.1 is to find an upper bound U (Proposition 1.1) on the limiting weak survival critical value $\limsup_{m\to\infty} \lambda_1$, a lower bound L (Proposition 1.2) on the limiting strong survival critical value $\liminf_{m\to\infty} \lambda_2$, and then to show that $U < L$.

To obtain the two bounds, we will compare the contact process to the branching random walk which has births like the contact process, but allows more than one particle at a site. More explicitly, each individual lives for one unit of time and sends offspring to itself and to its short-range neighbors with probability $\alpha/(2m + 1)^d$ and to long-range neighbors with probability β. Birth events are independent, and the state of a given site on the next time step is the total number of births there.

The number of particles of the branching random walk at a given site stochastically dominates the number of particles of the contact process at a given site. Therefore weak (strong) survival of the contact process on $\mathcal{B}_m$ implies weak (strong) survival of the branching random walk on $\mathcal{B}_m$. However, in order to get an upper bound on the weak survival critical value of the contact process, we need the opposite implication. This is achieved by showing that in the limit as $m \to \infty$, the behavior of the contact process is essentially the same as that of the branching random walk. The following proposition, proved in Section 2, shows that the trivial necessary condition for weak survival gives the asymptotic boundary of the survival regime.
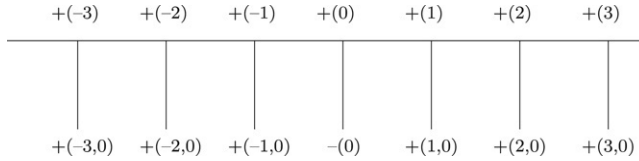
$$+(-3)\qquad +(-2)\qquad +(-1)\qquad +(0)\qquad +(1)\qquad +(2)\qquad +(3)$$

$$+(-3,0)\quad +(-2,0)\quad +(-1,0)\quad -(0)\quad +(1,0)\quad +(2,0)\quad +(3,0)$$

Fig. 3. The comb $\mathcal{C}_1$. We have rotated the edge from $+(0)$ to $-(0)$ to make the picture better match the name.

Proposition 1.1. *If $\alpha + \beta > 1$ then there exists $m_0(\alpha, \beta)$ such that for all $m > m_0(\alpha, \beta)$, the contact process on $\mathcal{B}_m$ survives.*

To obtain a lower bound L on λ_2, we use the fact that strong survival of the contact process on $\mathcal{B}_m$ implies strong survival of the branching random walk on $\mathcal{B}_m$. Let $\lambda_2^{\text{brw}}(\mathcal{B}_m)$ be the strong survival critical value of the branching random walk on $\mathcal{B}_m$. To compute the limit in m of $\lambda_2^{\text{brw}}(\mathcal{B}_m)$, we first compute the phase transition on another graph.

We define the "comb" of degree m, $\mathcal{C}_m$, by restricting $\mathcal{B}_m$ to vertices of the form $\{+(z), +(z, 0), -(0)\}$ and all edges between any of these vertices. As before, $+(z)$ and $+(z, 0)$ are long-range neighbors as are $+(0)$ and $-(0)$. The short-range neighbors of $+(z)$ are $+(z + y)$ for $0 < \|y\|_\infty \le m$. The vertices $+(z, 0)$ and $-(0)$ have no short-range neighbors. To see the reason for the name look at Fig. 3 which gives a picture of the graph for $m = 1, d = 1$. The comb graph is not logically necessary in the proof of lower bound L, however, it gives intuition for the value of L.

Since the comb is like $\mathbb{Z}^d$ the weak survival and strong survival critical values are the same. It will be shown in Section 3 that for any m the branching random walk on the comb survives if the largest eigenvalue of

$$\begin{pmatrix} \alpha & \beta \\ \beta & 0 \end{pmatrix}$$

is larger than 1. Solving the quadratic equation $(\alpha - \lambda)(-\lambda) - \beta^2 = 0$ the largest root is

$$\frac{\alpha + \sqrt{\alpha^2 - 4\beta^2}}{2}.$$

A little algebra shows that this is larger than 1 exactly when $\alpha + \beta^2 > 1$.

We will show in Section 3,

$$\lim_{m \to \infty} \lambda_2^{\text{brw}}(\mathcal{B}_m) = \frac{\alpha + \sqrt{\alpha^2 - 4\beta^2}}{2}$$

which gives us the following proposition.

Proposition 1.2. *If $\alpha + \beta^2 < 1$ then there exists $m_1(\alpha, \beta)$ such that for all $m > m_1(\alpha, \beta)$, there is no strong survival for the contact process on $\mathcal{B}_m$.*

Comparing the above with Proposition 1.1 shows that we have separated the critical values for m large enough when $0 < r < \infty$.

We have not been able to generalize Stacey's elegant argument to the big world graph. However, we have been able to establish the following ingredient used in his argument, which was proved for the tree by Morrow, Schinazi and Zhang [12]. Here, and in what follows, we will write 0 as shorthand for $+(0)$.

Theorem 1.2. *There exist constants $C_1 > 0$ and C_2 such that*

$$\exp(C_2 t) \leq \mathbf{E}(|B_t^0|) \leq \frac{1}{C_1} \exp(C_2 t).$$

Moreover, $C_2 < 0$ when $\lambda < \lambda_1$, $C_2 = 0$ when $\lambda = \lambda_1$, and $C_2 > 0$ when $\lambda > \lambda_1$.

Let $\tau_{\mathcal{B}} = \min\{t : B_t^0 = \emptyset\}$ be the extinction time of the contact process on the big world. It follows from Theorem 1.2 that $\tau_{\mathcal{B}} < \infty$ with probability one when $\lambda \leq \lambda_1$ and when $\lambda < \lambda_1$, $\tau_{\mathcal{B}}$ has a geometric tail.

Having established the existence of two phase transitions on the big world, our next question is: How does this translate into behavior of the contact process on the small world? Let $\sigma_{\mathcal{B}} = \min\{t : B_t^0 = \emptyset \text{ or } 0 \in B_t^0\}$ be the first time that the infection either dies out or comes back to the origin starting from one infection there at time 0. When $\lambda_1 < \lambda < \lambda_2$, $\tau_{\mathcal{B}}$ and $\sigma_{\mathcal{B}}$ are both infinite with positive probability, and when $\lambda > \lambda_2$, $\sigma_{\mathcal{B}}$ is almost-surely finite. Let $\tau_{\mathcal{S}} = \min\{t : \xi_t^0 = \emptyset\}$ and $\sigma_{\mathcal{S}} = \min\{t \geq 1 : \xi_t^0 = \emptyset \text{ or } 0 \in \xi_t^0\}$ be the corresponding times for the contact process on the small world.

Theorem 1.3. *Writing $\Rightarrow$ for convergence in distribution as $R \rightarrow \infty$ we have*

(a) $\tau_{\mathcal{S}}$ *is stochastically bounded above by* $\tau_{\mathcal{B}}$ *and* $\tau_{\mathcal{S}} \Rightarrow \tau_{\mathcal{B}}$.
(b) $\sigma_{\mathcal{S}}$ *is stochastically bounded above by* $\sigma_{\mathcal{B}}$ *and* $\sigma_{\mathcal{S}} \Rightarrow \sigma_{\mathcal{B}}$.

Theorems 1.1 and 1.3 show that the small world contact process exhibits different behavior in the three regimes $(0, \lambda_1)$, (λ_1, λ_2), and (λ_2, ∞). In the first interval the process dies out. In the second, $\sigma_{\mathcal{B}} = \infty$ with positive probability, so the time for the infection to return to 0 is not tight (in the sense of weak convergence). In the third interval, $\sigma_{\mathcal{B}} < \infty$ with probability one, so the infection returns to 0 in a time that has a limit as the system size increases. This is especially interesting since other researchers have studied the contact process on the small world by simulation without having noticed this qualitative difference between (λ_1, λ_2), and (λ_2, ∞).

Since the small world is a finite graph, the infection will eventually die out. However, by analogy with results for the d-dimensional contact process on a finite set, we expect that if the process does not become extinct quickly, it will survive for a long time. Durrett and Liu [7] showed that the supercritical contact process on $[0, R]$ survives for an amount of time of order $\exp(cR)$ starting from all ones, while Mountford [13] showed that the supercritical contact process on $[0, R]^d$ survives for an amount of time of order $\exp(cR^d)$. At the moment we are only able to prove the last conclusion for the following modification of the small world contact process: each infected site infects its short-range neighbors with probability $\alpha/(2m + 1)^d$ and its long-range neighbor with probability β, but now in addition, at each time step each infected site may, according to a Bernoulli (γ) coin flip, infect a uniformly random vertex from the entire grid. In other words, at each time step a γ proportion of the infected sites choose a random site to infect (across all sites irrespective of the edges of the graph).

From a modelling point of view, this mechanism is reasonable. In addition to long-range connections with friends at school or work, one has random encounters with people one sits next to on airplanes or meets while shopping in stores. In the language of physics, the model with $\gamma = 0$ has a quenched (i.e., fixed) random environment, while the model with $\beta = 0$ has an annealed environment.

Our strategy for establishing prolonged survival is to show that if the number of infected sites drops below ηR^d, it will with high probability rise to $2\eta R^d$ before dying out. To do this we use the random connections to spread the particles out so that they can grow independently. Ideally

we would use the long-range connections (instead of the random connections) to achieve this; however, we have to deal with unlikely but annoying scenarios such as all infected individuals being long-range neighbors of sites that are respectively short-range neighbors of each other.

Theorem 1.4. *Consider the modified small world model on a d-dimensional torus of radius R with random infections with probability $\gamma > 0$. If $\lambda > \lambda_1$, the lower critical for the contact process on $\mathcal{B}_m$ and we start with all infected individuals then there is a constant $c > 0$ so that the probability the infection persists to time $\exp(cR^d)$ tends to 1 as $R \to \infty$.*

Consider a contact process where the birth probability to any neighbor (long-range or short-range) is λ/M where M is the total number of neighbors including itself. The probability that a particle does not give birth and dies in the next time step is

$$\rho = \left(1 - \frac{\lambda}{M}\right)^M .$$

Since on any graph with n vertices, the probability that all particles die by time 1 is greater or equal to ρ^n, the survival time is bounded above by a geometric with mean ρ^{-n}.

The calculations show that survival for a time that is exponential in the number of vertices is the best possible result on any graph. Stacey proved a result of this type for continuous-time contact processes on trees when $\lambda > \lambda_2$. In the intermediate regime, $\lambda_1 < \lambda < \lambda_2$ this is not true. The closest particle to the origin on the infinite tree grows linearly in time, so the process on a finite tree of radius R survives for a time of order R.

Berger, Borgs, Chayes and Saberi [5] have studied the contact process on Barabási and Albert's [3] preferential attachment graph, which has a power law degree distribution $\sim Ck^{-3}$ and a critical value for prolonged persistence $\lambda_c = 0$. For all fixed $\lambda > 0$, they were able to show survival up to time $\exp(c\lambda^2 n^{1/2})$ for the contact process on a graph with n vertices, but this is sub-exponential in volume. The proof of this result is easy: they combine an estimate for the survival time of the contact process on a star graph with large degree, with the fact that the largest degree in their random graph is $O(n^{1/2})$.

The last paragraph gives one of several open problems:

1. Prove $\lambda_1 < \lambda_2$ in the two dimensional nearest neighbor case.
2. Remove the assumption $\gamma > 0$ from Theorem 1.4.
3. Show that the extinction time τ starting from all sites occupied has $R^{-d} \log(\tau)$ converging to a limit.
4. Prove or disprove survival for time exponential in the volume for the contact process on Barabási and Albert's [3] preferential attachment graph.
5. For graphs with a power law degree distribution, $p_k \sim Ck^{-\alpha}$, see Chapter 4 of [6], show that $\lambda_c = 0$ if and only if $\alpha \leq 3$.

2. An upper bound on the weak survival critical value

To prove Proposition 1.1, consider the branching random walk B_t on the half-space $\mathcal{B}_m^+$ consisting of all points of the form $+(z_1, \ldots, z_n)$. We say a particle is at level l if it is at a point of the form $+(z_1, \ldots, z_l)$.

Clearly, the branching random walk is supercritical when $\alpha + \beta > 1$. The idea here is to run the supercritical process to time t large enough so that we get enough special particles at

different points all of the form $+(z_1, \ldots, z_{l-1}, 0)$ on some level l. We can then consider the half-spaces formed by all points at level greater than or equal to l treating points $+(z_1, \ldots, z_{l-1}, 0)$ as the origins. Each of the special particles starts its own branching random walk on these disjoint half-spaces, and by construction none of these branching random walks will ever run into each other (although a given branching random walk can run into itself). If the expected number of special particles at time t is greater than 1, then a comparison with a branching process shows that the total number of special particles on levels nl, $n \in \mathbb{N}$ at times nt is infinite with positive probability. We then choose m large so that the contact process looks like a branching random walk.

Let us now carry out this plan more rigorously. Letting

$$\tilde{\alpha} = \alpha \left(1 - \frac{1}{(2m+1)^d} \right),$$

we have that $\mathbf{E}(|B_t|) \geq \tilde{\alpha}(\tilde{\alpha} + \beta)^{t-1}$. One way to see this is by disallowing short-range births onto points of the form, $+(z_1, \ldots, z_{\ell(T)-1}, 0)$. The particles in B_t are on levels $0, \ldots, t$ so there is a level $\ell(t)$ where the expected number of particles is at least $\tilde{\alpha}(\tilde{\alpha} + \beta)^{t-1}/(t+1)$. When $\alpha + \beta > 1$ we can choose m and T large enough so that

$$\frac{\tilde{\alpha}(\tilde{\alpha} + \beta)^{T-1}}{T+1} > \frac{3}{\tilde{\alpha}\beta}.$$

Modify the dynamics so that the only births occurring at time $T + 1$ are those across short-range edges on level $\ell(T)$ (excluding births to points of the form $+(z_1, \ldots, z_{\ell(T)-1}, 0)$), and the only births occurring at time $T + 2$ are those across long-range neighbors from level $\ell(T)$ to level $\ell(T) + 1$. These birth at time $T + 2$ will lead to special particles on level $\ell(T) + 1$, and the half-spaces obtained by eliminating the long-range edges just crossed are all disjoint.

The expected number of special particles in the modified branching random walk on level $\ell(T) + 1$ at time $T + 2$ is at least 3. We claim that if m is large then the expected number of special particles on level $\ell(T) + 1$ at time $T + 2$ in the contact process on the half-space is at least 2.

The expected number of particles in the branching process at any time $t \leq T$ is at most $(\alpha + \beta)^T$. Let E_K be the event that $|B_t| \leq K(\alpha + \beta)^T$ for all $t \leq T$. For $\epsilon > 0$ choose K large enough so that

$$\mathbf{E}(|B_T|1_{E_K}) > (1 - \epsilon)\mathbf{E}(|B_T|).$$

We now compare the contact process to the branching random walk. To go from time t to time $t + 1$ in the contact process, we first let births occur across long-range edges. This cannot lead to two generation $t + 1$ particles landing on the same site. We then let the short-range births occur one at a time. On the event E_K, the probability of a collision on any given birth is at most $K(\alpha + \beta)^T/(2m+1)^d$, so the probability of a collision on some birth at some time $t \leq T$ is at most $TK^2(\alpha + \beta)^{2T}/(2m+1)^d$.

Now taking m large enough, the expected number of special particles at level $\ell(T) + 1$ at time $T + 2$ in the contact process is at least 2. Since these particles sit atop half-spaces that are disjoint, the number of special particles on level $2(\ell(T) + 1)$ at time $2(T + 2)$ for these processes in their disjoint half-spaces are independent. By comparing with a branching process we conclude that the contact process survives.

3. A lower bound on the strong survival critical value

The first step is to compute the critical value for the branching random walk on a comb. As mentioned earlier, the comb is not necessary for the proof of the lower bound on strong survival, but it explains the expression for the bound that we obtain. Additionally, it motivates us for the latter part of this section. Here we will return to our original notation and fix $r = \alpha/\beta$ and vary $\lambda = \alpha + \beta$.

Lemma 3.1. *The critical value for the branching random walk on the comb $\mathcal{C}_m$ is independent of m and is equal to*

$$\frac{2(r+1)}{r + \sqrt{r^2 + 4}}$$

Proof. We consider sites $+(z)$ to be level 1 and all other sites on the comb to be level 2. The level transition probabilities for a random walk on $\mathcal{B}_m$ restricted to the comb and killed when it jumps off, are given by

$$\begin{pmatrix} \dfrac{\alpha}{\alpha+\beta} & \dfrac{\beta}{\alpha+\beta} \\ \dfrac{\beta}{\alpha+\beta} & 0 \end{pmatrix} = \begin{pmatrix} \dfrac{r}{r+1} & \dfrac{1}{r+1} \\ \dfrac{1}{r+1} & 0 \end{pmatrix}.$$

Note that the second row does not add up to one since there is a killing rate. Solving the quadratic equation

$$0 = \left(\frac{r}{r+1} - x \right)(-x) - \frac{1}{(r+1)^2} = x^2 - \frac{r}{r+1}x - \frac{1}{(r+1)^2}, \tag{2}$$

the largest eigenvalue of the matrix is

$$\frac{r + \sqrt{r^2 + 4}}{2(r+1)}.$$

This shows that the expected number of particles at 0 tends to 0 exponentially fast if

$$\lambda < \frac{2(r+1)}{r + \sqrt{r^2 + 4}}$$

and gives the lower bound on the critical value. For the other direction, we note that if the random walk is on level 1 after n steps then there have been k down steps, k up steps, and $n - 2k$ sideways steps. The sideways steps are independent of the up and down steps so the probability of being at 0 after n steps, given that the walk is on level 0 is greater than $c/n^{d/2}$. This shows that the expected number of particles at 0 grows exponentially fast if

$$\lambda > \frac{2(r+1)}{r + \sqrt{r^2 + 4}}.$$

The proof can now be completed using techniques of Madras and Schinazi [10] or Pemantle and Stacey [17]. □

Since the branching random walk η_t^B on $\mathcal{B}_m$ dominates the branching random walk η_t^C on the comb of degree m, we have

$$\lambda_2^{\mathrm{brw}}(\mathcal{C}_m) \geq \lambda_2^{\mathrm{brw}}(\mathcal{B}_m).$$
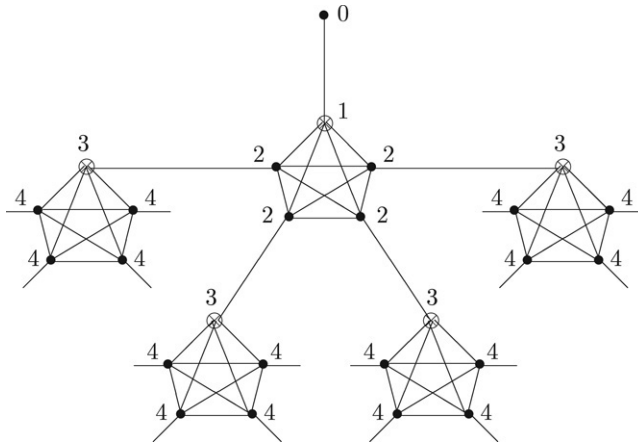
Fig. 4. Comparison graph $\mathcal{K}_M$ when $M = 5$. Numbers next to the vertices indicate the corresponding states in the birth and death chain.

However, this inequality is in the opposite direction of what we want. To get a result in the other direction, we introduce another graph structure $\mathcal{K}_M$ which is similar to $\mathcal{B}_m$ except that 0 has no short-range neighbors, and we replace copies of $\mathbf{Z}^d$ by copies of the complete graph on $M = (2m + 1)^d$ vertices. Again, each point can be described algebraically by a vector $(z_1, \ldots, z_n)$ with $n \geq 1$ integer components, but now we have $z_1 = 0$ and $0 \leq z_j < M$ where $z_j \neq 0$ for $1 < j < n$. Fig. 4 shows a picture of $\mathcal{K}_M$ with $M = 5$.

Consider two branching random walks η_t^B and η_t^K on the respective graphs $\mathcal{B}_m$ and $\mathcal{K}_M$ starting from one particle at the origin. In the second process, individuals that would be sent to short-range neighbors of 0 stay at 0. We can couple η_t^B and η_t^K so that $\eta_t^B(0) \leq \eta_t^K(0)$ for all t.

Let $\lambda_2^{\mathrm{brw}}(\mathcal{K}_M)$ denote the strong survival critical value of η_t^K. In order to get a lower bound L for the strong survival critical value in terms of λ and r, it is enough to show that

$$\frac{2(r + 1)}{r + \sqrt{r^2 + 4}} \leq \liminf_{M \to \infty} \lambda_2^{\mathrm{brw}}(\mathcal{K}_M). \tag{3}$$

If we then interpret this in terms of α and β, Proposition 1.2 will have been proved.

Let S_k be the random walk on $\mathcal{K}_M$ which jumps to short-range neighbors with probability $\alpha/(M\alpha + M\beta)$ and long-range neighbors with probability $\beta/(\alpha + \beta)$. Jumps from 0 to short-range neighbors make no change in the location of the particle. Define

$$\tau = \inf\{k \geq 1 : S_k = 0\} \quad \text{and} \quad F^\lambda = \sum_{k \geq 1} \lambda^k \mathbf{P}(\tau = k).$$

Lemma 3.2. $\lambda_2^{\mathrm{brw}}(\mathcal{K}_M) \geq \sup\{\lambda : F^\lambda < 1\}$.

Proof. For the branching random walk started with one particle at 0:

$$\mathbf{E}(\eta_k^K(0)) = \lambda^k \mathbf{P}(S_k = 0).$$

Summing over k we define

$$G^\lambda = \sum_{k \geq 0} \lambda^k \mathbf{P}(S_k = 0).$$

We need only show that $F^\lambda < 1$ implies $G^\lambda < \infty$, since this in turn implies that the branching random walk dies out locally. Breaking things down according to the value of τ we have for $k \geq 1$

$$\mathbf{P}(S_k = 0) = \sum_{l=1}^{k} \mathbf{P}(\tau = l)\mathbf{P}(S_{k-l} = 0).$$

Multiplying by λ^k and summing over $1 \leq k < \infty$ we get

$$G^\lambda - 1 = F^\lambda G^\lambda$$

so that $G^\lambda = 1/(1 - F^\lambda)$.   □

To examine the behavior of S_k we will compare it to a birth and death chain $R_k = \phi(S_k)$ on the nonnegative integers. ϕ maps 0 in $\mathcal{K}_M$ to 0, while for $n \geq 2$, a point of the form $(z_1, \ldots, z_{n-1}, 0)$ maps to $2n - 3$ and a point of the form $(z_1, \ldots, z_n)$ with $z_n \neq 0$ maps to $2n - 2$. See Fig. 4 for a picture of the mapping. Let $u = 1/(1 + r)$ and $M = (2m + 1)^d$. It is clear from the symmetries of $\mathcal{K}_m$ that R_k is a Markov chain with transition probabilities: $r(0, 0) = 1 - u$ and $r(0, 1) = u$. When j is odd, the last coordinate is 0 so:

$$r(j, j - 1) = u, \qquad r(j, j) = (1 - u)/M, \qquad r(j, j + 1) = (1 - u)(1 - 1/M).$$

When $j \geq 2$ is even the last coordinate is nonzero so:

$$r(j, j + 1) = u, \qquad r(j, j - 1) = (1 - u)/M, \qquad r(j, j) = (1 - u)(1 - 1/M).$$

It follows from our definitions that $\tau = \inf\{k \geq 1 : R_k = 0\}$. For $x > 0$, let

$$h(x) = \sum_{k \geq 1} \lambda^k \mathbf{P}_x(\tau = k)$$

and define $h(0) = 1$. By considering what happens on one step, if $x > 0$

$$h(x) = \lambda \sum_{y} r(x, y)h(y). \tag{4}$$

Using this equation for $x = 2n, 2n - 1$ and $2n + 1$, we have for $n \geq 1$

$$
\begin{aligned}
h(2n)\left(1 - \frac{\lambda(1 - u)}{M}\right) &= \lambda^2 h(2n - 2)r(2n, 2n - 1)r(2n - 1, 2n - 2)\\
&\quad + \lambda^2 h(2n + 2)r(2n, 2n + 1)r(2n + 1, 2n + 2)\\
&\quad + h(2n)\left[\lambda r(2n, 2n)\left(1 - \frac{\lambda(1 - u)}{M}\right)\right.\\
&\quad + \lambda^2 r(2n, 2n - 1)r(2n - 1, 2n)\\
&\quad \left. + \lambda^2 r(2n, 2n + 1)r(2n + 1, 2n)\right].
\end{aligned}
$$

This can be rewritten as $ah(2n + 2) - bh(2n) + ch(2n - 2) = 0$ where

$$a = \lambda^2 u(1 - u)\left(1 - \frac{1}{M}\right)$$

$$b = 1 - \frac{\lambda(1 - u)}{M} - \left(\lambda - \frac{\lambda^2(1 - u)}{M}\right)(1 - u)\left(1 - \frac{1}{M}\right)$$
$$- \frac{\lambda^2(1 - u)^2\left(1 - \frac{1}{M}\right)}{M} - \lambda^2 u^2$$

$$c = \frac{\lambda^2 u(1 - u)}{M}.$$

The solutions to the homogeneous difference equation are of the form $h(2n) = C_1\theta_1^n + C_2\theta_2^n$ where θ_1 and θ_2 are the roots of $a\theta^2 - b\theta + c = 0$, or

$$\theta_1 = \frac{b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \theta_2 = \frac{b - \sqrt{b^2 - 4ac}}{2a}.$$

When $\lambda < 1$, $h(2n)$ is decreasing so $h(2n) = \theta_2^n$. Since $h(x)$ is an analytic function of λ, this is the formula for all λ inside the radius of convergence. When $M \to \infty$, $c \to 0$ while a and b have positive limits so using the Maclaurin expansion of $\sqrt{1 - x}$ we get

$$\theta_2 = b \cdot \frac{1 - \sqrt{1 - \frac{4ac}{b^2}}}{2a} \sim \frac{c}{b} \to 0$$

and it follows that $h(2) \to 0$.

Intuitively what we have shown is that particles that reach level 2 (i.e., fall off the comb) and their descendants can be ignored. To complete the calculation now we observe that using (4) with $x = 1$ and letting $M \to \infty$ gives $h(1) \to \lambda u$. By considering what happens on the first step starting from 0 we have

$$F^\lambda = \lambda(1 - u + uh(1)) \to \lambda(1 - u) + u^2\lambda^2.$$

Setting $F^\lambda = 1$ and recalling $u = 1/(1 + r)$ gives the quadratic equation

$$\frac{1}{(1 + r)^2}\lambda^2 + \lambda\frac{r}{r + 1} - 1 = 0.$$

The change of variable $\lambda = 1/x$ gives (2) and establishes (3).

4. Exponential growth and decay

In this section we will prove Theorem 1.2. Since we have been writing 0 for $+(0)$, we will similarly write -0 for $-(0)$, and -1 for one of the nearest neighbors of $-(0)$. We begin with a well-known property of trees, see e.g., Lemma 6.2 in [16].

Lemma 4.1. *Let* $\mathbf{T}_r^3$ *be the rooted binary tree in which the root has degree one and all other vertices have degree* 3. *Suppose there are* I *infected sites on* $\mathbf{T}^3$. *There must be at least* $I + 2$ *copies of* $\mathbf{T}_r^3$, *disjoint except for possibly the root, where the only infected site is the root.*

Proof. We proceed by induction. If $I = 1$ there are $3 = I + 2$ copies. Each time we add a vertex we destroy one copy and create two more. □
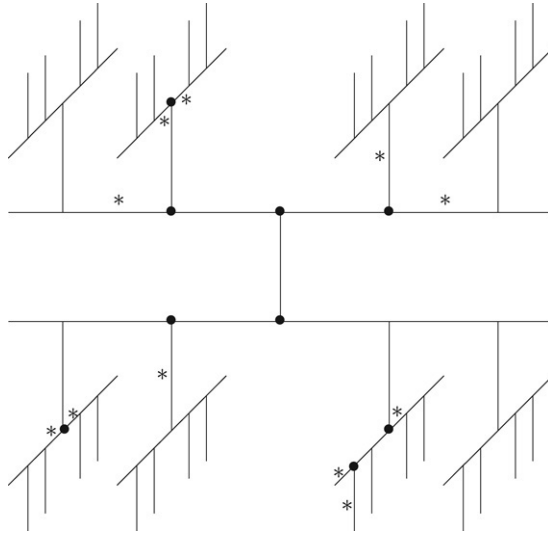
Fig. 5. Finding exterior edges (*'s) on the big world graph, $\mathcal{B}$. Black dots indicate infected sites.

See Fig. 5 for a picture of a case with $I = 9$. We have drawn $\mathcal{B}_1$ but it is the same as $\mathbf{T}^3$. As predicted there are 11 copies of $\mathbf{T}_r^3$, marked by *'s on the edges leading to their roots. We will call these edges, "exterior edges". The tree result extends to the following property of the big world.

Lemma 4.2. *Given a finite set $B \subset \mathcal{B}_m$, there is a set of vertices $D \subset B$ with $|D| \geq \frac{1}{3}|B|$ such that each vertex $x \in D$ is either* (a) *adjacent to a vacant translate of a copy of $\mathcal{B}_m^+$ with x sitting at -0, or* (b) *adjacent to a vacant translate of a copy of $\mathcal{B}_m^+ \cup \{-0\}$ with x sitting at a site that is a nearest neighbor of -0.*

Proof. Consider first the case $d = 1$. We can embed $\mathbf{T}^3$ into $\mathcal{B}_m$. Using Lemma 4.1 it follows that there are at least $(I + 2)/3$ completely disjoint copies of $\mathbf{T}_r^3$ where the only infected site is the root. If the exterior edge from the boundary vertex x is a long-range edge then we are in case (a). If the exterior edge from the boundary vertex x is a short-range edge we are in case (b). See Fig. 5.

In $d = 2$ we can still embed a tree but it has a variable degree greater or equal to 3. In each plane we connect $(0, 0)$ to its four nearest neighbors. For $k \geq 1$ we connect $(k, 0)$ to both $(k+1, 0)$ and $(k, 1)$ while we connect (k, j) to only $(k, j + 1)$ for $j \geq 1$. Then extend the construction to the other four quadrants so that it is symmetric under 90 degree rotations. See Fig. 6 for a picture. All points have at least two neighbors in their plane and one long distance neighbors so the result follows from the proof of Lemma 4.1. We leave the details of the construction for $d \geq 3$ to the reader, but it should be clear that one can embed a tree but it has a variable degree greater or equal to 3. $\square$

With Lemma 4.2 established, the rest of the proof is very similar to the proof of Theorem 2 in [12].

Lemma 4.3. *For the contact process B_t^0 on $\mathcal{B}_m$, there exists $C_1 > 0$, which depends on m, such that*

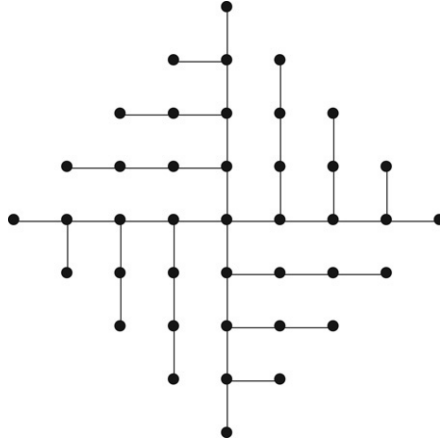$$\mathbf{E}(|B_{t+s}^0|) \geq C_1 \mathbf{E}(|B_t^0|)\mathbf{E}(|B_s^0|).$$

Fig. 6. Embedding a tree in $\mathbf{Z}^2$.

Proof. For $j = 0, 1$, let $|B_t^{+,-j}|$ be the number of points in $\mathcal{B}_m^+$ for the process started with a single infected individual at $-j$. Let $M_t = \min\{\mathbf{E}(|B_t^{+,-0}|), \mathbf{E}(|B_t^{+,-1}|)\}$. By additivity, the Markov property, and Lemma 4.2 we have

$$\mathbf{E}(|B_{t+s}^0|) \geq \frac{1}{c_d} \mathbf{E}(|B_t^0|) M_s.$$

The lemma will be proved when we show that there exists $C_1 > 0$ such that $M_s \geq C_1 \mathbf{E}(|B_s^0|)$.

To do this we begin by observing that for $j = 0, 1$

$$\mathbf{E}(|B_t^{+,-j}|) \geq \frac{\beta}{2} \cdot \frac{\alpha^2}{(2m+1)^{2d}} \cdot \mathbf{E}(|B_{t-2}^{\{0,-0\}}|) \tag{5}$$

where we are considering the event that on the first step $-j$ infects -0, and on the second step -0 infects itself and 0. The factor of $1/2$ comes from the fact that when starting from $\{-0, 0\}$, the expected number of points in the positive half-space is half the number in the whole space. To take care of the differences in starting configuration and time, we note

$$\mathbf{E}(|B_t^0|) \leq \mathbf{E}(|B_t^{\{0,-0\}}|) \leq \lambda^2 \mathbf{E}(|B_{t-2}^{\{0,-0\}}|).$$

From this it follows that for $j = 0, 1$

$$\liminf_{t \to \infty} \frac{\mathbf{E}(|B_t^{+,-j}|)}{\mathbf{E}(|B_t^0|)} \geq \frac{\beta}{2} \cdot \left(\frac{\alpha}{(2m+1)^d \lambda} \right)^2. \quad \square$$

Proof of Theorem 1.2. By the additivity of the contact process and the transitivity of $\mathcal{B}_m$,

$$\mathbf{E}(|B_{t+s}^0|) \leq \mathbf{E}(|B_t^0|)\mathbf{E}(|B_s^0|).$$

Subadditivity then implies that

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbf{E}(|B_t^0|) = \inf_{t > 0} \frac{1}{t} \log \mathbf{E}(|B_t^0|) = C_2 \tag{6}$$

exists so that $\exp(C_2 t) \leq \mathbf{E}(|B_t^0|)$. A similar argument using superadditivity of $C_1 \mathbf{E}(|B_t^0|)$ and Lemma 4.3 shows that

$$\lim_{t \to \infty} \frac{1}{t} \log C_1 \mathbf{E}(|B_t^0|) = \sup_{t > 0} \frac{1}{t} \log C_1 \mathbf{E}(|B_t^0|) = C_2 \tag{7}$$

so that $\frac{1}{C_1} \exp(C_2 t) \geq \mathbf{E}(|B_t^0|)$.

We turn now to the sign of C_2 in the different regimes. When $\lambda > \lambda_1$ we use the well-known fact that on the event

$$\{|B_t^0| \geq 1 \text{ for all } k \geq 0\},$$

$\lim_{t \to \infty} |B_t| = \infty$ almost surely. So by the upper bound $\frac{1}{C_1} \exp(C_2 k) \geq \mathbf{E}(|B_t^0|)$, we see that $C_2 > 0$ when $\lambda > \lambda_1$.

When $\lambda < \lambda_1$, the work of Aizenman and Barsky [1] gives

$$\sum_{t \geq 0} \mathbf{E}(|B_t^0|) < \infty$$

so that $C_2 < 0$ in this regime.

For $\lambda = \lambda_1$ we use a continuity argument. Note that for any fixed $t > 0$,

$$\lambda \to \frac{1}{t} \log \mathbf{E}(|B_t^0|)$$

is a continuous function of $\lambda > 0$. By (6) and (7), $\lambda \to C_2$ is lower and upper semi-continuous and thus continuous. Therefore when $\lambda = \lambda_1$ it must be that $C_2 = 0$. □

5. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. We start by viewing $\mathcal{B}_m$ as a "covering space" of $\mathcal{S}_m^R$. Given a realization of the random graph $\mathcal{S}_m^R$, we mark a distinguished vertex which we identify with the origin of $\mathcal{B}_m$. From there we identify the long-range edge of the distinguished vertex with the long-range edge of the origin in $\mathcal{B}_m$. Similarly, we identify the short-range edges of the two graphs together in such a way so that graph distances on the underlying structure given by $\mathcal{B}_m$ are preserved. Continuing in this manner we can identify each vertex in $\mathcal{B}_m$ with some vertex in our realization of $\mathcal{S}_m^R$.

Using this identification, couple together the processes B_t^0 and ξ_t^0 for each realization of the random graph $\mathcal{S}_m^R$. By additivity of the contact process, it is clear that $|B_t^0| \geq |\xi_t^0|$ which shows that $\tau_{\mathcal{S}}$ is bounded above by $\tau_{\mathcal{B}}$. Choose $\epsilon > 0$ and $N > 0$. Using the graph identification above, let $G_{R,N}$ be the event (on the random graph $\mathcal{S}_m^R$) that there is no cycle of length $2N$ which contains 0. For fixed N, we have that $\mathbf{P}(G_{R,N}) \to 1$ as $R \to \infty$ (if this is not clear to the reader, an explicit argument is given in the proof of Theorem 1.4 below). If there are no cycles of length $2N$ containing 0 in the graph $\mathcal{S}_m^R$ then no two vertices within distance N of 0 are identified together in $\mathcal{B}_m$.

Choose R_0 so that $R \geq R_0$ implies $\mathbf{P}(G_{R,N}) > 1 - \epsilon$. Using the coupling to identify points of $\mathcal{B}_m$ with $\mathcal{S}_m^R$, we have for all $R \geq R_0$,

$$\mathbf{P}(B_k^0 = \xi_k^0 \text{ for all } k \leq N) > 1 - \epsilon$$

which proves $\lim_{R \to \infty} \tau_{\mathcal{S}} = \tau_{\mathcal{B}}$. The proof of $\lim_{R \to \infty} \sigma_{\mathcal{S}} = \sigma_{\mathcal{B}}$ is similar and is therefore omitted.

Proof of Theorem 1.4. Let $M = (2m + 1)^d$ be the number of neighbors and let

$$\delta = \frac{\alpha \gamma}{8M}.$$

The contact process on the big world with $\gamma = 0$ is supercritical so we can pick T so that $\mathbf{E}(|\xi_T^0|) \geq 5/\delta$ and we can pick K so that if $\bar{\xi}_T^0$ is the contact process restricted to the ball of radius K around 0 then $\mathbf{E}(|\bar{\xi}_T^0|) \geq 4/\delta$. Let N_K be the number of points in the ball of radius K around 0 on the big world. Pick η so that

$$\eta \cdot \frac{\alpha \gamma N_{2K}}{4M} \leq \frac{1}{5}.$$

It suffices to show that if the number of occupied sites drops below ηR^d occupied sites then with probability greater than $1 - \exp(-bR^d)$ for some $b > 0$, it will return to having more than $2\eta R^d$ occupied sites. To see this we just choose $c < b$ giving us a high probability of having $\exp(cn^d)$ successful recoveries. Since each recovery takes at least one unit of time, the result follows.

The first step towards achieving the above is to show that the first time it drops below ηR^d it does not fall too far. To get a lower bound on how far it falls, we only look at births from sites onto themselves. These are independent events with probability α/M. A standard large deviations result for the binomial implies that if there are at least ηR^d particles alive at time t then the probability of less than $\eta R^d \alpha/2M$ alive at time $t + 1$ is less than $\exp(-b_1 R^d)$ for some $b_1 > 0$. Thus the probability this occurs at some time before $\exp(b_1 R^d/2)$ is less than $\exp(-b_1 R^d/2)$.

After the number of occupied sites falls below ηR^d, our next step is to randomize the locations by having one time step in which we only allow births induced by the parameter γ. If we start with $\eta R^d \alpha/2M$ particles, then after these particles give birth onto their randomly chosen neighbors (with no other births allowed) we will have, on average, $\eta R^d \gamma \alpha/2M$ particles. In fact, if $t + 1$ is the time at which the number of occupied sites falls below ηR^d (but not too far), then using the large deviations estimate for the binomial once more we have that the probability of there being less than $\eta R^d \gamma \alpha/4M$ particles at time $t + 2$ is less than $\exp(-b_2 R^d)$ for some $b_2 > 0$ (we will take care of double counting below). Having used the randomized births in this step, we ignore them for the rest of the proof and use the process with $\gamma = 0$ which we have assumed is supercritical.

We say that a site x is *good* if the small world is identical to the big world inside a ball of radius K around x. Fixing x, the probability of a self-intersection (i.e., the probability that x is not a good site) when we generate the ball of radius K around x is less than N_K^3/n. To see this, we grow the ball around x starting from just x and its short-range neighbors. Adding the long-range neighbor of x, we see that the probability it causes a self-intersection is bounded above by N_K/n. But we also have to be worried about the short-range neighbors of this long-range neighbor causing a self-intersection, so we increase the bound generously to N_K^2/n. Now since there are at most N_K neighbors to add, the bound N_K^3/n follows; therefore, the probability that x is a bad site tends to 0 as $n \to \infty$.

Now fix a realization of the small world graph such that the fraction of bad points does not exceed 1/20. We want to find a subset G of the randomized births that lie on good sites and for which the evolutions of $\bar{\xi}_t^x$, $x \in G$ are independent. To do this we imagine that the successful randomized births occur one at a time (successful with respect to γ). We accept the first birth if it lands on a good site. We accept the second birth if it lands on a good site and it does not fall within the ball of radius $2K$ around the first birth, and we continue in this manner.

When there are j successful randomized births accepted, the total number of sites that lie within the balls of radius $2K$ around them does not exceed jN_{2K}. When

$$j \le \eta R^d \gamma \alpha / 4M,$$

the probability of the next successful randomized birth landing in the forbidden zone is less than $1/5$ by our choice of η above. Since we are on a realization of $\mathcal{S}_m^R$ whose fraction of bad points does not exceed $1/20$, the probability of discarding a successful randomized birth conditioning on j prior acceptances is less than $1/4$. Thus the number of births discarded out of the first $\eta R^d \gamma \alpha / (8m+4)$ successful randomized births is bounded by a binomial with success probability $p = 1/4$. Our large deviations estimate implies that with probability less than $\exp(-b_3 R^d)$, the number of randomized births remaining, $|G|$, is larger than

$$\frac{\eta R^d \gamma \alpha}{8M} = \delta \eta R^d.$$

By our choice of G, the particles in G can grow independently up to a distance of K, or in other words, the processes $\bar{\bar{\xi}}_t^x$, $x \in G$ evolve independently. Since $\mathbf{E}(|\bar{\bar{\xi}}_T^0|) > 4/\delta$, a final large deviations estimate for bounded i.i.d. random variables tells us that the probability of ending up with fewer than $2\eta R^d$ particles at time T is less than $\exp(-b_4 R^d)$. The total of our error probabilities is less than $\exp(-b R^d)$ which completes the proof.	$\square$

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