COCYCLES IN TOPOLOGICAL DYNAMICS

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§1. INTRODUCTION

Let \((X, T)\) be a flow, \(K\) a topological group, and \(\sigma\) a cocycle on \(X\) to \(K\); i.e. \(\sigma: X \times T \to K\) continuous with \(\sigma(x, ts) = \sigma(x, t)\sigma(x, s)\) \((x \in X, t, s \in T)\). Then these data have been used to construct other flows related to \((X, T)\). Probably the most important of these constructions is the skew product flow \((K \times X, T)\). Here the underlying phase space is \(K \times X\) and \(T\) acts via the map \((k, x) \to (ks(x, t), xt): K \times X \to K \times X\) (see for example [5, 7]).

Another application of the cocycle \(\sigma\) occurs when in addition to the above data one is given a bitransformation group \((K, Y, T)\) and a homomorphism \(\pi: (Y, T) \to (X, T)\). Then \(\sigma\) is used to change the original action \((y, t) \to yr: Y \times T \to Y\) of \(T\) on \(Y\) to \((y, t) \to \pi(\sigma(y, t)^{-1}xt): Y \times T \to Y\). (See [2]). Indeed this construction includes the former \((Y = K \times X, k(l, x) = (kl, x), (l, x)t = (l, xt); \pi(k, x) = x, (k, l \in K, x \in X, t \in T)\)).

A third use for the cocycle \(\sigma\) is to change the velocity of the flow (see [1, 8]). This may also be subsumed under the above. Thus in the papers [1, 8] \(Y = X, T = R = K, \) and \(\pi\) is the identity.

The function \(\pi\) is called a cocycle because it may be identified with a one cocycle on \(T\) with coefficients in the \(Z(T)\)-module \(C(X, K)\), when \(K\) is abelian. Indeed the whole discussion may be carried on in the context of the cohomology theory of groups. This is done in [9] where Petersen discusses some applications of the latter to the study of the extensions of minimal sets. In this paper we make no use of this theory and there is no overlap with [9].

As can be seen from the above the study of cocycles on \(X\) is closely related to the study of extensions of \(X\). The extensions that occur are the so-called almost periodic and distal ones. (Let \(\pi: (Y, T) \to (X, T)\) be a homomorphism of minimal flows. Then \((Y, T)\) is an almost periodic extension of \((X, T)\) if there exists a bitransformation group \((H, Z, T)\) such that \((Z/H, T) = (X, T)\) and \((Z/L, T) = (Y, T)\) where \(H\) is a compact topological group and \(L\) is a closed subgroup of \(H\). Let \(y_1, y_2 \in Y\) with \(y_1 \neq y_2\). Then \(y_1\) and \(y_2\) are distal if there is no net \((l_n) \subset T\) with \(\lim l_n \tau_0 = \lim y_2l_n\). The extension \((Y, T)\) of \((X, T)\) is distal if \(y_1\) and \(y_2\) are distal for all pairs \(y_1, y_2\) in \(Y\) with \(y_1 \neq y_2\) and \(\pi(y_1) = \pi(y_2)\).)

Many of the dynamical problems involved in the study of cocycles are concerned with whether \(\sigma\) is a coboundary, i.e. whether there exists a continuous function \(f: X \to K\) such that \((1) f(xt) = f(x)\sigma(x, t)\) \((x \in X, t \in T)\). In general no such function exists. However, when \((X, T)\) is minimal and \(K\) is compact, there exists an almost periodic minimal extension \((Y, T)\) of \((X, T)\) such that equation \((1)\) is solvable on \(Y\). Moreover the solution \(f\) induces a homomorphism from the group of \(X\) to \(K\) with kernel the group of \(Y\). This fact is the starting point of a systematic study of cocycles in the category of minimal sets.

In order to make the above statement precise it is necessary to make several identifications. This is most conveniently done by working in the category of pointed minimal sets or equivalently with the collection of \(T\)-subalgebras of \(C(M)\) where \(M\) is a fixed universal minimal set (see [3]).

When this point of view is adopted it becomes apparent that one need only
consider cocycles on $M$ to $K$. It turns out that every such cocycle $\sigma$ is a coboundary and that any two solutions of the equation $df = \sigma$ "differ" by a constant. Thus there is a unique function $f_\sigma: M \to K$ with $df_\sigma = \sigma$ and $f_\sigma(e) = e$. (The equation $df = \sigma$ is just (1) above.)

For every cocycle $\sigma$ on $M$ there is a natural way of associating $T$-subalgebras, $al(\sigma)$ and $al(f_\sigma)$ with $\sigma$ and $f_\sigma$ respectively. Moreover the cocycles on $X$ to $K$ may be identified with those cocycles $\sigma$ on $M$ to $K$ with $al(\sigma) \subseteq C(X)$. These facts allow us to use the algebraic machinery developed in [3] to study cocycles on minimal sets.

In general $(K \times X, T)$ is not minimal but the minimal subsets thereof are all isomorphic and constitute a partition of $K \times X$. The algebra $ext(\mathcal{A}, \sigma)$ corresponding to the orbit closure of the point $(e, x_0)$ is just the supremum of the two algebras $\mathcal{A}$ and $al(f_\sigma)$ and the flow $[ext(\mathcal{A}, \sigma)]$ is the smallest extension of $X$ on which the cocycle $\sigma$ bounds. (Here $\mathcal{A}$ is the $T$-subalgebra of $C(M)$ which corresponds to the pointed flow $(X, x_0)$.)

The fundamental fact upon which this study rests is that $f_\sigma$ restricted to the group $A$ of $\mathcal{A}$ is a $T$-continuous homomorphism whose kernel is the group of $ext(\mathcal{A}, \sigma)$. This provides the link between the cohomology theory of skew products and the algebraic theory of minimal sets.

The principal results of the paper fall into three groups.

The ones in the first group show how the algebraic techniques may be used to obtain simple and unified proofs of old results and far reaching generalizations thereof. As an example of the former, 3.9 states that the flow $S \times X$ is minimal if and only if $\sigma^n$ is not cohomologous to 0 for all integers $n \neq 0$. (Here $S$ is the circle group.) (For other examples see 3.23 and 3.24.)

An example of the latter type is Proposition 3.20 which states that when the canonical map $\text{Hom}(G/E, K) \to \text{Hom}(AE/E, K)$ is onto then the cocycle $\sigma$ is cohomologous to a constant cocycle if and only if $ext(\mathcal{A}, \sigma)$ may be gotten from $\mathcal{A}$ by adding almost periodic functions. This generalizes the known result 3.23 which is essentially 3.20 with the additional assumptions that $T$ is the integers, $K$ is an $n$-torus and $(\mathcal{A}, T)$ is equicontinuous.

Moreover 3.19 shows that these theorems are not about almost periodic functions or equicontinuous flows but rather about how the group of the flow $(\mathcal{A}, T)$ is related to the group associated with a specified class of cocycles.

The second group of results has to do with the cohomology $H(\mathcal{A}, K)$ and the group $A$ of $\mathcal{A}$. Here the main result is 4.18 which states that when $[\mathcal{A}]$ is 0-dimensional $H(\mathcal{A}, K)$ is isomorphic to $\text{Hom}(A/A^n, K)$ (where $A^n$ is the group associated with the largest almost periodic extension of $\mathcal{A}$.) In general all that can be said is that there is an injection of $H(\mathcal{A}, K)$ into $\text{Hom}(A/A^n, K)$ (Proposition 3.18).

The last group has to do with the existence of a supplement. Let $\mathcal{A} \subset \mathcal{F} \subset \mathcal{B}$. Then $\mathcal{F}$ is an $\mathcal{A}$-supplement of $\mathcal{F}$ if $\mathcal{A} = \mathcal{F} \cap \mathcal{F}$ and $\mathcal{B} = \mathcal{F} \cup \mathcal{F}$. The principal result is Proposition 5.26 which states that if $\mathcal{B}$ is an almost periodic extension of $\mathcal{A}$ with $B \subset A$ and if the map from $H(\mathcal{A}, A/B)$ to $\text{Hom}(A/A^n, A/B)$ is onto then $\mathcal{F}$ has an $\mathcal{A}$-supplement in $\mathcal{B}$ if and only if $\mathcal{F} = ext(\mathcal{A}, \sigma)$ for some cocycle $\sigma$ on $\mathcal{A}$ to $A/B$. In this case $\mathcal{F}$ may be taken to be the perturbed flow per $(\mathcal{B}, \sigma)$. (See §5 for definitions.)

perturbed flow per $(\mathcal{B}, \sigma)$. (See §5 for definitions.)

The theory of cocycles as expounded below shows that the group $K$ plays several roles. It is the coefficient group of the cohomology theory, the group of the extension, $ext(\mathcal{A}, \sigma)$ mod the group of $\mathcal{A}$ and the group whose action on $[\mathcal{B}]$ is used to obtain the perturbed flow, per $(\mathcal{B}, \sigma)$. It is necessary to keep these roles distinct especially when considering the composition of perturbations (see 5.15).

Two other points about the group $K$ should be stressed. First, it need not be abelian, and second, it must be compact. Although the circle group is the most frequently occurring coefficient group, it is clear especially for the theory of perturbed flows that one has to be able to handle non-abelian coefficient groups. The compactness of the group $K$ is necessary in order to study cocycles on $X$ to $K$ by means of cocycles on $M$ to $K$. 
Standing Notation. In the paper $K$ denotes an arbitrary compact group, $T$ an arbitrary discrete group, $\beta T$ the Stone-Cech compactification of $T$. $M$ a fixed minimal right ideal in $\beta T$, $u$ a fixed idempotent in $M$, and $G$ the group $Mu$ (see [3] for details.)

$\S 2$. COCYCLES ON $M$

2.1 In this section the basic definitions and results are given which enable one to study cocycles on minimal sets within the context of the algebraic theory of minimal sets developed in [3].

2.2 Definition. A cocycle on $T$ to $K$ is a function $\sigma: T \times T \rightarrow K$ such that $\sigma(r, ts) = \sigma(r, t)\sigma(rt, s)(r, t, s \in T)$. The set of cocycles on $T$ to $K$ will be denoted $Z(T, K)$. The function associated with (T is the map $t \rightarrow \sigma(e, t): T \rightarrow K$. It is denoted $f_\sigma$.

2.3 Lemma. Let $f: T \rightarrow K$ with $f(e) = e$, and let $\sigma(t, s) = f(t)^{-1}f(ts)(t, s \in T)$. Then $\sigma \in Z(T, K)$ and $f = f_\sigma$.

Proof. $\sigma(r, t)\sigma(rt, s) = f(r)^{-1}f(rt)f(rt)^{-1}f(rts) = f(r)^{-1}f(rts) = \sigma(r, ts)$.

Also $f_\sigma(t) = \sigma(e, t) = f(e)^{-1}f(t) = f(t)$.

2.4 Proposition. Let $\mathcal{F} = \{f: T \rightarrow K, f(e) = e\}$. Then $\sigma \rightarrow f_\sigma: Z(T, K) \rightarrow \mathcal{F}$ is bijective; the inverse being the map $f \rightarrow \sigma_f: \mathcal{F} \rightarrow Z(T, K)$, where $\sigma_f(t, s) = f(t)^{-1}f(ts)(t, s \in T)$.

Proof. In view of 2.3 it suffices to show that $\sigma(t, s) = f_\sigma(t)^{-1}f_\sigma(ts)$ and that $f_\sigma(e) = e(\sigma \in Z(T, K))$. Both relations follow immediately from the definitions.

2.5 Remark. If $K$ is abelian, the set $C(T, K)$ of functions is an abelian group. It may be made into a $T$-module by defining $tf$ to be the function $s \rightarrow f(St)$: $T \rightarrow K$. Then it turns out that $Z(T, K)$ may be identified with the set $Z'(T, C(T, K))$ of one cocycles on $T$ with coefficients in $C(T, K)$. The cocycle $\sigma_f$ is just the coboundary of the function $f$. Thus $df_\sigma = \sigma$ and so every one cocycle on $T$ to $C(T, K)$ is a coboundary.

2.6 Remarks. 1. Let $\sigma \in Z(T, K)$ and $t \in T$. Then the function $s \rightarrow \sigma(s, t): T \rightarrow K$ is continuous and so may be extended continuously to $\beta T$ since $K$ is compact. Denote the extension by $\sigma_t$.

Now let $p \in \beta T$. Then $t \rightarrow \sigma_t(p): T \rightarrow K$ may be extended to a continuous function $\sigma^p: \beta T \rightarrow K$.

The function $(p, q) \rightarrow \sigma^p(q): \beta T \times \beta T \rightarrow K$ will also be denoted $\sigma$ since it coincides with $\sigma$ on $T \times T$.

Notice that the function $x \rightarrow \sigma(p, x): \beta T \rightarrow K$ is continuous ($p \in T$).

2. Let $s, t \in T$ and $x \in \beta T$. Then there exists a net $(r_n) \subset T$ with $r_n \rightarrow x$. Hence $\sigma(x, ts) = \lim \sigma(r_n, ts) = \lim \sigma(r_n, t)s = \sigma(x, t)s = \sigma(x, t)\sigma(x, s)$.

Similarly by taking appropriate nets in $T$ and using 1, it is easy to see that $\sigma(x, yz) = \sigma(x, y)\sigma(xy, z)(x, y, z \in \beta T)$.

3. Let $\sigma \subset Z(T, K)$. Then $f_\sigma$ has a continuous extension (also denoted $f_\sigma$) to $\beta T$ and it is immediate that $f_\sigma(p) = \sigma(e, p)(p \in \beta T)$ and that $\sigma(x, y) = f_\sigma(x)^{-1}f_\sigma(xy)(x, y \in \beta T)$.

Now I should like to restrict myself to the class of “minimal” cocycles.

2.7 Definition. The cocycle $\sigma$ is minimal if $\sigma(u, x) = \sigma(e, x)(x \in \beta T)$. The function $f: \beta T \rightarrow K$ is minimal if $f(ux) = f(x)(x \in \beta T)$.

2.8 Proposition. The cocycle $\sigma$ is minimal if and only if its associated function $f_\sigma$ is minimal.

Proof. Since $u^2 = u$, the cocycle equation implies that $\sigma(u, u) = e$. Thus if $\sigma$ is minimal $f_\sigma(up) = \sigma(e, up) = \sigma(u, up) = \sigma(u, u)\sigma(u, p) = \sigma(u, p) = \sigma(e, p) = f_\sigma(p)$.

On the other hand if $f_\sigma$ is minimal, $\sigma(u, x) = f_\sigma(u)^{-1}f_\sigma(ux) = f_\sigma(e)^{-1}f_\sigma(x) = \sigma(e, x)$.  

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2.9 Basic assumptions. Henceforth all cocycles, functions, and flows are assumed minimal. The set of minimal cocycles will be denoted by $Z(M, K)$.

2.10 Definitions. 1. Let $f: T \to K$ be a minimal function. Then the space of $f$, $sp(f) = M/R$, where $x = y(R_p)$ if $f(x) = f(y)$ $(x, y \in M, p \in \beta T)$. The space of $f$ is made into a pointed flow by taking the base point to be the equivalence class to which $u$ belongs. This allows us to speak of the $T$-subalgebra $al(f)$ of the flow and the group $g(f)$ of the flow.

2. Let $\sigma \in Z(M, K)$. Then the space of $\sigma$, $sp(\sigma) = M/R$, where $x = y(R_p)$ if $\sigma(x, p) = \sigma(y, p) (x, y \in M, p \in \beta T)$. Since $\sigma(x, pq) = \sigma(x, p)\sigma(p, x)$, $x = y(R_o)$ if and only if $xy = yp$ $(p \in \beta T)$. Thus $R_o$ is invariant.

Let $(x_o, y_o) \in R_o$ with $x_o \to x$, $y_o \to y$. Then $\sigma(x, t) = \lim \sigma(x_n, t) = \sigma(y, t)$ $(t \in T)$ whence $\sigma(x, p) = \sigma(y, p)$ $(p \in \beta T)$. This shows that $R_o$ is closed and so $sp(\sigma)$ is a compact Hausdorff minimal flow.

Again we "point" the flow $sp(\sigma)$ by choosing as base point the equivalence class to which $u$ belongs. Thereby the algebra, $al(\sigma)$, of the flow and $g(\sigma)$, the group of the flow $sp(\sigma)$ are well defined.

2.11 PROPOSITION. Let $u$ be a cocycle and $f_u$ its associated function. Then 1. $al(f_u) \subset al(f)$, 2. $g(\sigma) = \{a | a \in G, f_u(ax) = f_u(a)f_u(x), (x \in \beta T)\}$. 

Proof. 1. Let $p = q$ on $al(f_u)$, i.e. $p = q(R_o)$. Then $\sigma(p, x) = f_u(p)^{-1}f_u(px) = f_u(q)^{-1}f_u(qx) = \sigma(q, x)$ $(x \in \beta T)$. Thus $p = q$ on $al(\sigma)$.

2. Let $\alpha \in g(\sigma)$ and $x \in \beta T$. Then $f_u(ax) = \sigma(e, ax) = \sigma(e, a)\sigma(a, x) = \sigma(e, a)\sigma(e, \alpha)\sigma(e, x) = f_u(\alpha)f_u(x)$.

On the other hand let $\alpha \in G$ with $f_u(\alpha x) = f_u(\alpha)f_u(x)$ $(x \in \beta T)$. Then $\sigma(\alpha, x) = f_u(\alpha)^{-1}f_u(x) = \sigma(e, x)(x \in X)$.

2.12 PROPOSITION. Let $\sigma$ be a cocycle, $f_\sigma$ its associated function, and $F = g(\sigma)$. Then $f_\sigma: (F, \tau) \to K$ is a continuous homomorphism and so induces a continuous homomorphism from $F/H$ $(F, \tau)$ to $K$. (Recall that $H(F, \tau)$ is the intersection of the $\tau$-closed neighborhoods of $e \in F$).

Proof. That $f_\sigma$ restricted to $F$ is a homomorphism follows immediately from 2 of 2.12.

Let $\alpha \in F$ and $N$ a neighborhood of $f_\sigma(\alpha)$. Let $N_1$ and $N_2$ be neighborhoods of $e \in K$ and $f_\sigma(\alpha)$ respectively with $N_1N_2^{-1} \subset N$.

Since $f_\sigma(u) = e$ and $f_\sigma: M \to K$ is continuous there exist $U \in \alpha$ and $V \in u$ with $Vu = V, f_\sigma(p) \in N_1, f_\sigma(x) \in N_2$ $(x \equiv U, p \equiv V)$.

Let $\beta \in (U, V) \cap F$. There exists $t \in V$ with $U \in \beta t$. Then $f_\sigma(\beta t) = f_\sigma(\beta)f_\sigma(ut)$ implies that $f_\sigma(\beta) \in N_2N_1^{-1} \subset N$. (Recall that if $t \in V$ and $Vu = V$ then $V \subset ut$).

Since $H(F, \tau) = \cap \{c, V \cap F | V \text{ a } \tau\text{-neighborhood of } e \}$ and $K$ is Hausdorff, $H(F, \tau) \subset ker f_\sigma$.

2.13 LEMMA. Let $\sigma$ be a cocycle. Then $\sigma(xv, v) = e(x \in \beta T, M \in \beta T$ with $v^2 = v$).

Proof. $\sigma(x, v) = \sigma(xv, v) = \sigma(x, v)\sigma(xv, v)$.

2.14 PROPOSITION. Let $\alpha \in Z(T, K)$. Then $al(f_\sigma)$ is an almost periodic extension of $al(\sigma)$ and $g(f_\sigma)$ is the kernel of the homomorphism induced by $f_\sigma$ on $g(\sigma)$.

Proof. Since $H(F, \tau) \subset ker f_\sigma$ by 2.12 it suffices for the first part of 2.14 to show that $al(f_\sigma)$ is a distal extension of $al(\sigma)$.

To this end let $pv = p$ on $al(\sigma)$ with $v^2 = v \in M$. Then $f_\sigma(puv) = \sigma(e, puv) = \sigma(e, p)\sigma(pv, x) = \sigma(e, p)\sigma(p, x)$.

Now $\sigma(e, puv) = \sigma(e, p)\sigma(pv, v) = \sigma(e, p)\sigma(pv, v) = \sigma(e, p)$ by 2.13. Thus $f_\sigma(puv) = \sigma(e, p)\sigma(e, p)\sigma(p, x) = \sigma(e, pv) = f_\sigma(pv)$.

Now $\alpha \in g(f_\sigma)$ if and only if $f_\sigma(ax) = f_\sigma(x)$ $(x \in \beta T)$. Since $f_\sigma(ax) = f_\sigma(x)$, the latter holds if and only if $f_\sigma(\alpha) = e$.

2.15 Definition. Let $\sigma, \delta \in Z(T, K)$. Then $\sigma \delta$ is that element of $Z(T, K)$ determined by the function $f_\sigma f_\delta$; i.e., $f_\sigma \delta = f_\sigma f_\delta$. 
2.16 Remarks. 1. The cocycles, $Z(T, K)$ from a group with identity $e$, where $f_s(x) = e(x \in \beta T)$. The product $\sigma \delta$ is given by the equation $(\sigma \delta)(x, y) = f_s(x)^{-1} \sigma(x, y) f_s(x) \delta(x, y)$ and the inverse of $\sigma$ by $\sigma^{-1}(x, y) = f_s(x) \sigma(x, y)^{-1} f_s(x)^{-1} (x, y \in \beta T)$.

2. The group $Z(T, K)$ is in general not abelian. However, if $K$ is abelian, so is $Z(T, K)$.

3. The minimal cocycles $Z(M, K)$ form a subgroup of $Z(T, K)$.

The proof of the following proposition is straightforward and will be omitted.

2.17 Proposition. Let $\sigma, \delta$ be cocycles and $f_s, f_s$ their associated functions. Then

1. $\sigma(\delta) \subset \sigma(\sigma) \lor \sigma(f_s)$
2. $\sigma(f_s) \subset \alpha(\sigma) \lor \sigma(f_s)$
3. $\sigma(f_s) = \alpha(f_{s-1})$
4. $\sigma(f_s) \subset \alpha(f_{s}) \lor \sigma(f_s)$
5. $\sigma(f_s) \subset \alpha(f_{s+1}) \lor \sigma(f_s)$

If $K$ is abelian, then
6. $\sigma(\sigma) \subset \alpha(\sigma) \lor \sigma(\delta)$.

2.18 Definition. Let $\sigma \in \text{Hom}(T, K)$, the set of homomorphisms from $T$ to $K$. Then the map $(s, t) \mapsto \sigma(t) \cdot T \times T \to K$ is a cocycle (which I shall also denote by $\sigma$). Such cocycles will be called constant cocycles and $\text{Hom}(T, K)$ will be viewed as a subset of $Z(T, K)$.

2.19 Remarks. 1. It is easy to see that every constant cocycle is minimal and that $\sigma \in \text{Hom}(T, K)$ if and only if $\sigma(\sigma) = C$. Moreover when $\sigma$ is constant then $f_s = \sigma$.

2. The reason for calling the elements of $\text{Hom}(T, K)$ constant cocycles is that when $T = \mathbb{Z}$, $Z(T, K)$ may be identified with the set of functions from $T$ to $K$ and under this identification $\text{Hom}(T, K)$ becomes the set of constant functions.

2.20 Proposition. Let $\sigma$ be a constant cocycle. Then $\alpha(\sigma) \subset C$, the $T$-algebra of almost periodic functions on $T$.

Proof. This follow from 2.14 and the fact that $\alpha(\sigma) = C$.

§3. COCYCLES ON $\mathcal{A}$

3.1. In this section cocycles on a minimal flow $\mathcal{A}$ are identified with a subset of $Z(M, K)$. Then the results of §2 are used to study the skew product flow $K \times [\mathcal{A}]$.

3.2 Definition. Let $\mathcal{A}$ be a $T$-subalgebra of $[\mu]$. A cocycle $\sigma$ on $\mathcal{A}$ to $K$ is a continuous map $\sigma: [\mathcal{A}] \times T \to K$ such that $\sigma(x, ts) = \sigma(x, t)\sigma(xt, s)(x \in [\mathcal{A}], t, s \in T)$. The set of cocycles on $\mathcal{A}$ to $K$ will be denoted $Z(\mathcal{A}, K)$.

3.3 Proposition. For every $\sigma \in Z(\mathcal{A}, K)$ let $\tilde{\sigma}: T \to K$ be such that $\tilde{\sigma}(t, s) = \sigma(x_0, s)(t, s \in T)$ (here $x_0 = u[\mathcal{A}]$ is the base point of $[\mathcal{A}]$). Then $\tilde{\sigma} \in Z(M, K)$ and $\sigma \mapsto \tilde{\sigma}: Z(\mathcal{A}, K) \to Z(M, K)$ is an injective map with image $\{\tilde{\sigma} | \tilde{\sigma} \in Z(M, K), \alpha(\tilde{\sigma}) \subset \mathcal{A}\}$.

Proof. That $\tilde{\sigma}$ is a cocycle follows immediately from the fact that $\sigma$ is one. Since $x_0 u = x_0$, $\tilde{\sigma}$ is a minimal cocycle.

Let $\sigma = \delta$. Then $\sigma(x_0, s) = \delta(x_0, s)(t, s \in T)$, whence by continuity $\sigma = \delta$.

Let $p = q$ on $\mathcal{A}$. Then $\sigma(p, s) = \sigma(x_0, s) = \sigma(x_0, q, s) = \delta(q, s)$ whence $p = q$ ($R_\delta$). Thus $\alpha(\tilde{\sigma}) \subset \mathcal{A}$.

Finally suppose $\delta \in Z(M, K)$ with $\alpha(\delta) \subset \mathcal{A}$. Then $\delta$ induces a cocycle $\tilde{\sigma}$ on $\text{sp}(\delta)$ to $K$ and $\sigma(x, t) = \delta(\pi(x), t)(x \in [\mathcal{A}], t \in T)$ is a cocycle on $[\mathcal{A}]$ to $K$ with $\tilde{\sigma} = \delta$. (Here $\pi: |\mathcal{A}| \to \text{sp}(\delta)$ is the map induced by the inclusion $\alpha(\delta) \subset \mathcal{A}$).

3.4 Standing convention. Henceforth $Z(\mathcal{A}, K)$ will be identified with the set of minimal cocycles $\sigma$ with $\alpha(\sigma) \subset \mathcal{A}$.
3.5 Definition. Let $\sigma \in Z(\mathcal{A}, K)$. Then the skew product flow $(K \times |\mathcal{A}|, T)$ induced by $\sigma$ is the flow with phase space $K \times |\mathcal{A}|$, phase group $I$, and action $((k, x), t) \mapsto (k \sigma(x, t), x_t): (K \times |\mathcal{A}|) \times T \rightarrow K \times |\mathcal{A}|$.

3.6 Remarks. 1. In general $(K \times |\mathcal{A}|, T)$ is not minimal; however, it is pointwise almost periodic. Indeed it is partitioned by its minimal subsets. This follows from the fact that the map $(k, (l, x)) \mapsto (kl, x): (K \times K) \times K \times I_d$ defines an action of $K$ on $K = |\mathcal{A}|$ which makes $(K, K \times K, T)$ into a birtransformation group with $(K \times K, T) = (|\mathcal{A}|, T)$.

2. Let $p \in M$. Then $p$ induces a map on $K \times I_d$ (also denoted $p$) viz: $(k, x)p = \lim (k, x)t_n = \lim (k \sigma(x, t_n), x_{t_n}) = (k \sigma(x, p), x_p)(k \in K, X \in |\mathcal{A}|)$. (Here $(t_n)$ is a net in $T$ which converges to $p$ on $\beta T$.)

Now let $q \in M$ with $q \in M$ and $q |\mathcal{A}| = x$. Then $\sigma(x, p) = \sigma(q, p) = f_\sigma(q)^{-1}f_\sigma(qp)$. Thus $(k, q |\mathcal{A}|)(p) = (k \sigma(q)^{-1}f_\sigma(qp), qp |\mathcal{A}|)(k \in K, p, q \in \beta T)$.

3.7 Definition. Let $\sigma \in Z(\mathcal{A}, K)$. Then the extension of $\mathcal{A}$ by $\sigma$ (denoted $\text{ext}(\mathcal{A}, \sigma)$) is the $T$-subalgebra of $\mathcal{A}$ which corresponds to the pointed flow $\{(e, x_0)^{*}T, (e, x_0)^{t}T\}$. Here $x_0$ is the base point of $|\mathcal{A}|$ and $(e, x_0)^{t}T$ is the orbit closure in $K \times |\mathcal{A}|$ of the point $(e, x_0)$.

3.8 Proposition. Let $\sigma \in Z(\mathcal{A}, K)$. Then
1. $\text{ext}(\mathcal{A}, \sigma) = \mathcal{A} \vee \text{al}(f_\sigma)$
2. $g(\text{ext}(\mathcal{A}, \sigma)) = A \cap g(f_\sigma) = \text{ker}(f_\sigma |\mathcal{A})$
3. $L = \{k |k \in K, k(e, x_0) \in (e, x_0)^{t}T\}$ is a closed subgroup of $K$ such that $f_\sigma(A) = L$
4. $|\exp(\mathcal{A}, \sigma)| = K \times |\mathcal{A}|$ if and only if $f_\sigma(A) = K$. (Here $I$ identify $\text{ext}(\mathcal{A}, \sigma)$ with $(e, x_0)^{t}T$.)

Proof. 1. By 1 of 3.6 $\text{ext}(\mathcal{A}, \sigma)$ is an almost periodic extension of $\mathcal{A}$, and by 2.14 $\mathcal{A} \vee \text{al}(f_\sigma)$ is also an almost periodic extension of $\mathcal{A}$ since $\text{al}(\sigma) \subset \mathcal{A}$. Hence it suffices to show that $g(\text{ext}(\mathcal{A}, \sigma)) = g(\mathcal{A} \vee \text{al}(f_\sigma))$.

Now $g(\text{ext}(\mathcal{A}, \sigma)) = \{\alpha \in G. (e, x_0)\alpha = (e, x_0)\}$. Hence by 2 of 3.6, $\alpha \in A \cap g(\text{ext}(\mathcal{A}, \sigma))$ if $f_\sigma(\alpha) = (e, x_0)$; i.e. $f_\sigma(\alpha) = e$ and $\alpha \in A$. But this is just $A \cap \text{ker} f_\sigma = A \cap g(f_\sigma) = g(\mathcal{A} \vee \text{al}(f_\sigma))$.

This completes the proof of 1 and 2.

3. Let $k \in L$. Then there exists $p \in M$ with $(k, x_0) = (f_\sigma(p), x_0p)$. This implies that $p |\mathcal{A} = u |\mathcal{A}$ and $f_\sigma(p) = k$. Since $\mathcal{A} \vee \text{al}(f_\sigma)$ is an almost periodic extension of $\mathcal{A}$, $p = \alpha$ on $\mathcal{A} \vee \text{al}(f_\sigma)$ for some $\alpha \in A$. Thus $k = f_\sigma(p) = f_\sigma(\alpha) \in f_\sigma(A)$.

One the other hand if $k = f_\sigma(\alpha)$ for some $\alpha \in A$ then $k(e, x_0) = (k, x_0) = (f_\sigma(\alpha), x_0) = f_\sigma(\alpha), x_0) \in (e, x_0)^{t}T$.

Thus $f_\sigma(A) = L$, whence $L$ is a closed subgroup of $K$ since $f_\sigma$ induces a continuous homomorphism of $A/A^e$ into $K$.

4. If $|\text{ext}(\mathcal{A}, \sigma)| = K \times |\mathcal{A}|$ then $L = K$ and so $f_\sigma(A) = K$ by 3.

On the other hand if $f_\sigma(A) = K$ then $K|\text{ext}(\mathcal{A}, \sigma)| = |\text{ext}(\mathcal{A}, \sigma)|$ and so $|\text{ext}(\mathcal{A}, \sigma)| = K \times |\mathcal{A}|$.

3.9 Corollary. Let $K$ be the circle group and let $\sigma \in Z(\mathcal{A}, K)$. Then $K \times |\mathcal{A}|$ is minimal if and only if $\sigma^* \not\equiv 0(\text{mod}\, \mathcal{A})$ for all integers $n \neq 0$.

Proof. Let $K \times |\mathcal{A}|$ be minimal. Then $f_\sigma(A) = K$ whence $f_\sigma^*(A) = K$ for all integers $n \neq 0$. Since $f_\sigma^* = f_\sigma^*, \sigma^* \not\equiv 0(\text{mod}\, \mathcal{A})$ for all $n \neq 0$.

Conversely if $\sigma^* \not\equiv 0$ then $f_\sigma^*(A) \neq \{1\}$ for all integers $n$. Consequently $f_\sigma(A)$ cannot
be a finite subgroup of $K$. This implies that $f_\sigma(A) = K$ whence $|\text{ext}(A, \sigma)| = K \times |A|$ and so $K \times |A|$ is minimal.

3.10 Definitions. Let $\sigma \in Z(\mathcal{A}, K)$. Then $\sigma$ is a coboundary on $\mathcal{A}$ or $\sigma$ is cohomologous to 0 on $\mathcal{A}$ (denoted $\sigma \sim 0 \pmod{\mathcal{A}}$) if $\text{all}(f_\sigma) \subset \mathcal{A}$. The set of coboundaries on $\mathcal{A}$ will be denoted $B(\mathcal{A}, K)$.

Let $\sigma, \delta \in Z(\mathcal{A}, K)$. Then $\sigma$ and $\delta$ are cohomologous on $\mathcal{A}$ ($\sigma \sim \delta \pmod{\mathcal{A}}$) if $\text{all}(f_\sigma^{-1}) \subset \mathcal{A}$.

3.11 Lemma. Let $\sigma, \delta \in Z(\mathcal{A}, K)$. Then
1. $\sigma \sim 0 \pmod{\mathcal{A}}$ if and only if $A \subset \text{g}(f_\sigma)$
2. $\sigma \sim \delta \pmod{\mathcal{A}}$ if and only if $\sigma^{-1} \delta \in Z(\mathcal{A}, K)$ and $\sigma^{-1} \delta \sim 0 \pmod{\mathcal{A}}$.
3. $\sigma \sim \delta \pmod{\mathcal{A}}$ if and only if $f_\sigma(\alpha) = f_\delta(\alpha)$ ($\alpha \in A$). (Here $A = \text{g}(\mathcal{A})$).

Proof. 1. Let $\sigma \sim 0 \pmod{\mathcal{A}}$. Then $\text{all}(f_\sigma) \subset \mathcal{A}$ and so $A \subset \text{g}(f_\sigma)$.
On the other hand if $A \subset \text{g}(f_\sigma)$, $\text{g}(\text{ext}(A, \sigma)) = A \cap \text{g}(f_\sigma) = A$ and so $\text{all}(f_\sigma) \subset \mathcal{A}$.

2. Let $\sigma \sim \delta \pmod{\mathcal{A}}$. Then $\text{all}(\sigma^{-1} \delta) \subset \text{all}(f_\sigma^{-1} \delta) \subset \mathcal{A}$ shows that $\sigma^{-1} \delta \in Z(\mathcal{A}, K)$ and $\sigma^{-1} \delta \sim 0 \pmod{\mathcal{A}}$.

3. Let $\sigma \sim \delta \pmod{\mathcal{A}}$ and $\alpha \in A$. Then $\text{all}(f_\sigma) \subset \mathcal{A}$ where $\rho = \sigma^{-1} \delta$. Hence $\text{al}(\rho) \subset \mathcal{A}$, $A \subset \text{g}(\rho)$ and $A \subset \ker f_\rho$. Thus $e = f_\rho(\alpha) = f_\sigma(\alpha)^{-1} f_\delta(\alpha)$.

Now let $f_\rho(\alpha) = e$ ($\alpha \in A$) where $\rho = \sigma^{-1} \delta$. Then $\text{all}(f_\sigma) \subset \text{all}(f_\sigma^{-1}) \subset \text{all}(f_\sigma)$ by 1 of 2.17. Since $f_\delta = f_\sigma$ on $A$ and $\delta \in Z(\mathcal{A}, K)$, $\text{g}(\text{ext}(A, \sigma)) = \text{g}(\text{ext}(A, \delta))$. Hence $\text{ext}(A, \sigma) = \text{ext}(A, \delta)$. Thus $\text{all}(f_\sigma) \subset \text{all}(f_\sigma^{-1}) \subset \text{ext}(A, A, \sigma)$. But $\text{all}(f_\sigma^{-1}) = \text{all}(f_\sigma)$ by 3 of 2.17. Hence $\text{all}(f_\rho) \subset \text{ext}(A, \sigma)$.

4. Let $\alpha \in A$ and $x \in K$. Then $f_\rho(\alpha x) = f_\sigma(\alpha x)^{-1} f_\delta(\alpha x) = f_\sigma(\alpha x)^{-1} f_\sigma(\alpha)^{-1} f_\delta(\alpha) = f_\delta(\alpha) = f_\delta(\alpha) = f_\rho(\alpha)$. Thus $A \subset \text{g}(f_\rho)$.
Finally $\text{all}(f_\rho) \subset \text{ext}(\mathcal{A}, \sigma)$ implies that $\mathcal{A} \lor \text{all}(f_\rho)$ is an almost periodic extension of $\mathcal{A}$. Since $\text{g}(\mathcal{A} \lor \text{all}(f_\rho)) = A \cap \text{g}(f_\rho) = A$, $\mathcal{A} \lor \text{all}(f_\rho)$ and so $\text{all}(f_\rho) \subset \mathcal{A}$.

3.11 Proposition. 1. $B(\mathcal{A}, K) \subset Z(\mathcal{A}, K)$
2. $B(M, K) = Z(M, K)$
3. $B(\mathcal{A}, K)$ is a subgroup of $Z(M, K)$
4. $Z(\mathcal{A}, K) \lor B(\mathcal{A}, K) \subset Z(\mathcal{A}, K)$
5. If $K$ is abelian, then $Z(\mathcal{A}, K)$ is an abelian group.

Proof. 1. Follows from the definition.
2. Since $\text{all}(f_\sigma)$ is minimal and $\mathcal{U}(u)$ the algebra corresponding to $M$ is universal, $\text{all}(f_\sigma) \subset \mathcal{U}(u)$.
3. Let $\sigma, \delta \in B(\mathcal{A}, K)$. By 2 of 2.17 $\text{all}(f_\sigma) \subset \text{all}(f_\delta) \subset \mathcal{A} \lor \mathcal{A} = \mathcal{A}$. Hence $\mathcal{A} \lor \mathcal{A} \subset B(\mathcal{A}, K)$.
4. Let $\sigma \in B(\mathcal{A}, K)$ and $\delta \in Z(\mathcal{A}, K)$. Then $\text{all}(\delta \sigma) \subset \mathcal{A} \lor \mathcal{A} \lor \mathcal{A} = \mathcal{A}$ by 2.17.
5. This follows from 6 of 2.17.

3.12 Proposition. Let $\sigma, \delta \in Z(\mathcal{A}, K)$. Then $\sigma \sim \delta \pmod{\mathcal{A}}$ if and only if the bitransformation groups $(K, K \times \mathcal{A}, T)$ and $(K, K \times \mathcal{A}, T)$ are isomorphic under an isomorphism $\varphi$ with $\varphi(e, x_0)$. (Here $x_0 = u|\mathcal{A}$, the base point of $|\mathcal{A}|$.)

Proof. Assume such an isomorphism $\varphi$ exists. Then $f_\sigma(t) \varphi(e, x_0) = \varphi(\sigma(x_0, t), x_0) = \varphi((e, x_0), \sigma^t) = f_\sigma(t)(e, x_0)$. Let $\alpha \in A$. The above relations and the continuity of various functions involved allow us to conclude that $f_\sigma(\alpha)(e, x_0) = f_\sigma(\alpha) \varphi(e, x_0) = f_\delta(\sigma)(e, x_0)$. Thus $f_\sigma(\alpha) = f_\delta^\alpha(\alpha)$ ($\alpha \in A$) whence $\sigma \sim \delta \pmod{\mathcal{A}}$.

Now let $\sigma \sim \delta \pmod{\mathcal{A}}$. For $k \in K$ and $x \in |\mathcal{A}|$ set $\varphi(k, x) = (k f_\sigma^{-1}(p) f_\delta(p), x)$ where $p \in M$ with $p|\mathcal{A} = x$. The function $\varphi$ is well defined, because $f_\sigma^{-1} f_\delta = f_\sigma^{-1} \delta$ and all($f_\sigma^{-1} \delta$) $\subset \mathcal{A}$. It is readily checked that $\varphi$ has the desired properties.
3.13 Remarks. 1. Lemma 3.10 number 3 shows that ~ is an equivalence relation on \( Z(\mathcal{A}, K) \) and number 4 shows that the equivalence class to which \( \sigma \) belongs is just the orbit of \( \sigma \) under \( B(\mathcal{A}, K) \) (see 3 and 4 of 3.12).

The orbit space \( Z(\mathcal{A}, K)/B(\mathcal{A}, K) \) is denoted \( H(\mathcal{A}, K) \). When \( K \) is abelian, \( Z(\mathcal{A}, K) \) and \( H(\mathcal{A}, K) \) are abelian groups. In this case \( H(\mathcal{A}, K) \) is called the cohomology group of \( \mathcal{A} \) with coefficients in \( K \).

The symbol \([\sigma]\) will denote the element of \( H(\mathcal{A}, K) \) determined by \( \sigma \in Z(\mathcal{A}, K) \).

2. Proposition 3.8 and 3 of 3.11 show that if \( u - S \pmod{a} \) then \( \text{ext}(a, u) = \text{ext}(da, S) \). The converse is not true in general. Indeed 3.12 shows that \( u - S \pmod{a} \) is a much stronger statement than \( \text{ext}(Sa, a) = \text{ext}(a, S) \).

3.14 Notation. Henceforth \( \text{Hom}(K, K) \) will denote the set of continuous endomorphisms of \( K \).

If \( u \in Z(\mathcal{A}, K) \) and \( \varphi \in \text{Hom}(K, K) \), then \( \varphi u \) will denote the map \( (x, t) \rightarrow \varphi(\sigma(x, t)) : \mathcal{A} \times T \rightarrow K \). It is clear that \( \varphi u \in Z(\mathcal{A}, K) \).

3.15 Proposition. Let \( \sigma, \delta \in Z(\mathcal{A}, K) \). Then

1. If \( \delta \sim \varphi \sigma \pmod{a} \) for some \( \varphi \in \text{Hom}(K, K) \), then \( \text{ext}(\delta, \sigma) \subset \text{ext}(\delta, \sigma) \).

2. If \( K \) is the circle group and \( \text{ext}(\delta, \sigma) \subset \text{ext}(\delta, \sigma) \), then \( \delta \sim \varphi \sigma \pmod{a} \) for some \( \varphi \in \text{Hom}(K, K) \).

Proof. 1. It is clear that \( f_{\sigma} = \varphi \circ f_{\varphi} \). Since \( f_{\delta} = f_{\sigma} = \varphi \circ f_{\delta} \) on \( A \), \( \text{ker}(f_{\sigma}|A) \subset \text{ker}(f_{\delta}|A) \), whence \( \text{ext}(\delta, \sigma) \subset \text{ext}(\delta, \sigma) \).

2. If \( \text{ext}(\delta, \sigma) \subset \text{ext}(\delta, \sigma) \), \( \text{ker}(f_{\delta}|A) \subset \text{ker}(f_{\sigma}|A) \). Consequently there exists a continuous epimorphism \( : f_{\delta}(A) \rightarrow f_{\sigma}(A) \) such that \( \varphi(f_{\delta}(a)) = f_{\delta}(a) (a \in A) \). Then \( \varphi \) may be extended (if necessary) to a continuous endomorphism of \( K \). The above relation shows that \( \varphi \sigma \sim \delta \pmod{a} \).

3.16 Corollary. Let \( K \) be the circle group. \( \sigma, \delta \in Z(\mathcal{A}, K) \) with \( \text{ext}(\delta, \sigma) = \text{ext}(\delta, \sigma) \) and \( f_{\delta}(A) = K \). Then \( \sigma \sim \delta \pmod{a} \) or \( \sigma \sim \delta^{-1} \pmod{a} \).

Proof. By 3.15 \( \sigma \sim \varphi \delta \) and \( \delta \sim \psi \sigma \) for some \( \varphi, \psi \in \text{Hom}(K, K) \). Then \( f_{\sigma} = \psi \varphi f_{\delta} \) and \( f_{\delta} = \varphi \psi f_{\sigma} \). Since both \( f_{\sigma} \) and \( f_{\delta} \) are onto, \( \varphi \) and \( \psi \) are automorphisms. Thus \( \varphi(k) = k(k \in K) \) or \( \varphi(k) = k^{-1}(k \in K) \).

Recall that for any \( T \)-subalgebra \( \mathcal{A} \) of \( \mathbb{U}(u) \), the set of almost periodic functions on \( \mathcal{A} \) is a \( T \)-subalgebra \( \mathcal{A}^{*} \) of \( \mathbb{U}(u) \). This algebra \( \mathcal{A}^{*} \) is the largest almost periodic extension of \( \mathcal{A} \).

The compact topological group \( A/A^{*} \) acts freely (on the left) on \( |\mathcal{A}^{*}| \). The set of homomorphisms of \( (A/A^{*}, |\mathcal{A}^{*}|) \) into \( (k, K) \) will be denoted \( \text{Hom}(|\mathcal{A}^{*}|, K) \). An element of \( \text{Hom}(|\mathcal{A}^{*}|, K) \) is thus a pair of \( (f, g) \) where \( f : A/A^{*} \rightarrow K \) is a continuous homomorphism and \( g : |\mathcal{A}^{*}| \rightarrow K \) is a continuous map such that \( g(ax) = g(a)g(x) (a \in A/A^{*}, x \in |\mathcal{A}^{*}|) \). I shall also assume that \( g(u|\mathcal{A}^{*}|) = e \).

Now let \( \sigma \in Z(\mathcal{A}, K) \). Then \( (f_{\sigma}|A) \subset \mathcal{A}^{*} \) because \( \text{ext}(\mathcal{A}, \sigma) \) is an almost periodic extension of \( \mathcal{A} \). Thus \( f_{\sigma} \) induces a continuous function on \( |\mathcal{A}^{*}| \). Moreover \( f_{\delta}|A \) induces a continuous homomorphism of \( A/A^{*} \) into \( K \) because \( A^{*} \subset \text{ker}(f_{\delta}) \). The pair \( (f_{\delta}|A, f_{\sigma}) \) is an element of \( \text{Hom}(|\mathcal{A}^{*}|, K) \).

3.17 Proposition. The map \( \sigma \rightarrow (f_{\sigma}|A, f_{\sigma}) : Z(\mathcal{A}, K) \rightarrow \text{Hom}(|\mathcal{A}^{*}|, K) \) is bijective.

Proof. Since \( \sigma(x, y) = f_{\sigma}(x)^{-1}f_{\sigma}(xy) \), the above map is injective.

Let \( (f, g) \in \text{Hom}(|\mathcal{A}^{*}|, K) \). The map \( p \rightarrow g(p|\mathcal{A}^{*}|) : \beta T \rightarrow K \) (also denoted \( g \)) is continuous. Set \( \sigma(x, y) = g(xy)^{-1}g(x) (x, y \in \beta T) \).

Let \( x, y \in \beta T \) with \( x|\mathcal{A} = y|\mathcal{A} \). Then there exists \( a \in A \) such that \( y|\mathcal{A}^{*} = ax|\mathcal{A}^{*} \).

Hence \( \sigma(x, z) = g(y)^{-1}g(yz) = g(ax)^{-1}g(axz) = g(x)^{-1}f(\alpha A^{*})^{-1}f(\alpha A^{*})g(xz) = g(x)^{-1}g(xz) = \sigma(x, z) \), whence \( a(\sigma) \subset \mathcal{A} \). Thus \( \sigma \in Z(\mathcal{A}, K) \).

Finally \( f_{\sigma}(x) = \sigma(e, x) = g(x) (x \in \beta T) \) and \( f(\alpha A^{*}) = f(\alpha A^{*}) = f(\alpha A^{*})f(u|\mathcal{A}^{*}) = g(\alpha|\mathcal{A}^{*}) = f_{\sigma}(\alpha) (\alpha \in A) \).

The cohomology classes \( H(\mathcal{A}, K) \) are obtained from \( \text{Hom}(|\mathcal{A}^{*}|, K) \) by identifying \( (f_{1}, g_{1}) \) with \( (f_{2}, g_{2}) \) when \( f_{1} = f_{2} \).
3.18 Proposition. Let $\text{Hom}(A/A^*, K)$ denote the set of continuous homomorphisms from $A/A^*$ to $K$. Then the map $\sigma \mapsto f_{\sigma}[A]: Z(\mathcal{A}, K) \to \text{Hom}(A/A^*, K)$ induces an injective map from $H(S, K)$ into $\text{Hom}(A/A^*, K)$. When $K$ is abelian this map is a monomorphism. (Note: the map $f_{\sigma}[A]$ is identified with the one it induces on $A/A^*$.)

Proof. By 3 of 3.13 two cocycles $\sigma, \delta$ are cohomologous if and only if they induce the same map on $A$ (or $A/A^*$ since $A^* \subset \ker f_{\sigma} \cap \ker f_{\delta}$).

That the map induced on $H(S, K)$ is a monomorphism when $K$ is abelian follows directly from the definition of the group structures involved. The proof is completed.

It will be shown in the next section that the above map is onto when $|\mathcal{A}|$ is 0-dimensional.

An interesting problem is to determine "dynamically" when a given cocycle $\sigma$ on $\mathcal{A}$ is cohomologous to some element of a given subset $\mathcal{F}$ of $Z(S, K)$. Let $\mathcal{L} = \{\ker f_{\sigma}[\delta] \in \Delta \}$, then $\mathcal{L}$ is a $\tau$-closed subgroup of $G$. Let $\mathcal{F}$ be a $\tau$-subalgebra of $\mathcal{A}$ with $g(\mathcal{F}) = \mathcal{A}$. Let $\mathcal{L}$ be a $\tau$-subalgebra of $\mathcal{A}$ with $g(\mathcal{L}) = \mathcal{A}$. Then $\sigma \sim \delta (\text{mod } \mathcal{A})$ for some $\delta \in \Delta$ if and only if $ext(\mathcal{F}, \sigma) \subset \mathcal{A} \cup \mathcal{L}$. The next proposition is concerned with sufficient conditions.

3.19 Proposition. Let $\mathcal{F}$ be a $\tau$-subalgebra of $\mathcal{A}$ with $g(\mathcal{F}) = F$. Let $\mathcal{L}$ be a $\tau$-closed normal subgroup of $F$ with $\mathcal{L} \subset ker (f_{\sigma}|F)$ and let the image of $H(S, K)$ in $\text{Hom}(F/F^*, K)$ contain $\text{Hom}(F/F^*, K)$. Then $\sigma \sim \delta (\text{mod } \mathcal{A})$ for some $\delta \in \Delta$ if and only if $ext(\mathcal{F}, \sigma) \subset \mathcal{A} \cup \mathcal{L}$, where $\sigma \in Z(S, K)$ and $\mathcal{L}$ is any $\tau$-subalgebra of $\mathcal{A}$ with $g(\mathcal{L}) = \mathcal{L}$.

Proof. The necessity of the condition $ext(\mathcal{A}, \sigma) \subset \mathcal{A} \cup \mathcal{L}$ has already been remarked upon.

If $ext(\mathcal{A}, \sigma) \subset \mathcal{A} \cup \mathcal{L}$, $f_{\sigma}|A$ induces a homomorphism $\psi$ of $F/L = AL/L = A/A' \cap L$ into $K$ such that $\psi(aL) = f_{\sigma}(a)$ ($a \in A$). Let $\delta \in Z(S, K)$ with $f_{\sigma}(aL) = \psi(aL)$ ($a \in F = AL$). Let $\rho$ be the image of $\delta$ in $Z(S, A)$. Then $\rho \in \Gamma$ and $f_{\sigma}(aL) = f_{\sigma}(a)$ ($a \in A$). The proof is completed.

3.20 Proposition. Let $\text{Hom}(G/E, K) \to \text{Hom}(A/E, K)$ be onto, and let $\sigma \in Z(A, K)$. Then $\sigma = \delta (\text{mod } \mathcal{A})$ for some $\delta \in \text{Hom}(A, K)$ if and only if $ext(\mathcal{A}, \sigma) \subset \mathcal{A} \cup \mathcal{L}$.

Proof. Set $\mathcal{F} = \mathcal{A} \cap \mathcal{L}$. Then $F = AE$ and the hypothesis implies that the image of $H(S, K)$ in $\text{Hom}(F/E, K)$ contains $\text{Hom}(F/E, K)$. (Recall that $G/E$ is the Bohr compactification of the discrete group $T$ so that $T = \text{Hom}(G/E, K)$.)

The proof is completed by observing that if $\delta \in Z(S, K)$ with $E \subset ker f_{\sigma}$ then $\delta \sim \rho (\text{mod } \mathcal{F})$ for some $\rho \in \text{Hom}(T, K)$ because $\text{Hom}(\mathcal{A}/E, K) \to \text{Hom}(F/E, K)$ is onto.

3.21 Proposition. Let $T$ be abelian, $K$ an $n$-torus, and $\sigma \in Z(A, K)$. Then $\sigma$ is cohomologous to a constant cocycle if and only if $ext(\mathcal{A}, \sigma) \subset \mathcal{A} \cup \mathcal{L}$.

Proof. Since $T$ is abelian, so is $G/E$. Proposition 3.21 now follows from 3.20.

3.22 Remarks. 1. Let $T$ be the integers $Z$ and let $\sigma \in Z(Z, K)$. Then the cocycle equation $\sigma(r, s + 1) = \sigma(r, s)\sigma(r + s, t)$ ($r, s, t \in Z$) shows that $\sigma(n, m) = m^{-1} \prod_{k=0}^{n+k} \sigma(n+k, 1)$, $\sigma(n, 0) = e$, and $\sigma(n, m) = \sigma(n-m, m)^{-1}$ ($n \in Z, m \geq 1$). Thus the cocycle $\sigma$ is completely determined by the function $n \mapsto \sigma(n, 1): Z \to K$. This function will also be denoted, $\sigma$.

2. Now suppose $\sigma \in \text{Hom}(Z, K)$, i.e. $\sigma(n, m) = \psi(m)$ where $\psi$ is a homomorphism of $Z$ into $K$. Then the function $n \mapsto \sigma(n, 1)$ is a constant $k \in K$ and $\sigma(n, m) = k^m$ ($n, m \in Z$).

With these remarks in mind the following proposition (due to L. Shapiro) follows immediately from 3.21.
3.23 Proposition. Let $\phi$ be a homeomorphism of the compact Hausdorff $X$ such that the flow $(X, \phi)$ is minimal and equicontinuous, let $K$ be an $n$-torus, and let $\sigma \in Z(X, K)$ be such that $\text{ext}(X, \sigma)$ is equicontinuous. Then there exists $\alpha \in K$ such that $\text{ext}(X, \sigma)$ is isomorphic to the orbit closure of the point $(e, x_0)$ under the flow generated by the homeomorphism $(k, x) \to (ka, x \phi)$: $K \times X \to K \times X$, where $x_0$ is any point in $X$.

When $K$ is the circle group there are many results similar to the ones above which depend upon the fact that a compact abelian group is determined by its character group and vice-versa. The following result is typical.

3.24 Proposition. Let $K$ be the circle group. $\Gamma$ a subgroup of $Z(\mathcal{A}, K)$, $\mathcal{L} = V(\text{ext}(\mathcal{A}, \sigma)|\sigma \in \Gamma)$, and $\delta \in Z(\mathcal{A}, K)$ with $\text{ext}(\mathcal{A}, \delta) \subset \mathcal{L}$. Then there exists $\sigma \in \Gamma$ with $\delta \sim \sigma \mod \mathcal{A}$.

Proof. Since $L = \cap \{\ker(f_{\sigma}|A)| \sigma \in \Gamma\}$, $f_{\sigma}$ induces a continuous homomorphism $\tilde{f}_{\sigma}$ from $A/L$ to $K$ ($\sigma \in \Gamma$), and the collection $\{f_{\sigma}|\sigma \in \Gamma\}$ separates points of $A/L$. The assumption $\text{ext}(\mathcal{A}, \delta) \subset \mathcal{L}$ implies that $L \subset \ker f_{\sigma}$. Hence there exists $\sigma \in \Gamma$ with $f_{\sigma} = f_{\tilde{\sigma}}$. The proof is completed.

## 4. SECTIONS OF FLOWS

4.1. The problem of when the canonical map of $|\text{ext}(\mathcal{A}, \sigma)|$ onto $|\mathcal{A}|$ admits a continuous cross section is studied with the aim of determining conditions under which the map $[\sigma] \to f_{\sigma}: H(\mathcal{A}, K) \to \text{Hom}(A/A^\sigma, K)$ is onto.

4.2 Definition. Let $\mathcal{A}$, $\mathcal{B}$ be $T$-subalgebras of $U(u)$ with $\mathcal{A} \subset \mathcal{B}$. Then a (continuous) section of $\mathcal{B}$ over $\mathcal{A}$ is a continuous map $s: |\mathcal{A}| \to |\mathcal{B}|$ such that $s(x|\mathcal{A}) = x$ ($x \in |\mathcal{B}|$). For purposes of normalization I also assume that $s(u|\mathcal{A}) = u|\mathcal{B}$ (i.e. $s$ preserves base points).

4.3 Notation. The fact that $\mathcal{B}$ is an almost periodic extension of $\mathcal{A}$ with $B$ a normal subgroup of $A$ will be denoted $\mathcal{A} \subset B$.

Let $\mathcal{A} \subset \mathcal{B}$ and let $s$ be a section of $\mathcal{B}$ over $\mathcal{A}$. Then given $x \in |\mathcal{A}|$ and $t \in T$ there exists $\alpha \in A$ with $s(xt) = \alpha s(x)$. Moreover any two such $\alpha$ are congruent modulo $B$. Hence there exists $\delta: |\mathcal{A}| \times T \to \text{Hom}(A/A^\alpha, K)$ such that $\delta(x, t) = s(xt)/(x, t)$ ($x \in |\mathcal{A}|, t \in T$).

4.4 Proposition. Let $\mathcal{A} \subset \mathcal{B}$, $s$ a section of $\mathcal{B}$ over $\mathcal{A}$, and let $\delta: |\mathcal{A}| \times T \to \text{Hom}(A/A^\alpha, K)$ be such that $s(xt) = \delta(x, t)s(x)t$ ($x \in |\mathcal{A}|, t \in T$). Then

1. $\delta$ is a cocycle on $\mathcal{A}$ to $A/B$
2. $s(p|\mathcal{A}) = f_\delta(p)^{-1}(p|\mathcal{B})$ ($p \in M$)
3. $f_\delta(\alpha) = B\alpha$ ($\alpha \in \mathcal{A}$).

Proof. 1. That $\delta$ satisfies the cocycle equation follows immediately from the definitions. The continuity of $\delta$ follows from that of $s$ and the fact that $A/B$ is a compact topological group which acts freely on $|\mathcal{B}|$ with quotient $|\mathcal{A}|$.

2. Let $p \in M$ and $(t_n)$ a net in $T$ with $t_n \to p$. Then $ut_n \to up = p$ and $s(p|\mathcal{A}) = \lim s((u|\mathcal{A})t_n) = \lim s((u|\mathcal{A})u) = \delta(u, t_n)^{-1} = (s(u|\mathcal{A})t_n = \delta(u, p)^{-1} = (u|\mathcal{B})p = f_\delta(p)^{-1}(p|\mathcal{B})$.

3. Let $\alpha \in \mathcal{A}$. Then $s(\alpha|\mathcal{A}) = s(u|\mathcal{A}) = u|\mathcal{B}$ whence by $2 u|\mathcal{B} = f_\delta(\alpha)^{-1}(\alpha|\mathcal{B})$; or $f_\delta(\alpha)(u|\mathcal{B}) = \alpha|\mathcal{B}$.

Now let $f_\delta(\alpha) = B\gamma$. Then $B\gamma(u|\mathcal{B}) = \gamma|\mathcal{B}$, whence $\gamma|\mathcal{B} = \alpha|\mathcal{B}$. Thus $B\alpha = B\gamma = f_\delta(\alpha)$.

The cocycle $\delta$ is called the cocycle determined by the section $s$. Notice that 3 above implies that all the sections of $\mathcal{B}$ over $\mathcal{A}$ determine the same element of $H(\mathcal{A}, A/B)$.

4.5 Proposition. Let $\mathcal{A} \subset \mathcal{B}$. Then there exists a section of $\mathcal{B}$ over $\mathcal{A}$ if and only if there exists a cocycle $\delta$ on $\mathcal{A}$ to $A/B$ with $f_\delta(\alpha) = B\alpha$ ($\alpha \in \mathcal{A}$).
Proof. Necessity is just 3 of 4.4.
To prove sufficiency observe that if \( p = q \) on \( \mathcal{A} \) then \( p = aq \) on \( \mathcal{B} \) for some \( a \in A \).
Since \( B \subset \text{ker}(f_{\delta}|A) \), \( \text{ext}(\mathcal{A}, \delta) \subset \mathcal{B} \) whence
\[
f_{\delta}(p)^{-1}(p|\mathcal{B}) = f_{\delta}(aq)^{-1} \quad (aq|\mathcal{B}) = f_{\delta}(q)^{-1}f_{\delta}(a)^{-1}(aq|\mathcal{B}) = f_{\delta}(q)^{-1}B^{-1}(aq|\mathcal{B}) = f_{\delta}(q)^{-1}(q|\mathcal{B}).
\]
Thus \( s(x) = f_{\delta}(p)^{-1}(p|\mathcal{B}) \) where \( p \in M \) with \( p|\mathcal{A} = x \) is a well defined section of \( \mathcal{B} \) over \( \mathcal{A} \).

4.6 Remark. Let \( \mathcal{A} \subset \mathcal{B} \) and suppose that \( \mathcal{B} \) does not admit a section over \( \mathcal{A} \).
Then the homomorphism \( A^* \alpha \rightarrow B_\alpha : A/A^* \rightarrow B \) is not in the image of the map
\[
[\alpha] \rightarrow f_\alpha : H(\mathcal{A}, A/B) \rightarrow \text{Hom}(A/A^*, A/B); \text{i.e. the above map is in general not surjective.}
\]

4.7 Proposition. Let \( \sigma \in Z(\mathcal{A}, K) \). Then there exists a section of \( \text{ext}(\mathcal{A}, \sigma) \) over \( \mathcal{A} \) if and only if \( \sigma \sim \rho \) (mod \( \mathcal{A} \)) with \( f_{\rho}(M) = f_{\rho}(A) \).

Proof. Assume that there is a section of \( \text{ext}(\mathcal{A}, \sigma) \) over \( \mathcal{A} \). Then there is a cocycle \( \delta \) on \( \mathcal{A} \) to \( A/B \) with \( f_{\delta}(a) - Ba (\alpha \in A) \) where \( B - \{ a(\text{ext}(\mathcal{A}, \sigma)) = \text{ker}(f_{\rho}|A) \) Set \( \rho = f_{\delta}|\mathcal{A} \) where \( f_{\delta}: A/B \rightarrow K \) with \( f_{\delta}(Ba) = f_{\sigma}(a) (\alpha \in A) \). Then \( \rho \in Z(\mathcal{A}, K) \) with \( f_{\delta}(M) = f_{\delta}(A), f_{\delta}(M) = f_{\sigma}(A) \). Moreover \( f_{\rho}(a) = f_{\rho}(Ba) = f_{\rho}(a) (\alpha \in A) \) whence \( \rho \sim \sigma \) (mod \( \mathcal{A} \)).

Now suppose \( \sigma \sim \rho \) (mod \( \mathcal{A} \)) with \( f_{\delta}(M) = f_{\sigma}(A) \). Let \( \delta \in Z(M, A/B) \) be such that \( f_{\delta} - f_{\rho} \in f_{\delta} \) where \( f_{\delta} \) is the isomorphism of \( A/B \) onto \( f_{\sigma}(A) \) induced by \( f_{\rho} \) (recall that \( \text{ker}(f_{\rho}|A) = \text{ker}(f_{\delta}|A) = B \)). Then \( \delta = f_{\rho}^{-1} \cdot \rho \in Z(\mathcal{A}, A/B) \) with \( f_{\delta}(a) = f_{\rho}^{-1}(f_{\sigma}(a)) = Ba (\alpha \in A) \) The proof is completed.

4.8 Remarks. 1. Let \( \mathcal{A} \) be the flow determined on the circle \( S^1 \) by a rotation through the angle \( 2\pi \) where \( 2\pi \) is an irrational multiple of \( \alpha \) and let \( \delta \) be the cocycle on \( \mathcal{A} \) determined by \( \delta(x, 1) = \exp(2\pi ib) (x \in \mathcal{A}) \). Set \( \mathcal{B} = \text{ext}(\mathcal{A}, \delta) \). Then \( A/B \equiv Z_2 \) and \( \mathcal{B} \) is not topologically the product \( A/B \times \mathcal{A} \). (Indeed \( \mathcal{B} \) may be identified with the subset \( \{(x, x^2) : x \in S^1 \} \) of \( S^1 \times S^1 \). This implies that there does not exist a cocycle \( \sigma \) on \( \mathcal{A} \) to \( A/B \) with \( \text{ext}(\mathcal{A}, \sigma) = \beta \). (Since \( A/B \equiv Z_2, f_{\sigma} \neq e \) would mean that \( f_{\sigma} \) would have to be onto and so there would be a section of \( \mathcal{B} \) over \( \mathcal{A} \)).

2. The above is an example of an almost periodic extension \( \mathcal{B} \) of \( \mathcal{A} \) such that \( \mathcal{A} \neq \text{ext}(\mathcal{A}, \sigma) (\sigma \in Z(\mathcal{A}, A/B)) \) but \( \mathcal{B} = \text{ext}(\mathcal{A}, \delta) \) for some \( \delta \in Z(\mathcal{A}, K) \) where \( K \) is a group containing \( A/B \). Let me now show that this cannot happen when \( A/B \equiv S^1 \) and \( K \) is abelian.

If it were to happen \( f_{\delta} \) would induce a monomorphism \( f_{\delta} \) from \( A/B \) into \( K \) such that \( f_{\delta}(Ba) = f_{\delta}(a) (\alpha \in A) \). There would exist a homomorphism \( \psi : K \rightarrow A/B \) with \( \psi(f_{\delta}(a)) = Ba (\alpha \in A) \). Then the cocycle \( \sigma = \psi \circ \delta \) on \( \mathcal{A} \) to \( A/B \) would have the property that \( f_{\sigma}(a) = Ba (\alpha \in A) \) and so \( \mathcal{B} = \text{ext}(\mathcal{A}, \sigma) \), which we assumed could not happen.

The classical three dimensional nil flow over the two torus is an example wherein \( A/B \equiv S^1 \) and \( \mathcal{B} = \text{ext}(\mathcal{A}, \sigma) (\sigma \in Z(\mathcal{A}, A/B)) \). (See [3] p. 53 for example.)

I would now like to examine more closely the question as to when there exists a section of \( \mathcal{B} \) over \( \mathcal{A} \). It turns out that one always exists when \( \mathcal{B} \) is an almost periodic extension of \( \mathcal{A} \) and \( |\mathcal{A}| \) is 0-dimensional. The proof of this result is based upon the following theorem due to Gleason.

4.9 Proposition. Let the compact Lie group, \( \mathcal{H} \) act freely on the compact Hausdorff space \( X \). Then the canonical map \( \pi : X/H = Y \) admits local sections; i.e. given \( y \in Y \) there exist a neighborhood \( U \) of \( y \) and a continuous map \( s : U \rightarrow X \) such that \( \pi(s(v)) = y(v \in U) \).

4.10 Corollary. If in addition to the assumptions of 4.9 \( Y \) is 0-dimensional, then \( \pi \) admits a global section; i.e. there exists \( s : Y \rightarrow X \) continuous with \( \pi(s(y)) = y(y \in Y) \).
Proof. Let \( U_1, \ldots, U_n \) be a cover of \( Y \) consisting of open-closed subsets and let \( s_i: U_i \to X \) be continuous maps with \( \pi(s_i(y)) = y \) (\( y \in U_i \), \( 1 \leq i \leq n \)). Such exist by 4.9.

Set \( i(y) = \min \{ i \mid y \in U_i \} \) (\( y \in Y \)). Then \( s(y) = s_{i(y)}(y) \) (\( y \in Y \)) is the desired section.

I would now like to do without the assumption that \( H \) is a Lie group. Thus let \( H \) be a compact topological group and let it act freely on a compact Hausdorff space \( X \) and suppose that \( Y = X/H \) is 0-dimensional. I would like to show that the canonical map \( \pi: X \to Y \) admits a section.

To this end let \( \mathscr{K} \) denote the set of closed subgroups of \( H \) and \( S \) the set of pairs \( (K, s) \) where \( K \in \mathscr{K} \) and \( s \) is a section of the canonical map \( \pi_K: X/K \to Y \). The set \( S \) is non-empty since \( (H, 1) \in S \), where \( 1: X/H \to Y \) is the identity map.

Let \( (K_1, s_1), (K_2, s_2) \in S \). Then set \( (K_1, s_1) \leq (K_2, s_2) \) if \( K_2 \subseteq K_1 \) and the diagram

\[
\begin{array}{ccc}
X/K_2 & \xrightarrow{s_2} & X/K_1 \\
\downarrow & & \downarrow \\
Y & \xrightarrow{s_1} & Y
\end{array}
\]

is commutative where \( \varphi \) is the canonical map. The relation \( \leq \) makes \( S \) into a partially ordered set.

Now let \( \{ (K_i, s_i) \} \) be a simply ordered subset of \( S \). Then the intersection \( K \) of the \( K_i \) (\( i \in I \)) is a closed subgroup of \( H \). Let \( s: Y \to \prod_{i \in I} X/K_i \) be the map such that \( s(y) = (s_i(y)) \) (\( y \in Y \)). Since \( \{ (K_i, s_i) \} \) is simply ordered the image of \( S \) is just the inverse limit of the system \( \{ (X/K_i, \varphi_{ij}) \} \) (\( i, j \in I \) with \( K_i \subseteq K_j \)), where \( \varphi_{ij}: X/K_i \to X/K_j \) is the canonical map. This inverse limit is just \( X/K \) whence \( (K, s) \in S \) and \( (K_1, s_1) \leq (K, s)(i \in I) \). The partially ordered set \( (S, \leq) \) is thus inductive.

Let \( (L, r) \) be a maximal element of \( S \). The proof will be completed by showing that \( L = \{ e \} \). Indeed if this were not so there would exist a closed normal subgroup \( K \) of \( H \) such that \( L \cap K \neq L \) and \( H/K \) is a Lie group.

The composition \( L \to H \to H/K \) induces an isomorphism of \( N = L/L \cap K \) onto a closed subgroup of \( H/K \) whence \( N \) is a Lie group. Moreover \( N \) acts freely on \( X/L \) and \( (X/L \cap K)/N = X/L \). Hence the canonical map \( \varphi: X/L \cap K \to X/L \) admits local sections. Let \( U_1, \ldots, U_n \) be an open cover of \( X/L \) and \( t_i \) continuous maps of \( U_i \) into \( X/L \cap K \) such that \( \varphi(t_i(u)) = u (u \in U_i, 1 \leq i \leq n) \). Let \( V_1, \ldots, V_n \) be an open-closed cover of \( Y \) such that \( r(V_i) \subseteq U_i(1 \leq i \leq n) \), and set \( i(y) = \min \{ i \mid y \in V_i \} \) (\( y \in Y \)). Then the map \( s: Y \to X/L \cap K \) such that \( s(y) = t_{i(y)}(r(y)) \) (\( y \in Y \)) is a continuous section with \( \varphi(s(y)) = r(y) (y \in Y) \) a fact which contradicts the maximality of \( (L, r) \). This completes the proof of:

4.11 Proposition. Let the compact topological group \( H \) act freely on the compact Hausdorff space \( X \) and let \( X/H \) be 0-dimensional. Then the canonical map of \( X \) onto \( X/H \) admits a global section.

4.12 Corollary. In addition to the assumptions of 3.12 let \( K \) be a closed subgroup of \( H \). Then the canonical map of \( X/K \) onto \( X/H \) admits a global section.

Proof. Compose a global section of \( X \to X/H \) with the canonical map \( X \to X/K \).

A slight modification of the proof of 4.11 together with 4.12 yields:

4.13 Proposition. Let the compact topological group \( H \) act freely on the compact Hausdorff space \( X \) such that \( X/H \) is 0-dimensional, let \( K \) and \( L \) be closed subgroups of \( H \) with \( L \subseteq K \) and let \( s \) be a section of the canonical map of \( X/K \to X/H \). Then there exists a section \( t \) of the map \( X/L \to X/H \) such that the diagram

\[
\begin{array}{ccc}
X/L & \to & X/K \\
\downarrow & & \downarrow \\
X/H & \to & X/H
\end{array}
\]

(is commutative. Here \( X/L \to X/K \) is the canonical map.)
4.14 Remark. The relevance of the above propositions to our situation resides in the fact that $A/A^*$ is a compact topological group which acts freely on $|\mathcal{A}|$ with quotient $|\mathcal{A}|$. Moreover if $\mathcal{B}$ is an almost periodic extension of $\mathcal{A}$, $B/A^*$ is a closed subgroup of $A/A^*$ and $|\mathcal{B}|$ is the quotient of $|\mathcal{A}|$ by $B/A^*$. Thus 4.12 and 4.13 may be applied to our situation to yield:

4.15 Proposition. Let $|\mathcal{A}|$ be 0-dimensional and let $\mathcal{B}$ be an almost periodic extension of $\mathcal{A}$. Then there exists a section of $\mathcal{B}$ over $\mathcal{A}$.

4.16 Proposition. Let $|\mathcal{A}|$ be 0-dimensional, let $\mathcal{B}$ and $\mathcal{L}$ be almost periodic extensions of $\mathcal{A}$ with $\mathcal{A} \subset \mathcal{B} \subset \mathcal{L}$ and let $r$ be a section of $\mathcal{B}$ over $\mathcal{A}$. Then there exists a section $s$ of $\mathcal{L}$ over $\mathcal{A}$ such that $r(x) = s(x)|\mathcal{L}$ (x $\in$ $|\mathcal{A}|$).

4.17 Remarks. 1. The space $|[\Omega(u)]|$ of the universal flow is 0-dimensional. Hence one way of proving that $|\mathcal{A}|$ is 0-dimensional is to show that the canonical map of $|[\Omega(u)]|$ onto $|\mathcal{A}|$ is open.

2. Let $\mathcal{A}$ be a $T$ subalgebra of $\Omega(u)$. Then [3] and 1 above shows that the space $|[\Omega(A)]|$ of its maximal proximal extension is 0-dimensional. (Indeed it can be shown that $|[\Omega(A)]|$ is extremely disconnected.)

3. Again let $\mathcal{A}$ be a $T$-subalgebra of $\Omega(u)$, $L = \{p: p \in M, p|\mathcal{A} = u|\mathcal{A}\}$, and $X = \{L \circ p|p \in M\} \subset 2^M$. Then $X$ is the maximum strongly proximal extension of $\mathcal{A}$ and the canonical map of $M$ onto $X$ is open. Hence $X$ is 0-dimensional. (These results were communicated to me by S. Glazner, see [4].)

4. Using an argument similar to the one used in the proof of 4.11 one can show that when the Furstenberg structure theorem [3, 15.14] is applicable that if $\mathcal{B}$ is a distal extension of $\mathcal{A}$ with $|\mathcal{A}|$ 0-dimensional, then there exists a section of $\mathcal{B}$ over $\mathcal{A}$.

5. Let $\mathcal{B}$ be an almost periodic extension of $\mathcal{A}$ and let $T$ be countable. Then it can be shown that there are extensions $\mathcal{A}$ of $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{A}$ is 0-dimensional. $\mathcal{A}$ is an almost periodic extension of $\mathcal{A}$ and $\{x: x \in \mathcal{A} \text{ and } x|\mathcal{A} = u|\mathcal{A}\}$ is a singleton, (i.e. $\mathcal{A}$ is an almost one-one extension of $\mathcal{A}$.) In this case of course there exists a section of $\mathcal{B}$ over $\mathcal{A}$.

I would now like to apply the above results in some situations where the flow is 0-dimensional.

Recall that $\mathcal{A}^*$ is the largest almost periodic extension of $\mathcal{A}$ and that $A^* = g(\mathcal{A}^*)$.

If $\sigma \in Z(\mathcal{A}, K)$, then $\tilde{\sigma}$ will denote the homomorphism of $A/A^*$ into $K$ induced by $f_\sigma/A$.

The following is a strengthening of Proposition 3.18 in the case when $|\mathcal{A}|$ is 0-dimensional.

4.18 Proposition. Let $|\mathcal{A}|$ be 0-dimensional. Then the map $[\sigma] \rightarrow f_\sigma: H(\mathcal{A}, K) \rightarrow \text{Hom}(A/A^*, K)$ is bijective. If $K$ is abelian, then the above map is an isomorphism.

Proof. By 3.18 it suffices to show that the map is onto.

To this end let $\lambda \in \text{Hom}(A/A^*, K)$ with kernel $B/A^*$. Then $B$ is a $\tau$-closed normal subgroup of $A$ and $\mathcal{A} < \mathcal{B}$ where $\mathcal{B} = g(B) \cap \mathcal{A}$. Moreover, $g(\mathcal{A}) = B$.

By 4.15 and 4.5 there exists $\delta \in Z(\mathcal{A}, A/B)$ such that $f_\delta(\alpha) = B\alpha \ (\alpha \in A)$. Set $\sigma = \lambda \circ \delta$. Then $\sigma \in Z(\mathcal{A}, K)$ and $f_\sigma = \lambda \circ f_\delta$. Hence $\tilde{\sigma} = \lambda$. The proof is completed.

The results of §4 give a clear picture of the set of almost periodic extensions of $\mathcal{A}$ when $|\mathcal{A}|$ is 0-dimensional. Thus:

4.19 Proposition. Let $|\mathcal{A}|$ be 0-dimensional. Then there exists $\sigma \in Z(\mathcal{A}, A/A^*)$ such that the unique almost periodic extension $\mathcal{B}$ with group $B$ is given by $|\mathcal{B}| = A/B \cup |\mathcal{A}|$ for all $\tau$-closed subgroups $B$ with $A^* \subset B \subset A$. (The action of $T$ on
$A/B \times |\mathcal{A}|$ is given by $(Ba, x) t = (Ba \pi(\sigma(x, t), xt) \ (a \in A, \ x \in |\mathcal{A}|, t \in T)$ where $\pi: A/A^* \rightarrow A/B$ is the canonical map of $A/A^*$ onto the homogeneous space $A/B$.

**Proof.** The existence of a section of $\mathcal{A}^*$ over $\mathcal{A}$ (Proposition 4.15) implies that $|\mathcal{A}^*| = A/A^* \times |\mathcal{A}|$ for some $\sigma \in Z(\mathcal{A}, A/A^*)$. Moreover $(A/A^*, A/A^* \times |\mathcal{A}|, T)$ is a bitransformation group so that if one divides out by the action of the closed subgroup $B/A^*$ one obtains a flow with group $B$. Since there is only one almost periodic extension of $\mathcal{A}$ with group $B$, this must be it. The proof is completed.

§5. THE PERTURBED FLOW

5.1 Let $\mathcal{A} < \mathcal{B}$. Then $(A/B, |\mathcal{B}|, T)$ is a bitransformation group with quotient $(|\mathcal{A}|, T)$. A cocycle $\sigma$ on $\mathcal{A}$ to $A/B$ may be used to define a new action on $|\mathcal{B}|$ via the map $(x, t) \mapsto \sigma(x|\mathcal{A}|, t)^{-1}xt: |\mathcal{B}| \times T \rightarrow |\mathcal{B}|$. In this section the properties of this new flow and its relation to the flow $\text{ext}(\mathcal{A}, \sigma)$ are studied.

The new action of $T$ on $|\mathcal{B}|$ will be denoted $x \circ t$ or simply $x \cdot t$ $(x \in |\mathcal{B}|, t \in T)$ and the new flow denoted $|\mathcal{B}|_{\sigma}$.

5.2 Proposition. Let $\mathcal{A} < \mathcal{B}$ and $\sigma \in Z(\mathcal{A}, A/B)$. Then

1. $x \circ p = \sigma(x|\mathcal{A}|, p)^{-1}(xp) = f_\sigma(qp)^{-1}f_\sigma(q)(xp) \ (x \in |\mathcal{B}|, p, q \in M$ with $q|\mathcal{B} = x)$ and

2. The flow $|\mathcal{B}|_{\sigma}$ is pointwise almost periodic.

**Proof.** 1. Let $(t_n)$ be a net on $T$ which converges to $p \in M$. By definition $x \circ p = \lim (x \circ t_n) = \lim \sigma(x|\mathcal{A}|, t_n)^{-1}(xt_n)$ which converges to $\sigma(x|\mathcal{A}|, p)^{-1}(xp)$ by 2.6 and the continuity of the map $(Ba, x) \mapsto \sigma(x|\mathcal{A}|, t)^{-1}xt: A/B \times |\mathcal{B}| \rightarrow |\mathcal{B}|$.

Let $q/B = x$. Then $q|\mathcal{A} = x|\mathcal{A}$ and $\sigma(x|\mathcal{A}|, p) = \sigma(q, p) = f_\sigma(q)^{-1}f_\sigma(qp)$.

2. Let $x \in |\mathcal{B}|$. Since the flow $\mathcal{B}$ is minimal, there exists an idempotent $v \in M$ with $xv = x$. Then $(x|\mathcal{A}|)v = x|\mathcal{A}$ and $x \circ x^\circ v = \sigma(x|\mathcal{A}|, v)^{-1}xv = x$ by 2.13. Hence $x$ is an almost periodic point of the flow $|B|_\sigma$. The proof of 5.2 shows that if $x_0$ is the base point of the flow $|\mathcal{B}|$ then $x_0 \circ u = x_0$ and so $x_0$ may be used as the base point of the minimal subset $\{x_0 \circ p | p \in M\}$ of the flow $|\mathcal{B}|_\sigma$. The algebra of this new pointed minimal flow will be denoted by $\text{per}(\mathcal{B}, \sigma)$ and be referred to as the perturbed flow of $\mathcal{A}$ by means of the cocycle $\sigma$. Thus $(\text{per}(\mathcal{B}, \sigma), |\mathcal{B}|_{\sigma})$ is identified with $(\{x_0 \circ p | p \in M\}, x_0)$ via the map $p/B \rightarrow x_0 \circ p (p \in M)$. I shall use this identification without further comment.

5.3 Proposition. Let $\mathcal{A} < \mathcal{B}$ and $\sigma \in Z(\mathcal{A}, A/B)$. Then

1. $g(\text{per}(\mathcal{B}, \sigma)) = \{a | a \in A, f_\sigma(\alpha) = Ba\}$
2. $\text{per}(\mathcal{B}, \sigma)$ is a distal extension of $\mathcal{A}$
3. $\mathcal{B} \vee \text{ext}(\mathcal{A}, \sigma) = \mathcal{B} \vee \text{per}(\mathcal{B}, \sigma) = \text{ext}(\mathcal{A}, \sigma) \vee \text{per}(\mathcal{B}, \sigma) = \text{ext}(\text{per}(\mathcal{B}, \sigma), \sigma)$.

(Notes: (i) Since $\mathcal{A} \subset \text{per}(\mathcal{B}, \sigma)$, $\sigma$ determines a cocycle (also denoted by $\sigma$) on $\text{per}(\mathcal{B}, \sigma)$ to $A/B$. Thus $\text{ext}(\text{per}(\mathcal{B}, \sigma), \sigma)$ is defined.

(ii) Since $\mathcal{B} \vee \text{ext}(\mathcal{A}, \sigma)$ is an almost periodic extension of $\mathcal{A}$, 3 implies that $\text{per}(\mathcal{B}, \sigma)$ is an almost periodic extension of $\mathcal{A}$.

**Proof.** 1. Let $x_0$ be the base point of both $|\mathcal{B}|$ and $|\text{per}(\mathcal{B}, \sigma)|$. Let $a \in g$ (const of $(\mathcal{B}, \sigma)$) and $f_\sigma(\alpha) = B \gamma \omega \gamma$ with $\gamma \in A$. Then $x_\sigma = x_\alpha \circ \gamma = f_\sigma(\gamma) = f_\sigma(u\alpha)^{-1}f_\sigma(u)x_\alpha = (B \gamma)^{-1}(x_\alpha)$. Hence $x_\alpha = (B \gamma)(x_0) = x_\alpha$ from which it follows that $\sigma(\gamma)^{-1} \in B$. Consequently $\alpha \in B \gamma \subset A$ and $f_\sigma(\alpha) = B \gamma = Ba$.

On the other hand if $a \in A$ and $f_\sigma(\alpha) = Ba$, then $x_\alpha \circ a = f_\sigma(\alpha)^{-1}x_\alpha = (B \alpha)^{-1}(x_\alpha) = x_\alpha$ whence $\alpha \in g$ (const of $(\mathcal{B}, \sigma)$).
2. Let \( p, q \in M \) with \( p = q \) on \( \text{per}(\mathcal{A}, \sigma) \). Then \( x_0 \circ p = x_0 \circ q \) and \( p|\sigma = (x_0 p)|\mathcal{A} = (x_0 q)|\mathcal{A} = q|\mathcal{A} \) by 1 of 5.2. Thus \( \mathcal{A} \subset \text{per}(\mathcal{B}, \sigma) \).

Now let \( p = pu \) on \( \mathcal{A} \). Since \( \mathcal{B} \) and \( \text{ext}(\mathcal{A}, \sigma) \) are almost periodic extensions of \( \mathcal{A} \), \( p = pu \) on \( \mathcal{B} \) and \( f_\sigma(p) = f_\sigma(pu) \). Hence \( x_0 \circ p = f_\sigma(p)^{-1} x_0 p = f_\sigma(pu)^{-1} x_0 pu = x_0 pu \). Thus \( \text{per}(\mathcal{B}, \sigma) \) is a distal extension of \( \mathcal{A} \).

3. The various flows involved are all distal extensions of \( \mathcal{A} \) so that it suffices to prove that their groups are equal.

Let \( S = \text{g}((\text{ext}(\mathcal{A}, \sigma))) \) and \( P = \text{g}(\text{per}(\mathcal{B}, \sigma)) \). Then \( \text{g}(\mathcal{B} \vee \text{ext}(\mathcal{A}, \sigma)) = B \cap S \), \( \text{g}(\mathcal{B} \vee \text{per}(\mathcal{B}, \sigma)) = B \cap P \), and \( \text{g}(\text{per}(\mathcal{B}, \sigma)) = \text{ker}(f/B) = B \cap P \). I shall show that \( B \cap S \subset B \cap P \subset P \cap S \subset B \cap S \).

Let \( \alpha \in B \cap S \). Then \( \alpha \in B \) and \( f_\sigma(\alpha) = B \). Therefore \( \alpha \in B \cap P \).

Finally let \( \alpha \in P \cap S \). Then \( f_\sigma(\alpha) = B \alpha \) and \( f_\sigma(\alpha) = B \), whence \( \alpha \in B \cap S \). The proof is completed.

5.4 Corollary. Let \( \sigma, \delta \in Z(\mathcal{A}, A/B) \). Then
1. \( \delta - \sigma \) (mod \( \mathcal{A} \)) implies that \( \text{per}(\mathcal{B}, \sigma) = \text{per}(\mathcal{B}, \delta) \).
2. \( \sigma - 0 \) (mod \( \mathcal{A} \)) implies that \( \mathcal{A} = \text{ext}(\mathcal{A}, \sigma) \).

Proof. 1. follows from 1 and 2 of 5.3 and the fact that \( \sigma - \delta \) (mod \( \mathcal{A} \)) implies that \( f_\sigma = f_\delta \) on \( A \).

2. follows from 3 of 5.3 and the fact that \( \sigma - 0 \) (mod \( \mathcal{A} \)) implies \( \mathcal{A} = \text{ext}(\mathcal{A}, \sigma) \).

In general the converse of 2 of 5.4 is false. However something can be said in the case when \( A/B \) is isomorphic to the circle group.

5.5 Proposition. Let \( \mathcal{A} < \mathcal{B}, \sigma \in Z(\mathcal{B}, A/B) \), and \( A/B \) isomorphic to the circle group. Then \( \text{per}(\mathcal{B}, \sigma) = \mathcal{B} \) if and only if \( \sigma - 0 \) (mod \( \mathcal{A} \)) or \( \mathcal{B} \) is a double cover of \( \text{ext}(\mathcal{A}, \sigma) \); i.e. \( \text{ext}(\mathcal{A}, \sigma) \subset \mathcal{B} \) and \( \text{g}(\text{ext}(\mathcal{A}, \sigma))/B = Z_2 \).

Proof. By 2 of 5.4 \( \mathcal{B} = \text{per}(\mathcal{B}, \sigma) \) if \( \sigma - 0 \) (mod \( \mathcal{A} \)).

Now suppose \( \mathcal{B} \) is a double cover of \( \text{ext}(\mathcal{A}, \sigma) \). Then \( \text{all}(f_\sigma) \subset \mathcal{B} \). Hence \( B \subset \ker f_\sigma \) and so \( f_\sigma \) induces a homomorphism \( f_\sigma \) of \( A/B \) into \( A/B \). Since \( \text{g}(\text{ext}(\mathcal{A}, \sigma)) = \ker f_\sigma(A/B) \) and \( B \) is a subgroup of index 2 of \( \text{g}(\text{ext}(\mathcal{A}, \sigma)) \), \( \ker(f_\sigma|A) \) must be a subgroup of index 2 of \( A/B \). Hence \( \tilde{f}(Ba) = Ba^2 \) (\( \alpha \in A \)). (Recall that the continuous endomorphisms of the circle group are of the form \( k^k \) with \( k \in Z \)).

Let \( \alpha \in \text{g}(\text{per}(\mathcal{B}, \sigma)) \). Then \( f_\sigma(\alpha) = Ba \). But \( f_\sigma(\alpha) = f_\sigma(Ba) = Ba^2 \). Since the only solution of the equation \( k^k = k \) in the circle group is \( k = 1 \), the above relations imply that \( \alpha \in B \). Thus \( \text{g}(\text{per}(\mathcal{B}, \sigma)) \subset B \).

On the other hand \( \text{all}(f_\sigma) \subset \mathcal{B} \) implies that \( B \subset \text{g}(\text{per}(\mathcal{B}, \sigma)) \) by 1 of 5.3.

Thus in this case also \( \mathcal{B} = \text{per}(\mathcal{B}, \sigma) \).

Let \( \mathcal{B} = \text{per}(\mathcal{B}, \sigma) \). Then by 3 of 5.3 \( f_\sigma \) induces an endomorphism \( f_\sigma \) of \( A/B \). This must be of the form \( f_\sigma(Ba) = Ba^n \) (\( \alpha \in A \)) for some \( n \in Z \). If \( 0 \neq n \neq 2 \), then there exists \( \alpha \in B \) with \( Ba^n = Ba^2 \); i.e. \( f_\sigma(\alpha) = Ba^2 \) with \( \alpha \in B \). This can't happen because \( \alpha \in \text{g}(\text{per}(\mathcal{B}, \sigma)) = B \).

The proof is completed.

5.6 Remarks. 1. Before discussing this situation further it is necessary to recall the dynamical interpretation of the fact that two \( T \)-subalgebras \( \mathcal{F} \) and \( \mathcal{H} \) of \( \mathfrak{U}(u) \) are equal. This means that there is an isomorphism \( \phi \) of the flow \( \mathcal{F} \) onto the flow \( \mathcal{H} \) which carries the base point of \( \mathcal{F} \) onto the base point of \( \mathcal{H} \). Thus if \( \mathcal{F} = \mathcal{H} \), then the corresponding flows are isomorphic or conjugate. The converse is in general not true. Thus the two flows \( \mathcal{F} \) and \( \mathcal{H} \) may be conjugate without \( \mathcal{H} \) and \( \mathcal{F} \) being equal. The reason for this is that algebras correspond to pointed flows not flows. Thus there might be an isomorphism of \( \mathcal{F} \) onto \( \mathcal{H} \) which does not preserve the basepoint.

It is easily seen from the general theory in [3] that \( \mathcal{F} \) and \( \mathcal{H} \) are conjugate if and
only if there exists $\alpha \in G$ with $\mathcal{F} \alpha = \mathcal{H}$. Thus in case one of the flows involved is regular the two notions (conjugacy, and equality of algebras) coincide.

2. Let $\mathcal{A}$ be the algebra corresponding to the flow whose phase space is $\mathbb{R}^2/\mathbb{Z}^2$ and whose phase group is the integers with generator $\varphi: \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ such that $\varphi(z^2 + x) = z^2 + x + \alpha$ where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ with 1, $\alpha_1$ and $\alpha_2$ rationally independent. (The base point is $Z^2$).

Let $\mathcal{B}$ correspond to the flow on $\mathbb{R}^2/\mathbb{Z}^2$ defined by $\psi(z^2 + x) = z^2 + x + \beta$ where $\beta = (\alpha_1, \alpha_2, \alpha_3)$ with 1, $\alpha_1$, $\alpha_2$, $\alpha_3$ rationally independent. Then $\mathcal{A} < \mathcal{B}$ and $A/B$ is isomorphic to the circle group $\mathbb{R}/\mathbb{Z}$.

The simplest way to perturb $\mathcal{B}$ is by means of a constant cocycle, $\sigma$. In this case this amounts to picking a real number $\sigma$. The perturbed flow $\psi_{\sigma}$ is then given by $\psi_{\sigma}(z^2 + x) = z^2 + x + (\alpha_1, \alpha_2, \alpha_3 - \sigma)$. Then $\sigma \sim 0$ if and only if 1, $\alpha_1$, $\alpha_2$, $\alpha_3$ are rationally dependent.

In this case $\mathcal{B}$ is regular so that $\mathcal{B} = \text{per}(\mathcal{B}, \sigma)$ if and only if $|\mathcal{B}|$ and $|\text{per}(\mathcal{B}, \sigma)|$ are conjugate. Thus 5.5 gives that $|\mathcal{B}|$ and $|\text{per}(\mathcal{B}, \sigma)|$ are conjugate if and only if 1, $\alpha_1$, $\alpha_2$, $\alpha_3$ are rationally dependent.

3. Let $\mathcal{A}$ be as in 2. I shall now construct an almost periodic extension $\mathcal{L}$ of $\mathcal{A}$ such that $\mathcal{A} < \mathcal{L}$, $A/L \equiv \mathbb{R}/\mathbb{Z}$ but per ($\mathcal{L}, \sigma$) will behave quite differently than in 2.

The flow $\mathcal{L}$ is just the classical nil flow. (For a fuller discussion of some of the remarks in 3 see [7]).

For $x$ and $y$ in $\mathbb{R}^2$ set $xy = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1y_2)$ where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. (The definition of $xy$ has been changed slightly in order to conform with 2.)

Let $D = \{x|x \in \mathbb{R}^3, x_1, x_2, x_3 \in \mathbb{Z}\}$. Then $D$ is a closed cocompact subgroup of $\mathbb{R}^3$ and the phase space $N$ of the flow corresponding to $\mathcal{L}$ is just $\mathbb{R}^3/D$. The action is given by the map $p(Dx) = Dx/\beta$ where $\beta = (\alpha_1, \alpha_2, \alpha_3)$ with 1, $\alpha_1$, $\alpha_2$, $\alpha_3$ as in 2.

The subgroup $L = \{x|x_1 = 0 = x_2\}$ of $\mathbb{R}^3$ is its center, $LD/D < \mathbb{R}/\mathbb{Z}$, the section of $LD/D$ on $N$ commutes with that of $p$ and the quotient is just $|\mathcal{A}|$. Thus $\mathcal{A} < \mathcal{L}$ and $A/L < \mathbb{R}/\mathbb{Z}$.

Now suppose we use $\sigma \in \mathbb{R}$ to perturb $\mathcal{L}$ just as we did in 2. Then the perturbed flow is given by $p_\sigma(Dx) = Dx/\beta - \sigma$ and as before $\sigma \sim 0 \text{ (mod } \mathcal{A})$ if and only if 1, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\sigma$ are rationally dependent. (The possibility that $\mathcal{L}$ be a double cover of ext ($\mathcal{A}, \sigma$) can be ruled out on topological grounds.) However in this case the flows $[\mathcal{L}]$ and $|\text{per}(\mathcal{L}, \sigma)|$ are always conjugate. To see this pick $\gamma_1, \gamma_2, \gamma_3$ such that $\alpha_1\gamma_1 - \sigma = -\gamma_2\alpha_2$ set $\varphi(Dx) = Dx_\sigma$ and verify that $\varphi p = \varphi$.

This shows that $\mathcal{L}$ is not regular. In this example $A$ is a normal subgroup of $\mathcal{G}$; $L$ is a normal subgroup of $\mathcal{G}$; $L$ is a normal subgroup of $A$ but $L$ is not a normal subgroup of $G$.

5.7 Proposition. Let $\mathcal{A} < \mathcal{B}$ and $\sigma \in Z(\mathcal{A}, A/B)$. Then there is a section of $\mathcal{A} \lor \text{per}(\mathcal{B}, \sigma)$ over $\text{per}(\mathcal{A}, \sigma)$.

Proof. Set $s_\sigma(p|\text{per}(\mathcal{B}, \sigma)) = f_\sigma(p^{-1}(p|\mathcal{B} \lor \text{per}(\mathcal{B}, \sigma))) (p \in M)$. It suffices to show that $s$ is well defined. Let $p, q \in M$ with $p = q$ on $\text{per}(\mathcal{B}, \sigma)$. There exists $\alpha \in g(\text{per}(\mathcal{B}, \sigma))$ such that $p = q\alpha$ on $\mathcal{B} \lor \text{per}(\mathcal{B}, \sigma) = \mathcal{B} \lor \text{ext}(\mathcal{A}, \sigma)$. Then $f_\sigma(p^{-1}(p|\mathcal{B} \lor \text{per}(\mathcal{B}, \sigma)) = f_\sigma(q^{-1}(aq)|\mathcal{B} \lor \text{per}(\mathcal{B}, \sigma)) = f_\sigma(q^{-1}B\alpha^{-1}(aq|\mathcal{B} \lor \text{per}(\mathcal{B}, \sigma)) = f_\sigma(q^{-1}(aq|\mathcal{B} \lor \text{per}(\mathcal{B}, \sigma)))$. The proof is completed.

5.8 Corollary. Let $\mathcal{A} < \mathcal{B}$. Then there is a section of $\mathcal{B}$ over $\mathcal{A}$ if and only if there exists $\sigma \in Z(\mathcal{A}, A/B)$ with $\mathcal{A} = \text{per}(\mathcal{B}, \sigma)$.

Proof. Assume that there is a section of $\mathcal{B}$ over $\mathcal{A}$ and let $\sigma$ be the corresponding cocycle on $\mathcal{A}$ to $A/B$. Then the equation $f_\sigma(\alpha) = B\alpha$ ($\alpha \in A$) shows that $A = g(\text{per}(\mathcal{B}, \sigma))$, whence $\mathcal{A} = \text{per}(\mathcal{B}, \sigma)$.

The sufficiency part of 5.8 follows directly from 5.7.

The following results are closely related to those centering around Proposition 3.19.
5.9 Proposition. Let $\mathcal{A} < \mathcal{B}$ with $A/B$ abelian, let $L$ be a $\tau$-closed subgroup of $G$ such that $A \subset L$ and such that the image of $H(\mathcal{A}, A/B)$ in $H(\mathcal{A}/A^*, A/B)$ contains $\{\lambda | A \cap L \subset \ker \lambda\}$, and let $\sigma \in Z(\mathcal{A}, A/B)$ with $\per (\mathcal{A}, \sigma) \subset \mathcal{A} \vee A(L)$. Then there exists a section of $\mathcal{B}$ over $\mathcal{A}$ and $\gamma \in Z(\mathcal{A}, A/B)$ with $\gamma \sigma - \eta \mod \mathcal{A}$ where $\eta$ is the cocycle corresponding to the section.

Proof. Let $\rho(\alpha) = f_\sigma(\alpha)^{-1}B\alpha$ ($\alpha \in A$). Then because $A/B$ is abelian, $\rho$ is a homomorphism of $A$ into $A/B$ with $\ker \rho = \{\per (\mathcal{A}, \sigma) \cap A \cap L\}$. Hence there exists $\gamma \in Z(\mathcal{A}, A/B)$ such that $f_\sigma(\alpha) = \rho(\alpha) (\alpha \in A)$. Consequently if $\eta = \gamma \sigma$, $f_\sigma(\alpha) = f_\sigma(\alpha) f_\sigma(\alpha) = B\alpha$ ($\alpha \in A$). The proof is completed.

5.10 Proposition. Let $\mathcal{A} < \mathcal{B}$, $\sigma \in Z(\mathcal{A}, A/B)$ with $\per (\mathcal{A}, \sigma) \subset \mathcal{A} \vee \mathcal{A}$, and let $A/B$ be abelian and satisfy (1) every continuous homomorphism of a closed subgroup of $G/E$ into $A/B$ may be extended to a continuous homomorphism of $G/E$ into $A/B$. Then there exists a section of $\mathcal{B}$ over $\mathcal{A}$, and $\gamma \sigma - \eta \mod \mathcal{A}$ where $\gamma \in \Hom(T, A/B)$ and $\eta$ is the cocycle on $\mathcal{A}$ to $A/B$ corresponding to the section.

Proof. Let $\lambda: A/A \cap E \rightarrow A/B$ be a continuous homomorphism. Then there exists a continuous homomorphism $\rho: G/E \rightarrow A/B$ such that $\rho$ restricted to $A/E$ is the composite of $\lambda$ and the canonical isomorphism of $A/E$ onto $A/A \cap E$.

Set $\delta(t) = \rho(\lambda t)$ ($t \in T$). Then $\delta \in \Hom(T, A/B)$ and $f_\delta(\alpha) = \rho(E\alpha) = \lambda(A \cap E\alpha)$ ($\alpha \in A$). Proposition 5.10 now follows from 5.9.

5.11 Corollary. Let $\mathcal{A} < \mathcal{B}$, let $A/B$ be abelian and satisfy (1) of 5.10, and let $D = \{\sigma \in Z(\mathcal{A}, A/B), \text{ and } \per (\mathcal{A}, \sigma) \subset \mathcal{A} \vee \mathcal{A}\}$. Then $\{\sigma \mod D\}$ constitutes a single coset of $\Hom(T, A/B)$ in $H(\mathcal{A}; A/B)$. (Recall that since $A/B$ is abelian, $H(\mathcal{A}; A/B)$ is a group).

Proof. Let $\sigma \in D$, $\delta \in \Hom(T, A/B)$, $\rho = \sigma\delta$, and $\alpha \in A \cap E$. Then $f_\sigma(\alpha) = \epsilon$ and $f_\sigma(\alpha) = B\alpha$ implies that $f_\sigma(\alpha) = B\alpha$. Thus $A \cap E \subset (\per (\mathcal{A}, \rho))$ whence $\per (\mathcal{A}, \rho) \subset \mathcal{A} \vee \mathcal{A}$ and so $\rho \in D$. Consequently $\{\sigma \mod D\}$ is a union of cosets of $\Hom(T, A/B)$.

Now let $\sigma_1, \sigma_2 \in D$. By 5.10 there exist $\eta_1, \eta_2, \gamma_1, \gamma_2$ such that $[\sigma_1] = [\eta_1][\gamma_1]$, $[\sigma_2] = [\eta_2][\gamma_2]$, $\gamma_1, \gamma_2 \in \Hom(T, A/B)$ and $f_{\sigma_1}(\alpha) = B\alpha = f_{\sigma_2}(\alpha)$ ($\alpha \in A$). Thus $[\eta_1] = [\eta_2]$ whence $[\sigma_1] = [\sigma_2][\gamma_2^{-1}][\gamma_1] = [\sigma_2]'[\gamma_2^{-1}][\gamma_1] \in [\sigma_2] \Hom(T, A/B)$.

5.12 Remarks. 1. If $G/E$ is abelian (which is the case when $T$ is abelian) and $A/B$ is abelian and connected, then it can be shown that (1) holds.

2. If $A/E = G$, then $A/A \cap E$ is isomorphic to $G/E$, and the proof of 5.10 shows that condition (1) is not needed in this case. Thus if $T$ is abelian, $\mathcal{A}$ weak mixing, $\mathcal{A} < \mathcal{B}$ with $A/B$ abelian, then there exists a section of $\mathcal{B}$ over $\mathcal{A}$ if there exists $\sigma \in Z(\mathcal{A}, A/B)$ with $\per (\mathcal{A}, \sigma) \subset \mathcal{A} \vee \mathcal{A}$.

3. The usual application of 5.10 and 5.11 is to the groups $\mathcal{Z}$ and $\mathcal{R}$. Since $\mathcal{Z}$ is discrete we may apply 5.10 and 5.11 to it verbatim. However in the case of $\mathcal{R}$ we would like to take into account its topology. Thus we would like to be able to replace the set of all homomorphisms of $\mathcal{R}$ into $A/B$ (i.e. $\Hom(\mathcal{R}, A/B)$) by the continuous ones $\Hom_c(\mathcal{R}, A/B)$. That this is legitimate can be seen by replacing $E$ by $S$ in 5.10, where $S$ is the group of the algebra $[f]_u = f, f: \mathcal{R} \rightarrow \mathcal{R}$ is uniformly continuous). Then $\delta \in \Hom_c(\mathcal{R}, A/B)$ because $\mathcal{A}/S$ is the Bohr compactification of $\mathcal{R}$.

4. Let $T = \mathcal{Z}$ or $\mathcal{R}$ and $A/B = K$, the circle group, and provide $Z(\mathcal{A}, K)$ with the topology of uniform convergence on compact subsets of $[\mathcal{A}] \times T$. If $[\mathcal{A}]$ is metrizable then it is easy to see that $Z(\mathcal{A}, K)$ is a complete metric topological group. In [6] the authors prove a theorem (3) which combined with 5.11 implies that when $\mathcal{A}$ is weak mixing and $A/B = K$, then $D$ is of the first category in $Z(\mathcal{A}, K)$.

An examination of the proofs of 5.10, 5.11 and 3.20 shows that the following is valid.

5.13 Proposition. Let $\mathcal{A} < \mathcal{B}$, let $A/B$ be abelian and satisfy (1) of 5.10 and let
F = \{\sigma | \sigma \in Z(\mathcal{A}, A/B), \text{ and } \text{ext}(\mathcal{A}, \sigma) \subset \mathcal{A} \vee \mathfrak{C} \}. \text{ Then } \{[\sigma] | [\sigma] \in F \} \text{ constitutes a single coset of } \text{Hom}(T, A/B) \text{ in } H(\mathcal{A}, A/B). \]

It is not difficult to construct examples wherein the cosets involved in 5.11 and 5.13 are different. However if \( B \subset \mathfrak{C} \), then the fact that \( B \vee \text{ext}(\mathcal{A}, \sigma) = B \vee \text{per}(\mathcal{A}, \sigma) \) (Prop. 5.3) implies that \( F = D \).

5.14 Proposition. Let \( \sigma \in Z(\mathcal{A}, K) \) with \( K \) abelian, \( B = \text{ext}(\mathcal{A}, \sigma) \), and \( \delta \in Z(\mathcal{A}, A/B) \). Then \( \text{per}(B, \delta) = \text{ext}(\mathcal{A}, \gamma) \) where \( \gamma \) is the cocycle on \( \mathcal{A} \) to \( K \) given by \( \gamma = (f_\sigma \circ \delta) \sigma^{-1} \).

**Proof.** Let \( a \in A \). The relation \( f_\sigma(a)f_\sigma(a) = f_\sigma(f_\sigma(a)) \) shows that \( f_\sigma(a) = e \) if and only if \( f_\sigma(a) = Ba \). Thus the groups of the two flows involved are equal whence the flows themselves are equal.

There is a similar result for the composition of perturbations.

5.15 Proposition. Let \( \mathcal{A} \subset \mathcal{B} \) with \( A/B \) abelian, \( \sigma \in Z(\mathcal{A}, A/B) \), \( L = \text{per}(\mathcal{B}, \sigma) \), and \( \rho \in Z(\mathcal{A}, A/L) \). Then \( \text{per}(L, \rho) = \text{per}(\mathcal{B}, \sigma \circ \rho) \) \( \text{where } \sigma \circ \rho : A/L \to A/B \) is the homomorphism induced by \( a \to f_\sigma^{-1}(a)Ba : A \to A/B \). (Note that since \( A/B \) is abelian, this map is a homomorphism. Its kernel is \( L \), the group of the flow \( \text{per}(\mathcal{B}, \sigma) \).)

**Proof.** Let \( \delta = \sigma \circ \rho \). Then \( f_\delta(a) = f_\sigma(a) \circ \rho(f_\sigma(a)) = f_\sigma(f_\sigma(a))f_\rho^{-1}(c)Be \) where \( f_\sigma(a) = Ba \). Thus \( f_\delta(a) = Ba \) implies that \( Ba = f_\sigma(a)f_\rho^{-1}(c)Be \) whence \( f_\sigma(ac^{-1}) = Bac^{-1} \). Hence \( ac^{-1} \in L \) or \( La = Lc \), and therefore \( f_\delta(a) = La = Lc \).

On the other hand if \( f_\delta(a) = La \), \( f_\sigma(a)f_\rho^{-1}(a)Ba = Ba \). Thus \( g(\text{per}(L, \rho)) = g(\text{per}(\mathcal{B}, \sigma \circ \rho)) \). The proof is completed.

5.16 Definition. Let \( \mathcal{A} \) be a \( T \)-subalgebra of \( \mathcal{U}(u) \) and let \( \mathcal{A}^\ast \) be the maximum almost periodic extension of \( \mathcal{A} \). Then \( \mathcal{A}^\ast = A^\ast[A, A] \) and \( \mathcal{A} \) is the \( \tau \)-closed subgroup of \( G \) generated by the set \{\( aba^{-1}b^{-1}\) | \( a, b \in A \)\}.

5.17 Proposition. Let \( \mathcal{A} \subset \mathcal{B} \). Then \( A/B \) is abelian if and only if \( \mathcal{B} \subset \mathcal{A} \).

**Proof.** This follows immediately from the fact that the image of \( \mathcal{A} \) in \( A/A^\ast \) under the canonical map is the commutator subgroup of \( A/A^\ast \).

5.18 Proposition. Let \( \mathcal{A} \) be such that the map \( [\sigma] \to f_\sigma : H(\mathcal{A}, A/\hat{A}) \to \text{Hom}(A/\hat{A}/A^\ast, A/\hat{A}) \) is onto and let \( \mathcal{B} \subset \mathcal{A} \subset \mathcal{A} \). Then there exists a homeomorphism \( \varphi \) of \( |\mathcal{B}| \) onto a subset of \( |\mathcal{A}| \) with \( \varphi(x)|\mathcal{A} = \varphi(x)|\mathcal{A} \) \( (x \in |\mathcal{B}|) \) if and only if \( \mathcal{B} = \text{ext}(\mathcal{A}, \sigma) \) for some \( \sigma \in Z(\mathcal{A}, A/\hat{A}) \).

**Proof.** Let \( \sigma \in Z(\mathcal{A}, A/\hat{A}) \) and \( \mathcal{B} = \text{ext}(\mathcal{A}, \sigma) \). Then \( \sigma \to f_\sigma(\sigma)^{-1}A\alpha ; A \to A/\hat{A} \) is a homomorphism since \( A/\hat{A} \) is abelian. Hence there exists \( \delta \in Z(\mathcal{A}, A/\hat{A}) \) with \( f_\sigma(\sigma) = f_\sigma(\sigma)^{-1}A\alpha \). This implies that \( B = g(\text{per}(\mathcal{A}, \delta)) \) whence \( \mathcal{B} = \text{per}(\mathcal{A}, \sigma) \). The required map is then \( p \to x_0 \circ \rho : |\mathcal{B}| \to |\mathcal{A}| \).

On the other hand let \( \varphi \) be a homeomorphism of \( |\mathcal{B}| \) onto a subset \( Y \) of \( |\mathcal{A}| \) such that \( \varphi(x)|\mathcal{A} = x|\mathcal{A} \) \( (x \in |\mathcal{B}|) \). Then the map \( (y, t) \to \varphi(y) \) \( (y \in Y, t \in T) \) defines an action of \( T \) on \( Y \) such that \( \pi(y \circ t) = \pi(y)t \) \( (y \in Y, t \in T) \), where \( \pi : |\mathcal{A}| \to |\mathcal{A}| \) is the restriction map. Of course the two flows \( (|\mathcal{B}|, T) \) and \( (Y, T) \) are isomorphic.

Since \( \pi(y \circ t) = \pi(yt) \) \( (y \in Y, t \in T) \) there exists \( \delta \in Z(\mathcal{A}, A/\hat{A}) \) with \( y \circ t = \delta(y, t)^{-1}yt \). Thus \( (Y, T) \cong (|\text{per}(\mathcal{A}, \delta)|, T) \). Finally let \( \sigma \in Z(\mathcal{A}, A/\hat{A}) \) be such that \( \sigma \delta = e \), where \( f_\sigma(a) = \hat{A}a \) \( (a \in A) \). Then \( \text{per}(\mathcal{A}, \delta) = \text{ext}(\mathcal{A}, \sigma) = \mathcal{B} \). The proof is completed.

Finally I should like to discuss the notion of a supplement and its relation to extensions and perturbations.
5.19 Definition. Let \( \mathcal{A} \subset \mathcal{F} \subset \mathcal{B} \). Then \( \mathcal{F} \) has an \( \mathcal{A} \)-supplement in \( \mathcal{B} \) if there exists a \( T \)-subalgebra \( \mathcal{S} \) with \( \mathcal{F} \vee \mathcal{S} = \mathcal{B} \) and \( \mathcal{F} \cap \mathcal{S} = \mathcal{A} \).

5.20 Lemma. Let \( \mathcal{B} \) be an almost periodic extension of \( \mathcal{A} \) and let \( \mathcal{A} \subset \mathcal{F} \subset \mathcal{B} \). Then \( \mathcal{F} \) has an \( \mathcal{A} \)-supplement in \( \mathcal{B} \) if and only if there exists a \( T \)-closed subgroup \( \mathcal{S} \) of \( \mathcal{A} \) with \( \mathcal{F} \cap \mathcal{S} = \mathcal{A} \) and \( \mathcal{F} \vee \mathcal{S} = \mathcal{B} \).

Proof. Let \( \mathcal{F} \) be a \( T \)-subalgebra of \( \mathcal{B} \) with \( \mathcal{F} \cap \mathcal{S} = \mathcal{A} \) and \( \mathcal{F} \vee \mathcal{S} = \mathcal{B} \). Then \( \mathcal{F} \cap \mathcal{S} = \mathcal{B} \) by \([3, 11.4]\) and \( FS = A \) by \([3, 4] \) of \( 18.10 \) and \( 18.41 \).

On the other hand let \( \mathcal{S} \) be a \( T \)-closed subgroup of \( \mathcal{A} \) with \( \mathcal{F} \cap \mathcal{S} = \mathcal{A} \) and \( \mathcal{F} \vee \mathcal{S} = \mathcal{B} \).

5.21 Corollary. Let \( \mathcal{B} \) be an almost periodic extension of \( \mathcal{A} \), \( \mathcal{A} \subset \mathcal{F} \subset \mathcal{B} \), and let \( \mathcal{B} \) be a proximal extension of \( \mathcal{A} \) (i.e. \( \mathcal{A} \subset \mathcal{B} \) and \( R = A \)). Then \( \mathcal{F} \) has an \( \mathcal{A} \)-supplement in \( \mathcal{B} \) if and only if \( \mathcal{F} \vee \mathcal{S} \) has an \( \mathcal{A} \)-supplement in \( \mathcal{F} \vee \mathcal{S} \).

Proof. This follows from 5.23 and the fact that \( g(\mathcal{A}) = g(\mathcal{B}) \), \( g(\mathcal{F}) = g(\mathcal{F} \vee \mathcal{S}) \) and \( g(\mathcal{B}) = g(\mathcal{F} \vee \mathcal{S}) \).

5.22 Proposition. Let \( \varphi \) be an almost periodic extension of \( \mathcal{A} \) with \( B < A \), \( \mathcal{A} \subset \mathcal{F} \subset \mathcal{B} \), and let the map \( [\sigma] \rightarrow \tilde{f}_\sigma: H(\mathcal{A}, A/B) \rightarrow Hom(A/A^*, A/B) \) be onto. Then \( \mathcal{F} \) has an \( \mathcal{A} \)-supplement in \( \mathcal{B} \) if and only if there exists \( \sigma \in Z(\mathcal{A}, A/B) \) with \( \mathcal{F} = \text{ext}(\mathcal{A}, \sigma) \) and \( \text{per}(\varphi, \sigma) \) is an \( \mathcal{A} \)-supplement of \( \mathcal{F} \) in \( \mathcal{B} \).

Proof. Let \( \mathcal{F} \) be an \( \mathcal{A} \)-supplement of \( \mathcal{F} \) in \( \mathcal{B} \). Then by 5.24, \( A/B \) is the semidirect product of \( F/B \) and \( S/B \). Let \( \varphi: A \rightarrow A/B \) be the map such that \( \varphi(a) = B_\sigma(a) (a \in A) \) where \( a = b_\sigma \) for some \( b \in F \) and \( \sigma \in S \). Then \( \varphi \) is a well defined continuous homomorphism with \( \varphi \circ \varphi = \varphi \) and \( \ker \varphi = F \).

Let \( \psi \in Z(\mathcal{F}, A/L) \) with \( f_\sigma(a) = \psi(a) (a \in A) \). Then \( \ker(f_\sigma|A) = F \) implies that \( \mathcal{F} = \text{ext}(\mathcal{A}, \sigma) \). Moreover \( f_\sigma(a) = Ba \) if and only if \( a \in S \) shows that \( S = (\text{per}(\varphi, \sigma)) \) whence \( \mathcal{F} = \text{per}(\mathcal{B}, \sigma) \).

Finally let \( \delta = \tilde{f}_\sigma \circ \varphi \). Then \( f_\sigma = \tilde{f}_\sigma \circ \varphi \) and \( \varphi = \varphi \circ \varphi \) imply that \( f_\sigma = f_\sigma \) whence \( \delta = \sigma \mod(\mathcal{A}) \).

Now suppose that \( \mathcal{F} = \text{ext}(\mathcal{A}, \sigma) \) where \( \sigma \in Z(\mathcal{A}, A/B) \) with \( [\sigma] = [\tilde{f}_\sigma \circ \varphi] \). Set \( \mathcal{F} = \text{per}(\mathcal{B}, \sigma) \). Then \( \mathcal{F} \vee \mathcal{F} \) by 5.3 and it remains to be shown that \( FS = A \). This follows from the fact that \( F = \ker(f_\sigma|A) \) and \( \tilde{f}_\sigma \circ f_\sigma = f_\sigma \). (Write \( a = (ar^{-1})r \) if and only if \( f_\sigma(a) = Br \).

Combining 5.22, 5.21, 2 of 4.17, and 4.18 gives the following result.

5.23 Proposition. Let \( \mathcal{A} < \mathcal{B} \) and \( \mathcal{A} < \mathcal{F} \subset \mathcal{B} \). Then \( \mathcal{F} \) has an \( \mathcal{A} \)-supplement in \( \mathcal{B} \) if and only if there exists \( \sigma \in Z(\mathcal{F} \vee \mathcal{A}, A/B) \) with \( \mathcal{F} \vee \mathcal{F} = \text{ext}(\mathcal{F} \vee \mathcal{A}, \sigma) \) and \( [\sigma] = [\tilde{f}_\sigma \circ \varphi] \).

5.24 Proposition. Let \( \mathcal{A} < \mathcal{B} \), \( \sigma \in Z(\mathcal{A}, A/B) \), \( F = \text{per}(\mathcal{B}, \sigma) \), \( L = \{a | a \in A \} \) and \( H = \ker(f_\sigma|A) \). Then (i) \( HF = A \) if and only if \( L = BH \) and (ii) if \( HF = A \) then \( \text{ext}(\mathcal{A}, \sigma) \) has \( \mathcal{A} \)-supplement in \( \text{ext}(\mathcal{A}, \sigma) \) as \( \mathcal{A} \)-supplement in \( \mathcal{F} \vee \mathcal{A} \).

Proof. (i) Assume \( HF = A \). It is clear from the definition of \( L \) that \( BH \subset L \). Let
\[ \alpha \in L. \text{ Then } Ba = \phi_e(\beta)^{-1}B\beta \text{ for some } \beta \in A. \text{ There exist } h \in H \text{ and } \gamma \in F \text{ with } \beta = h\gamma. \text{ Hence } Ba = \phi_e(\gamma)^{-1}Bh\gamma = B\gamma^{-1}BHy = B\gamma^{-1}h\gamma. \text{ Since } H \text{ is a normal subgroup of } A, \gamma^{-1}h\gamma \in H, \text{ whence } \alpha \in BH. \]

Now suppose \( BH = L \) and let \( a \in A. \) Set \( Ba = \phi_e(a)^{-1}Ba. \) Then \( \alpha \in L \) and so \( \alpha = bh \) with \( b \in B \) and \( h \in H. \) Hence \( Bh = \phi_e(a)^{-1}Ba, \text{ or } \phi_e(a) = Bh^{-1} - Bh^{-1}a^{-1}a. \) Set \( l = ah^{-1}a^{-1}. \) Then \( l \in H \) and \( \phi_e(la) = \phi_e(a) = Bla, \) whence \( la \in F. \) Thus \( a = l^{-1}la \in HF. \)

Statement (ii) follows immediately from 5.20.

REFERENCES


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