A CHARACTERIZATION OF INVERSE LIMITS OF Q-BUNDLE MAPS

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In this paper we prove that a surjection between metric compacta is a hereditary shape equivalence if and only if it is an inverse limit of trivial Q-bundle maps. This result was conjectured by T.B. Rushing. Near-homeomorphisms are instrumental to the proof.


Q-bundle map

equivalence

inverse limit

near-homeomorphism

hereditary shape

Hilbert cube

1. Introduction and statement of the theorem

In [5], T.B. Rushing shows that cell-like maps between finite dimensional metric compacta are inverse limits of disk-bundle maps, and conjectures an analogous result for infinite dimensional metric compacta. In this paper we prove this conjecture to be correct. More precisely, we prove

Theorem 1. Let \( f : E \to B \) be a surjection between metric compacta. Then \( f \) is a hereditary shape equivalence if and only if \( f \) is the inverse limit of trivial Q-bundle maps.

The condition in Theorem 1 that \( f \) be a hereditary shape equivalence is necessary since the Taylor map [7] is an example of a cell-like map between (infinite dimensional) metric compacta which fails to be a shape fibration (see [4] for the definition of shape fibration). Hence it could not possibly be an inverse limit of Q-bundle maps since these always induce a shape fibration.

The technique of proof in [5] does not directly carry over to the infinite dimensional setting, so we develop a different technique, which is the content of

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the three propositions in Section 4. In order to establish these propositions, we use results of T.A. Chapman [1], R.D. Edwards [2], some properties of ANR's discovered by G. Kozlowski [3], and some techniques of T.B. Rushing [5], namely Proposition 1, whose proof is a slight variation of the argument given in the proof of Theorem 4 of [5]. The interested reader should read [5] and [6] for a broader treatment of the subject in the finite dimensional setting.

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2. Notation and definitions

The diameter of a subset $S$ of the metric space $X$ is denoted by $\text{diam}(S)$. The complement of $S$ in $X$ is denoted by $S^c$ or $X - S$. The largest open set in $X$ contained in $S$ is denoted by $\text{Int}(S)$. The graph of a map $g: S \rightarrow Y$ in $X \times Y$ is represented by $G(g)$. If $S$ is closed, $G(g)$ is homeomorphic to $S$ via the homeomorphism $\theta(x) = (x, g(x))$ for all $x \in S$. Note that $\pi_y \circ \theta_g = g$, where $\pi_y : X \times Y \rightarrow Y$ is the natural projection. Thus we shall identify $S$ with $G(g)$ and say that $\pi_y$ is an extension of $g$. Furthermore, if $U$ is any neighborhood of $G(g)$ in $X \times Y$, then $\pi_y|U : U \rightarrow Y$ will also be called an extension of $g$ to $U$. For $g$ as above we define a decomposition $\Gamma_g$ of $X \times Y$ by

$$\Gamma_g = \{g^{-1}(y) \times \{y\} \subseteq X \times Y : y \in g(S)\} \cup \{(x, y) : (x, y) \notin G(g)\},$$

and $P_g : X \times Y \rightarrow (X \times Y)/\Gamma_g$ will denote the quotient map. Note that $G(g) = P_g^{-1}(P_g(G(g)))$, $P_g|G(g)^c$ is one to one, and $P_g(G(g))$ is homeomorphic to $g(S)$ via the homeomorphism $\phi_g(y) = P_g(g^{-1}(y) \times \{y\})$ for all $y \in g(S)$. The symbol $1_x : X \rightarrow X$ represents the identity map.

A map $g : X \rightarrow Y$ is called a near-homeomorphism if for every $\epsilon > 0$ there is a homeomorphism $h_\epsilon : X \rightarrow Y$ such that $d(g(x), h_\epsilon(x)) < \epsilon$ for all $x \in X$. A proper surjection $g : X \rightarrow Y$ is called a hereditary shape equivalence (H.S.E.) provided $g|g^{-1}(K) : g^{-1}(K) \rightarrow K$ is a shape equivalence for every closed set $K$ in $Y$. A surjective map $p : E \rightarrow B$ is called a bundle map with fiber $F$ if there exists a space $F$ such that for each $b_0$ in $B$ there is a neighborhood $U$ of $b_0$ and a homeomorphism $h : U \times F \rightarrow p^{-1}(U)$ such that $p(h(b, f)) = b$ for every $b$ in $U$ and $f$ in $F$. If $U$ can be chosen to be all of $B$, we say that $p$ is a trivial bundle map with fiber $F$.

**Definition 1.** A map $f : E \rightarrow B$ is an inverse limit of Q-bundle maps if there is a map of inverse sequences

$$f = \{f_i\}_{i=1}^\infty : E = \{E_i, a_{ij}\} \rightarrow B = \{B_i, r_{ij}\}$$

such that

(i) $E = \lim E$, $B = \lim B$. 

(ii) $f$ induces $f$. This means that $f_i q_i = r_i f$, where $q_i : E_i \to E$ and $r_i : B_i \to B_i$ are the natural projections, $i = 1, 2, 3, \ldots$

(iii) $f_i : E_i \to B_i$ is a bundle map with fiber $Q$ (abbreviated $Q$-bundle map) for all $i = 1, 2, 3, \ldots$

In case each $f_i$ is a trivial $Q$-bundle map, we say that $f$ is the inverse limit of trivial $Q$-bundle maps.

The set of natural numbers is denoted by $\mathbb{N}$. The Hilbert cube is denoted by $Q$ with the usual metric. We shall often write $Q = I^n \times Q^{n+1}$, where $I^n = \prod_{i=1}^{n} [-1, 1]$, and $Q^{n+1} = \prod_{i=n+1}^{\infty} [-1, 1]$. The natural projections $Q \to I^n$ and $Q \to Q^{n+1}$ are denoted by $\Pi_n$ and $\Pi^{n+1}$ respectively.

For any positive number $\varepsilon < 1$ and $n \in \mathbb{N}$, we define $\varepsilon I^n = \prod_{i=1}^{n} [-\varepsilon, \varepsilon]$ as a subset of $I^n$, and $Q_{ne} = \varepsilon I^n \times Q^{n+1}$ as a subset of $Q$. If $x \in (1-\varepsilon)I^n$, we define

$$x + \varepsilon I^n = \{x + y \in I^n : y \in \varepsilon I^n\}.$$  

For $x \in (1-\varepsilon)I^n \times \{0\} \subseteq Q$,

$$x + Q_{ne} = \{x + y \in Q : y \in Q_{ne}\}.$$ 

Note that diam($Q_{ne}$) $\leq 2 \varepsilon + 1/2^{n-1}$, hence for any $\delta > 0$ there exist $(n, \varepsilon)$ such that diam $Q_{ne} < \delta$.

3. Basic lemma

Now we define a retraction $\omega_{ne} : I^n \to (1-\varepsilon)I^n$ and a retraction $\tilde{\omega}_{ne} : Q \to (1-\varepsilon)I^n \times \{0\} \subseteq Q$, for $n \in \mathbb{N}$ and $0 < \varepsilon < 1$. Then in Lemma 1 we prove some properties of these retractions which are used in the proof of the Basic Lemma.

**Definition of $\omega_{ne}$ and $\tilde{\omega}_{ne}$.** For $x \in I^n$, set $\omega_{ne}(x) = \{\omega_i(x)\}_{i=1}^{n}$ with

$$\omega_i(x) = \begin{cases} x_i & \text{if } |x_i| \leq 1-\varepsilon, \\ \frac{x_i(1-\varepsilon)}{|x_i|} & \text{if } |x_i| \geq 1-\varepsilon. \end{cases}$$

Clearly $\omega_{ne}(I^n) = (1-\varepsilon)I^n$, and $\omega_{ne}$ is continuous.

Now define $\tilde{\omega}_{ne} = J_{n(1-\varepsilon)} \circ \omega_{ne} \circ \pi_n$, where $J_{n(1-\varepsilon)}$ denotes the embedding $(1-\varepsilon)I^n \hookrightarrow I^n \times \{0\} \subseteq I^n \times Q^{n+1}$.

**Lemma 1.** Choose $n \in \mathbb{N}$ and $0 < \varepsilon < 1$.

(a) For each $y \in Q$ set $M_y = \tilde{\omega}_{ne}(y) + Q_{ne}$. Then for every $x \in Q$ there exists an open neighborhood $C_x$ of $x$ in $Q$ such that $C_x \subseteq M_y$ for all $y \in C_x$.

(b) If $g : B \to Q$ is a map, and $M = \bigcup_{s \in B} \{s\} \times M_{g(s)}$, then $G(g) \subseteq \text{Int } M$ relative to $R \times Q$. 

Proof. (a) For every \( x \) in \( Q \), \( C_x \) will be of the form \( C_x = C_x^n \times Q^{n-1} \), where \( C_x^n \) is an open cube in \( I^n \) containing \( \Pi_n(x) \). Set \( C_x^n = \prod_{i=1}^n \gamma_i \), where \( \gamma_i = (x_i - \frac{1}{3} \epsilon, x_i + \frac{1}{3} \epsilon) \) if \( x_i \in [-1+\epsilon, 1-\epsilon] \), and \( \gamma_i = (x_i/|x_i|) (1-\epsilon, 1) \) otherwise. Here \(-1(1-\epsilon, 1)\) means \([-1, -1+\epsilon)\) and \(+1(1-\epsilon, 1)\) means \((1-\epsilon, 1]\).

It is not difficult to see that \( C_x^n \subseteq \omega_{ne}(y) + \epsilon I^n \) for all \( y \in C_x^n \), hence \( C_x \subseteq M_y \) for all \( y \in C_x \).

(b) Choose a point \((s, g(s)) \in G(g)\) and let \( V_s = g^{-1}(C_{g(s)}^s) \). Then \((s, g(s)) \in V_s \times C_{g(s)}^s\), which is open in \( B \times Q \), and \( V_s \times C_{g(s)}^s \subseteq M \).

Basic Lemma. Suppose \( g: B \to Q \) is a map, and \( U \) is an open neighborhood of \( G(g) \) in \( B \times Q \). Then there is a compact neighborhood \( M \) of \( G(g) \) in \( U \) with \( G(g) \subseteq \text{Int } M \) relative to \( B \times Q \), and a homeomorphism \( H: B \times Q \to M \) which makes the following diagram commute:

\[
\begin{array}{ccc}
B \times Q & \xrightarrow{H} & M \\
\pi_B \downarrow & & \downarrow \pi_B|_M \\
B & \xrightarrow{} & M
\end{array}
\]

(Note that diagram (1) says that \( \pi_B|_M : M \to B \) is a trivial bundle map with fiber \( Q \). Also \( B \) is assumed to be compact.)

Proof. Set \( \delta = d(G(g), U^c) \) if \( U^c \neq \emptyset \) and \( \delta = 1 \) if \( U = B \times Q \). Choose \( n \in \mathbb{N} \) and \( 0 < \epsilon < 1 \) such that \( \text{diam}(Q_{ne}) < \delta \). Define \( M \) as in Lemma 1, part (b). Then \( G(g) \subseteq \text{Int } M \) relative to \( B \times Q \). If \((x, y) \in M \), then \( y \in M_{g(x)} \). Also \( g(x) \in M_{g(x)} \), therefore \( d(y, g(x)) < \delta \). (Note that \( \text{diam } M_{g(x)} = \text{diam } Q_{ne} \).) Hence \( d((x, y), (x, g(x))) < \delta \). Now \((x, g(x)) \in G(g)\) implies that \((x, y) \in U\) by the choice of \( \delta \) and we conclude that \( M \subseteq U \).

Now define a homeomorphism \( \tilde{\varepsilon}_n : I^n \to \varepsilon I^n \) by \( \tilde{\varepsilon}_n(x) = \varepsilon x \), where \( \varepsilon x = \{\varepsilon x_i\}_{i=1}^n \) for all \( x \in I^n \). Let \( H_1 : B \times Q \to B \times Q_{ne} \) be given by \( H_1 = \pi_B \times ((\tilde{\varepsilon}_n \circ \pi_n) \times \pi_n^{n+1}) \circ \pi_Q \) and let \( H_2 : B \times Q_{ne} \to M \) be given by \( H_2(b, q) = (b, \tilde{\omega}_{ne}(g(b)) + q) \). Now set \( H = H_2 \circ H_1 \).

Since both \( H_1 \) and \( H_2 \) are homeomorphisms, \( H \) is a homeomorphism and so \( M \) is compact. The commutativity of diagram (1) is clear since both \( H_1 \) and \( H_2 \) are the identity on the \( B \)-factor. This completes the proof of the Basic Lemma.

4. Proof of Theorem 1

The compact metric spaces \( E \) and \( B \) are embedded in copies of the Hilbert cube \( Q_1 \) and \( Q_2 \) respectively, and shall be treated as closed subsets of \( Q_1 \) and \( Q_2 \) respectively.

Proposition 1. Suppose \( f : E \to B \) is a H.S.E. Set \( A = (Q_1 \times Q_2)/\Gamma_f \), then \( P_f \times 1_Q : (Q_1 \times Q_2) \times Q \to A \times Q \) is a near-homeomorphism.
Proof. The map $P_f$ is a proper surjection. The closed set $G(f)$ in $Q_1 \times Q_2$ contains the non-degeneracy set of $P_f$. The map $P_f|G(f): G(f) \to P_f(G(f))$ is a H.S.E. by hypothesis. Hence by Lemma 8 of [3], $P_f$ is a H.S.E. Now, since $Q_1 \times Q_2$ is an ANR, by Theorem 9 of [3] we conclude that $A$ is an ANR. Thus Theorem 43.1 of [1] establishes the proposition.

Remark 1. Set

$$\tilde{G} = \{(f(e), (e, f(e)), 0) \in B \times (Q_1 \times Q_2) \times Q \mid e \in E\}$$

and

$$\tilde{H} = \{(b, [f^{-1}(b) \times \{b\}], 0) \in B \times A \times Q \mid b \in B\}.$$

Then $(1_B \times P_f \times 1_Q)^{-1}(\tilde{H}) = \tilde{G}$.

Set $Q_3 = (Q_1 \times Q_2) \times Q$. Let $U$ be any open neighborhood of $\tilde{G}$ in $B \times Q_3$. Then $W = [1_B \times P_f \times 1_Q(U^c)]^c$ is an open neighborhood of $\tilde{H}$ in $B \times A \times Q$. So we have $\tilde{H} \subset W \subset 1_B \times P_f \times 1_Q(U)$.

Proposition 2. Let $\overline{\pi}_B: B \times A \times Q \to B$ denote the natural projection. Given any open neighborhood $W$ of $\tilde{H}$ in $B \times A \times Q_3$, there is a compact neighborhood $N$ of $\tilde{H}$ in $W$, such that $\overline{\pi}_B|N: N \to B$ is a trivial $Q$-bundle map. Furthermore, $\tilde{H} \subset \text{Int} N$ relative to $B \times A \times Q$.

Proof. Let $h: A \times Q \to Q_3$ be any one of the homeomorphisms given by Proposition 1. Define $g: B \to Q_3$ by $g = h \circ (\phi_f \times \{0\})$. (Recall that $\phi_f: B = f(E) \to P_f(G(f)) \subset A$.) Note the following:

(i) $(1_B \times h)(\tilde{H}) = G(g) \subset B \times Q_3$,

(ii) $U = (1_B \times h)(W)$ is an open neighborhood of $G(g)$ in $B \times Q_3$.

Hence the hypotheses of the Basic Lemma are satisfied and we obtain a compact neighborhood $M$ of $G(g)$ in $U$ with $G(g) \subset \text{Int} M$ relative to $B \times Q_3$ and $\pi_B|M: M \to B$ is a trivial $Q$-bundle map.

Set $N = (1_B \times h)^{-1}(M)$. By (i) we have $\tilde{H} \subset \text{Int} N$ relative to $B \times (A \times Q)$. Since $M \subset U$, (ii) implies that $N \subset W$, and since $1_B \times h$ is a fiber preserving homeomorphism, $\overline{\pi}_B|N: N \to B$ is a trivial $Q$-bundle map.

Proposition 3. Given any open neighborhood $\tilde{U}$ of $\tilde{G}$ in $B \times Q_3$, there is a compact neighborhood $M^*$ of $\tilde{G}$ in $\tilde{U}$, such that

(i) $\tilde{G} \subset \text{Int} M^*$ relative to $B \times Q_3$ and

(ii) $\overline{\pi}_B|M^*: M^* \to B$ is a trivial $Q$-bundle map.

Proof. For any $\varepsilon > 0$, let $h_\varepsilon: Q_3 \to A \times Q$ denote a homeomorphism such that $d(h_\varepsilon(x), (P_f \times 1_Q)(x)) < \varepsilon$ for all $x \in Q_3 = (Q_1 \times Q_2) \times Q$, which exists by Proposition 1.
Let $W$ be the open set containing $\tilde{f}$ defined in Remark 1. Let $N$ be the compact $Q$-bundle neighborhood of $\tilde{f}$ contained in $W$ given by Proposition 2. Since $\tilde{f} \subset \text{Int } N$, there is a positive number $\delta_1$ such that $d(\tilde{f}, (\text{Int } N)^c) > \delta_1$. Hence $(1_B \times h_e)(\tilde{G}) \subset \text{Int } N$ for all $\varepsilon \leq \delta_1$, and we obtain:

1. $\tilde{G} \subset (1_B \times h_e)^{-1}(\text{Int } N)$ for every $\varepsilon \leq \delta_1$.

Now since $N \subset W$, there is a positive number $\delta_2$ such that

2. $d(W^c, N) > \delta_2$.

Hence $N \subset (1_B \times h_e)(\tilde{U})$ for all $\varepsilon \leq \delta_2$, for if not, then $(1_B \times h_e)(x) \in N$ for some $x \in \tilde{U}^c$. Now since

$$d((1_B \times P_\ell \times 1_Q)(x), (1_B \times h_e)(x)) < \varepsilon$$

we have $d(W^c, N) < \varepsilon \leq \delta_2$ which contradicts (2), so we get

3. $(1_B \times h_e)^{-1}(N) \subset \tilde{U}$ for all $\varepsilon \leq \delta_2$.

Let $\varepsilon = \min\{\delta_1, \delta_2\}$, and let $M^* = (1_B \times h_e)^{-1}(N)$, then (1) implies (i) and by (3) $M^* \subset \tilde{U}$. Since $1_B \times h_e$ is a fiber-preserving homeomorphism and $\pi_B|N: N \to B$ is a trivial $Q$-bundle map, we have (ii).

**Proof of Theorem 1.** Inductively we wish to define a nested sequence of compact neighborhoods $\{E_n: n \in \mathbb{N}\}$ of $\tilde{G}$ in $B \times Q_3$ such that $\tilde{G} = \bigcap_{n=1}^\infty E_n$, and $\pi_B|E_n: E_n \to B$ is a trivial $Q$-bundle map, for all $n$. Then we set $E = \{E_n, q_i\}$, where $q_i: E_i \to E_i$ denotes the inclusion map $(i \leq j), B = \{B, 1_B\}$ and $f = \{\pi_B|E_i\}_{i=1}^\infty$, to obtain a map $f: E \to B$ of inverse sequences which clearly satisfies Definition 1 for the given H.S.E. $f: E \to B$. (Note that $\pi_B|E_i$ extends $f_i$.)

So, let $E_0 = B \times Q_3$ and use Proposition 3 with $\tilde{U} = (\text{Int } E_{n-1}) \cap N(\tilde{G}, 1/n)$, to obtain $E_n = M^*$ for $n \geq 1$. By (i) and (ii) of Proposition 3, $\{E_n: n \in \mathbb{N}\}$ has the desired properties. (Here $N(\tilde{G}, 1/n)$ denotes the $(1/n)$-neighborhood of $\tilde{G}$ in $B \times Q_3$).

The other implication of the theorem is clear.

**References**


