Topology and its Applications 50 (1993) 11-33
North-Holland

CE equivalence and shape equivalence of LC^n compacta

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Received 6 May 1991
Revised 9 January 1992

Abstract

Mrozik, P., CE equivalence and shape equivalence of LC^n compacta, Topology and its Applications 50 (1993) 11-33.

It is proved that each connected LC^{n+1} compactum X such that $\pi_1(X)$ is infinite admits a connected LC^n compactum X' which is shape equivalent but not UV^{n+1} equivalent, and a fortiori not CE equivalent, to X.

Keywords: CE equivalence, shape equivalence, UV^k equivalence, locally n-connected compacta.

AMS (MOS) Subj. Class.: 55P55, 54C56.

Introduction

A map of compacta is cell-like (CE) if all point-inverses have trivial shape. The CE maps generate an equivalence relation on the class CM_f of finite-dimensional compacta: $X, Y \in CM_f$ are called CE *equivalent* if there exist spaces $X_1 = X, X_2, \ldots, X_{2s}, X_{2s+1} = Y$ in CM_f and CE maps $X_{2i} \to X_{2i\pm 1}$, $i = 1, \ldots, s$. It is well known that CE equivalence implies shape equivalence. The converse, however, fails to be true. The first counterexample was given by Ferry who constructed a 1-dimensional compactum which is shape equivalent but CE inequivalent to the circle S^1 (see [6]). In [14] we produced a "universal counterexample" by showing that each connected compactum X such that pro-$\pi_1(X)$ is not pro-finite admits uncountably many compacta X_α, $\dim X_\alpha \leq \max(\dim X, 3)$, which are all shape equivalent to X but pairwise CE inequivalent. However, all these spaces X_α, as well as Ferry's counterexample, are *not locally connected*, and therefore it is natural to ask whether shape equivalence implies CE equivalence if the spaces in question have suitable

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"local niceness" properties. For example, if "locally nice" means to be an ANR, then the answer is in the affirmative (see Ferry [8]).

The local niceness property considered in this paper is *local n-connectedness*. We recall the definition. Let Y be any space. A subset $Y_0 \subset Y$ is said to be a UV^n *subset of* Y if each neighbourhood U of Y_0 in Y admits a neighbourhood V of Y_0 in U such that each map $f: S^k \to V$ is inessential in U, $k = 0, \ldots, n$ (we shall later refer to V as a UV^n *shrinking of* U). Y is called *locally n-connected* (LC^n) if each point $y \in Y$ is a UV^n subset of Y.

Generalizing Ferry's counterexample, Daverman and Venema have constructed LC^n compacta X_n, $\dim X_n = n+2$, which are shape equivalent but CE inequivalent to S^1 (see [5]). That is, shape equivalence *does not* imply CE equivalence for finite-dimensional LC^n compacta.

The purpose of this paper is to discuss this phenomenon in a broader context. Let us denote by UV^m *equivalence* the equivalence relation generated by the UV^m maps on the class CM of all compacta (see [9, 14]). Recall that a compactum is UV^m if it can be embedded as a UV^m subset of an ANR (equivalently, if *all* embeddings into ANRs yield UV^m subsets), and that a map is UV^m if all point-inverses are UV^m. Clearly, CE equivalence implies UV^m equivalence.

Main Theorem. *Let X be a connected LC^{n+1} compactum, $n \geq 0$, such that $\pi_1(X)$ is infinite. Then there exists a connected LC^n compactum X', $\dim X' \leq \max(\dim X, n+2)$, such that X and X' are shape equivalent but not UV^{n+1} equivalent. In particular, X and X' are not CE equivalent.*

We remark that for connected LC^1 compacta, the condition that $\pi_1(X)$ be infinite is *equivalent* to the condition that pro-$\pi_1(X)$ be not pro-finite. See Corollary 4.3. Also observe that we may suppress basepoints since all spaces appearing here are path-connected.

The Main Theorem is best possible in the sense that one can neither drop the condition that $\pi_1(X)$ be infinite nor achieve that X' be LC^{n+1}. This follows from results by Ferry and Chigogidze. In fact, Ferry proved in [9] that if X is a connected compactum with pro-$\pi_1(X)$ pro-finite, then each connected compactum X' which is shape equivalent to X must also be UV^k equivalent to X for any $k \geq 0$, whereas Chigogidze proved in [4] that shape equivalent LC^{n+1} compacta are always UV^{n+1} equivalent (i.e., shape equivalence implies UV^{n+1} equivalence on the class of LC^{n+1} compacta).

The reconstruction of X' in the Main Theorem goes as follows. Choose a map $\varphi: [0, \infty) \to X$ which lifts to a map $\tilde{\varphi}: [0, \infty) \to \tilde{X}$ into some covering space $\tilde{X}$ of X such that $\tilde{\varphi}([0, \infty))$ is not contained in any compact subset of $\tilde{X}$ (see Proposition 4.2). Let a fixed compactum A "slide along φ" to produce copies A_t of A, intersecting X in $\varphi(t)$, such that "diameter$(A_t) \to 0$ as $t \to \infty$". This yields a space $X' = X \cup \bigcup_{t \geq 0} A_t$. See Section 3 how this can be made precise. If we take $A = H^{n+1} =$ $(n+1)$-dimensional Hawaiian earring, we are able to show that X' has the properties required in the Main Theorem. We note that the above-mentioned counterexamples due to Daverman and Venema arise precisely by such a construction (with $A = S^{n+1}$). Moreover, if we take $A = S^0$, we easily see that X' is obtained from X by adding an *irregular ray* in the sense of [14].

The basic problem in proving results like the Main Theorem is to find *invariants* that are sufficiently fine to *detect* UV^m *inequivalence* (or CE *inequivalence*). The invariants used in this paper are called "UV^m groups"; they are defined in Section 1. Roughly speaking, the *kth* UV^m *group* $\pi_k^{(m)}(Y, y_0)$ of a pointed space (Y, y_0) is a modification of the ordinary kth homotopy group $\pi_k(Y, y_0)$ which is *forced* to be invariant under UV^m equivalence. The proof of the Main Theorem relies on the computation of certain UV^m groups. Not *all* of them, however, can be expected to be useful for our purposes. In fact, the invariant $\pi_k^{(m)}$ only has a chance to distinguish between shape equivalent LC^n compacta when $n < k \leqslant m$. More precisely, we have $\pi_k^{(m)}(Y, y_0) = 0$ for $k > m$ and any space Y (see Proposition 2.8), whereas $\pi_k^{(m)}(Y, y_0)$ is isomorphic to the *kth shape group* $\check{\pi}_k(Y, y_0)$ provided $k \leqslant n, m$ and Y is an LC^n compactum (see Proposition 2.1 and Theorem 2.7).

1. A generalization of homotopy groups

Let $\mathcal{M}$ be a class of nonempty topological spaces having the following properties.

($\mathcal{M}$1) $\mathcal{M}$ contains a one-point space $*$.

($\mathcal{M}$2) If $C_i \in \mathcal{M}$ and $c_i \in C_i$, $i = 1, 2$, then the one-point union $(C_1, c_1) \vee (C_2, c_2)$ is contained in $\mathcal{M}$.

($\mathcal{M}$3) For each $k \geqslant 1$, each $C \in \mathcal{M}$ and each map $\alpha : S^{k-1} \to C$, $\mathcal{M}$ contains the mapping cylinder $M(\alpha) = (S^{k-1} \times I + C)/(x, 0) \sim \alpha(x)$ of α and the quotient space $C \times I/\alpha$ obtained from $C \times I$ by identifying all fibers $\{y\} \times I$, $y \in \alpha(S^{k-1})$, to points.

($\mathcal{M}$4) There exists a *set* $\mathcal{M}' \subset \mathcal{M}$ such that each map $\alpha : S^{k-1} \to C$ with $k \geqslant 1$ and $C \in \mathcal{M}$ admits $C' \in \mathcal{M}'$ and maps $\alpha' : S^{k-1} \to C'$, $\gamma' : C' \to C$ with $\gamma'\alpha' = \alpha$.

The basic examples in this paper are $\mathcal{M}$ = CE = CE compacta and $\mathcal{M}$ = UV^m = UV^m compacta.

For each space X and each $k \geqslant 1$ we let $\mathcal{M}_k(X)$ denote the class of all triples $\Delta = (C, \alpha, \beta)$ where $C \in \mathcal{M}$ and $\alpha : S^{k-1} \to C, \beta : C \to X$ are maps. Given two such triples $\Delta = (C, \alpha, \beta)$ and $\Delta' = (C', \alpha', \beta')$, we write $\Delta' \geqslant \Delta$ if there exists a map $\gamma : C' \to C$ such that commutativity holds in

$$\begin{array}{ccccc} & & C' & & \\ & \overset{\alpha'}{\nearrow} & & \overset{\beta'}{\searrow} & \\ S^{k-1} & & \downarrow{\scriptstyle\gamma} & & X \\ & \underset{\alpha}{\searrow} & & \underset{\beta}{\nearrow} & \\ & & C & & \end{array}$$

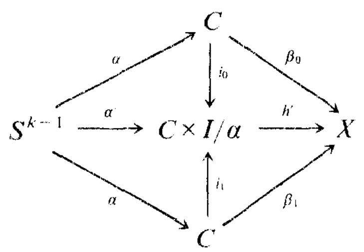
We let $\equiv$ denote the equivalence relation generated by $\geqslant$ (explicitly, we set $\Delta \equiv \Delta'$ if there exist triples $\Delta_1 = \Delta, \Delta_2, \ldots, \Delta_{2s+1} = \Delta'$ in $\mathcal{M}_k(X)$ such that $\Delta_{2i} \geqslant \Delta_{2i\pm1}$, $i = 1, \ldots, s$). We now define

$$\pi_k^{\mathcal{M}}(X) = \mathcal{M}_k(X)/\equiv. \tag{1}$$

By ($\mathcal{M}4$), this is a *set*. The equivalence class of $\Delta = (C, \alpha, \beta)$ in $\pi_k^{\mathcal{M}}(X)$ will be denoted by $[\Delta] = [C, \alpha, \beta]$.

Lemma 1.1. *Let $\Delta_i = (C_i, \alpha_i, \beta_i)$, $i = 0, 1$, and assume $C_0 = C_1 = C$ and $\beta_0\alpha_0 = \beta_1\alpha_1 = \Theta$. If there exist homotopies $g : \alpha_0 \simeq \alpha_1$ and $h : \beta_0 \simeq \beta_1$ such that the composed homotopy $h \circ g : \beta_0\alpha_0 \simeq \beta_1\alpha_1$ is stationary, then $\Delta_0 \equiv \Delta_1$.*

Proof. *Case* 1: *$\alpha_0 = \alpha_1 = \alpha$ and g is stationary.* Let $\alpha' : S^{k-1} \to C \times I/\alpha$, $\alpha'(x) = [x, 0]$. Since h must be stationary on $\alpha(S^{k-1})$, it induces a map $h' : C \times I/\alpha \to X$ and we obtain a commutative diagram



where $i_t(c) = [c, t]$.

Case 2: *General situation.* Let $G : (S^{k-1} \times I + C) \times I \to C$, $G(x, s, t) = g(x, 1 - s + st)$ for $(x, s) \in S^{k-1} \times I$, $G(c, t) = c$ for $c \in C$. Then $G(x, 0, t) = G(\alpha_1(x), t)$, i.e., G induces a homotopy $G' : M(\alpha_1) \times I \to C$. Let $i : S^{k-1} \to M(\alpha_1)$, $i(x) = [x, 1]$. Then $G_t'i = g_t$, and we infer $\Delta_0 \equiv (M(\alpha_1), i, \beta_0 G_0')$, $\Delta_1 \equiv (M(\alpha_1), i, \beta_1 G_1')$. The composed homotopy $h \circ G' : \beta_0 G_0' \simeq \beta_1 G_1'$ satisfies $(h \circ G')_t i = h_t G_t' i = h_t g_t = \Theta$. Using Case 1, we see that $(M(\alpha_1), i, \beta_0 G_0') \equiv (M(\alpha_1), i, \beta_1 G_1')$. □

To each $\Delta = (C, \alpha, \beta) \in \mathcal{M}_k(X)$ we associate the *total map* $\Theta_\Delta = \beta\alpha : S^{k-1} \to X$. It is clear that $\Theta_\Delta = \Theta_{\Delta'}$ provided $\Delta \equiv \Delta'$, and we can therefore define $\Theta_{[\Delta]} = \Theta_\Delta$ for $[\Delta] \in \pi_k^{\mathcal{M}}(X)$.

For $x_0 \in X$, let $\Delta_{x_0} = (*, \text{const}, \text{const}_{x_0}) \in \mathcal{M}_k(X)$. The following is obvious.

Observation 1.2. *Let $\Delta = (C, \alpha, \beta) \in \mathcal{M}_k(X)$. If at least one of the maps $\alpha : S^{k-1} \to C$, $\beta : C \to X$ is constant, then $\Delta \equiv \Delta_{x_0}$ where $\{x_0\} = \Theta_\Delta(S^{k-1})$.*

We are now ready to introduce the *fundamental $\mathcal{M}$-groupoid* of a space X. This is the category $\mathcal{P}^{\mathcal{M}}(X)$ whose objects are the points of X and whose morphisms from x_2 to x_1 are the elements $[\Delta] \in \pi_1^{\mathcal{M}}(X)$ such that $\Theta_{[\Delta]}(1) = x_2$ and $\Theta_{[\Delta]}(-1) = x_1$ (observe $S^0 = \{1, -1\}$). Composition of morphisms is defined as follows. Let $\kappa : S^0 \to (S^0, 1) \vee (S^0, -1)$, $\kappa(t) = t \in (S^0, -t) \subset (S^0, 1) \vee (S^0, -1)$. Moreover, for any pointed space (Y, y_0), let $\nabla : (Y, y_0) \vee (Y, y_0) \to Y$ denote the folding map. Given

$\Delta_i = (C_i, \alpha_i, \beta_i) \in \mathcal{M}_1(X)$, $i = 1, 2$, such that $\beta_1\alpha_1(1) = \beta_2\alpha_2(-1) = *$, we define

$$\Delta_1\Delta_2 = ((C_1, \alpha_1(1)) \vee (C_2, \alpha_2(-1)), (\alpha_1 \vee \alpha_2)\kappa, \nabla(\beta_1 \vee \beta_2)) \in \mathcal{M}_1(X), \quad (2)$$

$$S^0 \xrightarrow{\kappa} (S^0, 1) \vee (S^0, -1) \xrightarrow{\alpha_1 \vee \alpha_2} (C_1, \alpha_1(1)) \vee (C_2, \alpha_2(-1)),$$

$$(C_1, \alpha_1(1)) \vee (C_2, \alpha_2(-1)) \xrightarrow{\beta_1 \vee \beta_2} (X, *) \vee (X, *) \xrightarrow{\nabla} X.$$

It is easy to check that $\Delta_i \equiv \Delta_i'$, $i = 1, 2$, implies $\Delta_1\Delta_2 \equiv \Delta_1'\Delta_2'$. Hence, for $[\Delta_1] \in \mathcal{P}^{\mathcal{M}}(X)(x_3, x_2)$, $[\Delta_2] \in \mathcal{P}^{\mathcal{M}}(X)(x_2, x_1)$, we can define

$$[\Delta_2] \circ [\Delta_1] = [\Delta_1\Delta_2] \in \mathcal{P}^{\mathcal{M}}(X)(x_3, x_1). \quad (3)$$

It should be clear that this composition is associative and that the elements $[\Delta_{x_0}]$, $x_0 \in X$, are the identity morphisms. Moreover, an inverse for $[\Delta] = [C, \alpha, \beta] \in \mathcal{P}^{\mathcal{M}}(X)(x_2, x_1)$ is given by $[\Delta^{-1}] \in \mathcal{P}^{\mathcal{M}}(X)(x_1, x_2)$, where $\Delta^{-1} = (C, \alpha\nu, \beta)$ and $\nu : S^0 \to S^0$, $\nu(t) = -t$ (to see this, observe $\nabla(\beta \vee \beta) = \beta\nabla$ and apply Observation 1.2).

Next, for each $k \geq 1$, we shall define the *kth $\mathcal{M}$ group* of a pointed space (X, x_0). As a set, this is defined by

$$\pi_k^{\mathcal{M}}(X, x_0) = \{[\Delta] \in \pi_k^{\mathcal{M}}(X) \mid \Theta_{[\Delta]}(S^{k-1}) = \{x_0\}\}. \quad (4)$$

Since $\pi_1^{\mathcal{M}}(X, x_0) = \mathcal{P}^{\mathcal{M}}(X)(x_0, x_0)$, we already have a *group structure* for $k = 1$ (given by $[\Delta_1][\Delta_2] = [\Delta_2] \circ [\Delta_1] = [\Delta_1\Delta_2]$). For $k \geq 2$, we proceed as follows. Let $\kappa : S^{k-1} \to (S^{k-1}, *) \vee (S^{k-1}, *)$ denote the usual comultiplication map on the H-cogroup S^{k-1}. For $[\Delta_i] = [C_i, \alpha_i, \beta_i] \in \pi_k^{\mathcal{M}}(X, x_0)$, $i = 1, 2$, we define

$$\begin{aligned}[\Delta_1][\Delta_2] &= [(C_1, \alpha_1(*)) \vee (C_2, \alpha_2(*)), (\alpha_1 \vee \alpha_2)\kappa, \nabla(\beta_1 \vee \beta_2)] \\ &\in \pi_k^{\mathcal{M}}(X, x_0),\end{aligned} \quad (5)$$

$$S^{k-1} \xrightarrow{\kappa} (S^{k-1}, *) \vee (S^{k-1}, *) \xrightarrow{\alpha_1 \vee \alpha_2} (C_1, \alpha_1(*)) \vee (C_2, \alpha_2(*)),$$

$$(C_1, \alpha_1(*)) \vee (C_2, \alpha_2(*)) \xrightarrow{\beta_1 \vee \beta_2} (X, x_0) \vee (X, x_0) \xrightarrow{\nabla} X.$$

It is again easy to check that this is *well defined*. A few straightforward computations show that (5) actually defines a *group multiplication* on $\pi_k^{\mathcal{M}}(X, x_0)$. The neutral element is $[\Delta_{x_0}]$; and inverse for $[\Delta] = [C, \alpha, \beta]$ is given by $[\Delta^{-1}]$, where $\Delta^{-1} = (C, \alpha\nu, \beta)$ and $\nu : S^{k-1} \to S^{k-1}$ is the usual homotopy inverse on the H-cogroup S^{k-1}. The reader who wants explicit proofs is recommended to use Lemma 1.1. Moreover, the group $\pi_k^{\mathcal{M}}(X, x_0)$ is *Abelian* for $k \geq 2$. This follows from the fact that κ is homotopic to $\tau\kappa$, where τ is the switch map on $(S^{k-1}, *) \vee (S^{k-1}, *)$. Note that this is also true for $k = 2$ since we *do not need* the homotopy from κ to $\tau\kappa$ to be basepoint-preserving.

For our basic examples $\mathcal{M} = \mathrm{CE}$ and $\mathcal{M} = \mathrm{UV}^m$, we obtain the *kth* CE *group* $\pi_k^{\mathrm{CE}}(X, x_0)$ and the *kth* UV^m *group* $\pi_k^{\mathrm{UV}^m}(X, x_0)$ which will be abbreviated by $\pi_k^{(m)}(X, x_0)$.

Each pointed map $f : (X, x_0) \to (Y, y_0)$ induces a group homomorphism

$$f_* = \pi_k^{\mathcal{M}}(f) : \pi_k^{\mathcal{M}}(X, x_0) \to \pi_k^{\mathcal{M}}(Y, y_0)$$

which is defined by $f_*([C, \alpha, \beta]) = [C, \alpha, f\beta]$.

Proposition 1.3. *$\pi_k^{\mathcal{M}}$ is a functor from the pointed homotopy category of pointed spaces to the category of groups when $k = 1$ respectively Abelian groups when $k \geq 2$.*

Proof. Homotopy invariance follows from Lemma 1.1; the functorial properties are obvious. □

Next, we shall define a function

$$t_k : \pi_k(X, x_0) \to \pi_k^{\mathcal{M}}(X, x_0).$$

The elements of $\pi_k(X, x_0)$ can be regarded as homotopy classes rel S^{k-1} of maps $\beta : D^k \to X$ with $\beta(S^{k-1}) = \{x_0\}$. Hence, we may define (cf. Lemma 1.1)

$$t_k([\beta]) = [D^k, i, \beta]. \tag{6}$$

Here, $i : S^{k-1} \to D^k$ is the inclusion map. Observe that $D^k \in \mathcal{M}$, since it is the mapping cylinder of the constant map $S^{k-1} \to *$. It is easy to verify that t_k is a *group homomorphism.*

Remark. It is a nice exercise to prove that t_k is an *isomorphism* if all $C \in \mathcal{M}$ are *contractible.* For example, the class of nonempty spaces in which each point is a strong deformation retract satisfies ($\mathcal{M}$1)-($\mathcal{M}$4) and has this property. This shows that the *ordinary kth homotopy group* occurs as a special case of our general construction.

We are now going to study the question how the groups $\pi_k^{\mathcal{M}}(X, x_0)$ depend on the *basepoint* $x_0 \in X$. For that purpose, let us call a space X *$\mathcal{M}$-connected* if any two points $x, x' \in X$ admit $C \in \mathcal{M}$ and a map $\gamma : C \to X$ such that $x, x' \in \gamma(C)$. See [14] for the case $\mathcal{M} = \mathrm{UV}^m$. Obviously, each path-connected space is $\mathcal{M}$-connected (recall that $D^1 \in \mathcal{M}$).

Proposition 1.4. *If X is $\mathcal{M}$-connected, then $\pi_1^{\mathcal{M}}(X, x_1)$ and $\pi_1^{\mathcal{M}}(X, x_2)$ are isomorphic for all $x_1, x_2 \in X$.*

Proof. The groupoid $\mathcal{P}^{\mathcal{M}}(X)$ is connected whenever X is $\mathcal{M}$-connected. □

Let us now define an additional condition on $\mathcal{M}$.

($\mathcal{M}$5) For each $k \geq 2$, each map $\alpha : S^{k-1} \to C \in \mathcal{M}$ and each map $\lambda : S^0 \to D \in \mathcal{M}$, the adjunction space $M(\alpha, \lambda) = (S^{k-1} \times D + C)/(x, \lambda(-1)) \sim \alpha(x)$ is contained in $\mathcal{M}$.

Note that if λ is the inclusion of S^0 in D^1, then $M(\alpha, \lambda)$ is nothing but the mapping cylinder of α. For any λ, C can be regarded as a subspace of $M(\alpha, \lambda)$.

We remark that our above examples $\mathcal{M} = \mathrm{CE}$ and $\mathcal{M} = \mathrm{UV}^m$ satisfy ($\mathcal{M}$5). This may be seen as follows. Let $i : S^{k-1} \to D^k$ denote inclusion. Then $\mathrm{Sh}(M(i, \lambda)/D^k) = \mathrm{Sh}(M(i, \lambda))$ because D^k has trivial shape. Moreover, $M(\alpha, \lambda)/C = M(i, \lambda)/D^k$, so that $\mathrm{Sh}(M(\alpha, \lambda)/C) = \mathrm{Sh}(M(i, \lambda))$. In the case $\mathcal{M} = \mathrm{CE}$ both C and D have trivial shape; hence $\mathrm{Sh}(M(\alpha, \lambda)) = \mathrm{Sh}(M(\alpha, \lambda)/C)$ and $M(i, \lambda) = S^{k-1} \times D \cup D^k \times \{\lambda(-1)\}$ has trivial shape. This implies that $M(\alpha, \lambda)$ has trivial shape. In case $\mathcal{M} = \mathrm{UV}^m$, both the quotient map $M(\alpha, \lambda) \to M(\alpha, \lambda)/C$ and the canonical retraction $M(i, \lambda) \to D^k$ are UV^m maps; we easily infer that $M(\alpha, \lambda)$ must be a UV^m compactum (see e.g. [14, Section 1]).

Proposition 1.5. *Let $k \geq 2$. If $\mathcal{M}$ satisfies ($\mathcal{M}5$) and X is $\mathcal{M}$-connected, then $\pi_k^{\mathcal{M}}(X, x_1)$ and $\pi_k^{\mathcal{M}}(X, x_2)$ are isomorphic for all $x_1, x_2 \in X$.*

Proof. Let $[\Delta] = [C, \alpha, \beta] \in \pi_k^{\mathcal{M}}(X, x_1)$ and $[\Omega] = [D, \lambda, \mu] \in \mathcal{P}^{\mathcal{M}}(X)(x_2, x_1)$. Let us define $i_1 : S^{k-1} \to M(\alpha, \lambda)$, $i_1(x) = [x, \lambda(1)]$, $\mu * \beta : M(\alpha, \lambda) \to X$, $\mu * \beta([x, d]) = \mu(d)$ for $(x, d) \in S^{k-1} \times D$, $\mu * \beta([c]) = \beta(c)$ for $c \in C$. It is then easy to verify that

$$[\Delta] \cdot [\Omega] = [M(\alpha, \lambda), i_1, \mu * \beta] \in \pi_k^{\mathcal{M}}(X, x_2)$$

is well defined and that right multiplication by $[\Omega]$ is a homomorphism from $\pi_k^{\mathcal{M}}(X, x_1)$ to $\pi_k^{\mathcal{M}}(X, x_2)$. Moreover, if $[\Omega'] \in \mathcal{P}^{\mathcal{M}}(X)(x_3, x_2)$, then $([\Delta] \cdot [\Omega]) \cdot [\Omega'] = [\Delta] \cdot ([\Omega] \circ [\Omega'])$. □

Remark. If we do not assume ($\mathcal{M}5$), then the conclusion of Proposition 1.5 is nevertheless true for *path-connected* spaces X. In fact, for each equivalence class $[\omega]$ of paths from x_1 to x_2 we can define $[\Delta] \cdot [\omega]$ as in the above proof, using Lemma 1.1 to see that it is well defined. Since *constant* path equivalence classes are readily seen to operate trivially, we are finished.

Finally, we shall call a map $f : X \to Y$ *$\mathcal{M}$-regular* provided for each *pullback diagram*

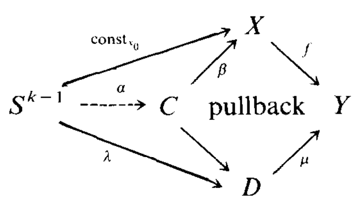
$$\begin{array}{ccc} C & \longrightarrow & X \\ \downarrow & & \downarrow f \\ D & \longrightarrow & Y \end{array}$$

the following holds true: If $D \in \mathcal{M}$, then also $C \in \mathcal{M}$.

For example, the hereditary shape equivalences between compacta (which include in particular the CE maps between finite-dimensional compacta) are CE regular and the UV^m maps between compacta are UV^m regular.

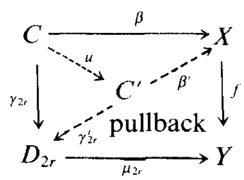
Theorem 1.6. *Let $f : X \to Y$ be an $\mathcal{M}$-regular map. Then for each $k \geq 1$ and each $x_0 \in X$, f induces an isomorphism $f_* : \pi_k^{\mathcal{M}}(X, x_0) \to \pi_k^{\mathcal{M}}(Y, f(x_0))$.*

Proof. (a) Surjectivity. Let $[\Omega] = [D, \lambda, \mu] \in \pi_k^{\mathcal{M}}(Y, f(x_0))$. Consider the following diagram



Here, α has been inserted using the pullback property. But now $[\Delta] = [C, \alpha, \beta] \in \pi_k^{\mathcal{M}}(X, x_0)$ and $f_*([\Delta]) = [\Omega]$.

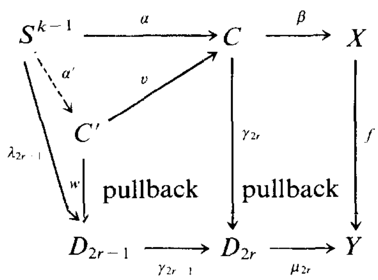
(b) Injectivity. Let $[\Delta] \in \ker f_*$, $\Delta = (C, \alpha, \beta)$. This means $f_*\Delta = (C, \alpha, f\beta) \equiv \Delta_{f(x_0)}$. We write $|\Delta| \leqslant r$ if there exist $\Delta_i = (D_i, \lambda_i, \mu_i) \in \mathcal{M}_k(X)$, $i = 0, \ldots, 2r+1$, such that $\Delta_0 = \Delta_{f(x_0)}$, $\Delta_{2r+1} = f_*\Delta$, $\Delta_{2i} \leqslant \Delta_{2i+1}$ via a map $\gamma_{2i}: D_{2i+1} \to D_{2i}$, $i = 0, \ldots, r$, and $\Delta_{2i+2} \leqslant \Delta_{2i+1}$ via a map $\gamma_{2i+1}: D_{2i+1} \to D_{2i+2}$, $i = 0, \ldots, r-1$. Clearly, there *exists* a number r such that $|\Delta| \leqslant r$. We shall show by induction on $|\Delta|$ that $\Delta \equiv \Delta_{x_0}$. For that purpose let us observe that we may always assume that the following solid arrow square is a pullback diagram.



(Otherwise we can replace Δ by $\Delta' = (C', u\alpha, \beta')$; then $\Delta' \equiv \Delta$ and Δ' has the desired property.)

If $|\Delta| = 0$, we have $D_{2r} = D_0 = *$, so that μ_{2r} and (by the pullback construction) β are injective. Since $\beta\alpha(S^{k-1}) = \{x_0\}$, α is constant, i.e., $\Delta \equiv \Delta_{x_0}$ by Observation 1.2.

Assume that $\Delta \equiv \Delta_{x_0}$ whenever $|\Delta| \leqslant r-1$. If $|\Delta| \leqslant r$, let us consider the following diagram.



Here, α' has been inserted using the pullback property. We have $C' \in \mathcal{M}$. Let $\Delta' = (C', \alpha', \beta v)$. Then $\Delta' \equiv \Delta$ and $\Delta_{2r-2} \leqslant f_*\Delta'$ via $\gamma_{2r-2}w$, i.e., $|\Delta'| \leqslant r-1$. This implies $\Delta' \equiv \Delta_{x_0}$. □

2. Some properties of LC^n spaces

In this section we collect some material on homotopy groups, shape groups and UV^m groups of LC^n compacta.

We begin by quoting a result due to Kozlowski and Segal (see [11]).

Proposition 2.1. *Let (X, x_0) be a pointed paracompact* LC^n *space. For each $k = 0, \ldots, n$, the shape functor induces an isomorphism from $\pi_k(X, x_0)$ to the kth shape group $\check{\pi}_k(X, x_0)$.*

Recall that $\check{\pi}_k(X, x_0)$ consists of all pointed shape morphisms from $(S^k, *)$ to (X, x_0).

The following result is implicitly contained in [11] and has been explicitly stated by Ferry in [7].

Proposition 2.2. *Let (X, x_0) be a pointed* LC^n *compactum. Then* pro-$\pi_k(X, x_0)$ *is stable for $k = 0, \ldots, n$ and Mittag-Leffler for $k = n+1$.*

Corollary 2.3. *Let (X, x_0) be a pointed* LC^n *compactum. For each $k = 0, \ldots, n$, the canonical morphism of pro-groups $\pi_k(X, x_0) \to$* pro-$\pi_k(X, x_0)$ *is an isomorphism of pro-groups.*

Proof. Since pro-$\pi_k(X, x_0)$ is stable, we infer that the canonical morphism of pro-groups $\check{\pi}_k(X, x_0) = \varprojlim$ pro-$\pi_k(X, x_0) \to$ pro-$\pi_k(X, x_0)$ is an isomorphism of pro-groups; cf. [12, Ch.I, § 5, Theorem 2]. Application of Proposition 2.1 yields the corollary. □

Corollary 2.4. *Let (X, x_0) be a pointed* LC^n *compactum. Then the homotopy groups $\pi_k(X, x_0)$, $k = 1, \ldots, n$, are countable.*

Proof. Since pro-$\pi_k(X, x_0)$ can be represented by an inverse sequence of kth homotopy groups of finite polyhedra, i.e., of *countable groups*, this is an immediate consequence of Corollary 2.3. □

We shall also need the following result on LC^0 spaces.

Lemma 2.5. *Let (X, x_0) be a pointed connected* LC^0 *space.*

(a) *The canonical morphism of pro-groups $\pi_1(X, x_0) \to$* pro-$\pi_1(X, x_0)$ *is an epimorphism of pro-groups.*

(b) pro-$\pi_1(X, x_0)$ *is Mittag-Leffler.*

(c) pro-$\pi_1(X, x_0)$ *is not pro-finite if and only if there exists a pointed* CW-*complex (Y, y_0) and a pointed map $f: (X, x_0) \to (Y, y_0)$ such that $f_*(\pi_1(X, x_0))$ is infinite.*

Proof. (a) Let $\underline{p} = \{p_\alpha\}: (X, x_0) \to \underline{X} = \{(X_\alpha, x_{0\alpha}), p_{\alpha\beta}\}_{\alpha \in A}$ be an HPol_*-expansion such that all X_α are connected CW-complexes (see [12, Ch.I, § 4.3]). We have to prove that $\pi_1(\underline{p}): \pi_1(X, x_0) \to \pi_1(\underline{X})$ is an epimorphism of pro-groups, i.e., that each α admits $\beta \geq \alpha$ such that $(p_{\alpha\beta})_*(\pi_1(X_\beta, x_{0\beta})) \subset (p_\alpha)_*(\pi_1(X, x_0))$; cf. [12, Ch.II, § 2, Theorem 4]. To show this, let $q: (Y, y_0) \to (X_\alpha, x_{0\alpha})$ be a covering projection such that $q_*(\pi_1(Y, y_0)) = (p_\alpha)_*(\pi_1(X, x_0))$. Since X is connected and LC^0, p_α can be lifted to a pointed homotopy class $r: (X, x_0) \to (Y, y_0)$ with $[q]r = p_\alpha$. But

Y is a CW-complex (cf. e.g. [16]) so that there exist $\gamma \in A$ and a pointed homotopy class $v:(X_\gamma, x_{0\gamma}) \to (Y, y_0)$ with $vp_\gamma = r$ (cf. [12, Ch.I, § 2, Theorem 1]). We may assume $\gamma \geq \alpha$. Then $[q]vp_\gamma = p_{\alpha\gamma}p_\gamma$, hence there is $\beta \geq \gamma$ such that $[q]vp_{\gamma\beta} = p_{\alpha\gamma}p_{\gamma\beta} = p_{\alpha\beta}$ (cf. again [12, Ch.I, § 2, Theorem 1]). We infer $(p_{\alpha\beta})_*(\pi_1(X_\beta, x_{0\beta})) \subset q_*(\pi_1(Y, y_0)) = (p_\alpha)_*(\pi_1(X, x_0))$.

(b) Each pro-group $\underline{H}$ which admits an epimorphism of pro-groups $G \to \underline{H}$, where G is a group, is easily seen to be Mittag-Leffler.

(c) pro-$\pi_1(X, x_0)$ is not pro-finite if and only if $\pi_1(\underline{X})$ is not pro-finite. It is easy to see that this is equivalent to the following condition: There exists $\alpha_0 \in A$ such that $(p_{\alpha_0\beta})_*(\pi_1(X_\beta, x_{0\beta}))$ is infinite for all $\beta \geq \alpha_0$. If this condition is satisfied, we know that $(p_{\alpha_0})_*(\pi_1(X, x_0))$ must be infinite; see the proof of (a). Conversely, if we are given a pointed map $f:(X, x_0) \to (Y, y_0)$ as in (c), we find $\alpha_0 \in A$ and a pointed homotopy class $u:(X_{\alpha_0}, x_{0\alpha_0}) \to (Y, y_0)$ such that $up_{\alpha_0} = [f]$ (cf. [12, Ch.I, § 2, Theorem 1]). Hence, $(p_{\alpha_0})_*(\pi_1(X, x_0))$ if infinite. Since $(p_{\alpha_0\beta})_*(\pi_1(X_\beta, x_{0\beta})) \supset (p_{\alpha_0\beta})_*(p_\beta)_*(\pi_1(X, x_0)) = (p_{\alpha_0})_*(\pi_1(X, x_0))$ for each $\beta \geq \alpha_0$, we see that the above condition is satisfied. □

In the lemma below we need the concept of an *approaching map*; the reader is referred to [3] or [15] for details.

Lemma 2.6. *Let X be a UV^m compactum contained in an AR M, and let $f: S^{k-1} \to X$ be a map, where $1 \leq k \leq m$. There exists an approaching map $\varphi: D^k \times [0, \infty) \to M$ from D^k to X which extends f, i.e., $\varphi(x, s) = f(x)$ for all $x \in S^{k-1}$ and $s \in [0, \infty)$.*

Proof. There exist open neighbourhoods U_n of X in M such that $\bigcap_{n=0}^\infty U_n = X$, $\mathrm{cl}(U_{n+1}) \subset U_n$, and such that each map $g: S^i \to U_{n+1}$, $0 \leq i \leq m$, is inessential in U_n. This allows us to find extensions $f_n: D^k \to U_{n+1}$ of f (note $f(S^k) \subset X \subset U_{n+2}$). Define $g_n: D^k \times \{n, n+1\} \cup S^{k-1} \times [n, n+1] \to U_{n+1}$, $g_n(x, t) = f_n(x)$ for $t = n$, $g_n(x, t) = f_{n+1}(x)$ for $t = n+1$ and $g_n(x, t) = f(x)$ for $x \in S^{k-1}$. There is an extension $\varphi_n: D^k \times [n, n+1] \to U_n$ of g_n. The maps φ_n determine a map $\varphi: D^k \times [0, \infty) \to M$ which is by construction an approaching map from D^k to X. □

Remark. As an application of Lemma 2.6 one can show that two points x_0, x_1 of a compactum X are *joinable* (cf. [12, Ch.II, § 8.2]) if there exist a UV^1 compactum C and a map $\gamma: C \to X$ such that $x_0, x_1 \in \gamma(C)$. Details are left to the reader. Note that the converse fails (there exist joinable compacta which are not UV^1 connected; an example is Ferry's compact spiral [6] which is not UV^1 connected by [14]).

Theorem 2.7. *Let (X, x_0) be a pointed LC^n compactum. Then the natural homomorphism $t_k: \pi_k(X, x_0) \to \pi_k^{(m)}(X, x_0)$ defined by (6) is an isomorphism provided $k \leq m, n$.*

Proof. (1) Surjectivity. Let $[C, \alpha, \beta] \in \pi_k^{(m)}(X, x_0)$. Choose a compact AR M containing the UV^m compactum C. By Lemma 2.6, there exists an approaching map

$\varphi: D^k \times [0,\infty) \to M$ from D^k to C which extends α. Let C' denote the *mapping cylinder* of φ (concerning this concept see [15]). Here are the properties of C' that are important for the present discussion (see [15]).

(a) C' is a compactum containing a copy of C;

(b) C' and C are shape equivalent; in particular, C' is a UV^m compactum;

(c) there exists a homeomorphism $h: D^k \times [0,\infty) \to C' \backslash C$ such that $C'' = C \cup h(S^{k-1} \times [0,\infty))$ is a copy of the ordinary mapping cylinder of α (where of course $h(S^{k-1} \times \{0\})$ is the "top").

Let $r: C'' \to C$ be the canonical retraction; then $\beta r: C'' \to X$ extends β. Since X is LC^n and $\dim(C' \backslash C'') \leq k+1 \leq n+1$, there is an extension $\omega: U \to X$ of βr on an open neighbourhood U of C'' in C' (see e.g. [2, Ch.III, Theorem (9.1)]). But U must contain $C' \backslash h(D^k \times [0,a)) = C \cup h(D^k \times [a,\infty))$ for some $a > 0$. There is a retraction $\rho: C' \to C'' \cup h(D^k \times [a,\infty))$ (induced by a retraction $D^k \times [0,a] \to S^{k-1} \times [0,a] \cup D^k \times \{a\}$), and we define $\beta': C' \to X$, $\beta'(c) = \omega\rho(c)$. This is an extension of βr. Let $\alpha': S^{k-1} \to C'$, $\alpha'(x) = h(x,0)$, and $\gamma: D^k \to C'$, $\gamma(x) = h(x,0)$. We obtain the following commutative diagram (where $\iota: C \to C'$ denotes inclusion).

$$
\begin{array}{ccccc}
 & & D^k & & \\
 & {}^{\text{incl}}\nearrow & \downarrow \gamma & \searrow {}^{\beta'\gamma} & \\
S^{k-1} & & & & \\
 & {}_{\alpha'}\searrow & & & \\
 & & C' & \xrightarrow{\beta'} & X \\
 & {}^{\iota\alpha}\nearrow & \uparrow \iota & \nearrow {}_{\beta} & \\
S^{k-1} & & & & \\
 & {}_{\alpha}\searrow & & & \\
 & & C & &
\end{array}
$$

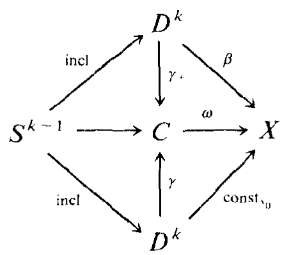
By Lemma 1.1, $(C', \iota\alpha, \beta') \equiv (C', \alpha', \beta')$, and we infer $[C, \alpha, \beta] = [D^k, \text{incl}, \beta'\gamma] \in \text{im } t_k$.

(2) Injectivity. Let $[\beta] \in \ker t_k$, where $\beta: D^k \to X$ with $\beta(S^{k-1}) = \{x_0\}$. We have to show that $\beta \simeq \text{const}_{x_0}$ rel S^{k-1}. But $[\beta] \in \ker t_k$ means $(D^k, \text{incl}, \beta) \equiv (D^k, \text{incl}, \text{const}_{x_0})$; cf. Observation 1.2. Hence we can find $\Delta_i = (C_i, \alpha_i, \beta_i)$, $i = 1, \ldots, 2r+1$, such that $\Delta_1 = (D^k, \text{incl}, \beta)$, $\Delta_{2r+1} = (D^k, \text{incl}, \text{const}_{x_0})$ and $\Delta_{2i} \leq \Delta_{2i\pm1}$ via a map $\gamma_{(i,\pm1)}: C_{2i\pm1} \to C_{2i}$, $i = 1, \ldots, r$.

Our *first step* is to show that *we may assume that each $\alpha_i: S^{k-1} \to C_i$ is an embedding.* Let C_i' denote the mapping cylinder of α_i, $\rho_i: C_i' \to C_i$ the canonical retraction onto the base and $\alpha_i': S^{k-1} \to C_i'$ the canonical embedding into the top. Of course, C_i' is a UV^m compactum. Moreover, we can easily find $\gamma_{(i,\pm1)}': C_{2i\pm1}' \to C_{2i}'$ such that $\rho_{2i}\gamma_{(i,\pm1)}' = \gamma_{(i,\pm1)}\rho_{2i\pm1}$ and $\gamma_{(i,\pm1)}'\alpha_{2i\pm1}' = \alpha_{2i}'$. Let $\Delta_i' = (C_i', \alpha_i', \beta_i\rho_i)$. Then $\Delta_{2i}' \leq \Delta_{2i\pm1}'$ via $\gamma_{(i,\pm1)}'$. If we identify C_1' and C_{2r+1}' in the obvious way with D^k, we see that $\Delta_1' = (D^k, \text{incl}, \beta')$, where $\beta' \simeq \beta$ rel S^{k-1}, and $\Delta_{2r+1}' = (D^k, \text{incl}, \text{const}_{x_0})$.

Our *second step* is to show that *we may assume that each* $\gamma_{(i,\pm1)}: C_{2i\pm1} \to C_{2i}$ *is an embedding.* Let $M_i = (C_{2i-1} \times I_{-1} + C_{2i} + C_{2i+1} \times I_{+1})/\sim$, where $I_{-1} = [-1, 0]$, $I_{+1} = [0, 1]$ and $\sim$ is the equivalence relation generated by $(x, 0) \sim \gamma_{(i,\pm1)}(x)$ for $x \in C_{2i\pm1}$; i.e., M_i is obtained by sewing together the two mapping cylinders $M(\gamma_{(i,\pm1)})$ along their common base C_{2i}. Of course, M_i is a compactum. There are canonical embeddings $e_{(i,\pm1)}: C_{2i\pm1} \to M_i$, $e_{(i,\pm1)}(x) = [x, \pm1]$, and $e_i: C_{2i} \to M_i$, $e_i(x) = [x]$, and $\mu_i: S^{k-1} \times [-1, 1] \to M_i$, $\mu_i(x, s) = [\alpha_{2i\pm1}(x), s]$ for $(x, s) \in S^{k-1} \times I_{\pm1}$ (recall that the α_i are embeddings after the first step). Moreover, there is a canonical retraction $\rho_i: M_i \to C_{2i}$, $\rho_i([x, s]) = \gamma_{(i,\pm1)}(x)$ for $(x, s) \in C_{2i\pm1} \times I_{\pm1}$, $\rho_i([x]) = x$ for $x \in C_{2i}$. Finally, let us define $H_i: M_i \times I \to M_i$, $H_i([x, s], t) = [x, st]$ for $[x, s] \in C_{2i\pm1} \times I_{\pm1}$, $H_i([x], t) = [x]$ for $x \in C_{2i}$. We have $H_i: e_i\rho_i \simeq \text{id}$. Let C_{2i}' denote the quotient space obtained from M_i by identifying all fibers $\mu_i(\{x\} \times [-1, 1])$, $x \in S^{k-1}$, to points. The quotient map $q_i: M_i \to C_{2i}'$ is easily seen to be a closed map, hence C_{2i}' is again a compactum. The maps $\gamma_{(i,\pm1)}' = q_i e_{(i,\pm1)}$ and the maps $\alpha_{2i}': S^{k-1} \to C_{2i}'$, $\alpha_{2i}'(x) = q_i\mu_i(x, 0)$, are embeddings; we have $\gamma_{(i,\pm1)}'\alpha_{2i\pm1} = \alpha_{2i}'$. There exist unique maps $\beta_{2i}': C_{2i}' \to X$ such that $\beta_{2i}\rho_i = \beta_{2i}'q_i$; they satisfy $\beta_{2i}'\gamma_{(i,\pm1)}' = \beta_{2i\pm1}$. Finally, let $\lambda_i = q_i e_i$ which embeds C_{2i} into C_{2i}'. There exist unique maps $\rho_i': C_{2i}' \to C_{2i}$ such that $\rho_i = \rho_i' q_i$ and $H_i': C_{2i}' \times I \to C_{2i}'$ such that $q_i H_i = H_i'(q_i \times 1_I)$. Then $\rho_i'\lambda_i = \text{id}$ and $\lambda_i\rho_i' \simeq \text{id}$ via H_i', i.e., C_{2i}' has the same homotopy type as C_{2i}, and is therefore a UV^m compactum.

Our *third step* is to show that *we may assume* $r = 1$, i.e., that there is a commutative diagram



where C is a UV^m compactum. In fact, when $r > 1$, we can shorten the sequence $\Delta_1, \ldots, \Delta_{2r+1}$ as follows. Let P be the pushout of $C_{2r-2} \xleftarrow{\gamma_{(r-1,+1)}} C_{2r-1} \xrightarrow{\gamma_{(r,-1)}} C_{2r}$, given together with maps $u: C_{2r-2} \to P$ and $v: C_{2r} \to P$. Then P is a compactum and u, v are embeddings, i.e., we may assume $C_{2r-2} \cap C_{2r} = C_{2r-1}$, $C_{2r-2} \cup C_{2r} = P$. The quotient map $C_{2r-2} \to C_{2r-2}/C_{2r-1}$ is a UV^m map, hence C_{2r-2}/C_{2r-1} is a UV^m compactum (see e.g. [14, Section 1]). But P/C_{2r} is homeomorphic to C_{2r-2}/C_{2r-1}, so that P/C_{2r} is a UV^m compactum. Since the quotient map $P \to P/C_{2r}$ is a UV^m map, we infer that P is a UV^m compactum. By the pushout property, there is a unique map $\pi : P \to X$ such that $\pi u = \beta_{2r-2}$ and $\pi v = \beta_{2r}$. Let $\Delta^*_{2r-2} = (P, u\alpha_{2r-2}, \pi)$. Then $\Delta^*_{2r-2} \leq \Delta_{2r+1}$ via the embedding $v\gamma_{(r,+1)}$ and $\Delta^*_{2r-2} \leq \Delta_{2r-3}$ via the embedding $u\gamma_{(r-1,-1)}$.

Now, given a commutative diagram as above, we define $\gamma : S^k \to C$ by $\gamma|$upper hemisphere $= \gamma_+$, $\gamma|$lower hemisphere $= \gamma_-$. Similarly, let $\beta^* : S^k \to X$ be defined by putting together β and const_{x_0}; then $\omega\gamma = \beta^*$. We wish to show that β^* is inessential. This clearly implies $\beta \simeq \text{const}_{x_0}$ rel S^{k-1}. Recalling Corollary 2.3, we see that $\pi_k(X, x_0) \to \text{pro-}\pi_k(X, x_0)$ is an isomorphism, and a fortiori a monomorphism, of pro-groups. Choose ANRs $M \supset C$ and $N \supset X$. Then pro-$\pi_k(X, x_0)$ is represented by $\{\pi_k(U_\lambda, x_0), (i_{\lambda\lambda'})_*\}$, where $\{U_\lambda\}$ is the set of open neighbourhoods of X in N and $i_{\lambda\lambda'} : U_{\lambda'} \to U_\lambda$ denotes inclusion (cf. [12, Ch.I, § 4, Theorem 4]). By the characterization of monomorphisms in [12, Ch.II, § 2, Theorem 2], we infer that the inclusions $i_\lambda : X \to U_\lambda$ induce monomorphisms $(i_\lambda)_* : \pi_k(X, x_0) \to \pi_k(U_\lambda, x_0)$ for $\lambda \geq \lambda_0$. For $\lambda \geq \lambda_0$, choose an extension $\omega' : V \to U_\lambda$ of $i_\lambda\omega$ to some open neighbourhood V of C in M. Since C is UV^m, $i_V\gamma$ is inessential where $i_V : C \to V$ denotes inclusion. This implies $(i_\lambda)_*([\beta^*]) = [i_\lambda\omega\gamma] = [\omega' i_V\gamma] = 0$, and we infer $[\beta^*] = 0$ in $\pi_k(X, x_0)$. □

Remark. The proof of Theorem 2.7 can easily be modified to show that for each pointed LC^n compactum (X, x_0), $t_k : \pi_k(X, x_0) \to \pi_k^{\mathcal{M}}(X, x_0)$ is an isomorphism provided $k \leq n$ and $\mathcal{M} = \text{CE}$.

Let us close this section by showing that the functors $\pi_k^{(m)}$ are *trivial* when $m < k$.

Proposition 2.8. *Let $m < k$. Then $\pi_k^{(m)}(X, x_0) = 0$ for every pointed space (X, x_0).*

Proof. Let $[\Delta] \in \pi_k^{(m)}(X, x_0)$, $\Delta = (C, \alpha, \beta)$. We may assume that $\alpha : S^{k-1} \to C$ is an embedding; cf. the proof of Theorem 2.7. Let $C' = C/\alpha(S^{k-1})$. Then the quotient map $\pi : C \to C'$ is a UV^{k-2} map, in particular a UV^{m-1} map. Hence π induces isomorphisms of pro-groups up to dimension $m-1$ and an epimorphism in dimension m. This shows that C' is again a UV^m compactum. Let $\Delta' = (C', \pi\alpha, \beta')$, where $\beta' : C' \to X$ is the unique map with $\beta'\pi = \beta$. Then $\Delta \geq \Delta'$ via π, and $\Delta' \equiv \Delta_{x_0}$ by Observation 1.2. □

3. The basic construction

Let X be a compactum, $\varphi : [0, \infty) \to X$ be a map and (A, a_0) be a pointed compactum. The *reduced cone* of (A, a_0) is the quotient space $C(A, a_0) = A \times [0, \infty]/(A \times \{\infty\} \cup \{a_0\} \times [0, \infty])$; it is again a compactum. For $s \in [0, \infty]$, let $A_s = p(A \times \{s\})$, where $p : A \times [0, \infty] \to C(A, a_0)$ is the quotient map. Clearly, A_s is a copy of A when $s<\infty$, whereas $A_\infty=\{*\}$. Let us define a subspace $X_\varphi(A, a_0)$ of $X\times C(A, a_0)$ by the following.

$$X_\varphi(A, a_0)=X\times A_\infty \cup \bigcup_{s\in[0,\infty)} \{\varphi(s)\}\times A_s. \tag{7}$$

Each pointed map $f:(A, a_0)\to(B, b_0)$ of pointed compacta induces a canonical map $C(f):C(A, a_0)\to C(B, b_0)$, and it is obvious that $1_X\times C(f)$ restricts to a map $f^*:X_\varphi(A, a_0)\to X_\varphi(B, b_0)$. Similarly, each pointed homotopy $F:(A, a_0)\times I\to(B, b_0)$ induces a homotopy $F^*:X_\varphi(A, a_0)\times I\to X_\varphi(B, b_0)$. Moreover, if there is no danger of confusion, we simply write $X_\varphi=X_\varphi(A, a_0)$. It can be readily verified that X_φ is closed in $X\times C(A, a_0)$; hence, X_φ is a *compactum.* Moreover, there is a canonical retraction $r_\varphi:X_\varphi\to X$ (where X has been identified with $X\times A_\infty\subset X_\varphi$); of course, $r_\varphi(x)=c^*(x)$ with the constant pointed map $c:(A, a_0)\to(A, a_0)$.

For technical purposes, we shall also need the following map.

$$i_\varphi:A\times[0,\infty)\to X_\varphi,\ i_\varphi(a, s)=(\varphi(s), p(a, s)). \tag{8}$$

Obviously, each $i_\varphi(A\times\{s\})$ is a copy of A such that $i_\varphi(A\times\{s\})\cap X=\{\varphi(s)\}$. Moreover, $r_\varphi i_\varphi(a, s)=\varphi(s)$ for all $(a, s)\in A\times[0,\infty)$. It is important to notice the following.

Observation 3.1. diam $i_\varphi(A\times\{s\})\to 0$ *as* $s\to\infty$.

Here, "diam" denotes the diameter with respect to a fixed metric d_φ on the space X_φ. Note that Observation 3.1 is evident if we choose d_φ to be a metric of the form $d_\varphi((x, c), (x', c'))=d_X(x, x')+d_C(c, c')$, where d_X is a metric on X and d_C a metric on $C(A, a_0)$. But then Observation 3.1 must be true for *any* d_φ because all metrics on compact spaces are uniformly equivalent. Finally, a routine verification yields the following.

Observation 3.2. i_φ *maps* $(A\backslash\{a_0\})\times[0,\infty)$ *homeomorphically onto* $X_\varphi\backslash X$.

We are now ready to study X_φ.

Proposition 3.3. *X is a shape strong deformation retract of X_φ (cf.* [3]*). In particular, X and X_φ have the same shape.*

Proof. By [3], we have to prove the following: Each map $f:X\to P$ into an ANR P has an extension $f':X_\varphi\to P$, and any two extensions $f'_0, f'_1:X_\varphi\to P$ of f are homotopic relative to X. Since X is a retract of X_φ, the first part is obvious. Now let us consider f'_0, f'_1 as above. Define $F:X_\varphi\times\{0, 1\}\cup X\times I\to P$ by $F(x, t)=f(x)$ for $x\in X$ and $F(x, i)=f'_i(x)$ for $i=0, 1$. There is an extension of F to an open $U\subset X_\varphi\times I$. Let V be an open neighbourhood of X in X_φ such that $V\times I\subset U$. Since $X_\varphi\backslash V$ is a compact subset of $X_\varphi\backslash X$, there exists $r\in[0,\infty)$ such that $X_\varphi\backslash V\subset i_\varphi((A\backslash\{a_0\})\times[0, r))$; cf. Observation 3.2. Then $X'=X_\varphi\backslash i_\varphi((A\backslash\{a_0\})\times[0, r))=X\cup i_\varphi(A\times[r,\infty))$ is a closed subset of X_φ with $X\subset X'\subset V$, and F has an extension

$H: X_\varphi \times \{0, 1\} \cup X' \times I \to P$. Consider the map $g: (A \times \{0, 1\} \cup \{a_0\} \times I) \times [0, r] \cup A \times I \times \{r\} \to P$, $g(a, t, s) = H(\alpha(a, s), t)$; it has an extension $G: A \times I \times [0, r] \to P$. Since $\beta: A \times I \times [0, r] \to i_\varphi(A \times [0, r]) \times I$, $\beta(a, t, s) = (i_\varphi(a, s), t)$, is a closed map (a fortiori a quotient map) and $G\beta^{-1}$ is single-valued, there is a unique map $H': i_\varphi(A \times [0, r]) \times I \to P$ such that $G = H'\beta$. By construction, H and H' can be pasted to a continuous $H'': X_\varphi \times I \to P$ which extends F. □

Remark. If a_0 has a closed neighbourhood $C \subset A$ which admits a homeomorphism $h: (\mathrm{bd}\, C) \times [0, 1) \to C \backslash \{a_0\}$ such that $h(a, 0) = a$ for all $a \in \mathrm{bd}\, C$ (=topological boundary of C in A), then X is even a *cylinder base* of X_φ (cf. [15]). In fact, $X_\varphi \backslash X \approx Z \times (0, 1]$ with $Z = (A \backslash \mathrm{int}\, C) \cup h((\mathrm{bd}\, C) \times [0, \frac{1}{2}])$.

Theorem 3.4. *Let X and A be* LC^n, *and let A be n-connected. Then* X_φ *is* LC^n.

Proof. There is a relatively simple proof for $n = 0$; however, we shall not treat this case separately. The general proof is lengthy and will be divided in two steps.

Step 1. Assume that there exists an open embedding $h: [0, 1) \to A$ such that $h(0) = a_0$.

Let $x_0 \in X_\varphi$ and U be an open neighbourhood of x_0 in X_φ. We have to construct a UV^n shrinking $V \subset U$ in X_φ (cf. Introduction). Since this is trivial for $x_0 \notin \mathrm{cl}\, \varphi([0, \infty))$, we only consider $x_0 \in \mathrm{cl}\, \varphi([0, \infty))$. Here, "cl" denotes closure. Let $U_0 = U \cap X$. This is an open neighbourhood of x_0 in X, hence there is a UV^n shrinking V_0 of U_0 in X. We may assume that V_0 is compact. Recalling Observation 3.1, we find $s_0 \in [0, \infty)$ such that $i_\varphi(A \times \{s\}) \subset U$ for $s \in \varphi^{-1}(V_0) \cap [s_0, \infty)$. We now choose a compact neighbourhood W_0 of x_0 in X such that $W_0 \subset \mathrm{int}_X V_0$. For each m, $\varphi^{-1}(\mathrm{int}_X V_0)$ is an open neighbourhood of $\varphi^{-1}(W_0) \cap [m, m+1]$; since the latter is compact, it can be covered by *finitely many* compact intervals $J_{m,i} \subset \varphi^{-1}(\mathrm{int}_X V_0)$. Let $J^* = \bigcup_{m,i} J_{m,i}$; then $\varphi^{-1}(W_0) \subset J^* \subset \varphi^{-1}(V_0)$. Moreover, let $J_0 = J^* \cap [0, s_0]$ and $J = J^* \cap [s_0, \infty)$. By construction, J_0 is compact and J is a closed locally contractible subset of $[0, \infty)$. Since $\{a_0\} \times J_0 \subset i_\varphi^{-1}(U)$, there is a neighbourhood L of a_0 in A such that $i_\varphi(L \times J_0) \subset U$. We may assume that $L = h([0, \Theta])$ for some $\Theta > 0$. Let $V = V_0 \cup i_\varphi(L \times J_0) \cup i_\varphi(A \times J)$. Then $r_\varphi^{-1}(W_0) \backslash i_\varphi((A \backslash h([0, \Theta))) \times J_0) \subset V \subset U$; in particular, V is a neighbourhood of x_0 in X_φ. We shall show that V is a UV^n shrinking of U in X_φ. Let $f: S^k \to V$ be any map, $k = 0, \ldots, n$. To prove that f is inessential in U, *it suffices to show* $f \simeq r_\varphi f$ *in* U (since $r_\varphi f(S^k) \subset V_0$). For this purpose, we proceed as follows. Let $r: A \to A$ be the map defined by $r(a) = a$ for $a \in L$ and $r(a) = h(\Theta)$ for $a \notin L$; moreover, we choose a homotopy $H: A \times I \to A$ such that $H(h(\Theta'), t) = h(t\Theta')$ for all $\Theta' \in [0, \Theta]$ (observe that the inclusion $L \to A$ is a cofibration). We obtain an induced map $r^*: X_\varphi \to X_\varphi$ and an induced homotopy $H^*: X_\varphi \times I \to X_\varphi$. Note that $r^*(V) = V_0 \cup i_\varphi(L \times J^*) \subset U$ and that $H^*(r^*f \times 1_I)$ is a homotopy from $r_\varphi f$ to r^*f in $r^*(V)$; hence $r^*f \simeq r_\varphi f$ in U. *It therefore suffices to show* $f \simeq r^*f$ *in* U. Since $X_L = X \cup i_\varphi(L \times [0, \infty))$ is compact (use Observation 3.2),

$V\backslash X_L$ is open in V and $P=f^{-1}(V\backslash X_L)$ is open in S^k. Let $A'=A\backslash h([0,\Theta))$. Then A' is a retract of A, hence n-connected and LC^n. We shall construct a homotopy $F:P\times I\to i_\varphi(A'\times J)\subset U$ from $f|_P$ to $r^*f|_P$ and compact $C_m\subset P$ such that $C_m\subset \operatorname{int} C_{m+1}$, $\bigcup_{m=1}^\infty C_m=P$ and $\operatorname{diam} F(\{x\}\times I)<1/m$ for $x\in P\backslash C_m$. This clearly proves $f\simeq r^*f$ in U (simply extend F by the stationary homotopy from $f|_{S^k\backslash P}$ to $r^*f|_{S^k\backslash P}$). To construct F, triangulate P by an infinite simplicial complex K. Choose compact subpolyhedra $P_m\subset P$, triangulated by finite subcomplexes $K_m\subset K$, such that $P_m\subset \operatorname{int} P_{m+1}$ and $\bigcup_{m=1}^\infty P_m=P$. Moreover, choose $\varepsilon_m>0$ such that $\operatorname{diam} f(M)<1/m$ for each $M\subset S^k$ with $\operatorname{diam} M<\varepsilon_m$. There are only finitely many k-simplices $\sigma^k\in K$ with $\operatorname{diam}\sigma^k\geqslant\varepsilon_m$; we may assume that they are already contained in K_m. This implies

$$\operatorname{diam} f(\sigma)<\frac{1}{m}\quad \text{for } \sigma\in K\backslash K_m. \tag{$*$}$$

Similarly, it is no restriction to assume

$$d_\varphi(f(x), r^*f(x))<\frac{1}{m}\quad \text{for } x\in P\backslash P_m. \tag{$**$}$$

Let $K^{(i)}$ denote the i-skeleton of K and $P^{(i)}\subset P$ the underlying polyhedron. We shall now inductively show the following.

For each i, there exist a strictly increasing function $\lambda_i:\mathbb{N}\to\mathbb{N}$ and a map $F^{(i)}:P\times\{0,1\}\cup P^{(i)}\times I\to i_\varphi(A'\times J)$ such that

(a$_i$) $F^{(i)}(x,0)=f(x)$, $F^{(i)}(x,1)=r^*f(x)$ for $x\in P$,

(b$_i$) $\operatorname{diam} F^{(i)}(\sigma\times I)<1/m$ for $\sigma\in K^{(i)}\backslash K_{\lambda_i(m)}$.

It is then clear that $F=F^{(k)}$ is a homotopy with the desired properties (take $C_m=P_{\lambda_k(m)}$).

The induction starts with $i=-1$; nothing has to be shown in this case.

Next, we show how to construct $F^{(i+1)}$ and λ_{i+1} if $F^{(i)}$ and λ_i are already given. For each $\sigma\in K^{(i+1)}\backslash K^{(i)}$, $F^{(i)}$ restricts to a map $g_\sigma:\partial(\sigma\times I)\to i_\varphi(A'\times J)$, where $\partial(\sigma\times I)$ denotes the boundary of the topological $(i+2)$-ball $\sigma\times I$. Observe that $\operatorname{diam} g_\sigma(\partial(\sigma\times I))<3/m$ when $\sigma\notin K_{\lambda_i(m)}$ (use ($*$), ($**$) and (b$_i$)). Moreover, let $J_\sigma\subset[0,\infty)$ denote the projection of $i_\varphi^{-1}g_\sigma(\partial(\sigma\times I))$ onto the second factor; it is a compact interval or a singleton (for $i=-1$, this follows from (a$_i$)). Clearly, $J_\sigma\subset J$. Since $i_\varphi(A'\times J_\sigma)$ is an n-connected space containing the image of g_σ, we see that $\delta(\sigma)=\inf\{\operatorname{diam}\psi(\sigma\times I)|\psi:\sigma\times I\to i_\varphi(A'\times J)\text{ extends } g_\sigma\}$ is a well-defined positive number. We choose an extension g'_σ of g_σ with $\operatorname{diam} g'_\sigma(\sigma\times I)<2\delta(\sigma)$. Now $F^{(i)}$ and the g'_σ, $\sigma\in K^{(i+1)}\backslash K^{(i)}$, can be pasted to a map $F^{(i+1)}:P\times\{0,1\}\cup P^{(i+1)}\times I\to i_\varphi(A'\times J)$ which satisfies (a$_{i+1}$). We wish to show that $\delta(\sigma)\leqslant 1/(2m)$ for $\sigma\notin K_{\mu(m)}$ with some sufficiently large $\mu(m)$; it is then obvious that we can construct $\lambda_{i+1}:\mathbb{N}\to\mathbb{N}$ such that (b$_{i+1}$) is fulfilled. Let $s_m\in[0,\infty)$ such that $\operatorname{diam} i_\varphi(A\times\{s\})<1/(6m)$ for $s\geqslant s_m$ (cf. Observation 3.1), and let Δ_m be the distance between the sets $i_\varphi(A'\times(J\cap[0,s_m]))$ and $(A'\times(J\cap[s_m+1,\infty)))$. Note that the second one is *closed* in $i_\varphi(A'\times J)$; hence $\Delta_m>0$. Finally, for each $x\in Z_m=i_\varphi(A'\times(J\cap[0,s_m+1]))$ let $U(x)=\{x'\in Z_m \mid d_\varphi(x',x)<1/(4m)\}$. But Z_m is an LC^n space (recall the definitions of J and A'), and we can choose a UV^n shrinking $V(x)$ of $U(x)$ in Z_m. Let $\delta_m>0$ be a Lebesgue number for the open cover $\{\operatorname{int} V(x)\}_{x\in M}$ of M, and let $\mu(m)$ be an integer such that diam $g_\sigma(\partial(\sigma\times I))<\min(\Delta_m,\delta_m,1/(6m))$ for $\sigma\notin K_{\mu(m)}$. We now consider a fixed $\sigma\notin K_{\mu(m)}$.

Case 1: $J_\sigma\subset[s_m,\infty)$. Let $\psi:\sigma\times I\to i_\varphi(A'\times J_\sigma)\subset i_\varphi(A'\times J)$ be an arbitrary extension of g_σ. Then $\delta(\sigma)\leq \operatorname{diam} i_\varphi(A'\times J_\sigma)\leq \operatorname{diam} g_\sigma(\partial(\sigma\times I))+2\sup\{\operatorname{diam}(A\times\{s\})\mid s\in J_\sigma\}<1/(6m)+2(1/(6m))=1/(2m)$.

Case 2: $J_\sigma\not\subset[s_m,\infty)$. Since diam $g_\sigma(\partial(\sigma\times I))<\Delta_m$, we infer $g_\sigma(\partial(\sigma\times I))\subset Z_m$. But $g_\sigma(\partial(\sigma\times I))$ has diameter $<\delta_m$, thus it is contained in some $V(x)$ and there exists an extension $\psi:\sigma\times I\to U(x)\subset i_\varphi(A'\times J)$. Then $\delta(\sigma)\leq \operatorname{diam} U(x)\leq 1/(2m)$.

This completes the proof.

Step 2. General case of the theorem.

Let $A'=\{(a,t)\in A\times I\mid t=1 \text{ or } a=a_0\}\subset A\times I$, $a_0'=(a_0,0)$; A' is the one-point union of A and I, and it is again n-connected and LC^n. But now (A',a_0') satisfies the assumption in Step 1, whence $X_\varphi(A',a_0')$ is LC^n. Consider the map $p:(A',a_0')\to(A,a_0)$, $p(a,t)=a$; it induces a surjective map $p^*:X_\varphi(A',a_0')\to X_\varphi(A,a_0)$. It is easy to see that the nondegenerate point-inverses of p^* are contractible (note that they are homeomorphic to the nondegenerate point-inverses of the canonical retraction $X_\varphi(I,0)\to X$). Hence, p^* is a CE map, and CE images of LC^n compacta are LC^n (see e.g. [1, Corollary 2.1.2(ii)]). □

4. Unbounded rays

Let X be an arbitrary space. A map $\varphi:[0,\infty)\to X$ is called an *unbounded ray in* X if there exists a covering projection $p:\tilde X\to X$ and a lift $\tilde\varphi:[0,\infty)\to\tilde X$ of φ such that no compact subset of $\tilde X$ contains $\tilde\varphi([0,\infty))$. The importance of this concept comes from the following result.

Theorem 4.1. *Let* $\varphi:[0,\infty)\to X$ *be an unbounded ray in a connected* LC^0 *compactum* X *and let* (A,a_0) *be a pointed connected* LC^0 *compactum. Let* $x_0=\varphi(0)\in X\subset X_\varphi(A,a_0)$, *and let* $i:(A,a_0)\to(X_\varphi(A,a_0),x_0)$, $i(a)=i_\varphi(a,0)$ (*cf.* (8)). *Then the homomorphism* $i_*:\pi_k^{(m)}(A,a_0)\to\pi_k^{(m)}(X_\varphi(A,a_0),x_0)$ *is injective for all* $k,m\geq 1$.

Proof. Let $[\Delta]\in\ker i_*$, $\Delta=(C,\alpha,\beta)$. This means that $i_*\Delta=(C,\alpha,i\beta)\equiv\Delta_{x_0}$; i.e., there exist $\Delta_j=(C_j,\alpha_j,\beta_j)$, $j=1,\dots,2r+1$, such that $\Delta_1=i_*\Delta$, $\Delta_{2r+1}=\Delta_{x_0}$ and $\Delta_{2j}\leq\Delta_{2j\pm1}$ via a map $\gamma_{(j,\pm1)}:C_{2j\pm1}\to C_{2j}$, $j=1,\dots,r$ (the C_j are of course UV^m compacta). Let us fix a covering projection $p:\tilde X\to X$ and a lift $\tilde\varphi:[0,\infty)\to\tilde X$ of φ such that no compact subset of $\tilde X$ contains $\tilde\varphi([0,\infty))$. We set $Y=X_\varphi(A,a_0)$ and form the pullback

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{r}_\varphi} & \tilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[r_\varphi]{} & X \end{array}$$

to obtain a covering projection $q:\tilde{Y}\to Y$. Observe that $\tilde{Y}$ must be metrizable (see [17]). Since r_φ is a retraction, we may assume that $\tilde{X}\subset\tilde{Y}$ and that $\tilde{r}_\varphi$ is a retraction. Let $\{\tilde{\varphi}_\lambda\}_{\lambda\in L}$ denote the set of all lifts of φ. Clearly, $\tilde{\varphi}_{\lambda_0}=\tilde{\varphi}$ for some λ_0. Using the pullback property, we see that there is a unique lift $i_\lambda: A\times[0,\infty)\to\tilde{Y}$ of $i_\varphi: A\times[0,\infty)\to Y$ such that $\tilde{r}_\varphi i_\lambda=\tilde{\varphi}_\lambda\pi$, where $\pi: A\times[0,\infty)\to[0,\infty)$ denotes projection. Each i_λ maps $(A\backslash\{a_0\})\times[0,\infty)$ homeomorphically onto the open subset $U_\lambda=i_\lambda((A\backslash\{a_0\})\times[0,\infty))$ of $\tilde{Y}$ (recall Observation 3.2 and observe that any lift of an open map is again an open map). The U_λ must be pairwise disjoint. To see this, consider $\lambda,\lambda'\in L$ and $(a,s),(a',s')\in(A\backslash\{a_0\})\times[0,\infty)$ such that $i_\lambda(a,s)=i_{\lambda'}(a',s')$. Since $qi_\lambda=qi_{\lambda'}=i_\varphi$, we infer $(a,s)=(a',s')$; hence $i_\lambda=i_{\lambda'}$ by connectedness. This yields $\tilde{\varphi}_\lambda=\tilde{\varphi}_{\lambda'}$, i.e., $\lambda=\lambda'$. Moreover, we have $\tilde{Y}\backslash\tilde{X}=\bigcup_{\lambda\in L}U_\lambda$. This can be shown as follows. First, we see that $q^{-1}(Y\backslash X)=\tilde{Y}\backslash\tilde{X}$, by the pullback construction. Next, let $\tilde{y}\in\tilde{Y}\backslash\tilde{X}$. Then $q(\tilde{y})=i_\varphi(\tilde{a},\tilde{s})$ with a unique $(\tilde{a},\tilde{s})\in(A\backslash\{a_0\})\times[0,\infty)$. Define $\psi:[0,\infty)\to Y$, $\psi(s)=i_\varphi(a,s)$. This map has a unique lift $\tilde{\psi}:[0,\infty)\to\tilde{Y}$ such that $\tilde{\psi}(\tilde{s})=\tilde{y}$. Obviously $\tilde{r}_\varphi\tilde{\psi}$ is a lift of φ, i.e., $\tilde{r}_\varphi\tilde{\psi}=\tilde{\varphi}_\lambda$ for some $\lambda\in L$. However, $\psi_\lambda:[0,\infty)\to\tilde{Y}$, $\psi_\lambda(s)=i_\lambda(a,s)$, is also a lift of ψ with $\tilde{r}_\varphi\psi_\lambda=\tilde{\varphi}_\lambda$, and the pullback property implies $\tilde{\psi}=\psi_\lambda$. Hence $\tilde{y}\in U_\lambda$. Let us now define $Z=\tilde{X}\cup U_{\lambda_0}$; this is a closed subset of $\tilde{Y}$ (the reader can show that Z can be naturally identified with $\tilde{X}_{\tilde{\varphi}}(A,a_0)$, but here we do not need this fact). There is a retraction $\rho:\tilde{Y}\to Z$ which agrees with $\tilde{r}_\varphi$ on $\tilde{X}\cup\bigcup_{\lambda\neq\lambda_0}U_\lambda$. By Theorem 3.4, Y is a connected LC^0 compactum; therefore $q:\tilde{Y}\to Y$ is an *overlay* in the sense of Fox (see [10, Theorem 3]). This means in particular that q can be extended to a covering projection $q':\tilde{W}\to W$ where $\tilde{W}\supset\tilde{Y}$ and $W\supset Y$ are ANRs; cf. [10, Theorem 13]. We infer that all $\beta_j: C_j\to Y$ can be uniquely lifted to maps $\tilde{\beta}_j: C_j\to\tilde{Y}$ such that $\tilde{\beta}_j\alpha_j(*)=\tilde{x}_0$, where $\tilde{x}_0=i_{\lambda_0}(a_0,0)$ and $*$ is a fixed basepoint of S^{k-1}. This is true by Fox's lifting theorem (see Theorem 17^0 in [10]), or can be shown directly by considering $q':\tilde{W}\to W$. The crucial point is that pro-$\pi_1(C_j,\alpha_j(*))=0$ because C_j is a UV^m compactum, $m\geqslant 1$. By uniqueness of liftings we obtain $\tilde{\beta}_{2j}\gamma_{(j,\pm1)}=\tilde{\beta}_{2j\pm1}$, $j=1,\ldots,r$. The set $K=\tilde{r}_\varphi(\bigcup_j\tilde{\beta}_j(C_j))\subset\tilde{X}$ is compact and thus does not contain $\tilde{\varphi}([0,\infty))$. We can therefore find $s_0,s_1\in[0,\infty)$, $s_0<s_1$ such that $\tilde{\varphi}((s_0,s_1))\cap K=\emptyset$. Let $Z_0=\tilde{X}\cup i_{\lambda_0}(A\times[0,s_0])\cup i_{\lambda_0}(A\times[s_1,\infty))\subset Z$. Clearly, we have $\rho\tilde{\beta}_j(C_j)\subset Z_0$. Define $\mu: Z_0\to A$ by $\mu(z)=a_0$ for $z\in\tilde{X}\cup i_{\lambda_0}(A\times[s_1,\infty))$ and $\mu(i_{\lambda_0}(a,s))=a$ for $(a,s)\in A\times[0,s_0]$. This is a well-defined continuous map (note that both pieces on which μ has been defined are *closed* in Z_0). It is therefore possible to define $\beta'_j: C_j\to A$, $\beta'_j(x)=\mu(\rho\tilde{\beta}_j(c))$. Then $\beta'_{2j}\gamma_{(j,\pm1)}=\beta'_{2j\pm1}$, $j=1,\ldots,r$. Hence, if we define $\Delta'_j=(C_j,\alpha_j,\beta'_j)$, we see that

$\Delta'_1 \equiv \Delta'_{2r+1}$. But now it is obvious that $\Delta'_{2r+1} = \Delta_{a_0}$ since C_{2r+1} is a one-point space. Moreover, we show that $\Delta'_1 = \Delta$. Define $i' : A \to \tilde{Y}$, $i'(a) = i_{\lambda_0}(a, 0)$. Then $i'\beta : C \to \tilde{Y}$ is a lift of $\beta_1 = i\beta$ such that $i'\beta(\alpha(*)) = \tilde{x}_0$, thus $\tilde{\beta}_1 = i'\beta$. The above construction shows that $\beta'_1 = \beta$ as required. We have now shown that $[\Delta] = 0 \in \pi_k^{(m)}(A, a_0)$ which completes the proof. □

The next result provides information about the *existence* of unbounded rays in LC^0 compacta.

Proposition 4.2. *Let* (X, x_0) *be a pointed connected* LC^0 *compactum. The following are equivalent.*

(i) *There exists a pointed connected semilocally* 1*-connected* LC^0 *space* (Y, y_0) *and a pointed map* $f : (X, x_0) \to (Y, y_0)$ *such that* $f_*(\pi_1(X, x_0))$ *is infinite.*

(ii) *There exists an unbounded ray in* X.

(iii) pro-$\pi_1(X, x_0)$ *is not pro-finite.*

Proof. (i) ⇒ (ii) Since $f_*(\pi(X, x_0))$ is infinite, there exists a sequence $b_0, b_1, b_2, \ldots$ in $f_*(\pi(X, x_0))$ such that $b_i \ldots b_j \neq$ neutral element for all $0 \leq i \leq j$ (i.e., an *irreducible sequence* in the sense of [14]). It is easy to construct a map $\varphi : [0, \infty) \to X$ such that $\varphi(n) = x_0$ and $f\varphi|_{[n,n+1]}$ represents b_n for all $n = 0, 1, 2, \ldots$. We shall show that φ is an unbounded ray. Let $q : \tilde{Y} \to Y$ be the universal covering. We form the pullback

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

and obtain a covering projection $p : \tilde{X} \to X$. Let $\tilde{\varphi} : [0, \infty) \to \tilde{X}$ be any lift of φ. Assume that $\tilde{\varphi}([0, \infty))$ is contained in a compact subset of $\tilde{X}$. Then $\tilde{f}\tilde{\varphi}([0, \infty))$ must be contained in a compact subset C of $\tilde{Y}$. But now $\tilde{f}\tilde{\varphi}(\mathbb{N})$ is a subset of $C \cap q^{-1}(y_0)$ and must therefore be a *finite* set (recall that the fibre $q^{-1}(y_0)$ is discrete). Choose $m, n \in \mathbb{N}$ such that $m < n$ and $\tilde{f}\tilde{\varphi}(m) = \tilde{f}\tilde{\varphi}(n)$. Then $\tilde{f}\tilde{\varphi}|_{[m,n]}$ is a closed path in $\tilde{Y}$, hence $q\tilde{f}\tilde{\varphi}|_{[m,n]} = f\varphi|_{[m,n]}$ represents the neutral element in $\pi_1(Y, y_0)$. On the other hand, $f\varphi|_{[m,n]}$ represents $b_m \ldots b_{n-1}$, a contradiction.

(ii) ⇒ (iii) Let φ be an unbounded ray in X and let $p : \tilde{X} \to X$ be an associated covering projection as in the definition of unbounded rays. Since X is LC^0, we can assume that $\tilde{X}$ is connected and that p extends to a covering projection $q : \tilde{W} \to W$, where $\tilde{W} \supset \tilde{X}$ and $W \supset X$ are ANRs (see again [10]). Let $i : X \to W$ denote inclusion. Choose $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) = x_0$. Then $G = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ has infinite index in $\pi_1(X, x_0)$, since p must have infinitely many sheets. Moreover, it is easy to see that $\ker i_* \subset G$, where $i_* : \pi_1(X, x_0) \to \pi_1(W, x_0)$ (cf. [13, Proposition 11.1]). Hence, im $i_* \approx \pi_1(X, x_0)/\ker i_*$ is infinite. Now (W, x_0) has the pointed homotopy type of a pointed CW-complex, so that pro-$\pi_1(X, x_0)$ is not pro-finite by Lemma 2.5(c).

(iii) $\Rightarrow$ (i) This follows from Lemma 2.5(c). □

Corollary 4.3. *Let (X, x_0) be a pointed connected semilocally 1-connected* LC^0 *compactum. Then* pro-$\pi_1(X, x_0)$ *is pro-finite if and only if $\pi_1(X, x_0)$ is finite.*

Proof. This follows from Lemma 2.5(c) and Proposition 4.2. □

Remark. If (X, x_0) is a pointed connected LC^1 compactum, then the conclusion of Corollary 4.3 can be derived more easily from Corollary 2.3.

5. Proof of the Main Theorem

We begin with an elementary observation.

Observation 5.1. *Let (X, x_0) and (Y, y_0) be two path-connected* UV^m *equivalent compacta. Then $\pi_k^{(m)}(X, x_0)$ and $\pi_k^{(m)}(Y, y_0)$ are isomorphic for each $k \geq 1$.*

In fact, this follows easily from Propositions 1.4, 1.5 and Theorem 1.6.

The next result seems to be well known. We shall nevertheless supply a proof since we did not succeed to find a reference.

Lemma 5.2. *Let H^n denote the n-dimensional Hawaiian earring (i.e., $H^n = \bigcup_{i=1}^{\infty} S_i^n$, where $S_i^n \subset \mathbb{R}^{n+1}$ is a sphere with radius $1/i$ and center $(0, \ldots, 0, (1/i) - 1)$). Then H^n is an $(n-1)$-connected* LC^{n-1} *compactum.*

Proof. Let $\exp : [0, \infty) \to S^1$, $\exp(s) = e^{i\pi s}$. By Theorem 3.4, the space $Y = S^1_{\exp}(S^n, *)$ is LC^{n-1}. Since $U = \exp((0, 1))$ is open in S^1, we see that $U' = r_{\exp}^{-1}(U)$ is open in Y and therefore an LC^{n-1} space. Since H^n is homeomorphic to a retract of U' (observe that $U' \approx (0, 1) \times H^n$), we infer that H^n is LC^{n-1}. In particular, $x_0 = (0, \ldots, 0, -1) \in H^n$ has a neighbourhood V such that each map $f: S^k \to V$, $k = 0, \ldots, n-1$, is inessential in H^n. We may assume $V = \bigcup_{i=i_0}^{\infty} S_i^n$ for some i_0. Then V is a retract of H^n and we conclude that V is $(n-1)$-connected. Since $V \approx H^n$, we are finished. □

Lemma 5.3. *$\pi_n^{(m)}(H^n)$ is uncountable for each $m \geq n$.*

Proof. Let $r_i : H^n \to S_i^n$ denote the retraction sending each S_j^n, $j \neq i$, to x_0. A homomorphism $\rho : \pi_n(H^n) \to \prod_{i=1}^{\infty} \pi(S_i^n)$ is defined by $\rho(a) = ((r_i)_*(a))$. Similarly, we obtain a homomorphism $\rho^{(m)} : \pi_n^{(m)}(H^n) \to \prod_{i=1}^{\infty} \pi_n(S_i^n)$, where we have used Theorem 2.7 to identify $\pi_n^{(m)}(S_i^n)$ with $\pi_n(S_i^n)$. Clearly, $\rho^{(m)} t_n = \rho$ with

$t_n : \pi_n(H^n) \to \pi_n^{(m)}(H^n)$ (cf. (6)). Hence, it suffices to show to im ρ is uncountable. We observe that $H^{n-1} \subset D^n$, where D^n is the standard closed ball in $\mathbb{R}^n$ with radius 1 and center 0. By identifying D^n with the lower hemisphere of S^n, we obtain a natural embedding $H^{n-1} \subset S^n$. Then H^n is obviously homeomorphic to the quotient space S^n/H^{n-1}, and we let $p:(S^n, x_0) \to (H^n, x_0)$ denote the "quotient map". For each $M \subset \mathbb{N}$, let $f_M : (H^n, x_0) \to (H^n, x_0)$ be defined by $f_M(x) = x$ for $x \in \bigcup_{i \in M} S_i^n$ and $f_M(x) = x_0$ otherwise. Then we obtain uncountably many maps $g_M = f_M p : (S^n, x_0) \to (H^n, x_0)$, and by construction we have $\rho([g_M]) = \rho([g_{M'}])$ if and only if $M = M'$. □

Remark. The above proof shows also that $\pi_n(H^n)$ is uncountable.

The general strategy to construct connected LC^n compacta that are shape equivalent but UV^{n+1} inequivalent is this.

Assume we are given a connected LC^n compactum X such that pro-$\pi_1(X)$ is not pro-finite ($n \geq 0$). By Theorem 4.1, there exists an unbounded ray $\varphi:[0, \infty) \to X$. Let us consider a pointed n-connected LC^n compactum (A, a_0). Then $X' = X_\varphi(A, a_0)$ is a connected LC^n compactum which is shape equivalent to X; see Section 3. Moreover, dim $X' = \max(\dim X, 1 + \dim A)$ (this follows from standard theorems in dimension theory; in case $A = \{a_0\}$ the equation holds because dim $X \geq 1$ by the assumption on pro-$\pi_1(X)$).

Proposition 5.4. *Assume that X' is UV^m equivalent to X. Then, for each $k \geq 1$, the following condition is satisfied.*

($C_k^{(m)}$) *There exists a split epimorphism $\varepsilon : \pi_k^{(m)}(X, \varphi(0)) \to \pi_k^{(m)}(X, \varphi(0))$ such that* ker ε *contains a subgroup isomorphic to $\pi_k^{(m)}(A, a_0)$.*

Remarks. (1) An epimorphism is *split* if it has a right inverse.

(2) It is useful to observe that ($C_k^{(m)}$) implies the following weaker condition.

($WC_k^{(m)}$) $\pi_k^{(m)}(X, \varphi(0))$ *contains a subgroup isomorphic to* $\pi_k^{(m)}(A, a_0)$.

Proof of Proposition 5.4. Let $r: X' \to X$ be the canonical retraction. Then, for each $k \geq 1$, we obtain a split short exact sequence of groups

$$0 \to \ker r_* \to \pi_k^{(m)}(X', \varphi(0)) \xrightarrow{r_*} \pi_k^{(m)}(X, \varphi(0)) \to 0$$

(a canonical splitting is given by the homomorphism induced by the inclusion $X \to X'$). We obviously have im $i_* \subset \ker r_*$, where $i:(A, a_0) \to (X', \varphi(0))$, is the map defined in Theorem 4.1. Since i_* is a monomorphism by Theorem 4.1, the proposition follows easily from Observation 5.1. □

We are now ready to prove the Main Theorem.

If X is a connected LC^{n+1} compactum such that $\pi_1(X)$ is infinite (i.e., pro-$\pi_1(X)$ is not pro-finite by Corollary 4.3), we can choose $(A, a_0) = (H^{n+1}, x_0)$ and the above construction yields a connected LC^n compactum X' which is shape equivalent to X and whose dimension is $\max(\dim X, n+2)$. Since $\pi_{n+1}^{(n+1)}(X, \varphi(0))$ is a countable group (cf. Theorem 2.7 and Corollary 2.4), the condition $(WC_{n+1}^{(n+1)})$ is not satisfied (recall Lemma 5.3). Hence X' and X are not UV^{n+1} equivalent.

Corollary 5.5. *Let X be a connected compactum such that* pro-$\pi_k(X)$ *is stable for $k \leqslant n+1$ and Mittag-Leffler for $k = n+2$. If* pro-$\pi_1(X)$ *is not pro-finite, there exists a connected* LC^n *compactum X' which is shape equivalent but* UV^{n+1} *inequivalent to X.*

Proof. By Ferry [7], X is shape equivalent to a connected LC^{n+1} compactum X''. If X'' is UV^{n+1} inequivalent to X, we are finished; otherwise we apply the Main Theorem to X''. □

Let us finally consider an example due to Daverman and Venema; see [5]. The map $\exp : [0, \infty) \to S^1$ considered in the proof of Lemma 5.2 is an unbounded ray in the circle S^1. Hence, $X'_n = S^1_{\exp}(S^{n+1}, *)$ is a connected LC^n compactum of dimension $n+2$ which is shape equivalent to S^1. In [5] it was shown that X'_n is not UV^{n+1} equivalent to S^1. Using the results of this paper, this can be seen as follows. For $n = 0$, condition $(C_1^{(1)})$ is not satisfied. In fact, $\pi_1^{(1)}(S^1) \approx \mathbb{Z}$ by Theorem 2.7, and the kernel of any epimorphism $\varepsilon : \pi_1^{(1)}(S^1) \to \pi_1^{(1)}(S^1)$ is trivial. For $n \geqslant 1$ not even condition $(WC_{n+1}^{(n+1)})$ is satisfied since $\pi_{n+1}^{(n+1)}(S^1) = 0$ and $\pi_{n+1}^{(n+1)}(S^{n+1}) \approx \mathbb{Z}$. Moreover, it should be clear that similar examples are obtained if S^{n+1} is replaced by any n-connected compact CW complex A such that $\pi_{n+1}(A) \neq 0$.

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