Symmetric triality relations and structurable algebras

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Abstract

In this paper, some properties of algebras satisfying local symmetric triality relations have
been studied, including both generalized symmetric composition algebras and the conjugate
algebras of any structurable algebras. We also discuss a general method of constructing Lie
algebras from such a system, which can permit a construction of the magic square.

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1. Symmetric triality Lie algebras

Let \( A \) be an algebra over a field \( F \) with the bilinear product denoted by juxtaposition \( xy \) \((x, y \in A)\). Let

\[
\text{stri}(A) = \{(d_0, d_1, d_2) \in (\text{End} A)^3, \ d_j(xy) = (d_{j+1}x)y + x(d_{j+2}y) \}
\]

for all \( x, y \in A \) and for \( j = 0, 1, 2 \), \( (1.1) \)

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where the indices \( j \) are taken modulo 3, i.e.,
\[
d_{j \pm 3} = d_j.
\]
\( (1.2) \)
This is a Lie algebra under componentwise commutation operation, and may be called the symmetric triality Lie algebra (abbreviated hereafter as STLA), which is endowed with a natural order 3 automorphism \( \theta \), given by
\[
\theta((d_0, d_1, d_2)) = (d_2, d_0, d_1).
\]
\( (1.3) \)
Note that the fixed subalgebra under \( \theta \) is then the Lie algebra of derivations Der \( A \). If \( A \) is a unital algebra over a field \( F \) of characteristic \( \neq 2 \), then it is easy to show that
\[
\text{str} (A) = \{ (d, d, d) ; d \in \text{Der} A \}.
\]
Also, given any \((d_0, d_1, d_2) \in \text{str}(A)\), \( A \) will be said to be a symmetric triality algebra (STA) with respect to the triple \((d_0, d_1, d_2)\). Before going into details, we first note the following: Let \( \alpha_j \in F \) with cyclic condition \( \alpha_j \pm 3 = \alpha_j \) be constants. If we set
\[
\tilde{d}_j := \sum_{k=0}^{2} \alpha_{j-k} d_k,
\]
\( (1.4) \) then it is simple to see that we have
\[
\tilde{d}_j (xy) = (\tilde{d}_{j+1} x) y + x (\tilde{d}_{j+2} y)
\]
\( (1.5) \) and hence that \( A \) is also a STA with respect to another triple \((\tilde{d}_0, \tilde{d}_1, \tilde{d}_2)\). Especially, if we choose \( \alpha_0 = \alpha_1 = \alpha_2 = 1 \), then
\[
D := d_0 + d_1 + d_2
\]
\( (1.6) \) is a derivation of \( A \), i.e., it satisfies
\[
D(xy) = (D x) y + x (D y)
\]
\( (1.7) \) so that \( A \) is a STA also with respect to a triple \( (d'_0, d'_1, d'_2) \) such that \( d'_0 = d'_1 = d'_2 = D \).

Given some STA's we can always construct a larger STA as follows.

**Proposition 1.1.** For any two STLA, \( \text{str}(A) \) and \( \text{str}(A') \), we have
\[
\text{str}(A) \otimes I' + I \otimes \text{str}(A') \subseteq \text{str}(A \otimes A'),
\]
where \( I' \) (respectively \( I' \)) denotes the centroid of \( A \) (resp. \( A' \)). More explicitly, given any \((d_0, d_1, d_2) \in \text{str}(A)\), the triple \((D_0, D_1, D_2) \in (\text{End}(A \otimes A'))^3 \) defined by
\[
D_j (x \otimes x') = (d_j x) \otimes (U' x') + (U x) \otimes (d'_j x')
\]
\( (1.8) \) for \( U \in I \) and \( U' \in I' \) and for \( x \in A \) and \( x' \in A' \) satisfies
\[
D_j ([x \otimes x'] \cdot (y \otimes y')) = [D_{j+1} (x \otimes x')] \cdot (y \otimes y')
\]
\[
+ (x \otimes x') \cdot D_{j+2} (y \otimes y').
\]
\( (1.9) \)
Proof. We calculate

\[ D_j[(x \otimes x') \cdot (y \otimes y')] = D_j[(xy) \otimes (x'y')] \]

\[ = d_j(xy) \otimes U'(x'y') + U(xy) \otimes d_j'(x'y') \]

\[ = (d_{j+1}x)y \otimes (U'x'y') + x(d_{j+2}y) \otimes x'(U'y') \]

\[ + (Ux)y \otimes (d_{j+1}'x')y' + x(Uy) \otimes x'(d_{j+2}'x') \]

\[ = (D_{j+1}(x \otimes x')) \cdot (y \otimes y') + (x \otimes x') \cdot D_{j+2}(y \otimes y'), \]

since we have

\[ \{D_{j+1}(x \otimes x') \} \cdot (y \otimes y') = \{(d_{j+1}x) \otimes U'x'y' + Ux \otimes d_{j+1}'x' \} \cdot (y \otimes y') \]

\[ = (d_{j+1}x)y \otimes (U'x'y') + (Ux)y \otimes (d_{j+1}'x')y' \]

and

\[ (x \otimes x') \cdot D_{j+2}(y \otimes y') = (x \otimes x') \cdot \{(d_{j+2}y) \otimes U'y' + Uy \otimes d_{j+2}'x' \} \]

\[ = x(d_{j+2}y) \otimes x'(U'y') + y(Uy) \otimes x'(d_{j+2}'x'). \]

□

Remark 1.2. Such a construction has been used in [4] to construct the so-called magic square involving exceptional Lie algebras (see Section 3).

Many STA so far known are also involutive, i.e., there exists an involution map \( x \rightarrow \overline{x} \) in \( A \) satisfying

(i) \( \overline{x} = x \), \hspace{1cm} (1.10a)

(ii) \( \overline{xy} = \overline{y} \overline{x} \). \hspace{1cm} (1.10b)

For any \( Q \in \text{End } A \), we define \( \overline{Q} \in \text{End } A \), as usual, by

\( \overline{Qx} := \overline{Q} \overline{x}. \) \hspace{1cm} (1.11)

Taking the involution of both sides of Eq. (1.1), it gives

\( \overline{d_j(xy)} = \overline{d_j} \overline{(d_{j+1}x)} + (d_{j+2}) \overline{y}. \) \hspace{1cm} (1.12)

Changing \( x \leftrightarrow \overline{y} \), this implies that \( A \) is also a STA with respect to a new triple \((\overline{d_0}, \overline{d_1}, \overline{d_2})\) given by

\( \overline{d_j} := \overline{d}_{-j} = \overline{d}_{j-3}. \) \hspace{1cm} (1.13)

Thus we see that \( \text{stri}(A) \) is closed under the map \((d_0, d_1, d_2) \rightarrow (\overline{d_0}, \overline{d_2}, \overline{d_1})\).

We next introduce the second bilinear product \( x \ast y \) in the same vector space of \( A \) by

\( x \ast y := \overline{xy} = \overline{y} \overline{x}. \) \hspace{1cm} (1.14)
Then, the resulting new algebra which we denote by $A^*$ is also involutive, i.e.,
\[ x \star y = y \star x (= xy). \]  
(1.15)
We call $A^*$ be the conjugate algebra of $A$. Then, Eq. (1.12) is rewritten as
\[ d_j(x \star y) = (d_{j+1}x) \star y + x \star (d_{j+2}y). \]  
(1.16)
In other words, $(d_j, d_{j+1}, d_{j+2})$ is a Lie related triple of $A^*$ (see [2]) and we say that $A^*$ is a Lie related triple algebra (abbreviated LRTA, hereafter) with respect to this triple. Conversely, let $A^*$ be a LRTA with respect to the triple $d_j$’s. Then, its conjugate algebra $A$ with the product $xy$ given by Eq. (1.15) is a STA with respect to the same $d_j$’s. As in [2], we are really dealing with the Lie algebra
\[ \text{lrt}(A^*, -) = \{(d_0, d_1, d_2) \in (\text{End } A^*)^3, \] 
\[ d_j(x \star y) = (d_{j+1}x) \star y + x \star (d_{j+2}y), \forall x, y \in A^*, \forall j = 0, 1, 2 \].

**Remark 1.3.** If $A^*$ is a structurable algebra (see [1,2]), then we can always find a triple $(d_0, d_1, d_2)$ to satisfy Eq. (1.16) (see [2] and Section 2 for details). Therefore, we can construct a STA from any structurable algebra $A^*$. Examples will be given in Section 2.

Before going into further details, we note the following Proposition.

**Proposition 1.4.** Let $A^*$ be a LRTA with respect to $d_j$’s. Setting
\[ D := d_0 + d_1 + d_2, \]  
(1.17a)
then
\[ DS := D + \overline{D} \]  
(1.17b)
is a derivation of $A^*$, while
\[ DA := D - \overline{D} \]  
(1.17c)
is an anti-derivation of $A^*$. In other words, we have
\[ DS(x \star y) = (DSx) \star y + x \star (DSy), \]  
(1.18a)
\[ DA(x \star y) = -(DAx) \star y - x \star (DAy). \]  
(1.18b)
In contrast, both $D$ and $\overline{D}$ (and hence $DS$ and $DA$ also) are derivations of the conjugate algebra $A$.

**Proof.** Summing over $j = 0, 1, 2$ in Eq. (1.16), it gives
\[ \overline{D}(x \star y) = (Dx) \star y + x \star (Dy). \]
Taking the involution of this relation, and letting $x \leftrightarrow \overline{y}$, we also have
\[ D(x \star y) = (\overline{D}x) \star y + x \star (\overline{D}y). \]
From these, we can readily derive Eqs. (1.18). □
**Remark 1.5.** Let $L_0$ and $L_1$ be vector spaces consisting of all derivations and anti-derivations of an algebra $A^*$ (or $A$). Then, $L = L_0 \oplus L_1$ is a graded Lie algebra with $L_0$ and $L_1$ being even and odd parts of $L$. Note that we have

$$[L_0, L_1] \subseteq L_1, [L_1, L_1] \subseteq L_0$$

as we can easily verify.

**Remark 1.6.** An example of anti-derivation can be obtained as follows. Let $A$ be an associative matrix algebra. Introducing $\overline{x}$ to be the transpose matrix of $x \in A$, then $A$ becomes involutive. Since $A$ is associative, $adv$ for any $v \in A$ is a derivation of $A$ so that $A$ is a STA with respect to $d_j$'s such that $d_0 = d_1 = d_2 = adv$. Moreover, we can easily verify

$$adv = -ad\overline{v}.$$

Then, $d_S := ad(v - \overline{v})$ is a derivation of $A^*$, but $d_A := ad(v + \overline{v})$ is an anti-derivation of $A^*$.

Returning to the original discussion, we introduce the left and right multiplication operators in $\text{End} A$ by

\begin{align*}
L(x)y & := xy, & (1.19a) \\
R(x)y & := yx & (1.19b)
\end{align*}

as usual. For $A^*$, we similarly set

\begin{align*}
\ell(x)y & := x * y (= \overline{y} \overline{x}), & (1.20a) \\
r(x)y & := y * x (= \overline{x} \overline{y}). & (1.20b)
\end{align*}

We then note the identity

\begin{align*}
L(x)R(y) & = r(\overline{x})r(y), & (1.21a) \\
R(x)L(y) & = \ell(\overline{x})\ell(y). & (1.21b)
\end{align*}

which will be relevant to discussions in Section 2. We now have

**Proposition 1.7.** Let $A$ be a STA with respect to $d_j$'s. We then have

\begin{align*}
[d_j, L(x)R(y)] & = L(x)R(d_{j+1}y) + L(d_{j+1}x)R(y), & (1.22a) \\
[d_j, R(x)L(y)] & = R(x)L(d_{j+2}y) + R(d_{j+2}x)L(y). & (1.22b)
\end{align*}

**Proof.** We rewrite (1.1) as

\begin{align*}
d_jL(x) & = L(x)d_{j+2} + L(d_{j+1}x), & (1.23a) \\
d_jR(y) & = R(y)d_{j+1} + R(d_{j+2}y). & (1.23b)
\end{align*}
Multiplying \( R(y) \) to Eq. (1.23a) from the right, and \( L(x) \) to Eq. (1.23b) from the left, this gives

\[
d_j L(x) R(y) = L(x) d_j R(y) + L(d_{j+1}x) R(y), \tag{1.24a}
\]
\[
L(x)d_j R(y) = L(x) R(y) d_j + L(x) R(d_{j+1}y). \tag{1.24b}
\]

We now let \( j \to j + 2 \) in Eq. (1.24b) and note \( d_{j+3} = d_j \), and \( d_{j+4} = d_{j+1} \) by Eq. (1.2). Adding it to Eq. (1.24a), this gives Eq. (1.22a). Similarly, we obtain

\[
R(y)d_j L(x) = R(y) L(x) d_j + R(y) L(d_{j+1}x),
\]
\[
d_j R(y) L(x) = R(y) d_{j+1} L(x) + R(d_{j+2}y)L(x),
\]

from Eqs. (1.23). Changing \( j \to j + 1 \) in the first relation and adding it to the second one, this yields Eq. (1.22b), when we let \( x \leftrightarrow y \).  

**Corollary 1.8.** If we set

\[
T_1(x, y) := R(y)L(x) - R(x)L(x),
\]
\[
T_2(x, y) := L(y)R(x) - L(x)R(y),
\]

we have

\[
[d_j, T_1(x, y)] = T_1(x, d_{j+2}y) + T_1(d_{j+2}x, y), \tag{1.26a}
\]
\[
[d_j, T_2(x, y)] = T_2(x, d_{j+1}y) + T_2(d_{j+1}x, y) \tag{1.26b}
\]

for \( j = 0, 1, 2 \).

**Proof.** This follows immediately from Eqs. (1.22).  

**Corollary 1.9.** Suppose that \( A \) is a flexible algebra, \((d_0, d_1, d_2) \in \text{stri}(A), \) and set

\[
U(x, y) = U(y, x) := L(x)R(y) + L(y)R(x) = R(x)L(y) + R(y)L(x).
\]

We then find

\[
[d_0, U(x, y)] = [d_1, U(x, y)] = [d_2, U(x, y)]
\]
\[
= U(d_jx, y) + U(x, d_jy) \tag{1.28}
\]

to be independent of \( j = 0, 1, 2 \).

Moreover, if we define \( T_0(x, y) \in \text{End} A \) by

\[
T_0(x, y)z := \lambda(U(x, z)y - U(y, z)x) \tag{1.29}
\]
for $\lambda \in F$, then $T_j(x, y)$ satisfies
\[ [d_j, T_k(x, y)] = T_k(d_j - kx, y) + T_k(x, d_j - ky) \] for $j, k = 0, 1, 2$.

**Proof.** We note first that the second relation in (1.27) is a consequence of the flexibility of $A$. Then, Eqs. (1.22) give
\[ [d_j, U(x, y)] = U(d_j + 1, x) + U(x, d_j + 1), \]
just as we have obtained Eqs. (1.26). Letting $j \rightarrow j + 1$ in the last relation in Eq. (2.31), this leads to the validity of Eq. (1.28). We then especially have
\[ [d_j, U(x, y)] = U(d_j, x) + U(x, d_j), \]
and hence
\[ d_j U(x, y)z - U(x, d_j y)z = U(x, y)d_j z + U(d_j x, y)z. \]
Changing $x \leftrightarrow z$ and subtracting the result from Eq. (1.33), it gives
\[ d_j T_0(x, z)y - T_0(x, z)d_j y = T_0(x, d_j z)y + T_0(d_j x, z)y, \]
which leads to
\[ [d_j, T_0(x, z)] = T_0(x, d_j z) + T_0(d_j x, z). \]
Changing $z \rightarrow y$, this together with Eqs. (1.26) yields Eq. (1.30). This completes the proof. □

Last, the following remark will be used in the next section.

**Remark 1.10.** Let $(d_0, d_1, d_2), \tilde{(d_0, d_1, d_2)} \in \text{stri}(A)$. Then, setting
\[ D_{j,k} := [d_j, \tilde{d}_k] = d_j \tilde{d}_k - \tilde{d}_k d_j, \]
it satisfies a generalized STA relation of
\[ D_{j,k}(xy) = (D_{j+1,k+1}x)y + x(D_{j+2,k+2}y). \]
Especially,
\[ \tilde{D} = \sum_{j=0}^{2} D_{i,j} = \sum_{j=0}^{2} [d_j, \tilde{d}_j] \]
is a derivation of $A$. If we identify $\tilde{d}_j$ with the one given in Eq. (1.4), it implies that
\[ \tilde{D} = \sum_{j,k=0}^{2} \alpha_{j-k}[d_j, d_k] \]
is also a derivation of $A$ for any $\alpha_j \in F$. 

\[ ]
2. Normal STA and examples

In the previous section, we considered the triple \( d_j \in \text{End } A \). However, the more interesting objects are

\[ d : A \otimes A \rightarrow \text{stri}(A) \]

\[ u \otimes v \mapsto (d_0(u, v), d_1(u, v), d_2(u, v)) \]

and may be called symmetric triality maps (abbreviated STM). Then, Eq. (1.1) is now rewritten as

\[ d_j(u, v)(xy) = (d_{j+1}(u, v)x)y + x(d_{j+2}(u, v)y). \]  \hspace{1cm} (2.1)

Then,

\[ D(u, v) := d_0(u, v) + d_1(u, v) + d_2(u, v) \]  \hspace{1cm} (2.2)

is a derivation of \( A \) as in Eq. (1.7). Moreover, if we set

\[ S_0 = S(u, v, w) := d_0(u, vw) + d_1(w, uv) + d_2(v, wu), \]  \hspace{1cm} (2.3a)

\[ S_1 = S(v, w, u) := d_1(u, vw) + d_2(w, uv) + d_0(v, wu), \]  \hspace{1cm} (2.3b)

\[ S_2 = S(w, u, v) := d_2(u, vw) + d_0(w, uv) + d_1(v, wu), \]  \hspace{1cm} (2.3c)

then we have (with \( S_{j+1} = S_j \))

\[ S_j(xy) = (S_{j+1}x)y + x(S_{j+2}y). \]  \hspace{1cm} (2.4)

so that \( A \) is also a STA with respect to the triple \( (S_0, S_1, S_2) \). However, for most of STA’s to be discussed in this note, we have \( S(u, v, w) = 0 \) identically, i.e.,

\[ d_0(u, vw) + d_1(w, uv) + d_2(v, wu) = 0. \]  \hspace{1cm} (2.5)

In that case, \( D(u, v) \) given by (2.2) satisfies

\[ D(u, vw) + D(w, uv) + D(v, wu) = 0. \]  \hspace{1cm} (2.6)

We note that any algebra \( A \) whose derivation \( D \) satisfies Eq. (2.6) has been called by Kamiya [7] to be a generalized structurable algebra, which includes Lie, Jordan, and structurable algebras.

Hereafter in this note, we shall assume

\[ d_j(u, v) = -d_j(v, u) \]  \hspace{1cm} (2.7)

unless it is otherwise stated. We now prove first the following Proposition.

**Proposition 2.1.** Let \( A \) be an algebra and let

\[ d : A \otimes A \rightarrow \text{stri}(A) \]

be a STM such that

\[ d_1(u, v) = R(v)L(u) - R(u)L(v), \]  \hspace{1cm} (2.8a)

\[ d_2(u, v) = L(v)R(u) - L(u)R(v), \]  \hspace{1cm} (2.8b)

although we do not specify yet the form of \( d_0(u, v) \).
Then, introducing a $2 \times 2$ matrix $A(x)$ by

$$A(x) = \begin{pmatrix} 0 & L(x) \\ R(x) & 0 \end{pmatrix},$$

(2.9a)

it satisfies a Lie relation of

$$[A(z), [A(x), A(y)]] = A(d_0(x, y)z).$$

(2.9b)

Moreover, if $A$ satisfies the following condition (C):

$$(C) : xA = 0 \ (or \ Ay = 0) \ for \ some \ x \in A \ (or \ y \in A) \rightarrow x = 0 \ (or \ y = 0),$$

we then have first

$$[d_j(u, v), d_k(x, y)] = d_k(x, d_{j-k}(u, v)y) + d_k(d_{j-k}(u, v)x, y)$$

$$= -d_j(u, d_{k-j}(x, y)v) - d_j(d_{k-j}(x, y)u, v)$$

(2.10)

for any $j, k = 0, 1, 2$ and second

$$d_0(x, y)z + d_0(y, z)x + d_0(z, x)y = 0$$

(2.11)

here for any $x, y, z, u, v \in A$. 

**Proof.** We note that special cases of $j = 1$ and $j = 2$ in Eq. (2.1) together with Eqs. (2.8) are rewritten as

$$\begin{align*}
(R(v)L(u) - R(u)L(v))R(y) - R(y)(L(v)R(u) - L(u)R(v)) &= R(d_0(u, v)y), \\
(L(v)R(u) - L(u)R(v))L(x) - L(x)(R(v)L(u) - R(u)L(v)) &= L(d_0(u, v)x),
\end{align*}$$

which is equivalent to the validity of Eq. (2.9b), when we change the notation of variables, suitably. If we now set

$$w := d_0(x, y)z + d_0(y, z)x + d_0(z, x)y,$$

and if we note that $A(z)$'s satisfy a Jacobi identity, Eq. (2.9b) then leads to

$$A(w) = 0$$

which gives $w = 0$ under the condition (C). This proves Eq. (2.11).

In order to show the validity of Eq. (2.10), it is sufficient to prove

$$[d_j(u, v), d_k(x, y)] = d_k(x, d_{j-k}(u, v)y) + d_k(d_{j-k}(u, v)x, y)$$

(2.12)

for any $j$ and $k$. However, comparing Eqs. (1.25) with Eqs. (2.8a) and (2.8b), we see

$$T_1(x, y) = d_1(x, y) \quad \text{and} \quad T_2(x, y) = d_2(x, y).$$

Therefore, identifying $d_j = d_j(u, v)$ in Eqs. (1.26), it proves Eq. (2.12) for any $j$ and for $k = 1$, and 2. Thus, it remains to prove Eq. (2.12) for $k = 0$. To this end, we set
\[ D_{j,k} := [d_j(u, v), d_k(x, y)] - d_k(x, d_{j-k}(u, v)y) - d_k(d_{j-k}(u, v)x, y) \]

(2.13)

for \( j, k = 0, 1, 2 \). Then, as in Remark 1.10, it satisfies

\[ D_{j,k}(zw) = (D_{j+1,k+1} + 1)w + z(D_{j+2,k+2} + 1). \]

(2.13′)

Moreover, we have already established \( D_{j,1} = D_{j,2} = 0 \). If we set \( k = 1 \) and \( k = 2 \) in Eq. (2.13′), we find then \( zD_{j,0}w = 0 \) and \( (D_{j+1,0} + 1)w = 0 \) for any \( j \) and for any \( z, w \in A \). This shows \( D_{j,0} = 0 \), provided that the condition (C) is satisfied. (If we wish, we can replace the condition (C) by another ansatz of \( AA = A \) in order to prove \( D_{j,0} = 0 \) by setting \( k = 0 \) in Eq. (2.13′).) At any rate, this completes the proof of Eq. (2.10) and of Proposition 2.1. □

Remark 2.2. For the special case of \( j = k \), Eq. (2.10) yields the same Lie algebra

\[ [d_j(u, v), d_j(x, y)] = d_j(d_0(u, v)x, y) + d_j(x, d_0(u, v)y) \]

(2.14)

for all \( j = 0, 1, 2 \), which has been already noted in [6] for generalized symmetric composition algebras.

Moreover, if \( A \) is flexible, and if we can identify \( d_0(u, v) = T_0(u, v) \) by

\[ d_0(u, v)w = \lambda [U(u, w)v - U(v, w)u] \quad (\lambda \in F) \]

(2.15)

as in Eq. (1.29), then the condition (C) is unnecessary for the validity of Eq. (2.10) in view of Eq. (1.30). We also remark that \( d_0(u, v) \) is in principle determined uniquely by Eqs. (2.8) and the triality relation Eq. (2.1), provided that the condition (C) holds valid. Suppose that we have the second solution \( d'_0(u, v) \). Then, \( A \) will be also a STA with respect to a triple \( d'_j(u, v) \)'s such that \( d'_0(u, v) = d'_0(u, v) - d_0(u, v) \) but \( d'_1(u, v) = d'_2(u, v) = 0 \). Then, utilizing the same argument used in the proof of Eq. (2.10) in Proposition 2.1, we must have \( d_0(u, v) = 0 \) i.e., \( d'_0(u, v) = d_0(u, v) \).

Partly due to Proposition 2.1 but mostly in view of the result of Section 3 for a construction of a larger Lie algebra (see Theorem 3.1), we introduce now the notion of a normal STA. We call \( A \) to be a normal symmetric triality algebra with respect to the symmetric triality map \( d \) satisfying Eq. (2.1) if we have the following additional conditions:

1. \( d_1(u, v) = R(v)L(u) - R(u)L(v), \)
2. \( d_2(u, v) = L(v)R(u) - L(u)R(v), \)
3. \( d_0(u, v)w + d_0(v, w)u + d_0(w, u)v = 0, \)
4. \( d_0(u, vw) + d_1(w, uv) + d_2(v, wu) = 0, \)
5. The validity of the Lie equation, Eq. (2.10).
Note that we do not specify the explicit form of $d_0(u, v)$. Moreover, the conditions, (3) and (5), i.e. Eqs. (2.15c and e) will be unnecessary by Proposition 2.1, if the condition (C) is satisfied. Also see Remark 2.2.

As we shall see shortly, many of known STA’s are normal. Also, in most cases, $d_0(u, v)$ has a particular form given by

$$d_0(u, v)w = A(u, w)v - A(v, w)u \tag{2.16}$$

for some $A(u, w) \in \text{End} A$. Moreover, if it satisfies the symmetry condition

$$A(u, w) = A(w, u) \tag{2.17}$$

just as $U(x, y)$, then the condition, Eq. (2.15c) is identically satisfied. We shall now give some examples of normal STA’s.

**Example 2.3 (Lie and Jordan algebras).** Both Jordan and Lie algebras are normal STA’s with respect to the triple, given by

$$d_0(u, v) = d_1(u, v) = d_2(u, v) = \epsilon [L(u), L(v)] \tag{2.18}$$

for $\epsilon = +1$ for Lie and $\epsilon = -1$ for Jordan with $uv = -\epsilon vu$ and $L(u) = -\epsilon R(u)$. Note that

$$d(u, v) := \epsilon [L(u), L(v)]$$

is a inner derivation of both Jordan and Lie. This is obvious for the case of Jordan, while we have

$$d(u, v) = L(uv)$$

for the Lie in view of the Jacobi identity

$$(uv)w + (vw)u + (wu)v = 0.$$ 

The validity of Eq. (2.15d) or Eq. (2.6) is well known for these algebras (see e.g., [7]).

Then, since $d(u, v)$ is a inner derivation for both cases, it satisfies

$$[d(u, v), d(x, y)] = d(d(u, v)x, y) + d(x, d(u, v)y), \tag{2.19}$$

which guarantees the validity of the condition (5) in Eq. (2.15e). Moreover, Eq. (2.15c) is nothing but the Jacobi identity for the case of Lie, while it is trivially satisfied for Jordan in view of the commutativity law. We also note that we can express $d_0(u, v)$ as in Eq. (2.16) with

$$A(u, w)v = -u(wv) - w(\epsilon uv)$$

for the Jordan and

$$A(u, w)v = -u(wv)$$

for Lie. Especially, Eq. (2.17) is satisfied for the case of $A$ being Jordan.
Example 2.4 (Generalized symmetric composition algebras). Let $A$ be a flexible algebra over the field $F$ of characteristic not 2, and introduce $U(x, y)$ by Eq. (1.27). Suppose that it satisfies
\begin{align}
(i) & \quad U(xy, z) = U(x, yz), \\
(ii) & \quad U(u, v)(xy) = (U(u, v)x)y = x(U(u, v)y).
\end{align}
Then the resulting algebra $A$ is called a generalized symmetric composition algebra.

It has been proven in [6] that it is a STA with respect to $d_j$'s, where $d_1(u, v)$ and $d_2(u, v)$ are given by Eqs. (2.15a) and (2.15b), respectively with \[ d_0(u, v)w = 2[U(u, w)v - U(v, w)u]. \] (2.21)
However, our definition of $d_j$'s differs from those of [6] by a factor of $-2$. We shall show here that $A$ is also normal. First, we note that Eq. (2.21) corresponds to Eq. (2.15) with $\lambda = 2$. Then, it satisfies the condition (5) in Eqs. (2.15e) in view of Remark 2.2. The validity of Eq. (2.15c) follows immediately from Eq. (2.21) when we note $U(x, y) = U(y, x)$. Therefore, it suffices to prove Eq. (2.15d). However, its proof will be given in Section 3.

Suppose that $U(x, y)$ has a special form of
\[ U(x, y) = 2\langle x|y\rangle Id, \] (2.22)
where $\langle .|\rangle$ is a bilinear symmetric associative non-degenerate form and where $Id$ stands for the identity map in $\text{End} A$. Then, $A$ is called a symmetric composition algebra since we can prove the validity of the composition law
\[ \langle xy|xy\rangle = \langle x|x\rangle \langle y|y\rangle. \]
It is known (see [3,8,12]) that any symmetric composition algebra is either a para-Hurwitz algebra or a 8-dimensional pseudo-octonion algebra. Here, the para-Hurwitz algebra is the conjugate of a Hurwitz algebra $A^*$ (i.e., unital composition algebra, see [13]) with the bilinear product $x * y$ by setting
\[ xy = \overline{x \overline{y}} = \overline{x} * \overline{y}. \] (2.23)
Note that if $e$ is the unit of the Hurwitz algebra $A^*$, then it is a para-unit of the para-Hurwitz algebra $A$ satisfying
\[ ex = xe = \overline{x}. \] (2.24)

Example 2.5. Let $A$ and $A'$ be two generalized symmetric composition algebras as in Example 2.4 with $U(u, v), (u, v \in A)$ and $U'(u', v'), (u', v' \in A')$ to satisfy conditions Eqs. (2.20). Then, if we set
\[ D_j(u \otimes u', v \otimes v') := \frac{1}{2}[d_j(u, v) \otimes U'(u', v') + U(u, v) \otimes d'_j(u', v')], \] (2.25)
the tensor product algebra $A \otimes A'$ is also a normal STA with respect to $D_j$'s as we shall demonstrate below. First, it is a STA by Proposition 1.1. To show it to be normal, we shall first prove that we can rewrite Eq. (2.25) as

$$\begin{align*}
D_1(u \otimes u', v \otimes v') &= R(v \otimes v')L(u \otimes u') - R(u \otimes u')L(v \otimes v'), \\
D_2(u \otimes u', v \otimes v') &= L(v \otimes v')R(u \otimes u') - L(u \otimes u')R(v \otimes v'), \\
D_0(u \otimes u', v \otimes v')(x \otimes x') &= A(u \otimes u', x \otimes x')(v \otimes v') \\
&\quad - A(v \otimes v', x \otimes x')(u \otimes u')
\end{align*}$$

with

$$\begin{align*}
A(u \otimes u', x \otimes x')(v \otimes v') &= U(u, x)v \otimes \{U(u', v')x + U'(v, v')u\} \\
&\quad + \{U(u, v)x + U(x, v)u\} \otimes U'(u', x')v'.
\end{align*}$$

To prove these, we write

$$R(v \otimes v')L(u \otimes u') = R(v)R(u) \otimes R(v')R(u')$$

and calculate

$$R(v)R(u) = \frac{1}{2}[R(v)L(u) - R(u)L(v)] + \frac{1}{2}[R(v)L(u) + R(u)L(v)]$$

so that we have

$$R(v \otimes v')L(u \otimes u') - R(u \otimes u')L(v \otimes v')$$

$$= \frac{1}{4}[(d_1(u, v) + U(u, v)) \otimes (d'_1(u', v') + U'(u', v'))$$

$$- (d_1(v, u) + U(v, u)) \otimes (d'_1(v', u') + U'(v', u'))]$$

$$= \frac{1}{2}[(d_1(u, v) \otimes U'(u', v') + U(u, v) \otimes d'_1(u', v'))]$$

$$= D_1(u \otimes u', v \otimes v').$$

This proves Eq. (2.26a). Similarly when we note

$$L(v)R(u) = \frac{1}{2}[d_2(u, v) + U(u, v)],$$

we find the validity of Eq. (2.26b) in a similar way. Finally, we calculate

$$A(u \otimes u', x \otimes x')(v \otimes v') - A(v \otimes v', x \otimes x')(u \otimes u')$$

$$= U(u, x)v \otimes \{U'(u', v')x + U'(v', v')u\}$$
\[ + \{ U(u, v)x + U(x, v)u \} \otimes U'(u', x')v' \\
- U(v, x)u \otimes \{ U'(v', u')x' + U'(x', u')v' \} \\
- \{ U(v, u)x + U(x, u)v \} \otimes U'(v', x')u' \\
= \{ U(u, x)v - U(v, x)u \} \otimes U'(u', v')x' \\
+ \frac{1}{2} d_{0}(u, v)x \otimes U'(u', v')x' + \frac{1}{2} U(u, v)x \otimes d'_{0}(u', v')x' \\
= D_{0}(u \otimes u', v \otimes v')(x \otimes x'), \]

proving Eq. (2.27a).

Since \( A \) given by Eq. (2.27b) satisfies the symmetric condition
\[ A(u \otimes u', x \otimes x') = A(x \otimes x', u \otimes u'), \]

it satisfies the condition (2.15c), i.e.,
\[ D_{0}(U, V)W + D_{0}(V, W)U + D_{0}(W, U)V = 0 \] (2.28)
for \( U = u \otimes u', V = v \otimes v' \), and \( W = w \otimes w' \). We next note
\[ D_{j}(u \otimes u', (v \otimes v') \cdot (w \otimes w')) \\
= D_{j}(u \otimes u', (vw) \otimes (v'w')) \\
= \frac{1}{2} \{ d_{j}(u, v), U(x, y) \} \otimes U'(u', v')U'(x', y') + \frac{1}{2} U(u, v)x \otimes d'_{j}(u', v')x' \]

Then, this gives the validity of Eq. (2.15d), i.e.,
\[ D_{0}(u \otimes u', (v \otimes v') \cdot (w \otimes w')) \\
+ D_{1}(w \otimes w', (u \otimes u') \cdot (v \otimes v')) \\
+ D_{2}(v \otimes v', (w \otimes w') \cdot (u \otimes u')) = 0, \] (2.29)

when we note Eq. (2.20a). Finally, in order to prove the validity of Eq. (2.15e), we first note (see [6])
\[ [d_{j}(u, v), U(x, y)] = [U(u, v), U(x, y)] = 0 \] (2.30)
so that we calculate
\[ [D_{j}(u \otimes u', v \otimes v'), D_{k}(x \otimes x', y \otimes y')] \\
= \frac{1}{4} \{ [d_{j}(u, v), d_{k}(x, y)] \otimes U'(u', v')U'(x', y') + U(u, v)U(x, y) \otimes [d'_{j}(u', v'), d'_{k}(x', y')] \}. \] (2.31)

On the other sides, we also note the validity (see [6]) of the following relations:
\[ d_{j}(U(u, v)x, y) = d_{j}(x, U(u, v)y), \] (2.32a)
\[ U(x, dj(u, v)y) = -U(dj(u, v)x, y), \quad (2.32b) \]

\[ U(x, U(u, v)y) = U(U(u, v)x, y) = U(x, y)U(u, v), \quad (2.32c) \]

e tc. In this way, we can verify the validity of Eq. (2.15e) i.e.,

\[ [D_j(U, V), D_k(X, Y)] = D_k(D_j(U, V)X, Y) + D_k(X, D_j(U, V)Y) \]

for \( U = u \otimes u', V = v \otimes v', X = x \otimes x', \) and \( Y = y \otimes y', \) although we will not go into detail. This completes the proof that \( A \otimes A' \) is normal. This example is relevant for the construction of the magic square by Elduque [4], as we will see in Section 3. Also, we note that \( A \otimes A' \) is not in general flexible except for the case of both \( A \) and \( A' \) being one or two dimensional.

Some other examples of normal STA will also be given in Section 4. However, major source of obtaining normal STA comes from structurable algebras (see Allison–Faulkner [2]). For this purpose, we first define the notion of normal LRTA for the conjugate algebra \( A^* \) of \( A, \) assuming \( A \) to be involutive. In Section 1, we have already seen that if \( A \) is a involutive STA, and \( A^* \) denotes its conjugate, then \( stri(A) = lrt(A^*, -). \) We now define a normal LRTA to satisfy the following conditions. Let \( A^* \) be an involutive algebra and let \( d : A^* \otimes A^* \to lrt(A^*, -) \) be a linear map, so that

\[ dj(u, v)(x \ast y) = (dj+1(u, v)x) \ast y + x \ast (dj+2(u, v)y), \quad (2.33) \]

for any \( u, v, x, y \in A^* \) and \( j = 0, 1, 2. \) These will be called Lie related triple maps \( (or LRTM). \) Then, \( A^* \) is said to be a normal LRTA if the following conditions are satisfied:

1. \( d_1(u, v) = \ell(\overline{\nu})\ell(u) \ast \ell(\overline{\nu})\ell(v), \quad (2.34a) \)
2. \( d_2(u, v) = r(\overline{\nu})r(u) \ast r(\overline{\nu})r(v), \quad (2.34b) \)
3. \( d_0(u, v)w + d_0(v, w)u + d_0(w, u)v = 0, \quad (2.34c) \)
4. \( d_0(\overline{\nu}, w \ast v) + d_1(\overline{\nu}, v \ast u) + d_2(\overline{\nu}, u \ast w) = 0, \quad (2.34d) \)
5. \( the \ validity \ of \ the \ Lie \ equation, \ Eq. (2.10)), \quad (2.34e) \)
6. \( d_j(u, v) = d_{3-j}(\overline{\nu}, \overline{\nu}). \quad (2.34f) \)

We note that Eqs. (2.34a)–(2.34f) are simple rewritings of the corresponding conditions Eqs. (2.15a)–(2.15e). Here, \( \ell(x) \) and \( r(x) \) are defined by Eqs. (1.20) and we noted the validity of Eqs. (1.21) to rewrite Eqs. (2.15a and b) into Eqs. (2.34a and b). Similarly, to obtain Eq. (2.34d) from Eq. (2.15d), we used \( uv = u \ast v = \overline{\nu} \ast \overline{\nu}, \) and changed \( u, v, w \) there into \( \overline{\nu}, \overline{\nu}, \overline{\nu}, \) respectively. However, a new addition is the imposition of the extra condition Eq. (2.34f) by the following reason. First, we know that \( A \) is also a STA with respect to \( d_j(u, v) = d_{3-j}(\overline{\nu}, \overline{\nu}) \) by Eq. (1.13). Moreover, Eqs. (2.34f) holds automatically valid for \( j = 1 \) and 2 because of Eqs. (2.34a) and (2.34b). Then, if the condition (C) is valid, we must have \( d_j(u, v) = d_j(\overline{\nu}, \overline{\nu}) \) by the
same reasoning as in the proof of Proposition 2.1. This is the reason why we assume Eq. (2.34f). It is really a single condition only for \( j = 0 \):

\[
d_0(u, v) = d_0(\pi, \nu).
\]  

(2.35)

If \( A^* \) is unital as in the structurable algebras, the condition (C) is automatically satisfied, so that we can simply omit three conditions of Eqs. (2.34c, e, and f) from the definition of the normal LRTA, since they are derivable from other postulates.

The major source for a normal LRTA is the structurable algebra and we will briefly sketch its definition (see [2]). Let \( A^* \) be an involutive unital algebra with the unit element \( e \). Define a multiplication operator \( L(x, y) \) in \( \text{End} A^* \) by

\[
L(x, y)z := (z * y) * x - (z * x) * y + (x * y) * z := xyz
\]

and suppose that it satisfies a commutation relation

\[
[L(x, e), L(y, z)] = L(xey, z) - L(y, exz),
\]

then \( A^* \) is called a structurable algebra [2], provided that the underlying field \( F \) is of characteristic neither 2 nor 3. For the other case of \( F \) being of characteristic 2 or 3, we need one more condition (see [2]) which will not be given here. In what follows, we assume that \( d_0(u, v) \) as in [2] has the following specific form:

\[
d_0(u, v) = r(u * v - v * u) + \ell(v)\ell(u) - \ell(u)\ell(v) = \ell(v^* u - u^* v) + r(v)r(u) - r(u)r(v).
\]  

(2.36)

Note that the 2nd relation in Eq. (2.36) has been imposed in order to satisfy Eq. (2.35). If \( A^* \) is unital, Eq. (2.36) follows automatically from Eqs. (2.34a), (2.34b), and (2.34d) as in the proof of Proposition 4.1 in Section 4 by setting \( w = e \) or \( v = e \) in Eq. (2.34d). We can rewrite the second relation of Eq. (2.36) as

\[
[w, \pi, \nu] = [w, \pi, u]^* = [u, \pi, \nu]^* - [v, \pi, w]^*,
\]  

(2.37)

where we have set

\[
[x, y, z]^* = (x * y) * z - x * (y * z)
\]  

(2.38)

for the associator of \( A^* \). Eq. (2.37) is automatically satisfied for any structurable algebra (see Eq. (A.1) on p. 6 of [2]), as it should be so.

Under these preparations, we now prove the following theorem.

**Theorem 2.6.** Let \( A^* \) be a unital involutive algebra. A necessary and sufficient condition that \( A^* \) is a normal LRTA is that \( A^* \) is structurable.

**Proof.** Suppose that \( A^* \) is structurable. Then, it is known from the result of [2] that \( A^* \) is a LRTA with respect to \( d_j \)'s given by Eqs. (2.34a and b) and (2.36). Since \( A^* \) is unital, conditions (3), (5), and (6) in Eqs. (2.34) are automatically satisfied. Moreover, if we rewrite Eq. (2.36) as

\[
d_0(u, v)w = \Lambda(u, w)v - \Lambda(v, w)u,
\]

...
with
\[ A(u, w)v := -w \ast (v \ast u) - u \ast (v \ast w) \]
and note \( A(u, w) = A(w, u) \), this also guarantees the validity of Eq. (2.34c). Therefore, we need only verify the validity of Eq. (2.34d), i.e.,
\[ d_0(\overline{w}, w \ast v)z + d_1(\overline{v}, v \ast u)z + d_2(\overline{u}, u \ast w)z = 0 \quad (2.39) \]
which is rewritten explicitly as
\[ z \ast \{u \ast (w \ast v) - (v \ast w) \ast u\} + (w \ast v) \ast (u \ast z) - u \ast \{(v \ast u) \ast z\} + (z \ast w) \ast (w \ast \overline{v}) - \{z \ast (u \ast w)\} \ast v = 0. \quad (2.40) \]
If we identify
\[ a = \overline{w}, \quad b = \overline{u}, \quad c = \overline{v}, \quad d = z, \]
then Eq. (2.40) can be readily seen to be equivalent to Eq. (X) of [2–p. 7], with \( a \ast b := ab \), which holds for any structurable algebra. This proves \( A^* \) to be normal.

Conversely suppose now that \( A^* \) is a unital normal LRTA and hence with \( d_0(u, v) \) given by Eq. (2.36). Then, this implies first the validity of Eq. (A) of [2] from Theorem 3.7 given there. Moreover, Eq. (2.39) leads to Eq. (X) as we have already observed. Combining these facts, \( A^* \) is structurable by Theorem 5.5 of [2–p. 13]. This completes the proof of the present Theorem. □

We shall now give a few examples of normal STA derivable from LRTA. We shall assume hereafter that \( d_0(u, v) \) is given as in Eq. (2.36) which may be rewritten as
\[ d_0(u, v)w = (\overline{w}v - uv)\overline{w} + (\overline{u}w)v - (\overline{v}w)\overline{u} = \overline{w}(v\overline{u} - u\overline{v}) + \overline{u}(w\overline{v}) - \overline{v}(w\overline{u}) \quad (2.41) \]
in \( A \).

**Example 2.7.** Any involutive associative algebra \( A^* \) is a normal LRTA by the following reason. If \( A^* \) is unital in addition, this is obvious since then any unital involutive associative algebra is known to be structurable (see [2]). However if \( A^* \) is not unital, we extend the algebra into a larger one \( B^* = A^* \oplus Fe \) by adding a formal unit element \( e \), making \( B^* \) to be a unital involutive associative algebra. Since \( A^* \) is a subalgebra of \( B^* \), this proves \( A^* \) to be a normal LRTA. This fact can also be verified by direct computations. Then, the conjugate algebra \( A \) of \( A^* \) is a normal STA, although it is not associative. Note that the associative law \((x \ast y) \ast z = x \ast (y \ast z)\) in \( A^* \) is translated in \( A \) as the para-associative law of
\[ \overline{z}(xy) = (yz)\overline{x} \quad (2.42) \]
by $xy = \bar{x} \bar{y} = \bar{y} \bar{x}$. We call $A$ to be a para-associative algebra. In this case, in view of Eq. (2.42), we can rewrite Eq. (2.41) as a relation
\begin{equation}
\tilde{d}_0(u, v)w = [\tilde{u}, w, \tilde{v}] - [\tilde{v}, w, \tilde{u}],
\end{equation}
where
\begin{equation}
[\tilde{u}, w, \tilde{v}] := (\tilde{u}w)v - u(\tilde{v}w)
\end{equation}
is the associator of $A$. Also, if $A^*$ has the unit element $e$, then $e$ is the para-unit of $A$, i.e., it satisfies $e x = xe = x$ for any $x \in A$.

This case can be further generalized as follows (see Example 6.5 of [2]). Let $B^*$ be a unital involutive associative algebra with product $x \ast y$, and let $W$ be a left $B^*$-module. Suppose that $k : W \times W \rightarrow B^*$ satisfies
\begin{enumerate}
\item $k(a, b) = k(b, a),$
\item $x \ast k(a, b) = k(xa, b),$
\item $k(a, b) \ast \bar{x} = k(a, xb)$
\end{enumerate}
for $a, b \in W$ and $x \in B^*$. Then, $A^* = B^* \oplus W$ with product and involution given by
\begin{equation}
(x \oplus a) \ast (y \oplus b) = (x \ast y + k(b, a)) \oplus (\bar{x}b + ya),
\end{equation}
\begin{equation}
\bar{x} \oplus a = \bar{x} \oplus a
\end{equation}
is structurable. Note that the unit element $E$ of $A^*$ is given by $E = e \oplus 0$ for the unit $e$ of $B^*$.

**Example 2.8.** Let us now assume that the underlying field $F$ is of characteristic neither 2 nor 3. We can then generalize Example 6.4 of [2] as follows. Suppose that $B$ is a cubic-admissible algebra [5], i.e., it is a commutative algebra with product $xy = yx$, satisfying
\begin{equation}
x^2x^2 = N(x)x
\end{equation}
for some cubic norm $N(x)$. Assuming $N \neq 0$, it has been shown in [5] first that there exists a unique bilinear symmetric associative form $\langle \cdot, \cdot \rangle$ in $B$ such that
\begin{equation}
N(x) = \langle x | x^2 \rangle.
\end{equation}
Secondly we have also an additional identity of
\begin{equation}
4x(x^2y) = 3(x|y)x^2 + (x|x^2)y
\end{equation}
for any $x, y \in B$. These facts are sufficient to generalize Example 6.4 of [2]. Let us now consider a vector space $A^*$ consisting of all linear combination of Zorn’s vector matrix of form
for $\alpha, \beta \in F$ and $x, y \in B$. We introduce a product in this space by
\[
X_1 \ast X_2 = \left( \begin{array}{cc} \alpha_1 & x_1 \\ y_1 & \beta_1 \end{array} \right) \ast \left( \begin{array}{cc} \alpha_2 & x_2 \\ y_2 & \beta_2 \end{array} \right)
\]
for $\alpha_j, \beta_j \in F$ and $x_j, y_j \in B$. Then, the resulting algebra $A^\ast$ is structurable with
\[
E := \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)
\]
and the involution given by
\[
\begin{pmatrix} \alpha \\ x \\ y \\ \beta \end{pmatrix} := \begin{pmatrix} \beta \\ x \\ y \\ \alpha \end{pmatrix}.
\]

The algebra $A^\ast$ is neither commutative nor flexible in general. Moreover, if we introduce a symmetric bilinear form in $A^\ast$ by
\[
\langle X_1 | X_2 \rangle := \text{Tr}(X_1 \ast X_2) = \alpha_1 \beta_2 + \beta_1 \alpha_2 + 3(\langle x_1 | y_2 \rangle + \langle x_2 | y_1 \rangle),
\]
it satisfies
\[
\langle X_3 | X_1 \ast X_2 \rangle = \langle X_1 | X_2 \rangle = \langle X_2 | X_1 \rangle, \quad \langle X_3 | X_1 \ast X_2 \rangle = \langle X_1 | X_2 \ast X_3 \rangle = \langle X_3 | X_1 \ast X_2 \rangle.
\]

Further, $\langle X | Y \rangle$ is non-degenerate, if $\langle x | y \rangle$ is non-degenerate in $B$. From $A^\ast$, we can construct a normal STA by
\[
X_1 X_2 := X_2 \ast X_1 = \left( \begin{array}{cc} \beta_1 \beta_2 + 3 \langle x_2 | y_1 \rangle & \alpha_1 x_2 + \beta_2 x_1 \pm 2 y_1 y_2 \\ \alpha_2 y_1 + \beta_1 y_2 \pm 2 x_1 x_2 & \alpha_1 \alpha_2 + 3 \langle x_1 | y_2 \rangle \end{array} \right),
\]
which gives the associative law
\[
\langle X_3 | X_1 \ast X_2 \rangle = \langle X_1 | X_2 X_3 \rangle = \langle X_2 | X_3 X_1 \rangle.
\]

Any cubic-admissible algebra $B$ has been classified in [5] when $\langle . | . \rangle$ is non-degenerate. See also Example 4.6 with Remark 4.7 in Section 4.

Finally, we simply note that a sub-algebra $A^\ast_0$ of $A^\ast$ consisting of all $X$ restricted to $\alpha = \beta$ and $x = y$ in Eq. (2.44) is isomorphic to the quartic Jordan algebra associated with the cubic-admissible algebra $B$ as in [5].
Example 2.9. Let $B$ be a commutative Jordan–Lie algebra [11], i.e., the algebra satisfying
\[ x^2x = xx^2 = 0, \]
assuming $F$ to be of characteristic not 2.

Instead of Eq. (2.45) we now impose
\[
\begin{pmatrix}
\alpha_1, x_1 \\
y_1, \beta_1
\end{pmatrix} \ast \begin{pmatrix}
\alpha_2, x_2 \\
y_2, \beta_2
\end{pmatrix} = \begin{pmatrix}
\alpha_1\alpha_2, \\
\alpha_1x_2 + \beta_2x_1 \pm 2y_1y_2
\end{pmatrix}.
\]

(2.45\textsuperscript{′})

The resulting algebra $A^\ast$ with Eqs. (2.46) is now a structurable algebra. Note that $x^3 = 0$ also implies the validity of $x^2(xy) = (x^2y)x = 0$. However, we will not go into detail.

Example 2.10. Any unital involutive alternative algebra is structurable [2]. So are any Hurwitz algebra and its tensor products. However, these last cases simply reproduce the normal STA’s of Example 2.4 and 2.5 when $A$ is a para-Hurwitz algebra. Similarly, the case of $A^\ast$ being a Jordan is equivalent to $A$ being a unital Jordan in Example 2.3.

In concluding this section, let $A^\ast$ again be a structurable algebra and set
\[
D(u, v) := d_0(u, v) + d_1(u, v) + d_2(u, v).
\]

(2.51)

Then, Eqs. (2.34f) and (2.34d) imply
\[
\begin{align*}
\overline{D}(u, v) &= D(\overline{u}, \overline{v}), \\
D(\overline{u}, w \ast v) + D(\overline{v}, v \ast u) + D(\overline{u}, u \ast w) &= 0.
\end{align*}
\]

Moreover, from Eqs. (2.34a), (2.34b), and (2.36), it is not difficult to calculate
\[
D_A(u, v) := D(u, v) - \overline{D}(u, v) = D(u, v) - D(\overline{u}, \overline{v})
\]

to satisfy
\[
D_A(u, v)w = [w, u, \overline{v}]^* - [w, v, \overline{u}]^* + [w, \overline{v}, u] - [w, \overline{u}, v]^* = 0,
\]

(2.52)

which is identically zero by Eq. (sk1) of [2]. Therefore, we have $D_A(u, v) = 0$ and hence there is no anti-derivation $D_A$ in this case. At any rate, this implies
\[
\overline{D}(u, v) = D(\overline{u}, \overline{v}) = D(u, v)
\]

for any structurable algebra. Further we calculate
\[ D_S(u, v)w := \{D(u, v) + \overline{D(u, v)}\}w \]
\[ = 2[[\overline{v}, u]^* + [v, \overline{u}]^*, w]^* + 3[[w, u, \overline{v}]^* - [w, \overline{v}, u]^* + [w, \overline{v}, v]^* - [w, v, \overline{u}]^*] \]
\[ = 2[[\overline{v}, u]^* + [v, \overline{u}]^*, w]^* + 6[[w, u, v]^* - [w, v, u]^*], \quad (2.53) \]

where we have used Eq. (2.52) with
\[ [x, y]^* := x \ast y - y \ast x. \]

Therefore, if the underlying field \( F \) is not of characteristic 3 nor 2, then
\[ D_0(u, v) := -\frac{1}{6} D_S(u, v), \]

given by
\[ D_0(u, v)w = \frac{1}{3} [[u, v]^* + [\overline{u}, \overline{v}]^*, w]^* + [w, v, u]^* - [w, \overline{v}, \overline{u}]^* \]
\[ (2.54) \]
is a derivation of \( A^* \) and satisfies
\[ D_0(u, v \ast w) + D_0(v, w \ast u) + D_0(w, u \ast v) = 0, \quad (2.55) \]
reproducing the results of [1] (see also [7]). On the other side, if \( F \) is of characteristic 3, then Eq. (2.53) gives
\[ D_S(u, v)w = [[u, v]^* + [\overline{u}, \overline{v}]^*, w]^* \]
to yield a derivation of \( A^* \).

3. Construction of Lie algebras

The major reason for introducing normal STA is that it enables us to construct a larger Lie algebra as follows. Our method is basically a generalization of the construction given in [2]. As in [2], we introduce 3 copies of an algebra \( A \) which we denote by \( \rho_j(A) (j = 0, 1, 2) \) instead of \( A([1, 2]), A([2, 3]), \) and \( A([3, 1]) \). Moreover, we use the symbol \( T_j(u, v) \) satisfying
\[ T_j(u, v) = -T_j(v, u) = T_{j+3}(u, v) \]

instead of the triple \( T(d_j(u, v), d_{j+1}(u, v), d_{j+2}(u, v)) \) and consider
\[ L = \rho_0(A) \oplus \rho_1(A) \oplus \rho_2(A) \oplus T, \quad (3.1) \]
where \( T \) is a vector space spanned by \( T_j(u, v) \)’s (or \( T(d_j(u, v), d_{j+1}(u, v), d_{j+2} (u, v)) \) if we wish). Also, as in [2], we shall assume hereafter that the symbol \( (i, j, k) \) refer to any cyclic permutation of indices \((0,1,2)\) unless it is stated otherwise. We then note
\[ j - i = 1(\text{mod } 3), \]
\[ k - i = 2(\text{mod } 3), \]
so that we have
\[ d_{j-i}(u, v) = d_1(u, v), \quad (3.2a) \]
\[ d_{k-i}(u, v) = d_2(u, v), \quad (3.2b) \]
which will be useful in what follows.

Assuming \( \rho_j(x) \) to be \( F \)-linear in \( x \), we now impose the following commutation relations in \( L \):

1. \( [\rho_i(x), \rho_i(y)] := \gamma_j \gamma_k^{-1} T_{3-i}(x, y), \quad (3.3a) \)
2. \( [\rho_i(x), \rho_j(y)] := -[\rho_j(y), \rho_i(x)] := -\gamma_j \gamma_k^{-1} \rho_i(x), \quad (3.3b) \)
3. \( [T_\ell(u, v), \rho_j(x)] := -[\rho_j(x), T_\ell(u, v)] := \rho_j(d_{\ell+j}(u, v)x), \quad (3.3c) \)
4. \( [T_\ell(u, v), T_m(x, y)] := T_m(x, d_{\ell-m}(u, v)y) + T_m(d_{\ell-m}(u, v)x, y) \\
\quad := -T_\ell(u, d_{m-\ell}(x, y)v) - T_\ell(d_{m-\ell}(x, y)u, v). \quad (3.3d) \)

5. \( T_i(x, yz) + T_j(z, xy) + T_k(y, zx) = 0. \quad (3.3e) \)

Here, \( \gamma_j \in F \) are some non-zero constants, while \( \ell \) and \( m \) stand for any integers 0, 1, 2. We note that if we identify \( T_j(x, y) \) with \( d_j(x, y) \) itself or by

\[ T_j(x, y) = T(d_j(x, y), d_{j+1}(x, y), d_{j+2}(x, y)) \quad (3.4) \]
as in \([2,4]\), then Eqs. (3.3a) and (3.3e) are automatically satisfied by Eqs. (2.10) and (2.15d), respectively. However, we will not do so here in order to maintain a generality. We notice then that the consistency between Eqs. (3.3c) and (3.3e) requires the validity of Eq. (2.15d). We now prove the following theorem.

**Theorem 3.1.** Let \( A \) be a normal STA. Then, \( L \) is a Lie algebra.

**Proof.** We introduce the Jacobian
\[ J(X, Y, Z) = [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] \]
and demonstrate \( J(X, Y, Z) = 0 \) identically for any \( X, Y, Z \in L \), if \( A \) is a normal STA.

1. We calculate
\[ [[\rho_i(x), \rho_i(y)], \rho_j(z)] = [\gamma_j \gamma_k^{-1} T_{3-i}(x, y), \rho_j(z)] \]
\[ = \gamma_j \gamma_k^{-1} \rho_j(d_3(x, y)z) = \gamma_j \gamma_k^{-1} \rho_j(d_0(x, y)z). \]

Therefore, we find
\[ J(\rho_i(x), \rho_j(y), \rho_j(z)) = \gamma_j \gamma_k^{-1} \rho_i(w), \]

Together with a constraint relation of

\[ T_i(x, yz) + T_j(z, xy) + T_k(y, zx) = 0. \]

as in \([2,4]\), then Eqs. (3.3a) and (3.3e) are automatically satisfied by Eqs. (2.10) and (2.15d), respectively. However, we will not do so here in order to maintain a generality. We notice then that the consistency between Eqs. (3.3c) and (3.3e) requires the validity of Eq. (2.15d). We now prove the following theorem.

**Theorem 3.1.** Let \( A \) be a normal STA. Then, \( L \) is a Lie algebra.

**Proof.** We introduce the Jacobian
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1. We calculate
\[ [[\rho_i(x), \rho_i(y)], \rho_j(z)] = [\gamma_j \gamma_k^{-1} T_{3-i}(x, y), \rho_j(z)] \]
\[ = \gamma_j \gamma_k^{-1} \rho_j(d_3(x, y)z) = \gamma_j \gamma_k^{-1} \rho_j(d_0(x, y)z). \]

Therefore, we find
\[ J(\rho_i(x), \rho_j(y), \rho_j(z)) = \gamma_j \gamma_k^{-1} \rho_i(w), \]
where
\[ w = d_0(x, y)z + d_0(y, z)x + d_0(z, x)y = 0 \]
because of Eq. (2.15c). Hence, \( J(\rho_i(x), \rho_i(y), \rho_i(z)) = 0 \).

(2) We similarly note
\[
[[\rho_i(x), \rho_i(y)], \rho_j(z)] = [\gamma_j \gamma_k^{-1} T_{3-i}(x, y), \rho_j(z)] \]
\[ = \gamma_j \gamma_k^{-1} \rho_j(d_{3-i+j}(x, y)z) = \gamma_j \gamma_k^{-1} \rho_j(d_1(x, y)z), \]
where we used Eq. (3.2a). Also, we calculate
\[
[[\rho_i(x), \rho_i(y)], \rho_i(z)] = [-\gamma_i \gamma_k^{-1} \rho_k(yz), \rho_i(x)] \]
\[ = (-\gamma_i \gamma_k^{-1})(-\gamma_j \gamma_k^{-1})\rho_j((yz)x) = \gamma_j \gamma_k^{-1} \rho_j((yz)x) \]
so that we find
\[ J(\rho_i(x), \rho_i(y), \rho_j(z)) = \gamma_j \gamma_k^{-1} \rho_j(w), \]
\[ w = d_1(x, y)z + (yz)x - (xz)y \]
\[ = [d_1(x, y) + R(x)L(y) - R(y)L(x)]z = 0 \]
by Eq. (2.15a) i.e., \( J(\rho_i(x), \rho_i(y), \rho_j(z)) = 0 \).

(3) We similarly note
\[
[[\rho_i(x), \rho_i(y)], \rho_k(z)] = [\gamma_j \gamma_k^{-1} T_{3-i}(x, y), \rho_k(z)] \]
\[ = \gamma_j \gamma_k^{-1} \rho_k(d_{3-i+k}(x, y)z) = \gamma_j \gamma_k^{-1} \rho_k(d_2(x, y)z), \]
by Eq. (3.2b), while
\[
[[\rho_i(y), \rho_k(z)], \rho_i(x)] \]
\[ = [-\gamma_i \gamma_k^{-1} \rho_k(yz), \rho_i(x)] \]
\[ = (-\gamma_i \gamma_k^{-1})(-\gamma_j \gamma_k^{-1})\rho_j((yz)x) = \gamma_j \gamma_k^{-1} \rho_j((yz)x) \]
In this way, we obtain
\[ J(\rho_i(x), \rho_i(y), \rho_k(z)) = \gamma_j \gamma_k^{-1} \rho_k(w), \]
where
\[ w = d_2(x, y)z + x(yz) - y(zx) = [d_2(x, y) + L(x)R(y) - L(y)R(x)]z = 0 \]
in view of Eq. (2.15b). Thus, \( J(\rho_i(x), \rho_i(y), \rho_k(z)) = 0 \).

(4) We calculate now
\[
[[\rho_i(x), \rho_j(y)], \rho_k(z)] = [-\gamma_j \gamma_i^{-1} \rho_k(xy), \rho_k(z)] \]
\[ = -\gamma_j \gamma_i^{-1} \gamma_i \gamma_j^{-1} T_{3-k}(xy, z) = -T_{3-k}(xy, z) = T_{3-k}(z, xy) \]
and hence,
\[ J(\rho_i(x), \rho_j(y), \rho_k(z)) = T_{3-k}(z, xy) + T_{3-i}(x, yz) + T_{3-j}(y, zx) = 0 \]
in view of Eq. (3.3e). Note that \((i, j, k)\) is a cyclic permutation of \((0, 1, 2)\).

(5) We simply compute
\[
\begin{align*}
[[\rho_i(x), \rho_i(y)], T_\ell(u, v)] &= [\gamma_j \gamma_k^{-1} T_{3-i}(x, y), T_\ell(u, v)] \\
&= -\gamma_j \gamma_k^{-1} \{T_{3-i}(d_{\ell+i}(u, v)x, y) + T_{3-i}(x, d_{\ell+i}(u, v)y)\}
\end{align*}
\]
and
\[
\begin{align*}
[[\rho_i(y), T_\ell(u, v)], \rho_i(x)] &= [-\rho_i(d_{\ell+i}(u, v)y), \rho_i(x)] \\
&= \gamma_j \gamma_k^{-1} T_{3-i}(x, d_{\ell+i}(u, v)y).
\end{align*}
\]
Then, it is easy to see
\[ J(\rho_i(x), \rho_i(y), T_\ell(u, v)) = 0. \]

(6) However, we note
\[
\begin{align*}
[[\rho_i(x), \rho_j(y)], T_\ell(u, v)] &= [-\gamma_j \gamma_k^{-1} \rho_i(x), T_\ell(u, v)] \\
&= \gamma_j \gamma_k^{-1} \rho_k(d_{\ell+i}(u, v)xy)
\end{align*}
\]
while
\[
\begin{align*}
[[\rho_j(y), T_\ell(u, v)], \rho_i(x)] &= [-\rho_j(d_{\ell+i}(u, v)y), \rho_i(x)] \\
&= -\gamma_j \gamma_i^{-1} \rho_k(x \{d_{\ell+i}(u, v)y\})
\end{align*}
\]
and
\[
\begin{align*}
[[T_\ell(u, v), \rho_i(x)], \rho_j(y)] &= \rho_i(d_{\ell+i}(u, v)x, \rho_j(y)] \\
&= -\gamma_j \gamma_i^{-1} \rho_k((d_{\ell+i}(u, v)x)y).
\end{align*}
\]
Therefore, we find
\[ J(\rho_i(x), \rho_j(y), T_\ell(u, v)) = \gamma_j \gamma_i^{-1} \rho_k(w), \]
where
\[ w = d_{\ell+i}(u, v)xy - x[d_{\ell+i}(u, v)y] - [d_{\ell+i}(u, v)x]y. \]
Then, \(w = 0\) by the triality relation Eq. (2.1) since \((i, j, k)\) is a cyclic permutation of \((0, 1, 2)\). Therefore, we have
\[ J(\rho_i(x), \rho_j(y), T_\ell(u, v)) = 0. \]

(7) We similarly calculate
\[ J(\rho_k(z), T_\ell(u, v), T_m(x, y)) = \rho_k(w), \]
with
\[ w = d_{k+m}(d_{e-m}(u, v)x, y)z + d_{k+m}(x, d_{e-m}(u, v)y)z \]
\[ -d_{e+k}(u, v)d_{k+m}(x, y)z + d_{m+k}(x, y)d_{e+k}(u, v)z = 0 \]
in view of Eq. (2.10)).

(8) Finally, after some computations, we find
\[ J(T_{\ell}(u, v), T_{m}(x, y), T_{p}(z, w)) = T_{m}(V_{x,y}) + T_{m}(x, V_{y}) \]
where \( V \) is given by
\[ V = [d_{e-m}(u, v), d_{p-m}(z, w)] - d_{p-m}(d_{e-p}(u, v)z, w) \]
\[ -d_{p-m}(z, d_{e-p}(u, v)w) = 0 \]
again by Eq. (2.10)). Therefore, we have
\[ J(T_{\ell}(u, v), T_{m}(x, y), T_{p}(z, w)) = 0. \]
In conclusion, we have shown \( J(X, Y, Z) = 0 \) identically for all cases. This completes the proof. The converse statement also follows, if \( \rho_{j}(w) = 0 \) implies \( w = 0 \).

If we choose \( \gamma_{0} = \gamma_{1} = \gamma_{2} = 1 \), then the Lie algebra \( L \) constructed in Theorem 3.1 admits an automorphism of order 3, \( P : L \to L(P^{3} = 1) \) by \( \rho_{j}(x) \to \rho_{j+1}(x) \), but \( T_{j}(u, v) \to T_{j-1}(u, v) \) for \( j = 0, 1, 2 \). Note that \( d_{j}(u, v) \) remains unchanged under \( P \).

**Remark 3.2.** As we see from the proof given above, all relations in Eqs. (2.1) and (2.15) are necessary for the validity of the theorem. With the correspondence, (see Eq. (3.4) and a comment following it),
\[ T_{j}(u, v) = T(d_{j}(u, v), d_{j}(u, v), d_{k}(u, v)) \]
and
\[ \rho_{0}(x) = x[1, 2], \rho_{1}(x) = x[2, 3], \rho_{2}(x) = x[3, 1]. \]
the construction of \( L \) in Theorem 3.1 essentially gives that of Theorem 4.1 of [2] when the conjugate algebra \( A^{*} \) of \( A \) is structurable. Also, if \( A \) is the tensor product algebra of two symmetric composition algebras \( A \) and \( A' \) as in Example 2.4, then our construction reproduces that given by Elduque [4]. Especially, if we vary two symmetric composition algebras \( A \) and \( A' \) by various choices of para-Hurwitz and pseudo-octonion algebras, it gives the Freudenthal’s magic square (see [4]). Also, if we do not assume Eq. (3.3e), then the only non-zero Jacobian is
\[ D(x, y, z) := J(\rho_{0}(x), \rho_{1}(y), \rho_{2}(z)) = T_{0}(x, yz) + T_{1}(z, xy) + T_{2}(y, zx). \]
However, Eqs. (3.3e) and (3.3a) imply the validity of
\[ [D(u, v, w), \rho_{j}(x)] = 0 = [D(u, v, w), T_{m}(x, y)] \]
in view of Eq. (2.15d) so that $D(u, v, w)$ is a center element of $L$. Therefore, the quotient algebra $L/D$ is now a Lie algebra, where $D$ is a vector space spanned by linear combinations of $D(u, v, w)$.

Remark 3.3. If $A$ is either Jordan or Lie as in Example 2.3, then we have $d_0(u, v) = d_1(u, v) = d_2(u, v) = \epsilon [L(u), L(v)]$. In that case, Eqs. (3.3) may be simplified to

\begin{enumerate}
  \item $[\rho_i(x), \rho_j(y)] = \gamma_j \gamma_i^{-1} T(x, y)$,
  \item $[\rho_i(x), \rho_j(y)] = -[\rho_j(y), \rho_i(x)] = -\gamma_j \gamma_i^{-1} \rho_k(xy)$,
  \item $[T(u, v), \rho_j(x)] = -[\rho_j(x), T(u, v)] = \rho_j(d(u, v)x)$,
  \item $[T(u, v), T(x, y)] = T(x, d(u, v)y) + T(d(u, v)x, y)$
    \[- T(u, d(x, y)v) - T(d(x, y)u, v),
  \item $T(x, yz) + T(z, xy) + T(y, zx) = 0$
\end{enumerate}

where we have set

$$d(x, y) = \epsilon [L(x), L(y)], \quad \text{and} \quad T_j(x, y) := T(x, y)$$

with $\epsilon = +1$ for Lie and $\epsilon = -1$ for Jordan.

As we have seen in the proof of Theorem 3.1, the validity of all conditions Eqs. (2.15a)–(2.15e) is crucial. However, for many cases, they are really not independent of each other, as we will see in Proposition 3.4. To this end, we first suppose that an algebra $A$ possesses a symmetric bilinear associative form $\langle \cdot | \cdot \rangle$, which is moreover assumed to be non-degenerate. Especially, this implies

\begin{align}
\langle x | y \rangle &= \langle y | x \rangle, \tag{3.5a} \\
\langle xy | z \rangle &= \langle x | yz \rangle. \tag{3.5b}
\end{align}

Proposition 3.4. Let $A$ be as in the above with the non-degenerate associative form $\langle \cdot | \cdot \rangle$, which needs however not be a normal STA. Suppose that $d_j(u, v) = -d_j(v, u) \in \text{End } A(j = 0, 1, 2)$ satisfy

$$\langle d_j(u, v)x | y \rangle = -\langle x | d_j(u, v)y \rangle = \langle d_{3-j}(x, y)u | v \rangle. \tag{3.6}$$

Then, any one of the following four statements implies the validity of all others:

\begin{enumerate}
  \item We have
    $$d_j(u, v)(xy) = (d_{j+1}(u, v)x)y + x(d_{j+2}(u, v)y) \tag{3.7}$$
    for all $j = 0, 1, 2$.
  \item Eq. (3.7) holds valid only for one value of $j$. For example, we may assume only
    $$d_0(u, v)(xy) = (d_1(u, v)x)y + x(d_2(u, v)y). \tag{3.8}$$
  \item $\langle d_0(u, v)z | xy \rangle + \langle d_1(u, v)x | yz \rangle + \langle d_2(u, v)y | zx \rangle = 0. \tag{3.9}$
\end{enumerate}
\[ d_0(x, yz) + d_1(z, xy) + d_2(y, zx) = 0. \] (3.10)

**Proof.** We first suppose that Eq. (3.8) holds valid. It is equivalent to
\[ \langle z | d_0(u, v)(xy) \rangle = \langle z | (d_1(u, v)x)y \rangle + \langle z | x(d_2(u, v)y) \rangle \]
because of the non-degeneracy of \( \langle . | . \rangle \). However, by the first relation in Eq. (3.6), it gives Eq. (3.9). Since Eq. (3.9) is invariant under cyclic permutations of \( 0 \rightarrow 1 \rightarrow 2 \rightarrow 0 \) and \( x \rightarrow y \rightarrow z \rightarrow x \), it implies also the validity of Eq. (3.7) for all values of \( j \).

We next utilize the last relation in Eq. (3.6) to note
\[ \langle d_j(u, v)z | xy \rangle = \langle d_{3-j}(z, xy)u | v \rangle. \]

Then, Eq. (3.9) is rewritten as
\[ \langle d_0(z, xy)u | v \rangle + \langle d_2(x, yz)u | v \rangle + \langle d_1(y, zx)u | v \rangle = 0, \]
which yields Eq. (3.10) again by non-degeneracy of \( \langle . | . \rangle \) with \( x \rightarrow y \rightarrow z \rightarrow x \). Reversing the argument, the validity of Eq. (3.10) leads to all other relations, etc. This completes the proof of Proposition 3.4. \( \square \)

**Remark 3.5.** It is easy to verify the validity of (3.6) for the case of symmetric composition algebras.

For the conjugate algebra \( A^* \) of \( A \), we must impose
\[ \langle x | y \rangle = \langle y | x \rangle = \langle \overline{x} | \overline{y} \rangle, \] (3.11a)
\[ \langle \overline{x} | y \ast z \rangle = \langle \overline{y} | z \ast x \rangle = \langle \overline{\langle x \ast y \rangle} \rangle \] (3.11b)
instead of Eqs. (3.5). Then, these conditions as well as Eq. (3.6) can be readily verified for example for both \( A^* \) being a Hurwitz algebra and Example 2.8 when \( \langle x | y \rangle \) for the underlying cubic-admissible algebra \( B \) is non-degenerate.

**Corollary 3.6.** Let \( A^* \) be a unital involutive LRTA, algebra endowed with a LRTM \( d : A^* \otimes A^* \to \text{lrt}(A^*, -) \) satisfying Eqs. (2.34a and b) and Eq. (2.36). If \( A^* \) possesses a non-degenerate symmetric form \( \langle . | . \rangle \) satisfying Eq. (3.7), then \( A^* \) is structurable.

**Proof.** This follows from Proposition 3.4 and from the fact that Eq. (3.6) holds for this case. \( \square \)

Next, we shall prove that the generalized symmetric composition algebra of Example 2.4 satisfies the condition Eq. (2.15d) by showing the validity of the following Proposition.

**Proposition 3.7.** Let \( A \) be a flexible algebra and let \( U(x, y) \in \text{End} A \) be given by Eq. (1.27). Suppose that \( d_j(x, y) \in \text{End} A \) have a specific form of
\[ d_1(u, v) = R(v)L(u) - R(u)L(v), \]  
\[ d_2(u, v) = L(v)R(u) - L(u)R(v), \]  
\[ d_0(u, v)w = \lambda [U(u, w)v - U(v, w)u] \]  
(3.12a)  
(3.12b)  
(3.12c)

for some \( \lambda \in F. \) Then, we have
\[ d_0(x, yz) + d_1(z, xy) + d_2(y, zx) = 0, \]  
(3.13)

provided that \( U(x, y) \) satisfies
\[ (U(y, zw) + U(z, wy) - \lambda U(w, yz))x \]
\[ = y[U(x, w)z] + [U(x, w)y]z - \lambda U(x, w)(yz). \]  
(3.14)

Moreover, if \( A \) possesses a non-degenerate bilinear form, \( \langle \cdot, \cdot \rangle \) satisfying Eqs. (3.5), then the condition Eq. (3.6) holds and \( A \) is a normal STA.

**Proof.** We first prove the validity of
\[ (zx)(wy) + (zw)(xy) \]
\[ = U(y, z)w + U(z, w)w - y[U(x, w)z] \]
\[ = U(x, y)z + U(x, w)z - y[U(x, w)z] \]
\[ = U(z, xy)w + U(z, wy)w - \lambda U(x, w)(yz). \]  
(3.15)

For it, we note
\[ U(x, y)z = (xz)y + (yz)x = x(zy) + y(zx) \]

since \( U(x, y) = R(x)L(y) + R(y)L(x) = L(x)R(y) + L(y)R(x) \). We then calculate
\[ (zx)(xy) = U(zx, y)x - y[U(x, w)z] = U(z, xy)x - \{U(x, w)y\}z. \]
Linearizing this, we find Eq. (3.15). Further, we note
\[ 2(zx)(wy) = 2L(xz)R(y)w = \{U(y, z) + d_2(y, zw)\}w, \]
\[ 2(zw)(xy) = 2R(xy)L(z)w = \{U(xy, z) + d_1(z, xy)\}w. \]
Adding these together with Eq. (3.15), we obtain
\[ \{d_1(z, xy) + d_2(y, zw)\}w \]
\[ = U(y, z)w + U(z, w)w - y[U(x, w)z] - \{U(x, w)y\}z, \]  
(3.16)

which leads to Eq. (3.13) in view of Eq. (3.14).

We will now assume the validity of Eqs. (3.5). It is then not difficult to prove
\[ \langle U(x, y)u|v \rangle = \langle U(u, v)x|y \rangle = \langle u|U(x, y)v \rangle, \]  
(3.17)

from which we can show the validity of Eq. (3.6). Then, \( A \) is a normal STA, since other conditions, Eqs. (2.15c) and (2.15e) will be automatically satisfied. □
Remark 3.8. Any generalized symmetric composition algebra satisfies Eq. (3.14) for $\lambda = 2$ because of Eqs. (2.20). This proves the validity of Eq. (3.13) for the algebra. We also note that this fact has been already noted and utilized in [4] for the special case of the symmetric composition algebra. Finally, Eq. (3.14) also holds valid for any commutative algebra $A$ satisfying $x^3 = 0$ with $\lambda = -1$, since then this implies $U(x, y)z = -(x y)z$. However, such an algebra is a Jordan.

In ending this section, we will give examples of STA which are not normal.

Remark 3.9. Let $A$ be a commutative algebra and set

$$d_0(x, y) = d_1(x, y) = d_2(x, y) := D(x, y)$$

(3.18a)

with

$$D(x, y)z := x(yz) - y(xz) + \lambda \{\langle y | z \rangle x - \langle x | z \rangle y\}$$

(3.18b)

for some bilinear symmetric form $\langle \cdot | \cdot \rangle$. If we choose the constant $\lambda \in F$ to be given by

1. $\lambda = 0$ for Jordan,
2. $\lambda = 2$ for pseudo-composition algebra [9], satisfying $x^3 = \langle x | x \rangle x$,
3. $\lambda = \frac{3}{4}$ for cubic-admissible algebra [5], satisfying $x^2 x^2 = \langle x | x^2 \rangle x$,

then we can prove that they satisfy Eqs. (2.1) as well as Eqs. (2.15c–e) but not Eqs. (2.15a and b) if $\lambda \neq 0$. Hence these algebras ($\lambda \neq 0$) are not normal STA.

For these cases, the triple product defined by

$$xyz := D(x, y)z$$

gives, however, a Lie triple system, i.e., we have

1. $xyz = -yxz$,
2. $xyz + yzx + zxy = 0$,
3. $uv(xyz) = (uvx)yz + x(uyv)z + xy(uvz)$.

4. Miscellaneous comments

(1) Let $A^*$ be a involutive LRTA which satisfies Eqs. (2.33) and (2.34) with possible exception of Eqs. (2.34c) and (2.36). We then call $A^*$ be a quasi-normal LRTA. We first note:

Proposition 4.1. Let $A^*$ be a unital quasi-normal LRTA. Then, $A^*$ is a structurable algebra.

Proof. This is almost evident except for the validity of Eq. (2.36), since then the condition (C) in Proposition (2.1) is automatically satisfied.

If we set \( w = e \) or \( v = e \) in Eq. (2.34d) where \( e \) is the unit element of \( A^* \), then we have

\[
\begin{align*}
&d_0(\mathcal{P}, v) = -d_1(\mathcal{P}, v * u) - d_2(\mathcal{P}, u), \\
&d_0(\mathcal{P}, w) = -d_1(\mathcal{P}, u) - d_2(\mathcal{P}, u * w).
\end{align*}
\]

Inserting expressions of \( d_1 \) and \( d_2 \) given by Eqs. (2.34a) and (2.34b), and changing the notations suitably, these give Eq. (2.36) which also satisfies Eq. (2.34c). This proves Proposition 4.1 in view of Theorem 2.6. □

Theorem 4.2. Let \( A^* \) be a quasi-normal LRTA over a field \( F \) of characteristic not 2, and set

\[
L(x, y) := \frac{1}{2}\{\ell(x * y + y * x) - d_0(x, y) - d_2(x, y)\}. \tag{4.1}
\]

If we introduce a triple product in \( A^* \) by

\[
xyz := L(x, y)z, \tag{4.2}
\]

then it defines a generalized Jordan triple system, i.e., we have

\[
uv(xyz) = (uvx)yz - x(vuy)z + xy(uvz). \tag{4.3}
\]

Remark 4.3. If \( d_0(x, y) \) is given as in Eqs. (2.36), then Eqs. (4.1) and (4.2) give

\[
xyz = (z * y) * x - (z * x) * y + (x * y) * z,
\]

which reproduces the triple product defining the structurable algebra.

For proof of Theorem 4.2, we need the following Lemma.

Lemma 4.4. Let \( A^* \) be a quasi-normal LRTA. We then have

\[
\begin{align*}
&[d_0(u, v) + d_2(\mathcal{P}, \mathcal{Q}), \ell(x)] = \ell((d_1(\mathcal{P}, \mathcal{Q}) + d_2(u, v))x), \tag{4.4a} \\
&[d_0(u, v) + d_1(\mathcal{P}, \mathcal{Q}), r(x)] = r((d_1(u, v) + d_2(\mathcal{P}, \mathcal{Q}))x). \tag{4.4b}
\end{align*}
\]

Proof. In view of Eq. (2.34f), we can rewrite Eq. (2.33) as

\[
\begin{align*}
&d_{3-j}(\mathcal{P}, \mathcal{Q})\ell(x) = \ell(d_{j+1}(u, v)x) + \ell(x)d_{j+2}(u, v), \tag{4.5a} \\
&d_{3-j}(\mathcal{P}, \mathcal{Q})r(y) = r(d_{j+2}(u, v)y) + r(y)d_{j+1}(u, v), \tag{4.5b}
\end{align*}
\]

from which we can derive Eqs. (4.4). □
We are now in position to prove Theorem 4.2. We calculate

\[
[L(u, v), L(x, y)] = \frac{1}{4} \left[ d_0(u, v) + d_2(\overline{u}, \overline{v}), d_0(x, y) + d_2(\overline{x}, \overline{y}) \right]
- \frac{1}{4} \left[ d_0(u, v) + d_2(\overline{u}, \overline{v}), \ell(x \ast \overline{y} + y \ast \overline{x}) \right]
+ \frac{1}{4} \left[ d_0(x, y) + d_2(\overline{x}, \overline{y}), \ell(u \ast \overline{v} + v \ast \overline{u}) \right]
+ \frac{1}{4} \left[ \ell(u \ast \overline{v} + v \ast \overline{u}), \ell(x \ast \overline{y} + y \ast \overline{x}) \right].
\] (4.6)

For the first term in Eq. (4.6), we utilize Eq. (2.10), while we use Eq. (4.4a) for the reduction of the 2nd and 3rd commutators. This gives

\[
[L(u, v), L(x, y)] = \frac{1}{4} \left[ d_0((d_0(u, v) + d_2(\overline{u}, \overline{v}))x, y)
+ d_0(x, (d_0(u, v) + d_2(\overline{u}, \overline{v}))y)
+ d_2((d_1(u, v) + d_0(\overline{u}, \overline{v}))\overline{x}, \overline{y})
+ d_2(\overline{x}, (d_1(u, v) + d_0(\overline{u}, \overline{v}))\overline{y}) \right]
+ \frac{1}{4} \left[ \ell((d_1(\overline{x}, \overline{y}) + d_2(x, y))(u \ast \overline{v} + v \ast \overline{u}))
- \ell((d_1(\overline{u}, \overline{v}) + d_2(u, v))(x \ast \overline{y} + y \ast \overline{x})) \right]
+ \frac{1}{4} \left[ \ell(u \ast \overline{v} + v \ast \overline{u}), \ell(x \ast \overline{y} + y \ast \overline{x}) \right].
\]

If we similarly compute \(L(uvx, y)\) and \(L(x,vuy)\), and note Eqs. (2.33) and (2.34d), we can prove the validity of

\[
[L(u, v), L(x, y)] = L(uvx, y) - L(x,vuy),
\] (4.7)

which is equivalent to Eq. (4.3). \(\square\)

**Remark 4.5.** If \(A^*\) is a quasi-normal LRTA satisfying Eq. (2.36) (but not necessarily unital), we can show moreover that the triple product defines a generalized Jordan triple system of the second order or equivalently \((-1,1)\) Freudenthal–Kantor triple system [14], by noting that Eq. (A) of [2] is then satisfied. However, we will not go into detail here.

(2) We have also found the following examples of nontrivial non-unital normal STA of some interest. First,

**Example 4.6.** Let \(A\) be a three-dimensional algebra with basis vectors \(e_0, e_1, e_2\) satisfying the multiplication table of

\[
(1) \quad e_0e_0 = e_0, \quad e_1e_1 = e_1, \quad e_2e_2 = e_2.
\] (4.8a)
If we set
\[ d_1(u, v) := R(v)L(u) - R(u)L(v), \]
\[ d_2(u, v) := L(v)R(u) - L(u)R(v), \]
\[ d_0(u, v)w := \langle u | w \rangle v - \langle v | w \rangle u, \]
then \( A \) is a normal STA, where \( \langle . | . \rangle \) is defined by
\[ \langle e_j | e_k \rangle = \delta_{jk}, \ (j, k = 0, 1, 2). \] (4.10)

We note that \( A \) is not flexible since we have \( xx^2 \neq x^2x \) for a generic element \( x \in A \).

**Remark 4.7.** This algebra possesses many interesting properties. First, \( \langle . | . \rangle \) is a bilinear symmetric associative non-degenerate form. Second, we calculate
\[ d_0(e_1, e_2) = d_1(e_2, e_0) = d_2(e_0, e_1), \]
\[ d_0(e_0, e_1) = d_1(e_1, e_2) = d_2(e_2, e_0), \]
\[ d_0(e_2, e_0) = d_1(e_0, e_1) = d_2(e_1, e_2), \]
so that Eqs. (4.8) and (4.11) are cyclically invariant under \( 0 \to 1 \to 2 \to 0 \). Third, there exist three distinct involutions in \( A \). For example, we may consider
\[ \bar{e}_0 = e_0, \quad \bar{e}_1 = e_2, \quad \bar{e}_2 = e_1, \] (4.12)
which defines an involution in \( A \). However, its conjugate algebra \( A^* \) is not structurable since it is not unital. Fourth, \( A \) is Lie-admissible with its associated Lie algebra being so (3). Fifth, it satisfies a special relation of
\[ (yx)(xz) = \langle x | zy \rangle x \] (4.13)
for \( x, y, z \in A \). Especially, if we set \( y = z = x \), then Eq. (4.13) gives
\[ x^2x^2 = \langle x | x^2 \rangle x, \] (4.14)
i.e., the cubic-admissible relation of [5]. If we express,
\[ x = \lambda_0 e_0 + \lambda_1 e_1 + \lambda_2 e_2 \] (4.15)
for \( \lambda_j \in F \), then we have
\[ N(x) = \langle x | x^2 \rangle = \lambda_0^3 + \lambda_1^3 + \lambda_2^3 - 3\lambda_0\lambda_1\lambda_2. \] (4.16)
Then, the relation of \( N(x^2) = [N(x)]^2 \) (see [5]) gives an amusing identity of
\[ (\lambda_0^3 + \lambda_1^3 + \lambda_2^3 - 3\lambda_0\lambda_1\lambda_2)^2 = (\lambda_0')^3 + (\lambda_1')^3 + (\lambda_2')^3 - 3\lambda_0\lambda_1\lambda_2. \]
with
\[ \lambda'_0 = \lambda_0^2 - \lambda_1 \lambda_2, \lambda'_1 = \lambda_1^2 - \lambda_2 \lambda_0, \lambda'_2 = \lambda_2^2 - \lambda_0 \lambda_1. \]

Since the new commutative algebra \( A^+ \) defined by a product \( x \cdot y = \frac{1}{2}(xy + yx) \) is a cubic-admissible algebra, we can use it to construct a larger normal STA by Example 2.8.

Last, we have a realization of Duffin–Kemmer–Petiau (DKP) algebra (see [10] and earlier references quoted therein) as follows. Let \( A(x) \) be as in Eq. (2.9a), i.e.,
\[ A(x) = \begin{pmatrix} 0 & L(x) \\ R(x) & 0 \end{pmatrix} \tag{4.17} \]
for \( x \in A \). We can then verify the validity of the DKP relation of
\[ A(x)A(y)A(z) + A(z)A(y)A(x) = \langle x|y\rangle A(z) + \langle z|y\rangle A(x) \tag{4.18} \]
for \( x, y, z \in A \). We may remark that in contrast, any symmetric composition algebra (see Example 2.4) satisfies the Clifford relation of
\[ A(x)A(y) + A(y)A(x) = 2\langle x|y\rangle E, \]
where \( E \) is the unit matrix. These relations are, of course, consistent with the validity of Eq. (2.9b), i.e., they are special examples of Eq. (2.9b).

Example 4.8. We have also found the following peculiar example of normal STA. Let \( A \) be an anti-associative algebra, i.e., it satisfies \( (xy)z = -x(yz) \) for \( x, y, z \in A \) over the field \( F \) of characteristic not 2. It has been shown in [11] that any product involving more than 3 elements in \( A \) is identically zero then. For example we have \( (xy)(zw) = 0 = \{(xy)z\}w \) etc. More generally, let \( A \) be any similar algebra satisfying \( AAAAA = 0 \). Then, setting
\[ d_0(x, y)z = \lambda[(xz + zx)y - (yz + zy)x] \]
for \( \lambda \in F \), \( A \) becomes trivially a normal STA. Note that we also have
\[ [d_j(u, v), d_k(x, y)] = 0. \]
However, the right side of Eq. (3.3d) needs not be zero, unless we make the identification of Eq. (3.4) for \( T_j(x, y) \).

(3) Let \( d_j \in \text{End} \ A \) define a STA as in Eq. (1.1). We then find from Eqs. (1.23) the following commutation relations,
\[ [d_j, L(x)L(y)L(z)] = L(d_{j+1}x)L(y)L(z) + L(x)L(d_jy)L(z) + L(x)L(y)L(d_{j+1}z), \tag{4.19a} \]
\[ [d_j, R(x)R(y)R(z)] = R(d_{j+1}x)R(y)R(z) + R(x)R(d_{j+1}y)R(z) + R(x)R(y)R(d_{j+1}z) \tag{4.19b} \]
in addition to Eqs. (1.22). If $A$ is finite dimensional, we can then define a tri-linear form $\phi: A \otimes A \otimes A \to F$ by

$$\phi(x, y, z) := \text{Tr}(R(x)R(y)R(z)),$$

where Tr stands for the trace. It satisfies

1. $\phi(x, y, z) = \phi(y, z, x) = \phi(z, x, y),$
2. $\phi(d_j x, y, z) + \phi(x, d_{j+1} y, z) + \phi(x, y, d_{j+2} z) = 0.$

Here, Eq. (4.21b) follows by taking the trace of both sides of Eq. (4.19b) and then letting $j \to j + 1.$ □

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