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Strict colourings for classes of steiner triple systems

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Abstract

We investigate the largest number of colours, called *upper chromatic number* and denoted $\bar{\chi}(\mathscr{H})$, that can be assigned to the vertices (points) of a Steiner triple system $\mathscr{H}$ in such a way that every block $H \in \mathscr{H}$ contains at least two vertices of the same colour. The exact value of $\bar{\chi}$ is determined for some classes of triple systems, and it is observed further that optimal colourings with the same number of colours exist also under the additional assumption that no monochromatic block occurs. Examples show, however, that the cardinalities of the colour classes in the latter case are more strictly determined.

1. Introduction

In the general definition of upper chromatic number, introduced by Voloshin [5,6], vertex colourings of the so-called 'mixed hypergraphs' $\mathscr{H} = (X, \mathscr{S})$ are considered in which two kinds of sets are distinguished: the *edges* and the *co-edges* (originally called anti-edges in [5]). In the *strict colourings* of $\mathscr{H}$, every edge has at least two vertices coloured differently and every co-edge has at least two vertices of the same colour. The maximum number of colours that can occur simultaneously in such a colouring is the *upper chromatic number* $\bar{\chi}(\mathscr{H})$. If $\mathscr{H}$ is uniform and contains co-edges only, then $\bar{\chi}(\mathscr{H})$ is obtained as a particular case of a more general problem raised by Ahlswede et al. [1].

In this paper the concepts of strict colouring and upper chromatic number are applied to the Steiner triple systems (STSs). In particular, we study two different kinds of colourings for STSs. In one of them, we view all blocks of the triple system as co-edges; such systems will be termed CSTS, referring to the expression 'co-hypergraph' and to indicate that monochromatic blocks are allowed. The other type, where all blocks are edges and co-edges at the same time, will be called (Bi-Steiner triple systems)

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(BSTS). Note that every CSTS is colourable, in the sense that it always admits a strict colouring (trivially, for example, with one colour); this is not necessarily the case, however, with a BSTS that may as well be uncolourable, as shown in [4].

In our previous work [4], we have proved that for every CSTS(v) and BSTS(v) on $v \leqslant 2^k - 1$ vertices the upper chromatic number is at most k, and we have also constructed an infinite class of colourable BSTSs $\mathscr{H}$ of order $v = 2^k - 1$ and upper chromatic number $\bar{\chi}(\mathscr{H}) = k$.

In this paper, some further properties of the strict colourings of CSTSs and BSTSs are described, and it is investigated, to what extent the cardinalities of the colour classes are determined in a strict colouring with a maximum number of colours. In this respect, the BSTSs are more restricted than the CSTSs, because different colour distributions may define unequal numbers of blocks of given colour patterns.

We also study the CSTSs and BSTSs of order $v \neq 2^k - 1$, and construct an infinite class with $v = 2^k \times 10 - 1$ and $\bar{\chi} = k+3$. Moreover, we indicate a method to build further infinite classes of colourable BSTSs when a colouring of any one particular BSTS(v) is known.

1.1. Mixed hypergraphs and upper chromatic number

A *hypergraph* is a pair $(X, \mathscr{E})$ where X is a finite set whose elements are called *vertices*, and $\mathscr{E}$ is a family of subsets of X, called *edges*. In this subsection we recall some definitions from [5,6], some of which have already been mentioned above less formally. A *mixed hypergraph* $\mathscr{H}$ is a pair $(X, \mathscr{S})$ where X is a finite set of *vertices*, and $\mathscr{S}$ is a family of subsets of X, written as the union of two subfamilies: $\mathscr{S} = \mathscr{A} \cup \mathscr{E}$. Here $\mathscr{A}$ and $\mathscr{E}$ need not be disjoint, and one of them may be empty. The elements of $\mathscr{A}$ are called *co-edges* or anti-edges, and the elements of $\mathscr{E}$ are the *edges*. If $\mathscr{E}$ is empty, then $\mathscr{H} = (X, \mathscr{A})$ will be called a *co-hypergraph*, while in the case of $\mathscr{A} = \emptyset$, $\mathscr{H} = (X, \mathscr{E})$ is just a *hypergraph*, in the usual sense.

Definition 1. A *strict colouring* of a mixed hypergraph $\mathscr{H} = (X, \mathscr{S})$ with $k \geqslant 1$ colours is a mapping from the vertex set X *onto* the set $\{1, \ldots, k\}$ of 'colours' in such a way that

(1) every co-edge has at least two vertices of the same colour, and

(2) every edge has at least two vertices coloured differently.

The mixed hypergraph $\mathscr{H}$ is *colourable* if it has at least one strict colouring. In this case, the largest and smallest integer k, for which there exists a strict colouring of $\mathscr{H}$ with k colours, is called the *upper* and the *lower chromatic number*, denoted by $\bar{\chi}(\mathscr{H})$ and by $\chi(\mathscr{H})$, respectively.

If $\mathscr{H}$ admits no strict colouring (because of the collision between the two constraints above), then $\mathscr{H}$ is called *uncolourable*; for such a mixed hypergraph we put $\chi(H) = \bar{\chi}(H) = 0$ by definition.

Note that if the hypergraph has no co-edges, then its strict colourings are precisely the proper colourings (i.e., those containing no monochromatic edge), therefore, in this case the lower chromatic number equals the chromatic number in the usual sense; while the upper chromatic number is clearly equal to the number of vertices.

Definition 2. A vertex subset $L \subseteq X$ in a mixed hypergraph $\mathscr{H}$ is called *co-stable* if it contains no co-edges; and L is *stable* if it contains no edges. The *co-stability number* $\bar{\alpha}(\mathscr{H})$ is the maximum cardinality of a co-stable set in $\mathscr{H}$.

One of the basic facts, following directly from the definitions, is the inequality $\bar{\chi}(\mathscr{H}) \leqslant \bar{\alpha}(\mathscr{H})$ [5].

1.2. Steiner triple systems

A hypergraph $(X,\mathscr{B})$ is a *Steiner triple system* if $\mathscr{B}$ is a family of three-element subsets of X, called *blocks*, such that any two distinct vertices are contained in precisely one block. A Steiner triple system of *order* v (i.e., with $|X|=v$) will be denoted by STS(v). We shall also use the notation STS($X,\mathscr{B}$). It is well known that a STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$.

Definition 3. For a STS($X,\mathscr{B}$), a subset S of X is said to induce a *subsystem* if all those blocks which contain at least two vertices of S are entirely in S; $s := |S|$ is the *order* of the subsystem that consists of the blocks $B \subset S$, $B \in \mathscr{B}$.

Next, we recall a well known way to construct Steiner triple systems from smaller ones.

Doubling-plus-one construction for STS. One can obtain a Steiner triple system of order $2v+1$ using the following construction. Let $(X',\mathscr{B}')$ be a STS(v) with $|X'|=v$, and take a set X'' of vertices with $|X''|=v+1$ where $X' \cap X'' = \emptyset$. Recalling that $v+1$ is even, consider a 1-factorization $\mathscr{F}=\{F_1,F_2,\dots,F_v\}$ of a complete graph K_{v+1} on the vertex set X'', and define the collection $\mathscr{B}$ of triples on $X := X' \cup X''$ in the following way:
(1) Every triple belonging to $\mathscr{B}'$ belongs to $\mathscr{B}$;
(2) If $x_i \in X'$ $(i=1,2,\dots,v)$ and $y_1, y_2 \in X''$, then $\{x_i, y_1, y_2\} \in \mathscr{B}$ if and only if $\{y_1, y_2\} \in F_i$.

It is easy to see that $(X,\mathscr{B})$ is a STS($2v+1$), $(X',\mathscr{B}')$ is its subsystem, and X'' is a stable set.

2. Strict colourings for CSTS and BSTS

One may view a STS($X,\mathscr{B}$) as a mixed hypergraph where every block is an edge and a co-edge at the same time. Let us call such a system of order v a BSTS(v). Moreover, if every block is only a co-edge, call this co-hypergraph a CSTS(v). We shall apply the concepts of strict colouring and of upper chromatic number for CSTS(v) and BSTS(v). When the emphasis will be put on the order of such systems, where the actual systems are either understood or irrelevant, we shall simply write $\bar{\chi}(v)$ for their upper chromatic number.

For CSTS(v) and BSTS(v), the following upper bound has been proved:

Theorem 1 (Milazzo and Tuza [4]). *For any integer $k>0$, a CSTS($X,\mathscr{B}$) or BSTS($X,\mathscr{B}$) of order $v=|X|\leqslant 2^k-1$ has upper chromatic number at most k. Moreover, if $\bar{\chi}=k$, then*

(1) $v=2^k-1$,

(2) *in every strict colouring with k colours, the colour classes have respective cardinalities*

$$2^0, 2^1, \ldots, 2^{k-1},$$

and all of them are co-stable sets, and

(3) *the triple system is obtained by a sequence of $k-2$ doubling plus one constructions starting from* STS(3).

Consider a strict colouring $\mathscr{P}$ for the co-hypergraph CSTS(v). Let h be the number of colours used in $\mathscr{P}$, and denote by X_i the set of vertices assigned to colour i. Let $n_i := |X_i|$ for $1\leqslant i\leqslant h$. Every block can be coloured in one of the following two ways:

Type 1: Two vertices are coloured with one colour, and the third vertex is coloured with a different colour.

Type 2: All the three vertices are coloured with the same colour.

Since every BSTS contains a CSTS of the same order as a subhypergraph, the next observations formulated for the latter give relevant information for both, sometimes with even stronger consequences for the former.

Proposition 1. *If $\mathscr{P}$ is a strict colouring with h colours for* CSTS(v), *then there are*

(1) $(c/2)$ *blocks of Type* 1, *where*

$$c := \binom{v}{2} - \sum_{i=1}^{h} \binom{n_i}{2}$$

(2) $|\mathscr{B}| - (c/2)$ *blocks of Type* 2, *where $|\mathscr{B}| = \frac{1}{3}\binom{v}{2}$ is the number of blocks in* CSTS(v).

Proof. The number of pairs of vertices coloured with different colours is equal to c, and in every block of Type 1 there are precisely two distinct pairs of vertices coloured differently. This proves the first assertion, from which the second one is evident, too. □

Denoting $s_i := n_1 + n_2 + \cdots + n_i$, the argument given in Proposition 1 also yields

Lemma 1. *If $\mathscr{P}$ is a strict colouring for* CSTS(v), *then the number*

$$c_i := \binom{s_i}{2} - \sum_{j=1}^{i} \binom{n_j}{2}$$

is even for all $2 \leqslant i \leqslant h$.

From the above observations, the following set of inequalities (first proved in [4]) can be deduced easily.

Lemma 2. *If $\mathscr{P}$ is a strict colouring for* CSTS(v), *then the inequalities*

$$s_i(s_i - 1) \leqslant 3 \sum_{j=1}^{i} n_j(n_j - 1) \tag{1}$$

are valid for all $1 \leqslant i \leqslant h$.

Proof. The set $S_i := \bigcup_{j=1}^{i} X_j$ contains no more than $|\mathscr{B}_i| = s_i(s_i - 1)/6$ blocks, and at least $c_i/2$ blocks, therefore, $|\mathscr{B}_i| \geqslant c_i/2$, and so (1) follows by Lemma 1. □

In the case of a strict colouring $\mathscr{P}$ for BSTS(v), all the blocks must be of Type 1. Consequently, Lemmas 1 and 2 are valid, and, in particular, the further fact $|\mathscr{B}| = c/2$ yields:

Lemma 3. *If $\mathscr{P}$ is a strict colouring for* BSTS(v) *with h colours, then*

$$v(v - 1) = 3 \sum_{i=1}^{h} n_i(n_i - 1).$$

If we know the upper chromatic number $\bar{\chi}(v)$ for a CSTS(v) or BSTS(v) where $v \neq 2^k - 1$, then we can obtain estimates for an infinite family of Steiner systems as follows.

Theorem 2. *If a* CSTS(v) [BSTS(v)] *is colourable and has upper chromatic number $\bar{\chi}(v)$, then all the systems* CSTS(v') [BSTS(v')] *with $v' = 2^k(v + 1) - 1$ obtained by a sequence of $k \geqslant 1$ doubling plus one constructions starting from* CSTS(v) [BSTS(v)] *are colourable and*

$$\bar{\chi}(v) + k \leqslant \bar{\chi}(v') < t + k \tag{2}$$

where t is the smallest integer such that $v < 2^t - 1$.

Proof. Let us begin with the inequality on the left-hand side of (2). We apply induction on k. If $k = 0, 1$, then the inequality is true. Suppose that it holds for $k - 1$. Then, for every CSTS(v'') [BSTS(v'')] with $v'' = 2^{k-1}(v + 1) - 1$, a strict colouring $\bar{\mathscr{P}}$ with $k - 1 + \bar{\chi}(v)$ colours exists. If CSTS(v') [BSTS(v')] with $v' = 2^k(v + 1) - 1$ is obtained from CSTS(v'') [BSTS(v'')] by the doubling plus one construction, it is then possible to colour its vertices in the following way: all the vertices of the subsystem STS(v'') are coloured with the colouring $\bar{\mathscr{P}}$, and the other vertices with one new colour. It is easy to verify that this colouring is a strict colouring with $k+\bar{\chi}(v)$ colours, therefore, the CSTS(v') [BSTS(v')] is colourable, and the first inequality of (2) is valid.

The inequality on the right-hand side also holds, because

$$(v+1)2^k < 2^k + 2^k(2^t-1) = 2^{k+t},$$

and from Theorem 1 we obtain $\bar{\chi}(v') < k+t$. □

The following assertion concerning the growth speed of some increasing sequences will be essential.

Lemma 4. *Let $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_h \leqslant \cdots \leqslant n_k$ be an increasing sequence of k natural numbers, and denote $s_i := n_1 + n_2 + \cdots + n_i$ for $1 \leqslant i \leqslant k$. If*

$$s_h(s_h-1) = 3\sum_{j=1}^{h} n_j(n_j-1) \tag{h}$$

for some $h<k$, and

$$s_i(s_i-1) \leqslant 3\sum_{j=1}^{i} n_j(n_j-1) \tag{i}$$

for all $h<i\leqslant k$, then

$$n_i \geqslant 2^{i-h-1}(s_h+1).$$

Proof. With some modifications, we apply the ideas of [4] where a less general result without the condition (h) has been shown. We begin with two observations. First, it is easily seen by induction that for the particular sequence with $n_i = 2^{i-h-1}(s_h+1)$ for $h+1 \leqslant i \leqslant k$, equality holds in each (i). Second,

(∗) assuming $n_i = 2^{i-h-1}(s_h+1)$ for all $h<i\leqslant k-1$, the inequality (k) implies $n_k \geqslant 2^{k-h-1}(s_h+1)$.

We shall prove the lemma by induction on k. For $k=h+1$ the lemma is true by (∗). For $k>h+1$, let us suppose, for contradiction, that the assertion is false for some k, and let k be the smallest integer for which some counterexample exists. Among all counterexamples $n_1, n_2, \ldots, n_k$, consider those which contain the longest subsequence $n_{h+1}, n_{h+2}, \ldots, n_{p-1}$ with $n_l = 2^{l-h-1}(s_h+1)$ for all $h+1 \leqslant l \leqslant p-1$ and $p-1<k$ ($p-1=h$ is allowed, in this case the subsequence is empty). Finally, among these latter restricted counterexamples, let $n_1, n_2, \ldots, n_k$ be one in which n_p is minimum.

It is clear that $n_p \neq 2^{p-h-1}(s_h+1)$, and, by (∗), $n_p \geqslant 2^{p-h-1}(s_h+1)$, i.e., $n_p \geqslant 2^{p-h-1}(s_h+1)+1$. Now we have three possibilities.

Case 1: $k=p+1$, *or* $k \geqslant p+2$ *and* $n_{p+1} < n_{p+2}$. In this case, we modify the sequence as follows: $n'_p = n_p - 1$ and $n'_{p+1} = n_{p+1}+1$. The sequence $n_1, n_2, \ldots, n'_p, n'_{p+1}, \ldots, n_k$

cannot be a counterexample for k, because $n'_p < n_p$ and we assumed that n_p is as small as possible. On the other hand, the inequalities (i) for $h+1 \leqslant i \leqslant k$ hold. Indeed,

- $(h+1), (h+2), \ldots, (p-1)$ remain unchanged,
- (p) is valid since $n'_p \geqslant 2^{p-h-1}(s_h+1)$,
- every (i) is again valid for $i > p$ since its left-hand side remains unchanged, while its right-hand side has increased because the only change in it is that $n_p(n_p-1) + n_{p+1}(n_{p+1}-1)$ is replaced by the larger number $(n_p-1)(n_p-2) + (n_{p+1}+1)n_{p+1}$.

This contradiction completes the proof of Case 1 when $k \geqslant p+2$ or $n_{p+1} \geqslant 2^{p-h}(s_h+1)$; and one can verify by a simple calculation that the inequality (k) does not hold when $k = p+1$ and $n_{p+1} = 2^{p-h}(s_h+1) - 1$.

Case 2: $k = p+2$, *or* $k \geqslant p+3$ *and* $n_{p+1} = n_{p+2} < n_{p+3}$. Now we make the following modifications: $n'_p = n_p - 1$ and $n'_{p+2} = n_{p+2} + 1$. Again, to obtain a contradiction, it suffices to prove that (i) holds for all $h+1 \leqslant i \leqslant k$ in the modified sequence. Analogous to the proof of Case 1, we obtain that $(h+1), (h+2), \ldots, (p), (p+2), \ldots, (k)$ are valid; the only inequality to prove is $(p+1)$. Let us suppose that $(p+1)$ is not valid, i.e.,

$$(s_{p+1}-1)(s_{p+1}-2) > 3\sum_{j=1}^{p-1} n_j(n_j-1) + 3(n_p-1)(n_p-2) + 3n_{p+1}(n_{p+1}-1).$$

We recall that the inequality $(p+2)$ is valid for the original sequence by assumption, and since $n_{p+1} = n_{p+2}$, we can write it as follows:

$$(s_{p+1}+n_{p+1})(s_{p+1}+n_{p+1}-1) \leqslant 3\sum_{j=1}^{p+1} n_j(n_j-1) + 3n_{p+1}(n_{p+1}-1).$$

Taking the sum of these last two inequalities, we obtain

$$s_{p+1}n_{p+1} + s_{p+1} - 1 < n_{p+1}(n_{p+1}-1) + 3n_p - 3.$$

This is impossible, however, because the condition $n_{p+1} \geqslant n_p \geqslant 1$ implies

$$\begin{aligned} s_{p+1}n_{p+1} + s_{p+1} - 1 &\geqslant n_p n_{p+1} + n_{p+1}^2 + n_p + n_{p+1} - 1 \\ &= n_{p+1}(n_{p+1}-1) + n_p n_{p+1} + 2n_{p+1} + n_p - 1 \\ &\geqslant n_{p+1}(n_{p+1}-1) + 4n_p - 1 \\ &> n_{p+1}(n_{p+1}-1) + 3n_p - 3. \end{aligned}$$

So $(p+1)$ is true for the new sequence. This contradiction completes the proof of Case 2 when $k \geqslant p+3$ or $n_{p+2} \geqslant 2^{p+1-h}(s_h+1)$; and by a simple calculation one can show that the inequality (k) does not hold when $k = p+2$ and $n_{p+1} = n_{p+2} = 2^{p-h+1}(s_h+1) - 1$.

Case 3: $n_{p+1} = n_{p+2} = n_{p+3}$. In this case we obtain an immediate contradiction, showing that the inequality $(p+3)$ is not valid. Indeed, denoting $n := s_{p+3}$, we have $n_j \leqslant n/3$ for all $1 \leqslant j \leqslant p+3$, thus

$$3(n_1(n_1-1) + n_2(n_2-1) + \cdots + n_{p+3}(n_{p+3}-1)) \leqslant 9\left(\frac{n}{3}\left(\frac{n}{3}-1\right)\right) < n(n-1).$$

This final contradiction completes the proof of the lemma. □

We conclude this section with an observation that shows how a fairly strong lower bound on the cardinalities of 'large' colour classes in a strict colouring can be obtained if the cardinalities of the 'small' classes correspond to a strictly coloured triple system.

Let $\mathscr{C}$ be the set of strict colourings of a BSTS(v) with $\bar{\chi}(v)$ colours. Consider a BSTS(v') or CSTS(v') with $v' > v$, and let $\mathscr{P}$ be one of its strict colourings with, say, k colours. Denote by n_i, $1 \leqslant i \leqslant k$, the cardinalities of colour classes of $\mathscr{P}$.

Theorem 3. *With the notation above, if $\mathscr{C}$ contains a strict colouring in which the first $\bar{\chi}(v)$ colour classes have respective cardinalities n_j, $1 \leqslant j \leqslant \bar{\chi}(v)$, then for the larger subscripts*

$$n_i \geqslant 2^{i-\bar{\chi}(v)-1}(v+1)$$

holds for all $\bar{\chi}(v) + 1 \leqslant i \leqslant k$.

Proof. Since $\mathscr{P}$ is a strict colouring for CSTS(v') [BSTS(v')], the inequalities (1) are valid by Lemma 2. Moreover, applying Lemma 3 for a suitably chosen strict colouring of BSTS(v) with $\bar{\chi}(v)$ colours, we obtain

$$3 \sum_{j=1}^{\bar{\chi}(v)} n_j(n_j - 1) = v(v-1).$$

Now the theorem follows by Lemma 4. □

One interesting point in the above result is that it *does not require any structural relationship* between the systems of orders v and v'.

3. Infinite classes of colourable CSTS and BSTS

In this section we determine the upper chromatic number for an infinite class of colourable CSTSs and BSTSs.

Let us consider first the system STS(9). It is presented in Table 1 and it is known to be unique up to isomorphism [2].

For CSTSs or BSTSs in general, more than one strict colouring with the same number of colours may exist, even with different cardinalities for the colour classes. A strict colouring $\mathscr{P}$ with h colours, in which each colour i occurs precisely n_i times, will be associated with the ordered h-tuple $(n_1, n_2, \ldots, n_h)$ termed the *colouring sequence* in $\mathscr{P}$. For convenience, $n_i \leqslant n_{i+1}$ will always be assumed for all $i < h$.

Table 1

{1,2,3}	{4,5,6}	{7,8,9}
{1,4,7}	{2,5,8}	{3,6,9}
{1,5,9}	{2,6,7}	{3,4,8}
{1,6,8}	{2,4,9}	{3,5,7}

Proposition 2. *For* BSTS(9), $\bar{\chi}(9) = 3$ *and there is a unique colouring sequence*: $(1,4,4)$.

Proof. Let us note first that every colouring sequence (of an arbitrary BSTS(v) for any $v>3$) has length at least 3, because Steiner triple systems are well known to admit no blocking set (i.e., no 2-colouring). Now, for $v=9$, one can easily verify that the three sets $\{1\}$, $\{2,3,4,7\}$, $\{5,6,8,9\}$ as colour classes of the BSTS(9) exhibited in Table 1 form a strict colouring, i.e., $(1,4,4)$ is indeed a possible colouring sequence.

Beside $(1,4,4)$, there is one further sequence of length 3 satisfying the inequalities of Lemma 2 and the equality of Lemma 3, namely $(2,2,5)$. We are going to show that the latter is not a colouring sequence for any strict colouring, not even for CSTS(9). Suppose the contrary, and let $\{l_1,l_2\}$ and $\{m_1,m_2\}$ be the two 2-element colour classes. Since $\{l_1,l_2,m_1,m_2\}$ can contain at most one block of STS(9), at least one of the two distinct blocks containing the pairs $\{l_1,m_1\}$ and $\{l_2,m_2\}$ has a vertex from the third colour class, a contradiction.

Finally, no colouring sequence of length 4 or more can exist, because $\bar{\chi}(9)<4$ by Theorem 1. □

Proposition 3. *For* CSTS(9), $\bar{\chi}(9)=3$ *and there are two possible colouring sequences*: $(1,4,4)$ *and* $(1,2,6)$. *Moreover, in colourings of the latter type, precisely two monochromatic blocks occur.*

Proof. We have seen in the proof of Proposition 2 that $(1,4,4)$ is indeed a colouring sequence for CSTS(9), and that every colouring sequence has length at most 3. We have already excluded $(2,2,5)$; moreover, $(2,3,4)$ and $(3,3,3)$ contradict the case $i=3$ of Lemma 2. One can also exclude the sequence $(1,3,5)$ by applying a simple argument similar to that for $(2,2,5)$, considering the two small colour classes. Hence, beside $(1,4,4)$, the sequence $(1,2,6)$ remains the only possibility to consider for CSTS(9). In a colouring of this type, Proposition 1 yields that the number of monochromatic blocks equals $|\mathscr{B}| - c/2 = 2$. □

We next consider larger systems built from STS(9).

Theorem 4. *All the systems* CSTS(v') [BSTS(v')], *where* $v' = 10 \times 2^k - 1$, *obtained by a sequence of doubling plus one constructions starting from* CSTS(9), *are colourable and have* $\bar{\chi}(v') = k+3$.

Proof. We can apply the inequality (2) of Theorem 2 with $t=4$ (as $9<2^4-1$). Since $\bar{\chi}(\mathrm{CSTS}(9))=3$ by Propositions 2 and 3, we obtain $\bar{\chi}(v')=k+3$. □

Corollary 1. *If a* CSTS(v') [BSTS(v')] *with* $v' = 10 \times 2^k - 1$ *has a colouring with* $k+3$ *colours and the first three colour classes have cardinalities* $n_1=1$, $n_2=n_3=4$, *then the other colour classes have cardinalities*

$$10 \times 2^0,\ 10 \times 2^1, \ldots, 10 \times 2^k.$$

Proof. By Theorem 3 we have $n_i \geqslant 10 \times 2^{i-4}$ for $i \geqslant 4$, so

$$10 \times 2^k - 1 = \sum_{i=1}^{k+3} n_i \geqslant 9 + 10 \sum_{i=4}^{k+3} 2^{i-4} = 10 \times 2^k - 1,$$

therefore, equality must hold throughout. □

On applying Theorem 4, one can determine the upper chromatic number of an infinite class of colourable CSTSs and BSTSs.

The above procedure can be repeated. In fact, if we know the exact value of $\bar{\chi}(v)$ for a CSTS(v) or BSTS(v), with $v \neq 2^k - 1$ or $v \neq 9$, it is possible to find an infinite class of colourable CSTSs or BSTSs, and, under particular conditions, to determine the upper chromatic number for each of them.

We can use this procedure, for example, for CSTS(13) and BSTS(13). In fact, the first author [3] has verified that $\bar{\chi}(13) = 3$, and based on this fact, by a sequence of doubling plus one constructions starting from STS(13), it is possible to determine an infinite class of colourable CSTS(v') and BSTS(v'), where $v' = 14 \times 2^k - 1$, with $\bar{\chi}(v') = k + 3$.

Finally, we show a consequence of Theorem 3 for CSTS(v) and BSTS(v) where $v < 10 \times 2^k - 1$.

Theorem 5. *Suppose that a colourable* CSTS(v) [BSTS(v)] *of order* $v < 2^k \times 10 - 1$ *has a strict colouring* $\mathscr{P}$ *with* $\bar{\chi}(v)$ *colours where the first three colour classes are of cardinalities* $n_1 = 1$, $n_2 = n_3 = 4$. *Then* $\bar{\chi}(v) < k + 3$.

Proof. The assumptions together with Theorem 3 imply

$$10 \times 2^k - 1 > v = \sum_{i=1}^{\bar{\chi}} n_i \geqslant 9 + 10 \sum_{i=4}^{\bar{\chi}} 2^{i-4} = 9 + 10(2^{\bar{\chi}-3} - 1),$$

therefore, $\bar{\chi} < k + 3$ follows. □

It is worth noting that, without the condition of type (1, 4, 4) on the first three colour classes, the conclusion does not necessarily hold. This fact is shown, e.g., by the BSTS(v) and CSTS(v) with $v = 2^{k+3} - 1$, $\bar{\chi}(v) = k + 3$, obtained by the repeated application of the doubling plus one construction.

4. Concluding remarks

In [4] we found an upper bound on the upper chromatic number of CSTSs and BSTSs, and also described an infinite class of colourable BSTSs with an extremal upper chromatic number.

In this paper, we presented further infinite classes of colourable STSs whose upper chromatic number can be found explicitly. Moreover, by using Theorems 2 and 3, a method is shown that permits further tight estimates for other infinite (colourable) classes.

Also, from Theorem 5, under particular conditions on the types of colourings, it is possible to obtain an upper bound better than the one in [4] for the upper chromatic number.

One important tool is the investigation of the colour patterns of the blocks in the Steiner triple systems. In particular, it can be useful to determine which (or, how many) of the blocks are monochromatic. Information of this kind may lead to sharper results and a better description of the different colourings.

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