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Polygon vertex extremality and decomposition of polygons

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ABSTRACT

In this paper, we show that if we decompose a polygon into two smaller polygons, then by comparing the number of extremal vertices in the original polygon versus the sum of the two smaller polygons, we can gain at most two globally extremal vertices in the smaller polygons, as well as at most two locally extremal vertices. We then will derive two discrete Four-Vertex Theorems from our results.

1. Introduction

There are many notions of extremality in polygons, the earliest appeared circa 1813 in [2]. Recently, a very natural type of extremality was introduced in [5], one which very consistently adhered to that of curvature in the smooth case. A closely related global analogue had already appeared much earlier, and it as well has a smooth and discrete interpretation. While it is debatable to whom we attribute this discrete global notion of extremality, closely related ideas are presented in [1].

In this paper, we will expound on these two types of extremality by providing a few observations and facts to build intuition. We will then discuss the notion of decomposing a polygon and investigate how this impacts our two types of extremality. We then derive fresh results relating the number of extremal vertices of the larger polygon versus the two smaller polygons of decomposition. While our results will be relevant geometrically on their own, we will observe that they are closely tied to two discrete Four-Vertex Theorems pertaining to our two types of extremality, which follow almost immediately from our stronger results.

We note that we will skip proofs of the more simple results. All results in this paper are considered with much more detail in [4].

2. Global and local extremality

We denote by P a polygonal curve, which is a simple piecewise linear curve with vertices $V_1, V_2, \ldots, V_n$. When we speak of a closed polygonal curve, we will refer to it as a polygon. Also, we restrict our consideration simply to the planar case and all indices will be taken modulo the number of vertices of the polygonal curve. The following definition was coined in [6]:

Definition 2.1. We say that a polygonal curve is generic if the maximal number of vertices that lie on a circle is three and no three vertices are collinear.

Observe that all regular polygons are not generic.

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Definition 2.2. Let C_{ijk} be a circle passing through any three vertices V_i, V_j, V_k of a polygonal curve. We say that C_{ijk} is empty if it contains no other vertices of the polygonal curve in its interior, and we say that it is full if it contains all of the other vertices of the polygonal curve in its interior.

For simplicity, we will denote a circle passing through consecutive vertices V_{i-1}, V_i and V_{i+1} by C_i.

Definition 2.3. We call a full or empty circle C_i an extremal circle. We refer to the corresponding vertex V_i as a globally extremal vertex.

Some of our results will use triangulation arguments. Consider all of the empty circles passing through any three distinct points of a polygon. In [3] Delaunay shows that the triangles formed by each of the three points corresponding to an empty circle form a triangulation of the polygon P. This triangulation is called a *Delaunay triangulation*.

Analogously, if we assume convexity on our polygon and consider the full circles passing through any given three points, the triangles given by each of the three points corresponding to a full circle also form a triangulation. This triangulation is commonly known as the *Anti-Delaunay triangulation*.

Definition 2.4. A vertex V_i is said to be positive if the left angle with respect to orientation, $\angle V_{i-1}V_iV_{i+1}$, is at most π. Otherwise, it is said to be negative.

Definition 2.5 (*Discrete Curvature*)**.** Assume that a vertex V_i is positive. We say that the curvature of the vertex V_i is greater than the curvature at V_{i+1} ($V_i \succ V_{i+1}$) if the vertex V_{i+1} is positive and V_{i+2} lies outside the circle C_i or if the vertex V_{i+1} is negative and V_{i+2} lies inside the circle C_i.

By switching the word "inside" with the word "outside" in the above definition (and vice versa), we obtain that $V_i \prec V_{i+1}$, or that the curvature at V_i is less than the curvature at V_{i+1}. In the case that the vertex V_i is negative, simply switch the word "greater" with the word "less", and the word "outside" by the word "inside".

Definition 2.6. A vertex V_i of a polygonal line P is locally extremal if

$$V_{i-1} \prec V_i \succ V_{i+1} \quad \text{or} \quad V_{i-1} \succ V_i \prec V_{i+1}.$$

Remark 2.1. If we assume convexity on our polygon and observe the definition of locally extremal vertices closely, we simply are considering the position of the vertices V_{i-2} and V_{i+2} with respect to the circle C_i. Our vertex V_i will be locally extremal if and only if both vertices V_{i-1} and V_{i+2} lie inside or outside the circle C_i.

When defining global extremality, we discussed empty and full extremal circles. If a circle C_i is empty, then we say that the corresponding vertex V_i is maximal. If C_i is full, then we say V_i is minimal. Analogously for locally extremal vertices, we call a vertex maximal if $V_{i-1} \prec V_i \succ V_{i+1}$ and minimal if $V_{i-1} \succ V_i \prec V_{i+1}$.

We denote the number of globally maximal-extremal vertices of a polygonal curve P by $s_-(P)$ and globally minimal-extremal vertices by $s_+(P)$ to be consistent with [1]. For locally extremal vertices, we will attribute the notation $l_-(P)$ and $l_+(P)$, respectively.

Proposition 2.1. *Let P be a generic convex polygon. Then*

$$l_+(P) = l_-(P).$$

Remark 2.2. The proof of this fact immediately follows by carefully observing the definition of locally extremal vertices. Note that it was very important for us to include the assumption that our polygon is generic, since this eliminates the possibility of having two extremal vertices adjacent to each other. Also, it is easy to see that the equality $s_+(P) = s_-(P)$ does not hold. In fact, we cannot form any relationship between globally maximal-extremal and globally minimal-extremal vertices.

Proposition 2.2. *Let P be a generic convex polygon. If V_i is a globally extremal vertex, then V_i is a locally extremal vertex.*

This result follows immediately from the observation made in Remark 2.1.

Proposition 2.3. *Let P be a generic convex quadrilateral. Then P has four globally extremal and locally extremal vertices.*

Proof. For globally extremal vertices, we apply a Delaunay triangulation to P, which immediately yields two globally maximal-extremal vertices. We then apply an Anti-Delaunay triangulation to P, which yields two minimal-extremal vertices. Proposition 2.2 then yields the result for locally extremal vertices. □

While the following proposition is technical yet quite obvious, it will be a vital proposition that will be used frequently to prove our main results.

Proposition 2.4. *Let A, B, C and X be four points in the plane in a generic arrangement, C_B be the corresponding circle passing through A, B and C, and let C_A be the circle passing through the points X, A and B. We denote by $\widetilde{C}_A$ and $\widetilde{C}_B$ the open discs bounded by C_A and C_B, respectively. Denote by H^+_{AB} the half-plane formed by the infinite line AB containing the point C and by H^-_{AB} the half-plane formed by the infinite line AB not containing the point C. If X lies in $\widetilde{C}_B \bigcap H^+_{AB}$, then C lies in $H^+_{AB} \setminus \widetilde{C}_A$. If X lies in $H^+_{AB} \setminus \widetilde{C}_B$, then C lies in $\widetilde{C}_A$. Analogously, if X lies in $\widetilde{C}_B \bigcap H^-_{AB}$, then C lies in $\widetilde{C}_A$. If X lies in $H^-_{AB} \setminus \widetilde{C}_B$, then C lies in $H^+_{AB} \setminus \widetilde{C}_A$.*

Proof. The proof is a simple verification of the situation restricted around the origin and solving corresponding systems of equations. □

3. Globally extremal vertices and decomposition of polygons

Definition 3.1. We say an edge or diagonal of a polygon is Delaunay if there exists an empty circle passing through the corresponding vertices of that edge or diagonal. If there exists a full circle passing through the vertices of this edge or diagonal, then we say the edge or diagonal is Anti-Delaunay.

Remark 3.1. Note that a triangulation of a polygon where every edge and diagonal is Delaunay is a Delaunay Triangulation. Similarly, if every edge and diagonal of a triangulation is Anti-Delaunay, then we have an Anti-Delaunay triangulation.

So what exactly does it mean to decompose a polygon? Here the notion of decomposing a polygon will simply be the cutting of a polygon P by passing a line segment through any two vertices so that the line segment lies in the interior of the polygon. We will call this line segment a *diagonal*. Also, we denote the two new polygons formed by a decomposition by P_1 and P_2 and require that they each have at least four vertices. By this last requirement it automatically follows that P must have at least six vertices to successfully perform a decomposition.

Theorem 3.1. *Let P be a generic convex polygon with six or more vertices and P_1 and P_2 be the resulting polygons of a decomposition of P. Assume that the diagonal of this decomposition is Delaunay. Then*

$$s_-(P) \geq s_-(P_1) + s_-(P_2) - 2.$$

Analogously, if the diagonal is Anti-Delaunay, then

$$s_+(P) \geq s_+(P_1) + s_+(P_2) - 2.$$

Proof. We begin by applying a Delaunay triangulation to P, P_1 and P_2. Noticing that the diagonal is Delaunay for P_1 and P_2, as well as P by assumption, we obtain our first inequality. For the second inequality we mimic this argument, instead applying an Anti-Delaunay triangulation. □

It turns out that from the above result, we can derive a very nice geometric corollary. First, we need two small lemmas.

Lemma 3.1. *Let P be a convex polygon with seven or more vertices and let $T(P)$ be a triangulation of P. Then, there exists a diagonal of our triangulation such that, if we apply a decomposition of P using this diagonal, then both P_1 and P_2 have four or more vertices.*

This result is clear, and follows immediately by an induction argument on the number of vertices.

Remark 3.2. It is obvious that this result does not hold if $n = 6$. In fact, it is easy to find a convex polygon whose Delaunay Triangulation does not satisfy Lemma 3.1, hence the need for one more lemma.

Lemma 3.2. *Let P be a generic convex polygon with six vertices and let P_1 and P_2 be the resulting polygons of a decomposition. Then*

$$s_-(P) \geq s_-(P_1) + s_-(P_2) - 2$$

and

$$s_+(P) \geq s_+(P_1) + s_+(P_2) - 2.$$

Proof. Since we have no guarantee that our diagonal is Delaunay, we cannot mimic the proof of Theorem 3.1. We observe that, since P is generic, P_1 and P_2 are generic as well. Moreover, P_1 and P_2 are quadrilaterals. By applying Proposition 2.3 to P_1 and P_2, we prove our assertion. □

Corollary 3.1 (*The Global Four-Vertex Theorem*)**.** *Let P be a generic convex polygon with six or more vertices. Then*

$$s_+(P) + s_-(P) \geq 4.$$

Proof. We will prove the result by induction on the number of vertices of P. We first consider the base case $n = 6$, noticing if we apply a decomposition to P, then P_1 and P_2 are both quadrilaterals. By Proposition 2.3, we obtain that P_1 and P_2 each have four globally extremal vertices. It follows from Lemma 3.2 that P has four globally extremal vertices.

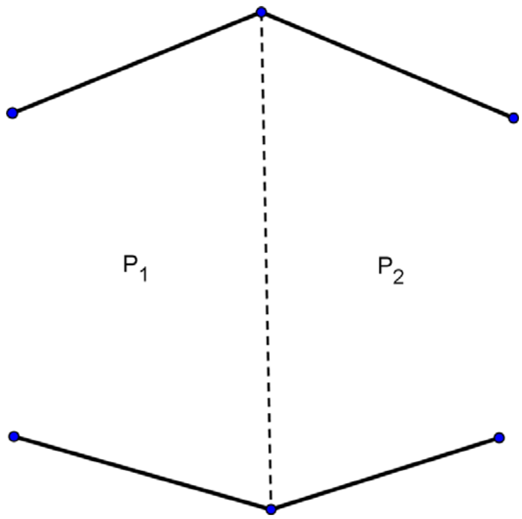

Fig. 1.

We now consider the case where $n \geq 7$. We begin by applying a Delaunay triangulation to P. By Lemma 3.1, it follows that there exists a diagonal d such that when we decompose P by this diagonal, P_1 and P_2 each have four or more vertices. Since our diagonal corresponds to a Delaunay triangulation, it follows that d is Delaunay. Since P_1 and P_2 have less vertices than P, we apply the inductive assumption to obtain $s_-(P_1) \geq 2$ and $s_-(P_2) \geq 2$. Applying this to Theorem 3.1, we obtain $s_-(P) \geq 2$. An analogous argument using an Anti-Delaunay triangulation and Theorem 3.1 yields $s_+(P) \geq 2$. So $s_+(P) + s_-(P) \geq 4$, proving the assertion. □

4. Locally extremal vertices and decomposition of polygons

When considering locally extremal vertices, it is easy to see that the only vertices affected by a decomposition of a polygon will be the vertices on the diagonal of decomposition and the neighboring vertices (see Fig. 1).

This means that we have a total of six vertices impacted by a decomposition, leading us to a feasible case-by-case analysis. Before proving our main result, we need a few lemmas.

Lemma 4.1. *Let P be a generic convex polygon and P_1 and P_2 the resulting polygons of a decomposition. Denote the vertices of the diagonal by B and D, the neighboring vertex of B in P_1 by A, and the neighboring vertex of B in P_2 by C. Assume that A is locally maximal-extremal in P_1 but not in P, and that C is locally maximal-extremal for P_2 but not in P. Then, B is a locally maximal-extremal vertex for P.*

Proof. Let X be the neighbor of A in P_1 and Y be the neighbor of C in P_2. Denote the circle passing through vertices A, B and C by C_B, the circle passing through vertices X, A and B by C_A, and the circle passing through vertices B, C and Y by C_C. Since A is not maximal-extremal in P, it follows that A lies inside the circle C_C. By Proposition 2.4, it follows that Y lies outside of the circle C_B. Since C is not maximal-extremal in P, it follows that C lies inside the circle C_A. By Proposition 2.4, it follows that X lies outside of the circle C_B. Fig. 2 illustrates the situation.

Since both X and Y lie outside of the circle C_B, B is maximal-extremal in P. □

Lemma 4.2. *Let P be a generic convex polygon and P_1 and P_2 the resulting polygons of a decomposition. Denote the vertices of the diagonal by B and D, the neighboring vertex of B in P_1 by A, and the neighboring vertex of B in P_2 by C. Assume that A is locally maximal-extremal in P_1 but not in P, and that B is locally maximal-extremal in P_2. Then, B is locally maximal-extremal in P.*

Proof. For simplicity, consider Fig. 3, which will illustrate our configuration of points and circles.

Let X be the neighbor of A in P_1 and Y be the neighbor of C in P_2. Denote by C_A the circle passing through vertices X, A, and B. Since A is maximal-extremal in P_1, it follows that D lies outside of the circle C_A. Since A is not maximal-extremal in P, it follows that C must lie inside the circle C_A. Now, denote the circle passing through vertices A, B, and C by C_B. Our goal is to show that vertices X and Y lie outside of the circle C_B.

A quick application of Proposition 2.4 to points X, C, A and B yields that X lies outside of C_B, so we need to show that Y lies outside of the circle C_B. Denote by C_B' the circle passing through the points C, B and D. We will show that if Y lies outside of C_B', then it lies outside of C_B. To do this, we first must show that A lies inside the circle C_B'.

Consider the circles C_A and C_B'. These circles intersect at two points, point B and some other point, say Z. Since D lies outside of the circle C_A, it follows by an application of Proposition 2.4 to points A, D, B and Z that A lies inside the circle C_B'.

Lastly, consider the circles C_B and C_B'. These two circles intersect at the points B and C. Since A lies inside the circle C_B', it follows from applying Proposition 2.4 to points A, D, B and C that D lies outside of the circle C_B.

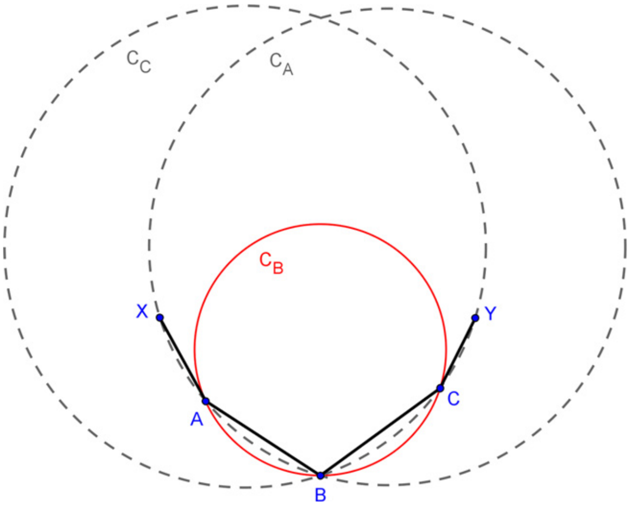

Fig. 2.

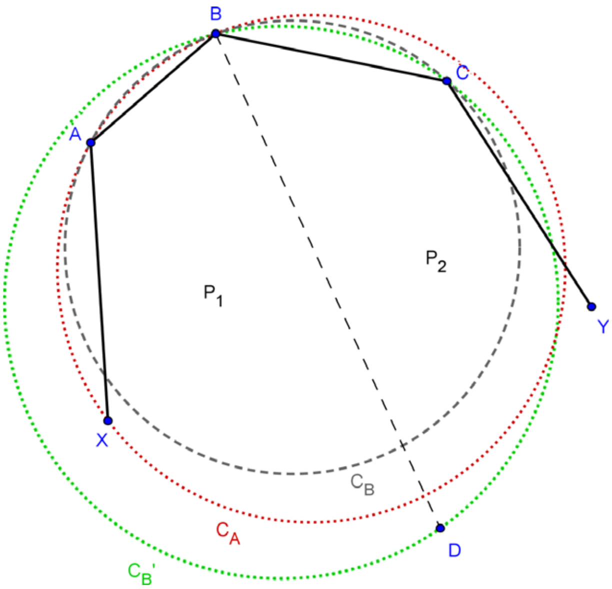

Fig. 3.

Now, since B is maximal-extremal in P_2, it follows that Y lies outside of C'_B. By our above observation, it follows immediately from Proposition 2.4 that Y lies outside of C_B. Since points X and Y both lie outside of the circle C_B, it follows that B is maximal-extremal in P. □

Lemma 4.3. *Let P be a generic convex polygon and P_1 and P_2 the resulting polygons of a decomposition. Denote the vertices of the diagonal by B and D, the neighboring vertex of B in P_1 by A, and the neighboring vertex of B in P_2 by C. Assume that A is locally maximal-extremal for P_1 and D is locally maximal-extremal for both P_1 and P_2, but not for P. Then A is locally maximal-extremal for P.*

Proof. Let X be the neighbor of A in P_1, E be the neighbor of D in P_1, and F be the neighbor of D in P_2. Denote by C_{D1} the circle passing through vertices B, D and E, by C_{D2} the circle passing through vertices B, E and F, and by C_A the circle passing through vertices X, A and B. Fig. 4 illustrates our configuration.

Our goal is to show that vertex C lies outside of the circle C_A. We will do this by showing that if C lies outside the circle C_{D2}, then it also lies outside of circle C_A. Since A is maximal-extremal in P_1, it follows that D lies outside of C_A. Since D is maximal-extremal in P_1, it follows that A lies outside of circle C_{D1}. By a similar argument used in the previous lemma, it follows that if C lies outside of C_{D1} then it lies outside of C_A. Fig. 5 illustrates this situation.

It remains to show that C lies outside of C_{D1}. Consider the circles C_{D1} and C_{D2}. Since D is maximal-extremal in P_2, it follows that C lies outside of the circle C_{D2}. If we show that C also lies outside of C_{D1}, then we are done. To do this, we will heavily

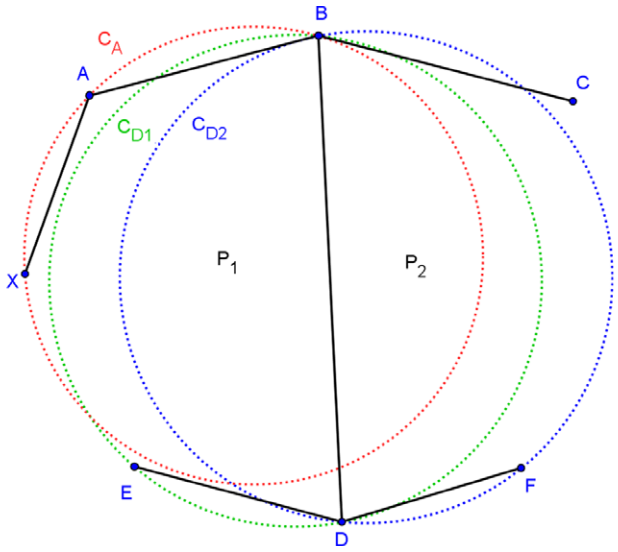

Fig. 4.

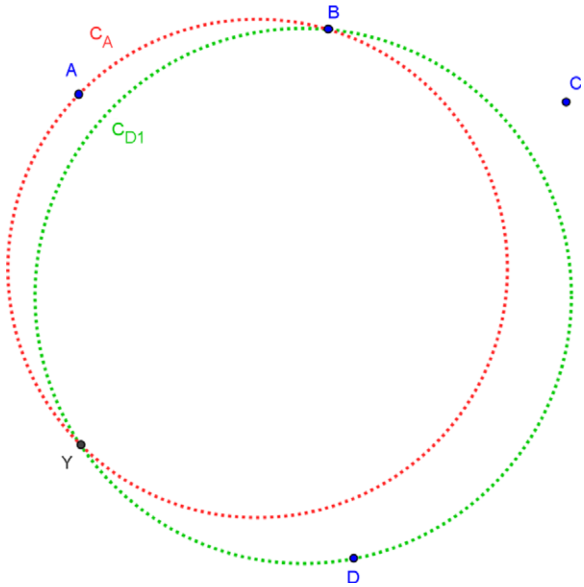

Fig. 5.

use the fact that D is not maximal-extremal in P. We will show that if E lies inside the circle C_{D2} or if F lies in C_{D1}, then D is maximal-extremal in P, contradicting our assumption.

It is enough just to check this for E. Denote the circle passing through vertices E, D and F by C_D. If E lies inside the circle C_{D2}, then applying Proposition 2.4 to points E, D, F and B yields that B lies outside of the circle C_D. Similarly, it follows by Proposition 2.4 that F lies inside the circle C_{D1}.

Now denote by E' the neighbor of E and by F' the neighbor of F. Fig. 6 illustrates the situation.

Since D is maximal-extremal in P_1, it follows that E' lies outside of the circle C_{D1}. Similarly, since D is maximal-extremal in P_2, it follows that F' lies outside the circle C_{D2}. Now, recall that B lies outside of the circle C_D. Proposition 2.4 applied to points E', B, D and D tells us that E' lies outside of C_D. A similar argument yields that F' also lies outside of C_D. So, we obtain that D is maximal-extremal in P, a contradiction.

So now we know that E must lie outside of the circle C_{D2}. Proposition 2.4 applied to points F, E, B and D now tells us that F lies outside of the circle C_{D1}. So, if C were to lie outside of circle C_{D2}, then it would also lie outside of the circle C_{D1}. But earlier we showed that if C would lie outside of circle C_{D1}, then C would lie outside of the circle C_A. Indeed, by assumption, C lies outside of C_{D2} and hence outside of C_A. Since A is maximal-extremal in P_1, it also follows that X lies outside of the circle C_A. Therefore A is maximal-extremal in P. □

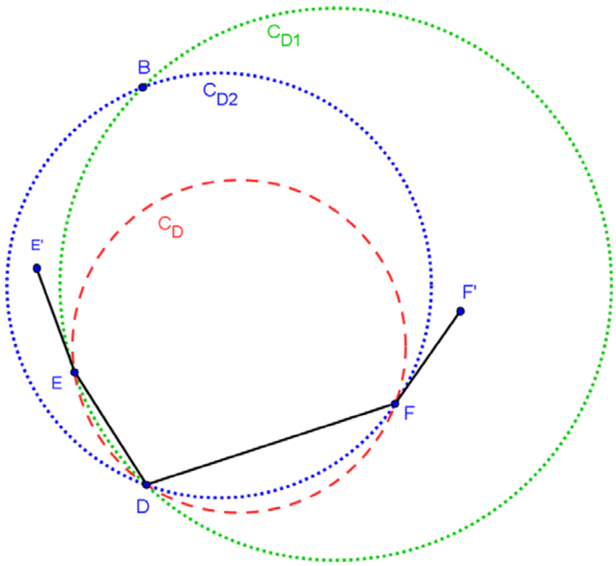

Fig. 6.

Theorem 4.1. *Let P be a generic convex polygon with at least 6 vertices and let P_1 and P_2 be the resulting polygons of a decomposition. Then*

$$l_-(P) \geq l_-(P_1) + l_-(P_2) - 2.$$

Proof. We note that only six vertices are affected by a decomposition from the local point of view: the vertices of the diagonal and the neighbors of those vertices. So, we eliminate the cases which violate our inequality. It is easy to check that by the symmetry of our cases, we only need to check three:

Case 1: We gain two maximal-extremal vertices in P_1, as well as P_2, but none of the six vertices are maximal-extremal in P.

Case 2: We gain two maximal-extremal vertices in P_1 and gain two maximal-extremal vertex in P_2, and one of the six vertices is maximal-extremal in P.

Case 3: We gain two maximal-extremal vertices in P_1 and gain one maximal-extremal vertex in P_2, and none of the six vertices is maximal-extremal in P.

By checking the possible configurations of vertices in each of the cases, we see that each case admits a configuration which is deemed not feasible by one of the three preceding lemmas. □

Corollary 4.1 (*The Local Four-Vertex Theorem*)**.** *Let P be a generic convex polygon with at least six vertices. Then*

$$l_+(P) + l_-(P) \geq 4.$$

Proof. We apply induction on the number of vertices of P. For the case where $n = 6$, we know that if we apply a decomposition to P, then both P_1 and P_2 will be quadrilaterals. Proposition 2.3 yields that $l_-(P_1) = l_-(P_2) = 2$. Applying this to Theorem 4.1 completes the proof for this case.

Now, assume that $n \geq 7$. We now apply induction to the smaller polygons P_1 and P_2 to obtain that $l_-(P_1) \geq 2$ and $l_-(P_2) \geq 2$. We now apply this to Theorem 4.1 to obtain that $l_-(P) \geq 2$. By Proposition 2.1, we obtain that $l_+(P) \geq 2$. Therefore $l_+(P) + l_-(P) \geq 4$, proving the assertion. □

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References

[1] R.C. Bose, On the number of circles of curvature perfectly enclosing or perfectly enclosed by a closed oval, Math. Ann. 35 (1932) 16–24.
[2] A.L. Cauchy, Recherche sur les polyèdres - premier mémoire, Journal de l'Ecole Polytechnique 9 (1813) 66–86.
[3] B. Delaunay, Sur la sphère vide, Izvestia Akademii Nauk SSSR, Otdelenie Matematicheskikh i Estestvennykh Nauk 7 (1934) 793–800.
[4] W.J. Mogilski, The Four-Vertex Theorem, The Evolute, and The Decomposition of Polygons. arXiv:0906.2388v2 [math.MG] (2009).
[5] O.R. Musin, Curvature extrema and four-vertex theorems for polygons and polyhedra, Journal of Mathematical Sciences 119 (2004) 268–277.
[6] Igor Pak, Lectures on Discrete and Polyhedral Geometry, 2008, pp. 183–197.