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Nature-inspired framework for measuring visual image resemblance: A near rough set approach

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ABSTRACT

The problem considered in this paper is how to determine the degree of nearness between complex visual objects. The proposed solution to this problem stems from a natural computing approach to solving the visual acuity problem in terms of a granular representation of visual information that is quantifiable as well as understandable for humans. This is accomplished via a near rough set framework in the approximation of a pair of disjoint sets and measurement of distances between sets using various fuzzy pseudometrics. Pseudometrics, in general, and fuzzy pseudometrics, in particular, are useful in measuring the distance between pairs of objects such as sets. Such distances are indicators of the nearness of (resemblance between) visual objects. These observations lead to a number of practical applications such as object recognition and object retrieval in digital image analysis. One such application is reported in this article. The contribution of this article is threefold: introduction of a nature-inspired framework for measurement of visual object resemblance, four different incarnations of the standard fuzzy metric and application of fuzzy metrics in content-based image retrieval experiments.

1. Introduction

The work reported in this article leads to a nature-inspired solution to the human visual acuity problem in representing visual images in an understandable way for humans. The visual acuity problem stems from the difference between *visual acuity* (what a person sees) and the actual visual object. This disparity between perceived and actual images is directly related to the frame problem in artificial intelligence and the need to offer different representations of visual images as a means of complementing (supplementing) the frailty and faultiness of visual perception (see, *e.g.*, the study of active image processing in [82, p. 304ff]). This separation between visual acuity and actual image features becomes evident when one considers similarities between regions of digital images that are rough sets.

Visual perception is characterized by a blending together of the parts of a visual image (*e.g.*, the coarse-grained view that *the grass is green* vs. the fine-grained view that there are many shades of green in blades of grass) and the actual separation of the parts of a visual image that is possible if one granulates visual images (see, *e.g.*, [84,82]). Evidence of a nature-inspired approach in image analysis in this work can be found in the early research by Zeeman and Buneman on tolerance spaces and the brain during the 1960s [85] as a result of the pioneering work by Zeeman [84].

This research is based on a set theoretic approach to image analysis where each image is viewed as a set of perceptual elements.

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Each perceptual element can be a pixel or a small subimage or image patch. An *image patch* is a collection of subimages that have similar average feature values (*e.g.*, colour) and that are not necessarily contiguous. This phenomenon was observed in a study of the nearness of objects in [56, p. 498, Fig. 1.2]. The rationale behind defining a perceptual element has both a practical and a physical aspect. From a practical point of view, it is easier to consider a small patch of adjacent pixels as a unit of perception and thus reducing the amount of information needed to represent the image as it is perceived by a human. From a physical point of view, we know that we do not see images in a pixel-based resolution and our local perception of the image forms by a group of pixels.

The problem considered in this paper is how to establish a useful framework for the study of resemblance of visual objects. Put another way, the problem considered in this article is how to measure the nearness (resemblance between) non-empty, disjoint sets and, in particular, the resemblance between rough sets. The solution to the problem results from a combination of the near rough set approach [65] to perceiving complex objects that stems from a nature-inspired computing approach [20] in solving the visual acuity problem in terms of a granular representation of visual information understandable for humans and the use of a number of nearness pseudometrics in the instantiation of what is known as the standard fuzzy metric (sfm) [13]. A pseudometric is a real-valued distance function useful in measuring the distance between pairs of objects such as sets.

A set of objects that is roughly classified is a rough set [46,50,49,48], *i.e.*, a set with a non-empty approximation boundary is a rough set. An sfm is defined in the context of fuzzy sets [83] and pseudometric spaces [17,11]. A pseudometric space $\langle X, \rho\rangle$ consists of a non-empty set X together with a function ρ satisfying a number of conditions (see Section 3 for the details). The function $\rho : X \times X \rightarrow \Re^{0+}$ returns a non-negative, real value representing the distance between points in a non-empty set X. The elements of X are called 'points' of the space and ρ is called a pseudometric or distance function. In this work, the focus is on a determination of nearness of digital images viewed as sets of points where each point corresponds to a part of an image such as a picture element (pixel). It is often the case that such sets of points are rough sets (see, *e.g.*, [31,42,45,54,44,65,62]). The feature space for sets of objects that are roughly classified is a set of feature vectors. A feature vector is an n-dimensional vector of real-valued features (*e.g.*, colour brightness, edge orientation, texture) that represent an object. In this article, the focus is on fuzzy metric-based approach in measuring the nearness of rough sets. A number of incarnations of the sfm are introduced in this article based on various distance functions. This makes it possible to quantize the nearness of disjoint sets.

This paper has the following organization. Related works for this article are presented in Section 2. The preliminaries concerning pseudometrics, a feature-based pseudometric and nearness of sets based on the gap functional are given in Section 3. The basics concerning fuzzy sets are given in Section 4. Various incarnations of the standard fuzzy metric are presented in Section 5. Near rough sets are defined and illustrated in Section 6. A proposed nature-inspired computing approach to solving the visual acuity problem in terms of understandable representations of granules of visual information for humans is given in Section 7. Image retrieval experiments using the four different fuzzy metrics are given in Section 8.

2. Related works

The proposed approach to comparing visual objects is related to a number of research streams. Interest in the nearness of sets of points can be traced back to a number of researchers that include J.H. Poincaré in the study of the apparent similarity between sets of sensations [59], Zeeman and Buneman on visual acuity, tolerance spaces and the brain and the separation between what we see and actual visual images [84,85], a problem formulated in the context of natural computing by Wörgötter et al. [82], Riesz on the spatial nearness of sets of points, Fréchet on metric spaces [12], Menger on statistical metric spaces and the t-norm [30], Zadeh on fuzzy sets [83], Naimpally on proximity spaces [34,33,7], Düntsch and Vakarelov on region-based theory of discrete spaces [9], Düntsch and Winter on mereotopological structures [10], Pawlak on the indiscernibility relation and the approximation of sets [46,47,50,49,48]. Of particular interest in this work is the recent study of nearness of objects considered within the context of approximation spaces [55,56], clustering [5], semantic similarity [1], approach spaces [57,64] and near sets [52,51,58,81,18,53,61,60,64,63,80].

This problem of the separation of objects in visual perception has been studied in terms of topography as a statistical property of visual input [19]. For simplicity, a digital image is granulated by organizing image pixels as a collection of visual information granules called subimages. Each subimage granule is a $p \times p$ square with p pixels on an edge. The choice of p is optional. A useful guideline in choosing the p is that it is small enough to represent details in an image and large enough to reduce the number of computational steps in arriving at a cover of an image as a collection of classes where each class contains subimages with similar descriptions. By granulating a digital image, the separation between a perceived image and an actual physical image can be readily observed. Two points in a visual field can be perceived as distinct if they stimulate two different cones on the retina that are separated by at least one other cone [26]. For example, there is considerable separation between in what one sees in looking at Lena's right eye and the 40,990 pixels represented by the 15×15 subimages in the projection of a total of 13×14 eye image granules in Fig. 1.

In this article, nearness of rough sets is considered in the context of a metric space. Briefly, a Fréchet form of a metric space is a non-empty set X supplied with a distance function $\rho : X \times X \rightarrow \Re^{0+}$ that returns non-negative real values ($\forall x, y \in X, \rho(x, y) \geq 0$) that satisfies several conditions: zero for indiscernibles ($\rho(x, y) = 0$, if and only if $x = y$), symmetry ($\forall x, y \in X, \rho(x, y) = \rho(y, x)$) and triangle inequality ($\forall x, y, z \in X, \rho(x, z) \leq \rho(x, y) + \rho(y, z)$). A function ρ is a metric if, and only if it satisfies the metric space conditions. To achieve generality, a pseudometric is introduced on

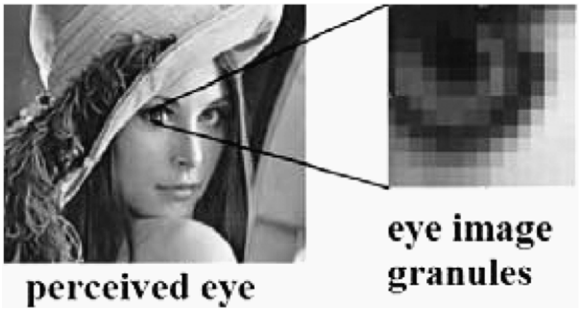


Fig. 1. Perception.

the set X by replacing the identity of indiscernibles condition with $\rho(x, x) = 0$ for every $x \in X$ [11]. It has been shown that pseudometric spaces provide a powerful means of studying the nearness of sets [79].

The extent of the separation between near rough sets can be measured employing what is known as a gap functional [34, p. 7] (see, also, [22]) that is defined in terms of the Hausdorff lower distance between points. Briefly, given a metric space (X, ρ), a gap functional $D_\rho(A, B)$ for $A, B \subset X$ is defined to be

$$D_\rho(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\},$$

i.e., the greatest lower bound of the distances between pairs of elements in A and B. This is explained in more detail in Section 3. The study of gap functionals [3,4] stems from work on the normality of metric spaces by Kuratowski [24, Section 15, pp. 99–122], [25, Section 4, p. 154] and the introduction of the triangle law by Fréchet in 1906 [12, Section 49, p. 30] in terms of the écart (neighbourhood) of two elements.[1] In general, the study of metric spaces depends upon the fundamental concept of the limit point of a set that can be described in terms of nearness of a point to a set, which was first suggested by Riesz [66].

3. Preliminaries

Fréchet observed that a distance function $\rho : X \times X \to \Re$ can be defined on any non-empty set X and called it a metric.

Definition 1 (*Distance Function [72]*)**.** A distance function $\rho : X \times X \to \Re^{0+}$ (or real-valued relation [75]) on a non-empty set X is a mapping from $X \times X$ to the non-negative reals.

Definition 2 (*Metric Space [12]*)**.** The pair $\langle X, \rho\rangle$ denotes a metric space that consists of a non-empty set X and distance function ρ (not necessarily continuous) from $X \times X$ into $\Re$ (the reals). Assuming non-negative values for ρ, this distance function satisfies the following conditions for all $x, y, z \in X$.

(M.1) $\rho(x, y) = 0$ if, and only if $x = y \in X$ (identity of indiscernibles),
(M.2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$ (symmetry),
(M.3) $\rho(x, z) \le \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$ (triangle inequality).

Remark 3 (*Pseudometric Space [11]*)**.** Let O denote a non-empty set. The elements of O are called points. A pseudometric is a distance function m defined on the set $O \times O$ if, and only if $m : O \times O \to \Re^{0+}$ (a mapping from $O \times O$ to the non-negative reals) satisfies the following conditions.

(PM.1) $m(x, x) = 0$ for every $x \in O$,
(PM.2) $m(x, y) = m(y, x)$ for all $x, y \in O$,
(PM.3) $m(x, z) \leqslant m(x, y) + m(y, z)$ for all $x, y, z \in O$.

A set O together with a pseudometric on it, is called a *pseudometric space* (denoted by $\langle O, m\rangle$). It has been observed that pseudometric spaces are direct generalizations of metric spaces, since there are pseudometric spaces in which it is the case that $x \neq y$ and $m(x, y) = 0$ for some x, y [79]. □

Remark 4 (*Historical Note*)**.** Fréchet introduced the idea of a metric space in connection with a study of function spaces in 1906 [12]. In his thesis, Fréchet introduced the triangle inequality axiom for metrics. The term *metric space* first appeared eight years later in Hausdorff's *Mengenlehre* [17]. Kuratowski proved that M.1 and M.3 are minimal conditions for a metric and that M.2 can be derived from M.1 and M.3 [24]. □

The notion of a metric space gives rise to the notion of the nearness of points.

[1] Ainsi, l'écart est un voisinage [12, loc. cit.].

Definition 5 (*Gap Distance [34]*). In a metric space $\langle X, \rho \rangle$, the *gap* $D_\rho(A, B)$ between two non-empty sets $A, B \subset X$ [34] is defined in terms of the greatest lower bound of a set of distance functional values, *i.e.*,

$$D_\rho(A, B) = \inf\{\rho(a, b) : a \in A, b \in B\}. \tag{1}$$

Remark 6 (*Norm*). Let X be a linear space over the reals with origin 0. A **norm** on X is a function $\|\cdot\| : X \to [0, \infty)$ satisfying several properties listed in Definition 11. Each norm on X induces a metric on X defined by $\rho(x, y) = \|x-y\|$ for $x, y \in \mathbb{R}$ [2]. For example, let $\vec{a}, \vec{b}$ denote a pair of n-dimensional vectors of numbers that are positive real values representing *perceived* intensities of light reflected from objects in a visual field, *i.e.*, $\vec{a} = (a_1, \ldots, a_i, \ldots, a_n)$, $\vec{b} = (b_1, \ldots, b_i, \ldots, b_n)$ such that $a_i, b_i \in \mathbb{R}^+$. Then, the distance function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is defined in terms of the l_1 norm (taxicab distance), *i.e.*,

$$\rho(\vec{a}, \vec{b}) = \|\vec{\mathrm{a}} - \vec{\mathrm{b}}\|_1 = \left(\sum_{i=1}^{n} |\mathrm{a}_i - \mathrm{b}_i|\right).$$

Notation 7. *Let O denote a set of points with measurable features represented by a set of probe functions of the form $\phi : O \to \Re$. Let $\mathcal{B} = \{\phi_1, \phi_2, \ldots, \phi_n\}$ be the set of n probe functions. A feature vector $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x) = (\phi_1(x), \ldots, \phi_n(x))$ is an n-dimensional vector of numerical features of an object $x \in O$ given by probe functions in $\mathcal{B}$. Let $\mathfrak{F}$ represent the set of feature vectors corresponding to elements of O where $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x) \in \mathfrak{F}$ denote a feature vector in $\mathfrak{F}$ obtained from some $x \in O$. Put $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{F}$ equal to sets of feature vectors corresponding to elements of the sets $X, Y \subseteq O$ respectively. Let $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x), \vec{\boldsymbol{\phi}}_{\mathcal{B}}(y)$ denote feature vectors in $\mathfrak{A}, \mathfrak{B}$, respectively for some $x \in X$ and $y \in Y$.*

$$\mathfrak{F} = \left\{\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x) \mid x \in O\right\}, \qquad \mathfrak{A} = \left\{\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x) \mid x \in X\right\}, \qquad \mathfrak{B} = \left\{\vec{\boldsymbol{\phi}}_{\mathcal{B}}(y) \mid y \in Y\right\}.$$

Definition 8 (*Feature Space-Based Distance*). Let O denote a set of points with measurable features represented by probe functions in $\mathcal{B}$. A feature space-based distance between points $x, y \in O$ is denoted by $\rho_{\mathfrak{F}}$ defined as the distance ρ between feature vectors $\vec{\boldsymbol{\phi}}(x)$ and $\vec{\boldsymbol{\phi}}(y)$

$$\rho_{\mathfrak{F}}(x, y) = \rho(\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x), \vec{\boldsymbol{\phi}}_{\mathcal{B}}(y)). \tag{2}$$

Definition 9 (*Feature Space-Based Gap Functional*). Let $\langle O, \rho_{\mathfrak{F}} \rangle$ denote a pseudometric space. Let $X, Y \subseteq O$ and let $\mathfrak{A}, \mathfrak{B} \subset \mathfrak{F}$ denote a feature space corresponding to X, Y, respectively. The feature space-based gap distance between X and Y is defined by

$$D_{\rho_{\mathfrak{F}}}(X, Y) = \inf\{\rho_{\mathfrak{F}}(x, y) : x \in X, y \in Y\} = D_\rho(\mathfrak{A}, \mathfrak{B}), \quad \text{where}$$

$$\rho_{\mathfrak{F}}(x, y) = \|\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x) - \vec{\boldsymbol{\phi}}_{\mathcal{B}}(y)\|_1 = \sum_{i=1}^{n} |\phi_i(x) - \phi_i(y)|.$$

Remark 10 (*Historical Note*). For a point x and a non-empty set B, Hausdorff [17] defines a lower distance

$$\rho(x, B) = \inf\{\rho(x, b), b \in B\}.$$

In other words, $\rho(x, B)$ is the greatest lower bound of the distances of x from points $y \in B$. For $\varepsilon \in [0, +\infty)$, a point x is close to a set B if the distance $\rho(x, B) < \varepsilon$. It was Hausdorff [17] who also suggested defining the distance between two sets and to treat sets as elements of a metric space. A set A is near a set B if the distance between A and B is small enough. The nearness of sets is formalized with the gap functional $D_\rho(A, B)$ [27]. □

Definition 11 (*Normed Space*). In a normed vector space X, a function $X \to \Re$ (written $x \mapsto \|x\| \in \Re$ for $x \in X$) is called a *norm* on X [74] such that

V.1 $\forall x \in X$, $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$,
V.2 $\forall x \in X$ and any scalar k, $\|kx\| = |k|\|x\|$,
V.3 $\forall x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$.

Proposition 12. *$\rho_{\mathfrak{F}}$ is a pseudometric.*

Proof. Immediate from Definition 11. □

Definition 13 (*Descriptively Near Sets*). Given a pseudometric space $\langle O, \rho_{\mathfrak{F}} \rangle$ with a set of feature vectors $\mathfrak{F}$ for objects in a set O. Let $X, Y \subset O$ denote non-empty, disjoint sets of objects and let $\mathfrak{A}, \mathfrak{B} \subset \mathfrak{F}$ represent sets of feature vectors for elements of X, Y respectively. The sets X, Y are descriptively ε-near sets (denoted by $X \underset{\Phi,\varepsilon}{\bowtie} Y$) if, and only if there exist $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x) \in \mathfrak{A}$, $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(y) \in \mathfrak{B}$ such that $\rho(\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x), \vec{\boldsymbol{\phi}}_{\mathcal{B}}(y)) < \varepsilon$ or equivalently $\rho_{\mathfrak{F}}(x, y) < \varepsilon$.

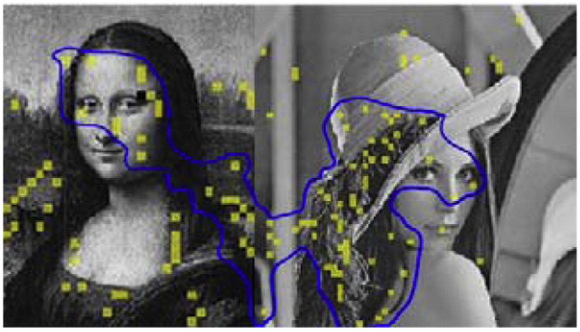


Fig. 2. Sample near eye images.

Proposition 14. *If $D_{\rho_{\mathfrak{F}}}(X, Y) < \varepsilon$, then $X \underset{\Phi,\varepsilon}{\bowtie} Y$.*

Remark 15 (*Application of $D_{\rho_{\mathfrak{F}}}(X, Y)$*)**.** To reduce the computational time, $D_{\rho_{\mathfrak{F}}}(X, Y)$ would normally be implemented over classes in coverings of X, Y determined by a tolerance relation, where $\mathfrak{A}, \mathfrak{B}$ are sets of feature vectors representing descriptions of objects in classes in coverings of X, Y, respectively. Let $\mathcal{B}$ denote a set of probe functions $\phi : O \rightarrow \Re$ representing features of objects in O. The assumption made here is that a covering of a set is determined by tolerance relation defined in (3) relative to pairs of feature vectors that describe pairs of objects in $O \times O$.

$$\cong_{\mathcal{B},\varepsilon} = \{(x, y) \in O \times O : \|\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x) - \vec{\boldsymbol{\phi}}_{\mathcal{B}}(y)\|_p \leq \varepsilon\}. \tag{3}$$

For the experimental results reported in this article, the norm $\|\cdot\|_1$ (taxicab distance in Remark 6) is used. One pair $(x, y) \in O \times O$ would be selected from each pair of classes, since we know that the descriptions of objects x, y will be similar to the descriptions of the remaining in the corresponding classes. $\square$

Example 16 (*Application of Descriptively Near Sets*)**.** Consider a pair of images X, Y. Let $\phi(x), \phi(y)$ denote the edge orientation for pixels $x \in X, y \in Y$. For simplicity, assume each feature vector in $\mathfrak{F}$ consists of a number representing a single feature value extracted from a pixel $x \in X$. For example, put $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x) = \phi(x)$ for $x \in X$ and put $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(y) = \phi(y)$ for $y \in Y$. Define

$$\rho_{\mathfrak{F}}(x, y) = \rho(\phi(x), \phi(y)) = \|\phi(x) - \phi(y)\|_1 = |\phi(x) - \phi(y)|.$$

Then X is ε-near Y if, and only if there exist $x \in X, y \in Y$ such that $|\phi(x) - \phi(y)| < \varepsilon$, *i.e.*, $D_\rho(\mathfrak{A}, \mathfrak{B}) \leq \varepsilon$. Each tiny box in the companion images in Fig. 2 represents a subimage (collections of pixels), where the average edge orientations of the pixels in the subimage are similar (within ε of each other). For a detailed explanation of the edge orientation of pixels using 3×3 Prewitt or Sobel masks, see, *e.g.*, [14] (for edge detection considered in the context of perceptually near sets using wavelets, see [18, Ap. B, p. 103ff]). The selection of each box was done using the NEAR system toolset (for a detailed explanation, see [18, 2.3.4, p. 21ff]). Notice that there are many regions in each portrait, starting with the edge orientation of Mona Lisa's right eyebrow and Lena's left eyebrow, where the average edge orientations are similar. Hence, $X \underset{\Phi,\varepsilon}{\bowtie} Y$, *i.e.*, the two portraits are considered descriptively ε-near each other. $\square$

Pseudometrics and feature-based gap functionals are next viewed within the context of fuzzy metric spaces.

4. Fuzzy sets

> ...A fuzzy set is characterized by a membership
> function which assigns to each object its grade of membership
> (a number lying between 0 and 1) in the fuzzy set.
> –A new view of system theory
> –L.A. Zadeh, 20–21 April 1965.

Definition 17 (*Fuzzy Set [83]*)**.** A fuzzy set A is a function with domain X and values in $[0, 1]$. Graded membership of the elements of A is determined by a specific membership function $\mu : U \rightarrow [0, 1]$.

Definition 18 (*Continuous t-norm [30,70]*)**.** A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if, and only if $*$ satisfies the following conditions.

t.1 $*$ is commutative and associative,
t.2 $*$ is continuous,

t.3 $a * 1 = a$ for all $a \in [0, 1]$,
t.4 $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for $a, b, c, d \in [0, 1]$.

Fuzzy sets have widely used in image analysis (see, *e.g.*, [68,38,39,35,40,36,37,41,43,32,8,29,73,16]). In the context of fuzzy sets, Pal and King [38,39] defined an image of $M \times N$ dimension and L levels as an array of fuzzy singletons, each with a value of membership function denoting the degree of having brightness or some property relative to some brightness level l, where $l = 0, 1, 2, \ldots, L-1$. The literature on fuzzy image analysis is based on the realization that the basic concepts of edge, boundary, region, relation in an image do not lend themselves to precise definition.

5. Fuzzy metrics

This section introduces a number of members of a family of standard fuzzy metrics. Fuzzy metrics are part of a growing research area in fuzzy set theory (see, *e.g.*, [23,13,15,76,67]).

Definition 19 (*Fuzzy Metric Space [23]*)**.** The tuple $\langle X, M, *\rangle$ denotes a fuzzy metric space that consists of a non-empty set X, continuous t-norm *, and function $M : X \times X \times]0, \infty[\to]0, 1]$ that satisfies the following properties for all $x, y, z \in X, t, s > 0$.

(FM.1) $M(x, y, t) > 0$,
(FM.2) $M(x, y, t) = 1$ if, and only if $x = y$,
(FM.3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X, t \in \Re$,
(FM.4) $M(x, z, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM.5) $M(x, z, \cdot) : (0, +\infty) \to [0, 1]$ is continuous.

Given a fuzzy metric space $\langle X, M, *\rangle$, the function M is a fuzzy metric on X. George and Veeramani [13] proved that every fuzzy metric M on X generates a topology τ_M on X that has as a base the family of open sets of the form

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r, t > 0\}, \quad \text{for every } r, \text{ such that } 0 < r < 1, t > 0.$$

For each specific pair of elements $x, y, M(x, y, t)$ is viewed as the fuzzy set that represents nearness of x and y. One, can use M_ρ to represent the membership function of the fuzzy distance (see Definition 20). Property FM. 2 indicates that there is no ambiguity in nearness of x and y, when $x = y$. This line of reasoning leads to a new formulation of a criterion for determining when disjoint sets are near sets.

Definition 20 (*Standard Fuzzy Metric*)**.** Let $\langle X, \rho\rangle$ be a metric space. The function M_ρ called the standard fuzzy metric induced by the distance function ρ [13]. M_ρ is defined by $M : X \times X \times]0, \infty[\to]0, 1]$ such that

$$M_\rho(x, y, t) = \frac{t}{t + \rho(x, y)}.$$

Definition 21 (*Feature Space-Based Fuzzy Gap Functional*)**.** Let $\langle O, \rho_{\mathfrak{F}}\rangle$ be a pseudometric space with feature space $\mathfrak{F}$ and pseudometric $\rho_{\mathfrak{F}}$ and let $\mathfrak{A}, \mathfrak{B} \subset \mathfrak{F}$ denote sets of feature vectors corresponding to $X, Y \subset O$. Let $M_{\rho_{\mathfrak{F}}}(x, y, t)$ denote the standard fuzzy metric defined in terms of $\rho_{\mathfrak{F}}$ from Definition 9. The feature space-based fuzzy gap distance between sets X, Y is defined by

$$D^t_{\rho_{\mathfrak{F}}}(X, Y, t) = \inf\{M_{\rho_{\mathfrak{F}}}(x, y, t) : x \in X, y \in Y\} \in [0, 1].$$

Definition 22 (*Fuzzy Descriptively Near Sets*)**.** Let $\langle O, \rho_{\mathfrak{F}}\rangle$ denote a pseudometric space with a normed space $\mathfrak{F}$ containing feature vectors of elements of O and metric $\rho_{\mathfrak{F}}$. Let $\mathfrak{A}, \mathfrak{B} \subset \mathfrak{F}$. Then $M_{\rho_{\mathfrak{F}}}(x, y, t)$ is defined in terms of the pseudometric $\rho_{\mathfrak{F}}$ from Definition 9. The sets X, Y are descriptively (t, ε)-near sets (denoted by $X \underset{\Phi,\varepsilon}{\overset{t}{\bowtie}} Y$) if, and only if there exist $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x) \in \mathfrak{A}$, $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(y) \in \mathfrak{B}$ such that $\mathfrak{A}$ is ε-near $\mathfrak{B}$ with respect to t such that $M_{\rho_{\mathfrak{F}}}(x, y, t) \leq \varepsilon$.

Proposition 23. *If $D^t_{\rho_{\mathfrak{F}}}(X, Y) \leq \varepsilon$, then $X \underset{\Phi,\varepsilon}{\overset{t}{\bowtie}} Y$.*

Proof. Immediate from Definitions 21 and 22. □

Example 24 (*Norm Fuzzy Metric $M_{\rho_{\mathfrak{F}}}$*)**.** Given the metric space $\langle \mathfrak{F}, \rho\rangle$, from Proposition 12, we know that $\rho_{\mathfrak{F}}$ is a pseudometric and hence $\langle O, \rho_{\mathfrak{F}}\rangle$ is a pseudometric space. To obtain the membership value of the standard fuzzy metric at distance t, put

$$M_{\rho_{\mathfrak{F}}}(x, y, t) = \frac{t}{t + \rho_{\mathfrak{F}}(x, y)}.$$

From Example 16, $\rho_{\mathfrak{F}}(x, y) = \rho(\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x), \vec{\boldsymbol{\phi}}_{\mathcal{B}}(y))$ returns the L_1 (taxicab) distance between the pair of feature vectors $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x), \vec{\boldsymbol{\phi}}_{\mathcal{B}}(y)$ for objects x, y. From [15], this instance of the standard fuzzy metric $M_{\rho_{\mathfrak{F}}}$ estimates the degree of nearness of the feature values $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x), \vec{\boldsymbol{\phi}}_{\mathcal{B}}(y)$ with a fuzzy set determined by a membership function $\mu_{x,y}(t) = M(x, y, t)$.

We may compare fuzzy distances by comparing $M(x, y, t)$ at a fixed value of t, (*e.g.*, $t = 1$).

$$M_{\rho_{\mathfrak{F}}}(x, y, 1) = \frac{1}{1 + \rho_{\mathfrak{F}}(x, y)}.$$

This leads to an instantiation of the standard fuzzy metric (denoted by $M_m(x, y, 1)$) defined in terms of a pseudometric m. □

Example 25 (*Tolerance Class Min–Max M_{tNM}*)**.** Let $\mathcal{B} \subset \mathbb{F}$ and let $\cong_{\mathcal{B},\varepsilon}$ denote the tolerance relation as defined in (3). Let $X, Y \subset \mathcal{P}(O)$ denote non-empty sets in $\mathcal{P}(O)$. Let $\mathrm{H}^{\varepsilon}_{\mathcal{B}}(X \cup Y)$ denote the family of all tolerance classes of relation $\cong_{\mathcal{B}}$ on the set $X \cup Y$. The $tNM_{\cong_{\mathcal{B},\varepsilon}}$ nearness measure [16] estimates the degree of resemblance between X and Y. A tNM-based distance function D_{tNM} that befits the study of pseudometrics is defined in (4) as the weighted average of the closeness between the cardinality (size) of sets $A \cap X$ and $A \cap Y$ where $A \in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(X \cup Y)$ and the cardinality of tolerance class A is used as the weighting factor.

$$D_{tNM}(X, Y) = 1 - \frac{\displaystyle\sum_{A \in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(X \cup Y)} \mathcal{T}(X, Y) \cdot |A|}{\displaystyle\sum_{A \in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(X \cup Y)} |A|}, \tag{4}$$

where

$$\mathcal{T}(X, Y) = \frac{\min\{\, |A \cap X|, |A \cap Y| \,\}}{\max\{\, |A \cap X|, |A \cap Y| \,\}}. \tag{5}$$

Assume all classes $A \in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(X \cup Y)$ are non-empty. That is, D_{tNM} is a pseudometric (see Proposition 26) for the pseudometric space $\langle \mathcal{P}(O), D_{tNM} \rangle$ and $\langle \mathcal{P}(O), M_{D_{tNM}}, * \rangle$ is a fuzzy metric space, where $\mathcal{P}(O)$ is the power set of O. □

Proposition 26. *D_{tNM} is a pseudometric.*

Proof. Assume $X, Y, Z \subset \mathcal{P}(O)$ are non-empty subsets of $\mathcal{P}(O)$ and that all classes $A \in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(X \cup Y)$ are non-empty. Then D_{tNM} defined over $\mathcal{P}(O) \times \mathcal{P}(O)$ satisfies the conditions for a pseudometric, *i.e.*, for any non-empty $X, Y \in \mathcal{P}(O)$, $D_{tNM}(X, Y) \geq 0$, since $\mathcal{T} \geq 0$ for any choice of a non-empty class $A \in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(X \cup Y)$.

PM.1 From (4), $D_{tNM}(X, X) = 0$.
PM.2 From (5), $D_{tNM}(X, Y) = D_{tNM}(Y, X)$ for all $X, Y \subset \mathcal{P}(O)$.
PM.3 From (5),

$$\begin{aligned} \mathcal{T}(X, Z) &= \frac{\min\{\, |A \cap X|, |A \cap Z| \,\}}{\max\{\, |A \cap X|, |A \cap Z| \,\}} \\ &\leq \frac{\min\{\, |A \cap X|, |A \cap Y| \,\}}{\max\{\, |A \cap X|, |A \cap Y| \,\}} + \frac{\min\{\, |A \cap Y|, |A \cap Z| \,\}}{\max\{\, |A \cap Y|, |A \cap Z| \,\}} \\ &\leq \mathcal{T}(X, Y) + \mathcal{T}(Y, Z). \end{aligned}$$

Hence, $D_{tNM}(X, Z) \leq D_{tNM}(X, Y) + D_{tNM}(Y, Z)$ for all $X, Y, Z \subset \mathcal{P}(O)$. □

Example 27 (*Tolerance Covering Distance M_{tcDM}*)**.** Assume $X, Y \subset \mathcal{P}(O)$ are non-empty subsets of $\mathcal{P}(O)$ and that all classes $A \in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(X \cup Y)$ are non-empty. A tolerance covering distance metric (tcDM) is defined in (6) as a measure of difference between the intersection of tolerance classes $A \in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(X \cup Y)$ and $X, Y \subset O$.

$$tcDM(X, Y) = \sum_{A \in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(X \cup Y)} \frac{||A \cap X| \;-\; |A \cap Y||}{|A \cap X| \;+\; |A \cap Y|}. \tag{6}$$

That is, tcDM is a pseudometric (see Proposition 28) for the pseudometric space $\langle \mathcal{P}(O), tcDM \rangle$ and $\langle \mathcal{P}(O), M_{tcDM}, * \rangle$ is a fuzzy metric space. □

Proposition 28. *tcDM is a pseudometric.*

Proof. Assume $X, Y, Z \subset \mathcal{P}(O)$ are non-empty subsets of $\mathcal{P}(O)$ and that all classes $A \in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(X \cup Y)$ are non-empty. Then $tcDM$ defined over the set $\mathcal{P}(O) \times \mathcal{P}(O)$ satisfies the conditions for a pseudometric. The details of the proof are omitted, since the proof is similar to that given in Proposition 26. □

Example 29 (*Hausdorff Fuzzy Metric M_{ρ_H}*). Hausdorff distance by definition is defined on non-empty, finite point sets in a metric space [17,11]. Assume $d(x, y)$ is a distance defined between points x and y in a pseudometric space $\langle O, d\rangle$. Assume $X, Y \subset \mathcal{P}(O)$ are non-empty subsets of $\mathcal{P}(O)$. The Hausdorff metric $\rho_H(X, Y)$ defined on subsets $X, Y \subset \mathcal{P}(O)$ of a pseudometric space $\langle \mathcal{P}(O), \rho_H\rangle$ is defined here, starting in (7).

$$\rho_H(X, Y) = \max\{d_H(X, Y), d_H(Y, X)\}, \tag{7}$$

where

$$d_H(X, Y) = \max_{x\in X}\{\min_{y\in Y}\{d(x, y)\}\}, \tag{8}$$

$$d_H(Y, X) = \max_{y\in Y}\{\min_{x\in X}\{d(x, y)\}\}. \tag{9}$$

$d_H(X, Y)$ and $d_H(Y, X)$ are directed Hausdorff distances from X to Y and from Y to X, respectively. In addition, ρ_H is a pseudometric (see Proposition 30) for the pseudometric space $\langle \mathcal{P}(O), \rho_H\rangle$ when d is a pseudometric for the pseudometric space $\langle O, d\rangle$. Hence, M_{ρ_H} is an incarnation of the standard fuzzy metric and $\langle \mathcal{P}(O), M_{\rho_H}, *\rangle$ is a fuzzy metric space. □

Proposition 30. *ρ_H is a pseudometric.*

Proof. Assume $X, Y, Z \subset \mathcal{P}(O)$ are non-empty subsets of $\mathcal{P}(O)$. Then ρ_H defined over $\mathcal{P}(O) \times \mathcal{P}(O)$ satisfies the conditions for a pseudometric. The details of the proof are omitted, since the proof is similar to that given in Proposition 26. □

6. Near rough sets

The term *tolerance space* was coined by E.C. Zeeman in 1961 in modelling visual perception with tolerances [84] (see, also, [85]). A Zeeman tolerance space $\langle X, \simeq\rangle$ consists of a set X supplied with a binary relation $\simeq$ (*i.e.*, a subset $\simeq \subset X \times X$) that is reflexive (for all $x \in X, x \simeq x$) and symmetric (for all $x, y \in X, x \simeq y$ implies $y \simeq x$) but transitivity of $\simeq$ is not required. Sets of similar elements in a tolerance relation are called preclasses [69]. A set $A \subset\simeq$ is a preclass if, and only if $\forall x, y \in A,\ x \simeq y$.

A tolerance class is a maximal preclass in a tolerance relation. Let $\mathfrak{O} \subset X$ denote a family of subsets of X. A family $\mathfrak{O}$ is a cover of a set X if, and only if every element of X belongs to some subset of $\mathfrak{O}$ [21]. A tolerance relation such as $\simeq$ determines a *covering* of X.

Definition 31 (*Tolerance Rough Set [28,71]*). Let $\langle O, \cong_{\mathcal{B},\varepsilon}\rangle$ denote a tolerance space with $X \subset O$, $\mathrm{H}^{\varepsilon}_{\mathcal{B}}(O)$ the set of tolerance classes in a covering of O determined by the tolerance relation $\cong_{\mathcal{B},\varepsilon}$ in (3). The lower approximation of X is denoted by $\mathcal{B}^{\varepsilon}_{*}X$ and the upper approximation of X is denoted by $\mathcal{B}^{*}_{\varepsilon}X$, where

$$\mathcal{B}^{\varepsilon}_{*}X = \bigcup_{A\in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(O):A\subseteq X} A,$$

$$\mathcal{B}^{*}_{\varepsilon}X = \bigcup_{A\in \mathrm{H}^{\varepsilon}_{\mathcal{B}}(O):A\cap X\neq\emptyset} A.$$

The B-boundary region of an approximation of a set X is denoted by $Bnd_{\mathcal{B}}X$, where

$$Bnd_{\mathcal{B}}X = \mathcal{B}^{*}_{\varepsilon}X \setminus \mathcal{B}^{\varepsilon}_{*}X = \{x \mid x \in \mathcal{B}^{*}_{\varepsilon}X \text{ and } x \notin \mathcal{B}^{\varepsilon}_{*}X\}.$$

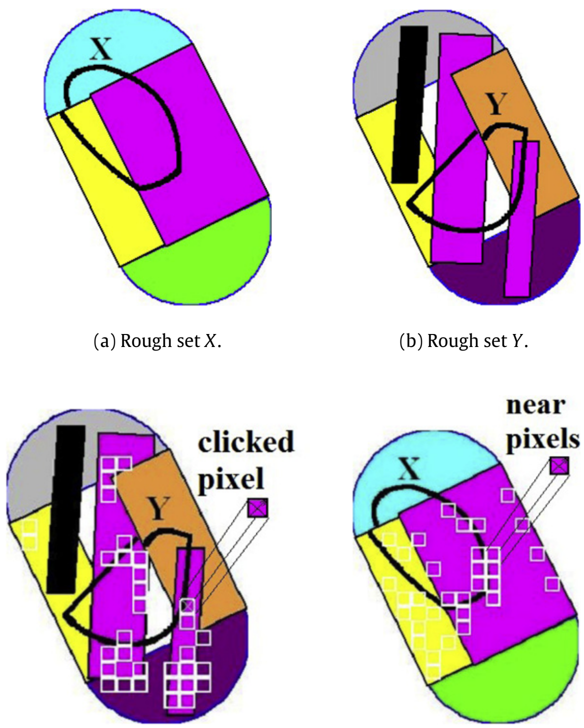
A set X is roughly classified whenever $Bnd_{\mathcal{B}}X$ is not empty. In that case, X is a rough set.

7. Solving the visual acuity problem

It is fairly easy to verify that sets X and Y in Fig. 3 are, in fact, rough sets. Visually, sets X and Y partially contain regions with differing hues. Appearance can be deceiving, since it would seem, for example, that the pixels in the pink regions of X and Y are entirely the same hue. If that were true, then set X appears to overlap classes in a partition in Fig. 3(a) and set Y appears to overlap classes in a cover in Fig. 3(b). Before we discuss how one can verify this in terms of determining covers for the coloured regions of the oblong shapes in Fig. 3, it is important to observe that one must first select values for ε (tolerance) and p for $p \times p$ subimages in a visual approximation of what we perceive. The proposed visual image representation approach has been implemented in what is known as the NEAR system, version 2.0.[2]

Using the NEAR system to load a pair of digital images, select the NI (near image) option. Next, it is necessary to select the visual representation parameters. For example, select $\varepsilon = 0.01$ for the relation $\cong_{\mathcal{B},\varepsilon}$ and $p = 15$ to obtain 15×15 subimages. Next, choose one or more features to be used in the description of each subimage for the covers in each of the

[2] The complete NEAR system and detailed research report are available at http://wren.ee.umanitoba.ca.



(a) Rough set X. (b) Rough set Y.

(c) Clicked pixel in Y. (d) Near X pixels.

Fig. 3. Sample near rough sets & near subimages.

selected images. After that, use the *run* option to create a cover for each image. The NEAR system has a touch feature that makes it possible to test the accuracy of an individual's visual acuity against the actual physical features of a visual image. To experiment with this touch feature, move the cursor to a visual region of interest (ROI) and click on a point (single pixel) within the ROI (the pixel you select will be the centre of a $p \times p$ subimage with average values of the features that have been chosen).

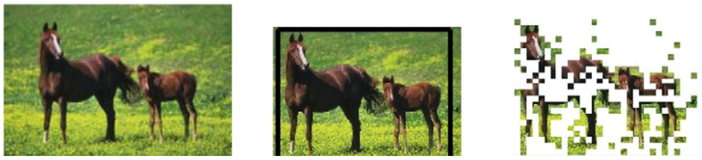
For simplicity, choose a single feature in the NEAR feature menu, *e.g.*, *normalizedR* (normalized red channel). In Fig. 3(c), observe that there is a box with ⊠ (for this example, cursor was used to click on the pixel at the centre of the ⊠). Then all subimages that are similar to the selected subimage shown in Fig. 3(c) as well as all similar (*near*) subimages in Fig. 3(d) will be displayed on the screen, *i.e.*, a single class in the cover of each image will be displayed.

It can be observed that some of the subimages are inside and some are outside for both X and Y. This indicates that both sets are, in fact, rough sets. Since there are subimages that are descriptively similar in both X and Y, this indicates that X is ε-near Y. Further, it can observed that this example is conceptually close to what E.C. Zeeman originally observed about visual acuity [84, p. 245]. That is, *what we look at* (pixel colours that are bunched together in the shapes in Fig. 3) and *what we perceive* (pixel colours that appear to be separated) are often different.

There is a glaring example of the difference between the *perceived* and actual colours in each of the coloured regions in both X (in Fig. 3(a)) and Y (in Fig. 3(b)). This is easy to see in Fig. 3(a), where each coloured region appears to be uniformly the same. Notice, however, that this is not the case. Evidence of the can be seen in Fig. 3(c). That is, one can observe a profusion of subimages (seemingly the same colour) that appear in markedly different regions of the image, where feature extraction has revealed subtle differences in the hues inside the tiny boxes that have similar but not identical descriptions. The proposed approach to solving the visual acuity problem provides a basis for measuring the degree of nearness between pairs of digital images. For disjoint rough sets, one can consider the descriptive nearness of rough sets in the context of fuzzy gap functionals.

Remark 32 (*Fuzzy Descriptively Near Rough Sets*)**.** Assume $X, Y \subset O$ are disjoint rough sets. Let $\langle \mathfrak{F}, \rho_{\mathfrak{F}} \rangle$ denote a pseudometric space with feature spaces $\mathfrak{A}, \mathfrak{B} \subset \mathfrak{F}$ for features of objects in the upper approximations $\mathcal{B}^*_\varepsilon X$ and $\mathcal{B}^*_\varepsilon Y$, respectively. Then the rough sets X and Y are fuzzy descriptively near rough sets (denoted by $X \underset{\Phi,\varepsilon}{\overset{t}{\bowtie}} Y$) if, and only if $D^t_{\rho_{\mathfrak{F}}}(\mathfrak{A}, \mathfrak{B}) \leq \varepsilon$. □

Example 33 (*Sample Fuzzy Descriptively Near Rough Sets*)**.** Consider sets $X, Y \subset O$ in Fig. 3(a) and (b), respectively. Sets X and Y are examples of rough sets, since the lower approximation is empty and there are several classes in each upper approximation.



(a) Sample query image. (b) ROI. (c) Lower approximation.

Fig. 4. Sample image region of interest (ROI).

(a) ROI1. (b) ROI2. (c) ROI3.

Fig. 5. Sample regions of interest (ROIs).

For both X and Y, the upper approximation contains classes filled with yellow $\square$ and $\square$ subimages. Let $A_{\square}$ denote a class (filled with $\square$ pink subimages) in the covering of X. Similarly, let $B_{\square}$ denote a class (filled with $\square$ pink subimages) in the covering of Y. Let $\mathfrak{A}$ denote a set of feature vectors for $A_{\square}$ and let $\mathfrak{B}$ denote a set of feature vectors for $B_{\square}$. Assume that $\mathfrak{F}$ is set of feature vectors for objects in O and that $\mathfrak{A}, \mathfrak{B} \subset \mathfrak{F}$. Assume that $\rho_{\mathfrak{F}}$ is the pseudometric given in Definition 9. If we assume both A, B contain subimages with identical colours, then $D_{\mathfrak{F}}(\mathfrak{A}, \mathfrak{B}) = 0$. Visually, this is the case in Fig. 3. In that case, there exist $\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x) \in \mathfrak{A}, \vec{\boldsymbol{\phi}}_{\mathcal{B}}(y) \in \mathfrak{B}$ such that $\rho_{\mathfrak{F}}(\vec{\boldsymbol{\phi}}_{\mathcal{B}}(x), \vec{\boldsymbol{\phi}}_{\mathcal{B}}(y)) = 0$. Then $M_{\rho_{\mathfrak{F}}}(x, y, 1) = 1$ and $D^t_{\rho_{\mathfrak{F}}}(\mathfrak{A}, \mathfrak{B}) \leq \varepsilon$ for $\varepsilon = 1$. From Proposition 23, $X \underset{\Phi,\varepsilon}{\overset{t}{\bowtie}} Y$, rough sets X and Y are fuzzy descriptively near sets. $\square$

8. Results of experiments

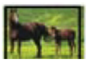
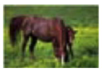
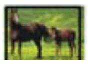
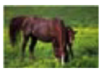
The basic approach in the experiments reported in this section is based on a comparison of the lower approximation of a region of interest (ROI) in a query image with various test images. For example, assume that the pair of horses in Fig. 4(a) is a query image (this image comes from the public Simplicity image archive [77]). Select a ROI from the query image (see, *e.g.*, Fig. 4(b)). Next, determine the lower approximation of the ROI and a given test image based on a selection of image features (*e.g.*, colour, texture, shape) and subimage size p. A sample lower approximation of the ROI in Fig. 4(b) is given in Fig. 4(c) with $p = 10$. Then any of the pseudometrics reported in this article can be used to measure the distance between each pair of query images based on their lower approximations. The images in Fig. 5 show three sample ROIs used in experiments reported in Table 1. This is explained in detail in the following two examples.

Example 34 (*Sample Fuzzy Metric Distances*)**.** For a number of particular fuzzy metric distances between the lower approximation of a ROI, see Table 1. The ROI (aircraft) in Fig. 5(a) is selected and compared with the 600 images in the Caltech imagebase [6] and ROI images from Fig. 5(b) and (c) are selected and compared with 1000 images in the SIMPLIcity imagebase [78]. It can be observed in Table 1 that $M_{\rho_{\mathfrak{F}}}$ gives better results than the remaining three fuzzy metrics. It should be also be observed that it is not possible to generalize (draw conclusions) from these measurements, since the results of only a few experiments are reported in Table 1.

Example 35 (*Image Retrieval Experiments*)**.** The three sample ROIs shown in Fig. 5 provide a basis for content-based image retrieval experiments. The content arises from classes in the coverings of each pair of ROI and test images. The basic approach in the sample cases shown in Table 1 is carried forward in comparisons of each ROI query image with each of the images in an image archive. Fuzzy nearness metric values make it possible to order the compared images relative to how close each test image is to a particular ROI query image. That is, for each pair of images $X, Y, M_{\rho}(X, Y, 1)$ (for a particular choice of a pseudometric ρ) is used to sort the test images based on their distance/nearness to a query image. Fig. 6 shows the number of related (similar) images versus the sorted retrieved images in each case.

The plots in Fig. 6(a) and (c) demonstrate the superiority of M_{ρ_H} (Hausdorff fuzzy metric). The important thing to notice in these two experiments is the crispness (uncluttered character of) the ROI for the aircraft in Fig. 5(a) and the horses in Fig. 5(c). In such images, M_{ρ_H} is a better choice for image retrieval. Also, notice that the image retrieval using M_{tcDM} in the case of the aircraft in Fig. 5(a) yields retrieval results in Fig. 6(a) that are almost as good as the retrieval results using M_{ρ_H}. In the case where there are two more objects to consider in a fairly crisp, non-noisy image (one with comparatively few objects), it can be observed that $M_{\rho_{\mathfrak{F}}}$ is marginally better than M_{tcDM} in the image retrieval results shown in Fig. 6(c). The surprising retrieval results occur in the plot in Fig. 6(b). The ROI in the very noisy (very cluttered) mountain scene in Fig. 5(b)

Table 1
ROI vs. test image nearness measurements.

X_{ROI}	Y	$\mathcal{B}_*^{0.5}X_{ROI}$	$\mathcal{B}_*^{0.5}Y$	$M_{\rho_{\mathfrak{F}}}$	$M_{D_{tNM}}$	M_{tcDM}	M_{ρ_H}	p
				0.967	0.615	0.654	0.644	20
				0.997	0.623	0.652	0.655	15
				0.999	0.633	0.660	0.707	10
				0.895	0.631	0.665	0.582	20
				0.953	0.592	0.641	0.630	15
				0.985	0.592	0.668	0.724	10
				0.943	0.687	0.731	0.654	20
				0.955	0.667	0.726	0.676	15
				0.989	0.694	0.769	0.716	10

contains buildings, roads, trees, uneven terrain, mountainous background. For images with lots of clutter, M_{ρ_H} (Hausdorff) and $M_{\rho_{\mathfrak{F}}}$ yield the poorest results. By contrast, the $M_{D_{tNM}}$ (best) and M_{tcDM} (only marginally different from $M_{D_{tNM}}$) yield the best image retrieval results. This suggests that $M_{D_{tNM}}$ is a good fuzzy metric to use for image retrieval if the query image contains a profusion of objects.

9. Conclusion

A nature-inspired framework for the study of resemblance between visual objects is introduced in this article. Specifically, visual object resemblance is quantified by measuring distances between descriptions of sets of objects. The standard fuzzy metric has been instantiated in terms of a number of pseudometrics used to measure the nearness of pairs of objects. The feature-based pseudometric $\rho_{\mathfrak{F}}$ measures the distance between feature vectors. By contrast, the pseudometrics D_{tNM}, $tcDM$, ρ_H measure the distance between sets. Image retrieval experiments with each of these pseudometrics have been presented. Different retrieval problems require pseudometrics tailored to the requirements embodied in a particular query image. These experiments reveal that the standard fuzzy metric defined using ρ_H (Hausdorff) works best for uncluttered query images. By contrast, for cluttered query images, D_{tNM} works best. Good retrieval results using the pseudometric $tcDM$ suggest that better choices of features $\mathcal{B}$ and ε for the tolerance relation $\cong_{\mathcal{B},\varepsilon}$ and image parameters such as subimage size p, will lead to improved image retrieval performance. The $\rho_{\mathfrak{F}}$ is important because it requires fewer comparisons to determine the nearness of a pair of sets. However, more work needs to be done before the utility of $\rho_{\mathfrak{F}}$ can be fully gauged.

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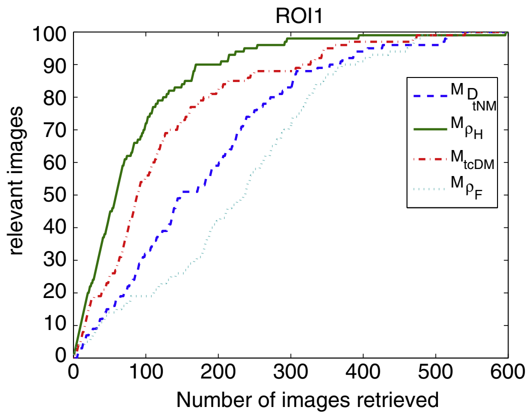
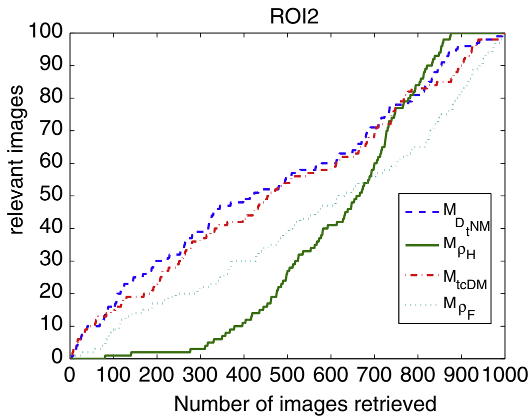
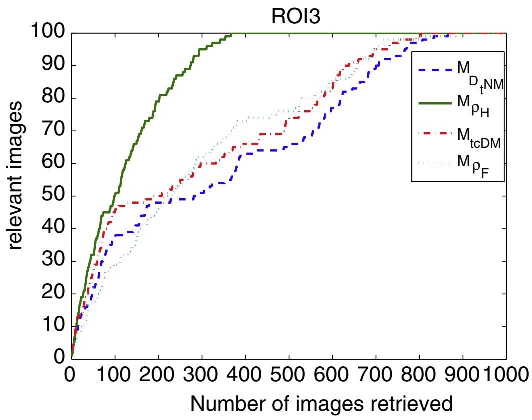


(a) Query 1.



(b) Query 2.



(c) Query 3.

Fig. 6. Image retrieval results with 4 fuzzy metrics.

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