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Operation-Preserving Functions and Autonomous Factors of Finite Automata

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The relationship between the structure of autonomous finite automata and their operation-preserving functions is considered. The results imply some ideas in the study of operation-preserving functions of arbitrary finite automata, because with each finite automaton the set of its autonomous factors is associated. Basing on the method of the investigation of operation-preserving functions of finite automaton A and by studying autonomous factors of A, the algorithm for determining operation-preserving functions of A is given.

INTRODUCTION

In this paper, the method of studying operation-preserving functions of the finite automaton is based on the investigation of its autonomous factors, i.e., autonomous automata.

The research of operation-preserving functions of autonomous automata is much easier than that of arbitrary automata, and it is possible to obtain more information on the structure of operation-preserving functions of such automata.

On the other hand, the knowledge of the structure of operation-preserving functions of autonomous factors of the finite automaton implies the knowledge of the structure of its operation-preserving functions. In fact, it implies an easy algorithm for determining the set of all its operation-preserving functions.

PRELIMINARY DEFINITIONS AND RESULTS

An automaton is a triple $A = (S, \Sigma, M)$, where S is a nonempty state set, Σ is a nonempty input set, and M is the next state function, M has the domain $S \times \Sigma$ and the range S. An automaton is finite if its state set is finite. The term "monadic algebra" is being reserved for unnecessary finite automata; furthermore, by "automaton" a finite automaton is meant.

A set of all possible, finite sequences from Σ will be denoted by I. Set I, together with the operation of concatenation, forms a free semigroup. We assume that

$$M(s, \sigma x) = M(M(s, \sigma), x)$$

for each $s \in S$, $\sigma \in \Sigma$ and $x \in I$.

Throughout this paper, it will be assumed that

$$\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_{n-1}\},$$

where n is a natural number. Moreover, B denotes the automaton (T, Σ, N), where T denotes the state set of B and N the next state function of B.

The following definitions are taken from [1]:

The automaton generated by s, denoted by $A(s)$, is a triple (S', Σ, M'), where $S' = \{M(s, x) : x \in I\}$, and M' is M restricted to $S' \times \Sigma$.

The set of generators of $A(s) = (S', \Sigma, M')$, denoted by gen $A(s)$, is a set $\{r \in S' : A(r) = A(s)\}$.

A subset R of S is a generating set of A, denoted by gen A, if and only if for each $s \in S$ there exists $r \in R$ such that s is in the state set of $A(r)$. In the family of generating sets of A all the generating sets with minimal cardinality will be called minimal generating sets.

By function $f : A \to B$ is meant a function from S into T.

A function $f : A(s) \to B$ is said to be transition-generated if and only if there exist $s' \in$ gen $A(s)$ and $t \in T$, and for each state r of $A(s)$ there exists $x \in I$ such that

$$r = M(s', x) \qquad \text{and} \qquad f(r) = N(t, x).$$

An (s, t)-transition generated function of $A(s)$ into B is a transition generated function of $A(s)$ into B with s and t specified.

A function $f : A \to B$ is said to be transition generated if and only if there exist an ordered minimal generating set $P = (s_0, s_1, \ldots, s_{m-1})$ of A and an ordered m-tuple $Q = (t_0, t_1, \ldots, t_{m-1})$ of states of B, and for each state r of A there exist $k \in \{0, 1, \ldots, m-1\}$ and $x \in I$ such that

$$r = M(s_k, x) \qquad \text{and} \qquad f(r) = N(t_k, x).$$

Bavel and Muller have defined in [2] the monadic algebra "onto". A monadic algebra (S, Σ, M) is onto if and only if $\{M(s, \sigma) : s \in S\} = S$ for each $\sigma \in \Sigma$.

LEMMA 1. *Let A and B be automata and let B be onto. Then for each s in S and each t in T there exists a transition generated function $f : A \to B$ such that $f(s) = t$.*

Proof. Let $\{s_0, s_1, \ldots, s_{m-1}\}$ be a minimal generating set of A and let s be in the state set of $A(s_0)$, i.e., for s_0 and s there exists x in I with $M(s_0, x) = s$. For such x in

I and for t in T there exists t_0 in T with $N(t_0, x) = t$, since B is onto; i.e., for each σ in Σ and t' in T there exists t'' in T such that $N(t'', \sigma) = t'$.

For $(s_0, s_1, ..., s_{m-1})$ and the ordered m-tuple $(t_0, t_1, ..., t_{m-1})$, where $t_1, ..., t_{m-1}$ are arbitrary states from T, there exists a transition generated function which maps s_k to t_k for each $k \in \{0, 1, ..., m-1\}$ and maps s to t, by Lemma 9 of [1].

A function $f: A \to B$ is operation-preserving [1, 3, 6, 10] if

$$f(M(s, \sigma)) = N(f(s), \sigma)$$

for each s in S and σ in Σ.

Note, that if the function is operation-preserving, then

$$f(M(s, x)) = N(f(s), x))$$

for each s in S and x in I.

An operation-preserving function is called a homomorphism if it is from A into B, endomorphism if it is from A into A, isomorphism if it is from A onto B and one to one, automorphism if it is from A onto A and one to one.

We denote the set of all homomorphisms of A into B by $H(A \to B)$, the set of all endomorphisms of A by $E(A)$, the set of all isomorphisms of A onto B by $Is(A \to B)$, and the set of all automorphisms of A by $G(A)$.

Autonomous Factors of A

An autonomous factor A_i, $i \in \{0, 1, ..., n-1\}$, of the automaton $A = (S, \Sigma, M)$ is a triple (S, σ_i, M_i), where M_i is the restriction of M to $S \times \sigma_i$.

The definition of the factor of the automaton was introduced by Hotz [7, 8]. Our definition slightly differs from that of Hotz and it corresponds to "einfacher Faktor" of Hotz.

We quote the next definitions from [4–6].

The graph (state diagram) $\Gamma(A_i)$ of the autonomous factor $A_i = (S, \sigma_i, M_i)$ of A is defined as follows: to each state s of S corresponds a vertex of $\Gamma(A_i)$ denoted by s, and to each ordered pair of vertices (s, s') with $M_i(s, \sigma_i) = s'$ there corresponds a branch of $\Gamma(A_i)$ oriented from s to s' and denoted by σ_i.

The graph $\Gamma(A_i)$ of A_i can be partitioned into subgraphs such that, viewing $\Gamma(A_i)$ as undirected, each subgraph is a connected subgraph, but no subgraph is connected to any other subgraph. These subgraphs will be called components of $\Gamma(A_i)$.

A finite sequence of not necessarily distinct branches of a graph $\Gamma(A_i)$, such that the k-th branch ends at the vertex from the $(k+1)$-th branch is coming out, will be called a path.

A cycle is a path which comes back to its first vertex.

If the branches of a path or cycle are all distinct, the path or cycle is said to be simple. For any path or cycle there exists uniquely a simple path or cycle which passes through the same vertices, respectively.

The length of a path or cycle is the number of branches in the corresponding simple path or cycle.

Each component of $\Gamma(A_i)$ contains only one cycle.

The set of all the paths reaching any vertex of the cycle of the component of $\Gamma(A_i)$, and such that there are no branches from the cycle in these paths, shall be called a tail of the component.

The set L of all vertices of the tail of the component of $\Gamma(A_i)$ such that the length of a path, formed from branches of this tail, from any vertex in L to the first vertex of the cycle is equal to ν constitutes the ν-th level of the component of $\Gamma(A_i)$. We assume, that any vertex of the cycle is in 0-level.

A finite, connected sequence of branches of $\Gamma(A_i)$, which can be traced on $\Gamma(A_i)$, will be called a chain.

The length of a chain between the vertices of an ordered pair (s, s'), and constituting one branch, will be equal to $+1$ when the branch is oriented from s to s' and -1 when the branch is oriented from s' to s. The length of an arbitrary chain between the vertices of an ordered pair (s, s') is the difference between the number of consistent and opposite branches examined along the chain from s to s'.

Let s and s' be vertices in the same component C of $\Gamma(A_i)$. By $| s, s' |_d$ we shall denote the length modulo d of an arbitrary chain between s and s', where d is a divisor of the cycle length of C. Note, that $| s, s' |_d$ is unique.

In the sequel, we shall identify a state of A with the corresponding vertex of A, since it does not cause ambiguity.

Operation-Preserving Functions of Autonomous Automata

We shall consider the operation-preserving functions of autonomous automata. We have

Lemma 2. *Let $A = (S, \sigma, M)$ and $B = (T, \sigma, N)$ be autonomous automata. Let f be a homomorphism of A into B. Let s belong to the component C_A of $\Gamma(A)$ and let s be in ν-level of C_A. Let $f(s)$ belong to the component C_B of $\Gamma(B)$ and let $f(s)$ be in ν'-level of C_B. Then $\nu' \leqslant \nu$.*

Proof. Let us assume that $\nu' > \nu$. Let the component C_A have a cycle length d. Then $M(s, \sigma^\nu) = M(s, \sigma^{\nu+d})$, where $\sigma^\nu = \sigma\sigma \cdots \sigma$ denotes the ν-fold concatenation of σ, and $M(s, \sigma^0) = s$. Furthermore, $N(f(s), \sigma^\nu)$ does not belong to the cycle of C_B, since $\nu' > \nu$. It follows that $N(f(s), \sigma^\nu) \neq N(f(s), \sigma^{\nu+d})$. But

$$N(f(s), \sigma^\nu) = f(M(s, \sigma^\nu)) = f(M(s, \sigma^{\nu+d})) = N(f(s), \sigma^{\nu+d}),$$

a contradiction.

LEMMA 3. *Let $A = (S, \sigma, M)$ and $B = (T, \sigma, N)$ be autonomous automata; let f be a homomorphism of A into B; let s belong to the component C_A of $\Gamma(A)$; let $f(s)$ belong to the component C_B of $\Gamma(B)$. Then the length d' of the cycle of C_B is the divisor of the cycle length d of C_A.*

Proof. Let s be in ν-level of C_A. Then $M(s, \sigma^\nu)$ is in the cycle of C_A, and from Lemma 2 it follows, that $N(f(s), \sigma^\nu)$ is in the cycle of C_B. Let d' be no divisor of d. Then $N(f(s), \sigma^\nu) \neq N(f(s), \sigma^{\nu+d})$. But $M(s, \sigma^\nu) = M(s, \sigma^{\nu+d})$, and

$$N(f(s), \sigma^\nu) = f(M(s, \sigma^\nu)) = f(M(s, \sigma^{\nu+d})) = N(f(s), \sigma^{\nu+d}),$$

a contradiction.

COROLLARY. *Let $A = (S, \sigma, M)$ and $B = (T, \sigma, N)$ be autonomous automata; let f be an isomorphism of A into B; let s belong to the component C_A of $\Gamma(A)$, and let s be in ν-level of C_A; let $f(s)$ belong to the component C_B of $\Gamma(B)$, and let $f(s)$ be in ν'-level of C_B. Then $\nu' = \nu$ and the length of the cycle of C_B is the same as the length of the cycle of C_A.*

Lemmas 2, 3, and the Corollary, in stronger forms, can be found in [6].

LEMMA 4. *Let $A = (S, \sigma, M)$ and $B = (T, \sigma, N)$ be autonomous automata; let f be a homomorphism of A into B; let s, s' be states of A such that both s and s' belong to the cycle of the same component C_A of $\Gamma(A)$. Then*

(i) *both $f(s)$ and $f(s')$ belong to the same component C_B of $\Gamma(B)$,*

(ii) *$| s, s' |_{d'} = | f(s), f(s')|_{d'}$, where d' is the length of the cycle of C_B.*

Proof. For A there exist nonnegative integers k and l such that $M(s, \sigma^k) = M(s', \sigma^l)$. since s and s' belong to the same component C_A of $\Gamma(A)$. Then

$$N(f(s), \sigma^k) = f(M(s, \sigma^k)) = f(M(s', \sigma^l)) = N(f(s'), \sigma^l),$$

and thus $f(s)$ and $f(s')$ belong to the same component, say C_B, of $\Gamma(B)$.

Let the length of the cycle of C_A be equal to d, and let the length of the cycle of C_B be equal to d'. By Lemma 3, d' is a divisor of d. Then

$$| s, s' |_d \equiv k - l \pmod d$$

and

$$|f(s), f(s')|_{d'} \equiv k - l \pmod{d'},$$

and hence

$$|s, s'|_{d'} = |f(s), f(s')|_{d'}.$$

LEMMA 5. *Let $A(s) = (S', \sigma, M')$ and $B = (T, \sigma, N)$ be autonomous automata. Let s be in S', s be in ν-level of $\Gamma(A(s))$, and let $\Gamma(A(s))$ have a cycle length equal to d; let t be an arbitrary state in T; let t be in the component C_B of $\Gamma(B)$; let C_B have a cycle length equal to d', and let t be in ν'-level of C_B, where $\nu' \leqslant \nu$ and d' is a divisor of d. Then a (s, t)-transition generated function $f : A(s) \to B$ is a homomorphism of $A(s)$ into B.*

Proof. Let $f : A(s) \to B$ be an (s, t)-transition generated function; let r be an arbitrary state in S'. Then for r there exists a nonnegative integer k, $k \leqslant \nu + d - 1$, such that $r = M(s, \sigma^k)$. Furthermore,

$$f(M(s, \sigma^k)) = N(t, \sigma^k)$$

for $k = 0, 1, \ldots, \nu + d - 1$, and

$$f(M(s, \sigma^{\nu+d})) = f(M(s, \sigma^{\nu})) = B(t, \sigma^{\nu}) = N(t, \sigma^{\nu+d}),$$

since $\nu' \leqslant \nu$ and d' is a divisor of d. Moreover,

$$\begin{aligned} f(M(r, \sigma)) &= f(M(M(s, \sigma^k), \sigma)) \\ &= f(M(s, \sigma^{k+1})) \\ &= N(t, \sigma^{k+1}) \\ &= N(N(t, \sigma^k), \sigma) \\ &= N(f(M(s, \sigma^k)), \sigma) \\ &= N(f(r), \sigma), \end{aligned}$$

i.e., f is a homomorphism of $A(s)$ into B.

Lemmas 2–4 give some restrictions for homomorphisms of autonomous automata. We shall compare it with the restriction for homomorphism, which results from Lemma 5 of Bavel [1].

Let $x \in I$, where $x = \sigma_0\sigma_1 \cdots \sigma_{l-1}$, $\sigma_k \in \Sigma$ for each $k \in \{0, 1, \ldots, l - 1\}$. The length of x is the number l, which will be denoted by $|x|$.

Let s be a state of $A = (S, \Sigma, M)$. Let $s \in S$ and $A(s) = (S', \Sigma, M')$. The length of s is equal to

$$\max_{r \in S'}\{\min_{x \in I}\{|x| : M(s, x) = r\}\},$$

and it will be denoted by $|s|$.

Let $A = (S, \sigma, M)$ and $B = (T, \sigma, N)$ be autonomous automata, and let the state s of A belong to the component C_A of $\Gamma(A)$. Let d be a cycle length of C_A, and let s be in ν-level of C_A. Then for a homomorphism $f: A \to B$ we have that $f(s)$ belongs to the component C_B of $\Gamma(B)$ such that the cycle length of C_B is equal to d' and $f(s)$ is in ν'-level of C_B, where d' is a divisor of d and $\nu' \leqslant \nu$. It implies that $|s| \geqslant |f(s)|$, since $|s| = \nu + d$ and $|f(s)| = \nu' + d'$. Obviously, if the state t of T is in the component C_B' of $\Gamma(B)$ with the cycle length of C_B' equal to d'', if t is in ν''-level of C_B', and if $|t| \leqslant |s|$, then it does not imply that $\nu'' \leqslant \nu$ and d'' is a divisor of d.

Let $A(s) = (S', \sigma, M')$ and $B = (T, \sigma, N)$ be autonomous automata. A function $f: A(s) \to B$ is regular transition generated if and only if for $f(s) = t$ and for each $k \in \{0, 1,..., \nu + d\}$

$$f(M(s, \sigma^k)) = N(t, \sigma^k),$$

where d is the cycle length of $\Gamma(A(s))$, and s is in the ν-level of $\Gamma(A(s))$.

Let $A = (S, \sigma, M)$ and $B = (T, \sigma, N)$ be autonomous automata. A regular transition generated function of A into B is a function $f: A \to B$, for which there exists a generating set $\{s_0, s_1,..., s_{m-1}\}$ of A such that any $f_p: A(s_p) \to B$ is regular transition-generated for each $p \in \{0, 1,..., m-1\}$, where f_p is f restricted to $A(s_p)$.

If, for f, this definition is performed for a certain generating set of A, then it is satisfied for any generating set of A.

Note, that for autonomous automata $A = (S, \sigma, M)$ and $B = (T, \sigma, N)$, function $f: A \to B$ is regular transition-generated if and only if f is a homomorphism.

Let $\{s_0, s_1,..., s_{m-1}\}$ be a generating set of $A = (S, \sigma, M)$ and let s_p belong to the ν_p-level of the component C_{i_p} of $\Gamma(A)$ with the cycle length of C_{i_p} equal to d_p, for $p \in \{0, 1,..., m-1\}$. If for an ordered set $(s_0, s_1,..., s_{m-1})$ we choose the ordered set $(t_0, t_1,..., t_{m-1})$, a subset of T, no matter which ones, and if we define a relation ρ by

$$\rho = \{(M(s_p, \sigma^q), N(t_p, \sigma^q)) : p \in \{0, 1,..., m-1\} \quad \text{and} \quad q \in \{0, 1,..., \nu_p + d_p\}\},$$

then ρ is a homomorphism of A into B if and only if ρ is a function. For similar result, see [9].

Operation-Preserving Functions of Automata

Let $A = (S, \Sigma, M)$ and $B = (T, \Sigma, N)$ be arbitrary automata. A function $f: A \to B$ is said to be regular transition generated if and only if f is a regular transition generated function of $A_i = (S, \sigma_i, M_i)$ into $B_i = (T, \sigma_i, N_i)$ for each $i \in \{0, 1,..., n-1\}$, where A_i and B_i are autonomous factors of A and B, respectively.

From previous Lemmas and remarks follows:

THEOREM. *A function $f: A \to B$ is regular transition generated if and only if f is a homomorphism of A into B.*

From this theorem follows

$$H(A \to B) = \bigcap_{i=0}^{n-1} H(A_i \to B_i),$$

or $f \in H(A \to B)$ if and only if there exist $f_0 \in H(A_0 \to B_0)$, $f_1 \in H(A_1 \to B_1),\ldots, f_{n-1} \in H(A_{n-1} \to B_{n-1})$ such that $f_0 \equiv f_1 \equiv \cdots \equiv f_{n-1}$,

$$| E(A)| \leqslant \min_{i=0,1,\ldots,n-1} | E(A_i)|,$$

and

$$| G(A)| \leqslant \min_{i=0,1,\ldots,n-1} | G(A_i)|,$$

where $| X |$ denotes the cardinality of set X.

Now we can quote an algorithm for determining $H(A_i \to B_i)$, where $i \in \{0, 1,\ldots, n-1\}$, and A_i and B_i are autonomous factors of A and B, respectively, since from it easily follows the algorithm for determining $H(A \to B)$.

Let $\Gamma(A_i)$ contain the components $C^0_{A_i}, C^1_{A_i},\ldots, C^{\alpha-1}_{A_i}$ with the cycle lengths $d_i^0, d_i^1,\ldots, d_i^{\alpha-1}$, respectively, and let $\Gamma(B_i)$ contain the components $C^0_{B_i}, C^1_{B_i},\ldots, C^{\beta-1}_{B_i}$ with the cycle lengths $\delta_i^0, \delta_i^1,\ldots, \delta_i^{\beta-1}$, respectively. Any $C^j_{A_i}$ determines automaton A_i^j, where $A_i^j = (S_i^j, \sigma_i, M_i^j)$, S_i^j is the state set of $C^j_{A_i}$, M_i^j is M restricted to S_i^j, and $j = 0, 1,\ldots, \alpha - 1$. By analogy, any $C^k_{B_i}$ determines automaton $B_i^k = (T_i^k, \sigma_i, N_i^k)$, with the corresponding changes, where $k = 0, 1,\ldots, \beta - 1$. Any homomorphism f_i of A_i into B_i is equal to the ordered set $(f_i^0, f_i^1,\ldots, f_i^{\alpha-1})$, where f_i^j is f_i restricted to S_i^j, or $f_i^j \in H(A_i^j \to B_i)$, for $j = 0, 1,\ldots, \alpha - 1$. Hence, our algorithm is reduced to finding all $H(A_i^0 \to B_i)$, $H(A_i^1 \to B_i),\ldots, H(A_i^{\alpha-1} \to B_i)$.

If among $d_i^0, d_i^1,\ldots, d_i^{\alpha-1}$ there exists a number d such that in $\{\delta_i^0, \delta_i^1,\ldots, \delta_i^{\beta-1}\}$ there does not exist a divisor of d, then $H(A_i \to B_i)$ is empty (by Lemma 3) and hence $H(A \to B)$ is empty, too. We assume that such number d does not exist.

First, we shall determine the members of $H(A_i^0 \to B_i)$.

Among the components of $\Gamma(B_i)$ we choose all the components $C^{k_0}_{B_i}, C^{k_1}_{B_i},\ldots, C^{k_{\gamma-1}}_{B_i}$ such that $\delta_i^{k_0}, \delta_i^{k_1},\ldots, \delta_i^{k_{\gamma-1}}$ are divisors of d_i^0.

We calculate a member of $H(A_i^0 \to B_i^{k_0})$. Therefore, we determine the minimal generating set $\{s_0, s_1,\ldots, s_{\epsilon-1}\}$ of A_i^0. For s_0 we choose an arbitrary state t_0 in $T_i^{k_0}$, with $\nu_0 \geqslant \nu_0'$, where s_0 is in ν_0-level of $C^0_{A_i}$ and t_0 is in ν_0'-level of $C^{k_0}_{B_i}$. Now, for $s_1,\ldots, s_{\epsilon-1}$ we determine $t_1,\ldots, t_{\epsilon-1}$ in $T_i^{k_0}$, respectively; $t_1,\ldots, t_{\epsilon-1}$ are arbitrary but such that $\nu_1 \geqslant \nu_1',\ldots, \nu_{\epsilon-1} \geqslant \nu'_{\epsilon-1}$, and

$$| s_0, s_1 |_{\delta_i^{k_0}} = | t_0, t_1 |_{\delta_i^{k_0}},\ldots, | s_0, s_{\epsilon-1} |_{\delta_i^{k_0}} = | t_0, t_{\epsilon-1} |_{\delta_i^{k_0}},$$

where s_1 is in ν_1-level of $C^0_{A_i}, \ldots, s_{\epsilon-1}$ is in $\nu_{\epsilon-1}$-level of $C^0_{A_i}$, t_1 is in ν_1'-level of $C^{k_0}_{B_i}, \ldots, t_{\epsilon-1}$ is in $\nu'_{\epsilon-1}$-level of $C^{k_0}_{B_i}$.

Next we define a relation $\rho_{t_0 t_1 \cdots t_{\epsilon-1}}$,

$$\rho_{t_0 t_1 \cdots t_{\epsilon-1}}$$
$$= \{(M_i^0(s_p, \sigma_i^q), N_i^{k_0}(t_p, \sigma_i^q)) : p \in \{0, 1, \ldots, \epsilon - 1\} \text{ and } q \in \{0, 1, \ldots, \nu_p + d_i^0\}\}.$$

If $\rho_{t_0 t_1 \cdots t_{\epsilon-1}}$ is a function then it is a member of $H(A_i^0 \to B_i^{k_0})$.

For any other possible ordered ϵ-tuple of states from $T_i^{k_0}$, satisfying the due conditions with regard to levels of $C^{k_0}_{B_i}$ and corresponding lengths modulo $\delta_i^{k_0}$, we check whether a relation ρ is a function. If it is, it is also a member of $H(A_i^0 \to B_i^{k_0})$.

By analogy, we calculate the members of $H(A_i^0 \to B_i^{k_1}), \ldots, H(A_i^0 \to B_i^{k_{\gamma-1}})$. Obviously,

$$H(A_i^0 \to B_i) = \bigcup_{k=k_0}^{k_{\gamma-1}} H(A_i^0 \to B_i^k).$$

Similarly, we calculate the members of $H(A_i^1 \to B_i), \ldots, H(A_i^{\alpha-1} \to B_i)$.

In the above algorithm for determining $H(A_i \to B_i)$ checking of operation-preserving does not appear.

Note, that to determine $H(A \to B)$, we need not calculate $H(A_i \to B_i)$ for all $i = 0, 1, \ldots, n-1$. Really, let $H(A_i \to B_i)$ be known for a certain i, say, for $i = 0$. Then $H(A \to B)$ can be determined by elimination of the members of $H(A_0 \to B_0)$. More specifically, we check the members of $H(A_0 \to B_0)$ whether they are members of $H(A_1 \to B_1), \ldots, H(A_{n-1} \to B_{n-1})$. First, we check the members of $H(A_0 \to B_0)$ whether they are members of $H(A_1 \to B_1)$. For it, for any automaton, implied by a certain component C_{A_1} of $\Gamma(A_1)$, we determine the minimal generating set, say $\{s_0, s_1, \ldots, s_{\xi-1}\}$. For $\{s_0, s_1, \ldots, s_{\xi-1}\}$ and for any $f \in H(A_0 \to B_0)$ we consider the set $\{f(s_0), f(s_1), \ldots, f(s_{\xi-1})\}$. If among $f(s_0), f(s_1), \ldots, f(s_{\xi-1})$ there exist states which belong to two different components of $\Gamma(B_1)$, then such f is eliminated (as not being a member of $H(A_1 \to B_1)$ according to Lemma 4).

Let all $f(s_0), f(s_1), \ldots, f(s_{\xi-1})$ be in the same component C_{B_1} of $\Gamma(B_1)$. If in the set $\{f(s_0), f(s_1), \ldots, f(s_{\xi-1})\}$ there exists a member $f(s_k)$; $k \in \{0, 1, \ldots, \xi - 1\}$; at a level higher than s_k (Lemma 2), or if the cycle length of C_{B_1} is not a divisor of the cycle length of C_{A_1} (Lemma 3), or if the corresponding lengths modulo the cycle length of C_{B_1} are not equal (Lemma 4), or if a relation ρ is not a function, then f must also be eliminated. In this way, we check f for any other component of $\Gamma(A_1)$. Similarly, we check all other members of $H(A_0 \to B_0)$, whether they are members of $H(A_1 \to B_1)$, and, then, whether they are members of $H(A_2 \to B_2)$ and so on.

The algorithms for determining $Is(A \to B)$, $E(A)$, and $G(A)$ are simple modifications of the algorithm for determining $H(A \to B)$, and hence are omitted.

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