Huygens' Principle for Wave Equations on Symmetric Spaces

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Let $X = G/K$ be a symmetric space, $L_X$ the Laplace-Betrami operator on $X$, and $2p$ the sum of the positive restricted roots of $X$ (with multiplicity). Using the Fourier transform on $X$ we write down the solution of the modified wave equation $\frac{\partial^2 u}{\partial t^2} = (L_X + |p|^2)u$ on $X$. It is proved, using the Paley-Wiener theorem for this Fourier transform (S. HELGASON, Ann. of Math. 98 (1973), 451-480) that the equation satisfies Huygens' principle if $\dim X$ is odd and all Cartan subgroups of $G$ are conjugate.

1. INTRODUCTION

In his work [5], Hadamard suggested the problem of deciding for which Riemannian manifolds $X$ the wave equation satisfies Huygens' principle. If $X = \mathbb{R}^{2n+1}$ it is well known that the wave equation satisfies this principle. Hadamard proved that the condition $\dim X = \text{odd}$ is a necessary condition. For a very readable report on the problem, see Günther's article [4a].

In their paper [10], Ölfsson and Schlichtkrull show that Huygens' principle holds for the wave equation on an odd-dimensional symmetric space $X$ of the noncompact type for which the isometry group has all Cartan subgroups conjugate. The proof uses the theory of the Radon transform on $X$. Such a result, using similar methods, had been indicated without proof by Solomatina [11].

In the present note we give another independent proof of essentially the same result (Theorem 4.1) using the Fourier transform on $X$. This proof was inspired by the paper [1] of Branson and Ölfsson on the energy partition problem.

The case $G$ complex had been settled in [6g] and [6h]; these papers also establish Huygens' principle for the modified wave equation on a

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compact semisimple Lie group of odd dimension. For the case 
\( G = \text{SO}_o(2n + 1, 1) \) explicit solution formulas are known [4b, 6g, 8, 9] implying, in particular, Huygens' principle.

I am greatly indebted to Ólafsson and Schlichtkrull for discussions about the problem and for sending me a copy of their paper [10] before publication.

2. HUYGENS' PRINCIPLE

Let \( X \) be a Riemannian manifold with distance function \( d \). As usual we denote the open ball and sphere, respectively, by

\[
B_r(x) = \{ y \in X : d(x, y) < r \}, \quad S_r(x) = \{ y \in X : d(x, y) = r \}.
\]

Let \( L_X \) be the Laplace–Beltrami operator on \( X \) and \( P(d/dt) \) a differential operator on \( \mathbb{R} \). Given \( f \in \mathcal{D}(X) = \mathcal{C}^\infty_c(X) \) we consider the Cauchy problem

\[
P \left( \frac{d}{dt} \right) u = L_X u, \quad u(x, 0) = 0, \quad u_t(x, 0) = f(x).
\]

Huygens' principle is said to hold for (1) if for each \( \varepsilon > 0 \)

\[
u(x, t) \text{ depends only on the restriction } f \mid V_{t, \varepsilon}(x),
\]

\( V_{t, \varepsilon}(x) \) denoting the shell \( B_{t+\varepsilon}(x) - B_{t-\varepsilon}(x) \). If \( X = \mathbb{R}^n \) and \( P(d/dt) = d^2/dt^2 \), (1) is the standard wave equation and the solution \( u \) is given by the Poisson–Tedone formula (for \( n \geq 2 \))

\[
u(x, t) = \frac{1}{(n-2)!} \frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t (M'f)(x) r(t^2 - r^2)^{(n-3)/2} dr
\]

(cf. [7, p. 33]), \( (M'f)(x) \) denoting the average of \( f \) on \( S_r(x) \). If \( n \) is odd the right hand side of (3) reduces to an expression

\[
u(x, t) = \sum_{k=0}^{(n-3)/2} a_k t^{k+1} \frac{d^k}{d t^k} (M'f)(x).
\]

so Huygens' principle does indeed hold.

3. RADON AND FOURIER TRANSFORMS
   ON RIEMANNIAN SYMMETRIC SPACES

Let \( X = G/K \) be a symmetric space of the noncompact type, i.e., \( G \) is a connected semisimple Lie group with finite center and \( K \) is a maximal com-
pact subgroup. Consider an Iwasawa decomposition $G = NAK$ where $N$ is nilpotent, $A$ abelian, and let $g = n + a + f$ be the corresponding Lie algebra decomposition. Let $M$ be the centralizer of $A$ in $K$. If $g \in G$ we determine $A(g) \in a$ by $g \in N \exp A(g) K$ and put $A(gK, kM) = A(k^{-1}g)$. Let $\rho \in a^*$ be determined in terms of the Jacobian of the map $n \rightarrow ana^{-1}$ by

$$\rho(\log a) = \frac{1}{2} \log \left( \frac{d(ana^{-1})}{dn} \right), \quad a \in A,$$

and let $B = K/M$.

The Fourier transform of a function $f$ on $X$ is defined by (cf. [6b])

$$\hat{f}(\lambda, b) = \int_X f(x) e^{-i\lambda \cdot x + \rho(A(x, b))} \, dx$$  \hspace{1cm} (1)

for all $\lambda \in a^*$, $b \in B$ for which the integral converges. Here $dx$ is the volume element on $X$. A horocycle in $X$ is by definition an orbit of a subgroup of $G$ conjugate to $N$. The group $G$ permutes the horocycles transitively; more precisely, if $o = \{K\}$ and $\xi_0 = N \cdot o$ the mapping

$$(kM, a) \rightarrow ka \cdot \xi_0$$

is a bijection of $K/M \times A$ onto the space $\Xi$ of horocycles. The Radon transform of a function $f$ on $X$ is defined by (cf. [6a])

$$\mathcal{F}(\xi) = \int_{\xi} f(x) \, dm(x)$$  \hspace{1cm} (2)

for all $\xi \in \Xi$ for which the integral exists, $dm$ being the volume element on the submanifold $\xi$. The two transforms are related by

$$\mathcal{F}(\lambda, kM) = \int_A \hat{f}(ka \cdot \xi_0) \, e^{-i\lambda \cdot x + \rho(\log a)} \, da,$$ \hspace{1cm} (3)

da being a suitably normalized Haar measure on $A$.

If $\phi$ is a continuous function on $\Xi$ its dual transform $\tilde{\phi}$ is defined by

$$\tilde{\phi}(gK) = \int_K \phi(gk \cdot \xi_0) \, dk,$$ \hspace{1cm} (4)

which is the average of $\phi$ over the set of horocycles passing through the point $gK$ in $X$. We then have for a fixed $G$-invariant measure $d\xi$ on $\Xi$,

$$\int_X f(x) \hat{\phi}(x) \, dx = \int_{\Xi} \hat{f}(\xi) \phi(\xi) \, d\xi.$$ \hspace{1cm} (5)
The Radon transform is inverted by the following formula: for a certain pseudodifferential operator $L$ on $\mathbb{Z}$,

$$f = (L\hat{f})'$$

$f \in \mathcal{D}(X)$ (6)

(cf. [6b, c]). Furthermore, $L$ is a differential operator exactly in the case when all Cartan subgroups of $G$ are conjugate. As proved in [6d], the range of $\mathcal{D}(X)$ under the Fourier transform consists of the functions $\psi \in \mathcal{C}^\infty(\mathfrak{a}_c^* \times B)$ satisfying the following two conditions: (i) $\lambda \to \psi(\lambda, b)$ is a holomorphic function on $\mathfrak{a}_c^*$ satisfying

$$\text{supp}_{(\lambda, b) \in \mathfrak{a}_c^* \times B} (1 + \lambda)^N e^{-R \lambda} |\psi(\lambda, b)| < \infty$$

(7)

for some fixed $R > 0$ and all $N \geq 0$. (Here $\text{Im} \lambda$ denotes the imaginary part of $\lambda$.) (ii) If $W$ denotes the Weyl group of $(\mathfrak{g}, \mathfrak{a})$,

$$\int_B e^{i(e - s \lambda + \rho)(A(x, b))} \psi(s\lambda, b) \, db = \int_B e^{i(e + \rho)(A(x, b))} \psi(\lambda, b) \, db$$

(8)

for all $s \in W$, $x \in X$.

The proof also gives the following support theorem for the Radon transform [6d, Lemma 8.1]. Let

$$\beta_r = \{ \xi \in \mathbb{Z} : d(o, \xi) < R \}, \quad \sigma_r = \{ \xi \in \mathbb{Z} : d(o, \xi) = r \}.$$

Then we have

$$f \in \mathcal{D}(X), \quad \text{supp}(\hat{f}) \subset \beta_r \iff \text{supp}(f) \subset \overline{B_r(o)}.$$

(9)

We also need the definition of the Fourier transform $\mathcal{F}$ and Radon transform $\hat{\mathcal{F}}$ for all $T$ in the space $\mathcal{E}'(X)$ of distributions of compact support. The analog of (1) is obviously

$$\mathcal{F}(\lambda, b) = \int_X e^{i(-s \lambda + \rho)(A(x, b))} \, dT(x),$$

(10)

and in accordance with (5) we define $\hat{T}(\phi) = T(\delta)$. The range $\mathcal{E}'(X)^\sim$ is described similarly to $\mathcal{D}(X)^\sim$ above except that condition (7) holds with one $N < 0$ (cf. [6d, Theorem 8.5] or [2]). We now need the extension of (9) to distributions.

**Lemma 3.1.** Let $T \in \mathcal{E}'(X)$ satisfy the condition

$$\text{supp}(\mathcal{F}) \subset \beta_A.$$


Then

\[ \text{supp}(T) \subseteq B_A(\sigma). \]

**Proof.** The proof is analogous to the one for the Euclidean case [6f, Chap. I, Theorem 2.20]. For \( F \in \mathcal{D}(G) \), \( \phi \in \mathcal{D}(\Xi) \), and \( \sigma \in \mathcal{D}'(\Xi) \) consider the mixed convolution

\[
(F \times \phi)(\xi) = \int_G F(g) \phi(g^{-1} \cdot \xi) \, dg, \quad (F \times \sigma)(\phi) = \int_G F(g) \sigma(\phi^g) \, dg,
\]

where \( \phi^g(\xi) = \phi(g^{-1} \cdot \xi) \). Then we have

\[
(F \times \phi)^g(x) = \int_G F(g) \phi(g^{-1} \cdot x) \, dg. \tag{11}
\]

By the definition of \( \hat{T} \) the support assumption on \( \hat{T} \) implies

\[
T(\hat{\phi}) = 0 \tag{12}
\]

for all \( \phi \in \mathcal{D}(\Xi) \) with \( \text{supp}(\phi) \cap \beta_A = \emptyset \). Let \( \varepsilon > 0 \) and let \( \phi \in \mathcal{D}(\Xi) \) satisfy \( \text{supp}(\phi) \cap \beta_{A+\varepsilon} = \emptyset \). Let \( N_\varepsilon \) be a symmetric neighborhood of \( e \) in \( G \) such that \( g \cdot \beta_A \subseteq \beta_{A+\varepsilon} \) for \( g \in N_\varepsilon \). Then

\[
g \cdot \text{supp}(\phi) \cap \beta_A = \emptyset \quad \text{for} \quad g \in N_\varepsilon.
\]

If \( F \) is symmetric and \( \text{supp}(F) \subseteq N_\varepsilon \) we have

\[
\text{supp}(F \times \phi) \cap \beta_A = \emptyset. \tag{13}
\]

Thus the distribution \( F \times T \) defined by

\[
(F \times T)(f) = \int_G F(g^{-1}) T(f^g) \, dg
\]

(where \( f^g(x) = f(g^{-1}x) \)) satisfies, since \( F(g^{-1}) = F(g) \),

\[
(F \times T)(\hat{\phi}) = \int_G F(g) T(\hat{\phi}^g) \, dg
\]

\[
= \int_G F(g) \left( \int_X \hat{\phi}(g^{-1}x) \, dT(x) \right) \, dg = T((F \times \phi)^g) = 0
\]

by (12)–(13). This means that

\[
(F \times T)^g(\phi) = (F \times T)(\hat{\phi}) = 0
\]
so \((F \times T)^\wedge\) has support in \(\overline{B}_{A+\varepsilon}\). On the other hand, \(F \times T\) is a function, as we see by lifting \(T\) to a distribution on \(G\). Thus the support theorem \((9)\) implies

\[\text{supp}(F \times T) \subset \overline{B}_{A+\varepsilon}(0)\].

Letting \(\varepsilon \to 0\) we get the desired conclusion.

**Remark.** The converse of Lemma 3.1 holds also as a result of \((9)\) and the relation \((F \times T)^\wedge = F \times \hat{T}\).

4. THE WAVE EQUATION ON \(X\)

With \(X\) as in Section 3 we consider the Cauchy problem for the modified wave equation on \(X\),

\[\frac{\partial^2 u}{\partial t^2} = (L_X + |\rho|^2)u, \quad u(x, 0) = 0, u_t(x, 0) = f(x),\]

and prove the following result.

**THEOREM 4.1.** Let \(X\) be an odd-dimensional symmetric space of the non-compact type and suppose \(G\) has all its Cartan subgroups conjugate. Then the modified wave equation \((1)\) satisfies Huygens principle and the solution is given by \((8)\) below.

**Proof.** We take \(f\) in \((1)\) in \(\mathfrak{D}(X)\) and apply the Fourier transform to the equation above. Then

\[\tilde{u}_{tt}(t, \lambda, b) + |\lambda|^2 \tilde{u}(t, \lambda, b) = 0\]

so, taking the initial data into account,

\[\tilde{u}(t, \lambda, b) = \tilde{f}(-\lambda, b) \frac{\sin |\lambda| t}{|\lambda|}\]

(cf. [1, Sect. 2]). Now \(|\lambda|^{-1} \sin |\lambda| t\) is the Euclidean Fourier transform of a distribution \(T\), on \(A\) of compact support:

\[\frac{\sin |\lambda| t}{|\lambda|} = \int_A e^{-i\lambda \log a} dT,(a)\].

(4)
For a quick proof of this in the present context consider the wave equation on $A$ (as a Euclidean space):

$$L_A v = \frac{\partial^2 v}{\partial t^2}, \quad v(a, 0) = 0, \quad \frac{\partial v}{\partial t}(a, 0) = F(a), \quad F \in \mathcal{D}(A).$$

Taking Fourier transform on $A$ we obtain (compare (3))

$$\tilde{v}(\lambda, t) = \tilde{F}(\lambda) \frac{\sin |\lambda| t}{|\lambda|}.$$  (4')

For $t$ fixed the map $F \rightarrow v(\cdot, t)$ commutes with the translations on $A$ so

$$v(a, t) = (F \ast T_t)(a), \quad \text{where } T_t \in \mathcal{D}'(A).$$  (4'')

From (3) Section 1 we deduce that

$$\text{supp}(T_t) \subseteq B_1(e) \quad \text{if dim } A \geq 2$$
$$\text{supp}(T_t) \subseteq S_1(e) \quad \text{if dim } A \geq 2, \text{ odd}.$$  (4')

Now (4) follows from (4') and (4''). Note also that if dim $A = 1$, (4) holds with $T_t$ as half the characteristic function of $B_1(e)$. On the other hand, the Paley–Wiener theorem for $\mathcal{E}''(X)$ mentioned in Section 3 implies that

$$\frac{\sin |\lambda| t}{|\lambda|} = \int_X e^{(-i \lambda + i \rho)(A(x, b))} d\tau_t(x),$$  (5)

where $\tau_t \in \mathcal{E}''(X)$. Integrating over $b$ we get

$$\frac{\sin |\lambda| t}{|\lambda|} = \int_X \phi(x) d\tau_t(x),$$

where $\phi(x)$ is the spherical function. By the uniqueness, $\tau_t$ is $K$-invariant. So is its Radon transform $\hat{\tau}_t$, and we now relate it to $T_t$.

**Lemma 4.2.** In terms of the product decomposition $\mathbb{Z} = K/M \times A$ we have

$$\hat{\tau}_t = 1 \otimes e^\rho T_t.$$

**Proof.** If $g = kan$ relative to the decomposition $G = KAN$ we put $a = \exp H(g)$. Let $e \in \mathcal{C}_c(K/M)$, $\beta(a) = e^{(-i \lambda - i \rho)(\log a)}$, and $a^\lambda$ the integral of $a$ over $K/M$. Then by $K$-invariance,

$$\hat{\tau}_t(\alpha \otimes \beta) = \alpha^\pi \hat{\tau}_t(1 \otimes \beta) = \alpha^\pi \tau_t((1 \otimes \beta) \circ)$$

$$(1 \otimes \beta) \circ (g \cdot a) = \int_K (1 \otimes \beta)(gk \cdot \xi) dk = \int_K \beta(\exp H(gk)) dk = \phi(\cdot, (g))$$
\[ \hat{\xi}(\alpha \otimes \beta) = \tilde{\alpha} \xi(\phi^{-1}) = \tilde{\alpha}^2 T_\alpha(e^{-i\cdot}) = (1 \otimes e^{\theta T_\alpha})(\alpha \otimes \beta). \]

Since these $\alpha \otimes \beta$ span a dense subspace of $C^\infty(\Sigma)$ the lemma is proved.

**Lemma 4.3.** Suppose $G$ has all its Cartan subgroups conjugate, and let $A > 0$. Then if $T \in \mathcal{E}'(X)$,

\[ \text{supp}(\hat{T}) \subset \sigma_A \Rightarrow \text{supp}(T) \subset S_A(o). \]  

**Proof.** Let $\varepsilon > 0$ and suppose $f$ satisfies $\text{supp}(f) \subset B_{A-\varepsilon}(o)$. Then $\text{supp}(\hat{f}) \subset \beta_{A-\varepsilon}$ and since $L$ in (6) Section 3 is now a differential operator, $\text{supp}(L\hat{f}) \subset \beta_{A-\varepsilon}$. Hence by (6) Section 3

\[ T(f) = T((L\hat{f})^\vee) = \hat{T}(L\hat{f}) = 0 \]

so $\text{supp}(T) \cap B_{A-\varepsilon} = \emptyset$. Combining this with Lemma 3.1, (6) follows.

**Remark.** This lemma is a distribution counterpart to Proposition 1 of [10].

Now if $\dim A$ is odd and $> 1$ we have as already noted $\text{supp}(\tau_i) \subset S_i(e)$. Hence, by Lemma 4.3 we deduce for $\dim X$ odd and rank $X > 1$,

\[ \text{supp}(\tau_i) \subset S_i(o). \]  

(7)

Now if $f \in \mathcal{D}(X)$ and the distribution $T \in \mathcal{E}'(X)$ is $K$-invariant we have for the convolution $f \times T$ on $X$

\[ (f \times T)(\lambda, b) = \bar{T}(\lambda, b) \hat{T}(\lambda, b). \]

Hence by (3) $u(x, t) = (f \times \tau_i)(x)$ so by the symmetry of $\tau_i$,

\[ u(g \cdot o, t) = \int_X f(g \cdot x) \, d\tau_i(x) \]  

(8)

and by (7) this proves Theorem 4.1 for rank $X > 1$.

If rank $X = 1$, then by the classification, $X$ is a hyperbolic space and the result known as mentioned in the introduction.

While the above proof of Theorem 4.1 required the classification at the last point, the following variation does not require the classification.
THEOREM 4.4. Let $X$ and $G$ satisfy the assumptions of Theorem 4.1. Then the modified wave equation
\[ \frac{\partial^2 v}{\partial t^2} = (L_X + |\rho|^2)v, \quad v(x, 0) = f(x), \quad v_t(x, 0) = 0 \tag{9} \]
satisfies Huygens' principle.

In fact, if $u$ satisfies (1) then the function
\[ v(g \cdot o, t) = u_t(g \cdot o, t) = \int_X f(g \cdot x) dx \tag{10} \]
satisfies (9). By Lemma 4.2
\[ (\tau^r_t)^k = 1 \otimes e^o T^r_t \]
and from the properties recalled about $T^r_t$, we have
\[ \text{supp}(T^r_t) \subseteq S^r_t(0) \]
both when $\dim A = 2k + 1 > 1$ and when $\dim A = 1$. Thus by Lemma 4.3, \[ \text{supp}(\tau^r_t) \subseteq S^r_t(0) \] and now (10) implies the theorem.

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