Journal of Pure and Applied Algebra 216 (2012) 643–661

Contents lists available at SciVerse ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Inductions and restrictions for stable equivalences of Morita type

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ARTICLE INFO

Article history:
Received 13 December 2010
Received in revised form 15 June 2011
Available online 10 August 2011
Communicated by B. Keller

MSC: 18E30; 16G10; 16S10; 18G15

ABSTRACT

In this paper, we present two methods, induction and restriction procedures, to construct new stable equivalences of Morita type. Suppose that a stable equivalence of Morita type between two algebras A and B is defined by a B-A-bimodule N. Then, for any finite admissible set Φ and any generator X of the category of A-modules, the Φ-Auslander–Yoneda algebras of X and $N\otimes_A X$ are stably equivalent of Morita type. Moreover, under certain conditions, we transfer stable equivalences of Morita type between A and B to ones between eAe and fBf, where e and f are idempotent elements in A and B, respectively. Consequently, for self-injective algebras A and B over a field without semisimple direct summands, and for any A-module X and B-module Y, if the Φ-Auslander–Yoneda algebras of $A\oplus X$ and $B\oplus Y$ are stably equivalent of Morita type for one finite admissible set Φ, then so are the Ψ-Auslander–Yoneda algebras of $A\oplus X$ and $B\oplus Y$ for *every* finite admissible set Ψ. Moreover, two representation-finite algebras over a field without semisimple direct summands are stably equivalent of Morita type if and only if so are their Auslander algebras. As another consequence, we construct an infinite family of algebras of the same dimension and the same dominant dimension such that they are pairwise derived-equivalent, but not stably equivalent of Morita type. This answers a question by Thorsten Holm.

1. Introduction

In the representation theory of algebras and groups, there are three fundamental equivalences: Morita, derived and stable equivalences. Roughly speaking, the first two are induced from tensor products of bimodules or two-sided complexes, thus there is a corresponding Morita theory for each (see [18,21,10]), while the last one seems still to be well understood in this way, and therefore a Morita theory for stable equivalences is missing. Recently, a special class of stable equivalences, called stable equivalences of Morita type, have been introduced by Broué in the modular representation theory of finite groups. They are induced by bimodules, have features of a Morita theory, and are shown to be of great interest in modern representation theory since they preserve many homological and structural invariants of algebras and modules (see, for example, [3,4,11,12,19,23,24]). In order to understand this kind of equivalence, one has to know, first of all, as many examples and basic properties of stable equivalences of Morita type as possible. So, one of the crucial questions in the course of studying these equivalences is:

Question: How to construct stable equivalences of Morita type for finite-dimensional algebras?

Until now, only a few methods using trivial extensions, one-point extensions and endomorphism algebras have been known in [20,15–17]. Of course, Rickard's result that the existence of derived equivalences for self-injective algebras implies the one of stable equivalences of Morita type provides another way to construct stable equivalences of Morita type. This

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0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2011.07.016

method, however, is no longer true for general finite-dimensional algebras (see [8] for some new advances in this direction). So, a systematical method for constructing stable equivalences of Morita type seems not yet to be available.

In this paper, we shall look for a more general and systematical answer to this question, and present two methods, called induction and restriction procedures, to construct new stable equivalences of Morita type for general finite-dimensional algebras. Here our induction procedure has two flexibilities, one is the choice of generators, and the other is the one of finite admissible sets. Thus this construction provides a large variety of stable equivalences of Morita type.

To state our first main result, let us recall the definition of Φ-Auslander–Yoneda algebras in [7]. Let A be a finite-dimensional algebra and X an A-module. Then, for an admissible set Φ of natural numbers, there is defined an algebra $\mathrm{E}_A^{\Phi}(X)$, called the Φ-Auslander–Yoneda algebra of X in [7], which is equal to $\bigoplus_{i\in\Phi}\mathrm{Ext}_A^i(X,X)$ as a vector space, and its multiplication is defined in a natural way (see Section 2.2 below for details). Our main result for inductions reads as follows:

Theorem 1.1 (*The Induction Procedure*)**.** *Suppose that A and B are finite-dimensional k-algebras over a field k. Assume that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. Let X be an A-module which is a generator for the category of A-modules. Then, for any finite admissible set Φ of natural numbers, there is a stable equivalence of Morita type between $\mathrm{E}_A^{\Phi}(X)$ and $\mathrm{E}_B^{\Phi}(N\otimes_A X)$.*

Note that if $\Phi = \{0\}$, then the above result was known in [17]. Thus Theorem 1.1 generalizes the main result in [17], and provides many more possibilities for constructing stable equivalences of Morita type through the choices of different Φ. Also, our proof of Theorem 1.1 is different from that in [17].

Next, we shall exploit certain kinds of restrictions to construct stable equivalences of Morita type. Our result along this line is the following theorem.

Theorem 1.2 (*The Restriction Procedure*)**.** *Suppose that A and B are finite-dimensional k-algebras over a field k such that neither A nor B has semisimple direct summands. Further, suppose that ${}_AM_B$ and ${}_BN_A$ are bimodules without projective bimodules as direct summands, and define a stable equivalence of Morita type between A and B. If $e^2 = e \in A$ such that $M\otimes_B Ne \in \mathrm{add}(Ae)$, and if $f^2 = f \in B$ such that $\mathrm{add}(Bf) = \mathrm{add}(Ne)$, then the bimodules eMf and fNe define a stable equivalence of Morita type between eAe and fBf. Moreover, if we define $\Lambda = \mathrm{End}_{eAe}(eA)$, $\Gamma = \mathrm{End}_{fBf}(fB)$, $N' = \mathrm{Hom}_{fBf}((fB)_\Gamma, fNe\otimes_{eAe}(eA)_\Lambda)$ and $M' = \mathrm{Hom}_{eAe}((eA)_\Lambda, eMf\otimes_{fBf}(fB)_\Gamma)$, then ${}_\Gamma N'_\Lambda$ and ${}_\Lambda M'_\Gamma$ define a stable equivalence of Morita type between Λ and Γ.*

In fact, under the assumptions of Theorem 1.2, we may have a more general formulation, namely, for any finite admissible set Φ of natural numbers and for any *eAe*-module X, the Φ-Auslander–Yoneda algebras of $eAe\oplus X$ and $fBf\oplus fNe\otimes_{eAe}X$ are stably equivalent of Morita type. This is a consequence of Theorems 1.1 and 1.2.

Also, from Theorems 1.1 and 1.2 we have the following characterization of stable equivalences of Morita type for representation-finite algebras as well as for self-injective algebras.

Corollary 1.3. *Suppose that A and B are finite-dimensional k-algebras over a field k such that neither A nor B has semisimple direct summands.*

(1) *Assume further that A and B are self-injective. Let X be an A-module and let Y be a B-module. If there is a finite admissible set Φ of natural numbers such that $\mathrm{E}_A^{\Phi}(A\oplus X)$ and $\mathrm{E}_B^{\Phi}(B\oplus Y)$ are stably equivalent of Morita type, then, for* any *finite admissible set Ψ of natural numbers, the algebras $\mathrm{E}_A^{\Psi}(A\oplus X)$ and $\mathrm{E}_B^{\Psi}(B\oplus Y)$ are stably equivalent of Morita type.*
(2) *Assume additionally that A and B are representation-finite. Then A and B are stably equivalent of Morita type if and only if so are their Auslander algebras.*

Note that the "only if" part of Corollary 1.3(2) follows from [17].

Of course, there are many important classes of algebras which are of the form $\mathrm{End}_A(A\oplus Y)$ with A self-injective and Y an A-module. For example, Schur algebras or q-Schur algebras. Thus, as a consequence of Corollary 1.3, we know that the global dimension of $\mathrm{End}_{k[S_n]}(k[S_n]\oplus\Omega^i(Y))$ is finite for $i\in\mathbb{Z}$, where $k[S_n]$ is the group algebra of the symmetric group S_n, Y is the direct sum of non-projective indecomposable Young modules, and Ω is the usual syzygy operator.

As another byproduct of our considerations in this paper, we can construct a family of derived-equivalent algebras with certain special properties.

Corollary 1.4. *Suppose that k is a field with a non-zero element that is not a root of unity. Then, there is an infinite series of k-algebras of the same dimension such that they have the same dominant and global dimensions, and are all derived-equivalent, but pairwise not stably equivalent of Morita type.*

The contents of this paper are organized as follows. In Section 2, we fix notations and prepare some basic facts for our proofs. In Sections 3 and 4, we prove our main results, Theorems 1.1 and 1.2, as well as Corollary 1.3(2), respectively. In Section 5, we concentrate our consideration on self-injective algebras, and establish some applications of our main results. In particular, in this section we prove Corollary 1.3(1) and supply a sufficient condition, which is used in Section 6, to verify when two algebras are not stably equivalent of Morita type. In Section 6, we apply our results in the previous sections to Liu–Schulz algebras and give a proof of Corollary 1.4 which answers a question by Thorsten Holm.

2. Preliminaries

In this section, we shall fix some notations, and recall some definitions and basic results which are needed in the proofs of our main results.

2.1. Some conventions and homological facts

Throughout this paper, k stands for a fixed field. All categories and functors will be k-categories and k-functors, respectively. Unless stated otherwise, all algebras considered are finite-dimensional k-algebras, and all modules are finitely generated left modules.

Let $\mathcal{C}$ be a category. Given two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we denote the composition of f and g by fg which is a morphism from X to Z, while we denote the composition of a functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ with a functor $G : \mathcal{D} \to \mathcal{E}$ between categories $\mathcal{D}$ and $\mathcal{E}$ by GF which is a functor from $\mathcal{C}$ to $\mathcal{E}$.

If $\mathcal{C}$ is an additive category and X is an object in $\mathcal{C}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of direct sums of finitely many copies of X. The object X is called an additive generator for $\mathcal{C}$ if $\mathrm{add}(X) = \mathcal{C}$.

Let A be an algebra. We denote by A-mod the category of all A-modules, by A-proj (respectively, A-inj) the full subcategory of A-mod consisting of projective (respectively, injective) modules, by D the usual k-duality $\mathrm{Hom}_k(-, k)$, and by ν_A the Nakayama functor $D\mathrm{Hom}_A(-, {}_AA)$ of A. Note that ν_A is an equivalence from A-proj to A-inj with the inverse $\mathrm{Hom}_A(D(A), -)$. We denote the global and dominant dimensions of A by gl.dim(A) and dom.dim(A), respectively.

As usual, by $\mathscr{D}^b(A)$ we denote the bounded derived category of complexes over A-mod. It is known that A-mod is fully embedded in $\mathscr{D}^b(A)$ and that $\mathrm{Hom}_{\mathscr{D}^b(A)}(X, Y[i]) \simeq \mathrm{Ext}^i_A(X, Y)$ for all $i \geq 0$ and all A-modules X and Y.

Let X be an A-module. We denote by $\Omega^i_A(X)$ the i-th syzygy, by $\mathrm{soc}(X)$ the socle, and by $\mathrm{rad}(X)$ the Jacobson radical of X.

Let X be an additive generator for A-mod. The endomorphism algebra of X is called the Auslander algebra of A. This algebra is, up to Morita equivalence, uniquely determined by A. Note that Auslander algebras can be described by two homological properties: An algebra A is an Auslander algebra if gl.dim(A)$\leq 2 \leq$ dom.dim(A).

An A-module X is called a generator for A-mod if $\mathrm{add}({}_AA) \subseteq \mathrm{add}(X)$; a cogenerator for A-mod if $\mathrm{add}(D(A_A)) \subseteq \mathrm{add}(X)$, and a generator–cogenerator for A-mod if it is both a generator and a cogenerator for A-mod. Clearly, an additive generator M for A-mod is a generator–cogenerator for A-mod since we have $\mathrm{add}(M) = A$-mod by definition for the additive generator M for A-mod. But the converse is not true in general.

Let T be an arbitrary A-module, and let B be the endomorphism algebra of T. We consider the following full subcategories of A-mod related to T.

$$\begin{aligned}
Gen({}_AT) &:= \{X \in A\text{-mod} \mid \text{ there is a surjective homomorphism from } T^m \text{ to } X \text{with } m \geq 1\}.\\
Pre({}_AT) &:= \{X \in A\text{-mod} \mid \text{there is an exact sequence } T_1 \to T_0 \to X \text{ with all } T_i \in \mathrm{add}({}_AT)\}.\\
App({}_AT) &:= \{X \in A\text{-mod} \mid \text{there is a homomorphism } g : T_0 \to X \text{ with } T_0 \in \mathrm{add}({}_AT) \text{ such that}\\
&\qquad \mathrm{Ker}(g) \in Gen({}_AT) \text{ and } \mathrm{Hom}_A(T', g) \text{ is surjective for all } T' \in \mathrm{add}(T)\}.
\end{aligned}$$

The following lemma is known, for a proof we refer for example to [25, Lemma 2.1].

Lemma 2.1. *Let T be an A-module and $B = \mathrm{End}({}_AT)$. Let X be an arbitrary A-module. Then:*

(1) *If Y is a right B-module, then the natural homomorphism δ: $Y \otimes_B \mathrm{Hom}_A(T, X) \to \mathrm{Hom}_B(\mathrm{Hom}_A(X, T), Y)$, given by $y \otimes f \mapsto \delta_{y\otimes f}$ with $\delta_{y\otimes f}(g) = y(fg)$ for $y \in Y, f \in \mathrm{Hom}_A(T, X), g \in \mathrm{Hom}_A(X, T)$, is an isomorphism if $X \in \mathrm{add}({}_AT)$.*
(2) *If $X' \in \mathrm{add}({}_AT)$, or $X \in \mathrm{add}({}_AT)$, then the composition map $\mu : \mathrm{Hom}_A(X', T) \otimes_B \mathrm{Hom}_A(T, X) \to \mathrm{Hom}_A(X', X)$ given by $f \otimes g \mapsto fg$ is bijective.*
(3) *If $X \in Gen({}_AT)$, then the evaluation map $e_X : T \otimes_B \mathrm{Hom}_A(T, X) \to X$ is surjective. If $X \in App({}_AT)$, then e_X is bijective. Conversely, if e_X is bijective, then $X \in App({}_AT)$.*

The next lemma is taken from [23, Lemma 2.1], which can also be verified directly.

Lemma 2.2 ([23]). (1) *Let A, B, C and E be k-algebras, and let ${}_AX_B$ and ${}_BY_E$ be bimodules with X_B projective. Put $X^* = \mathrm{Hom}_B(X, B)$. Then the natural homomorphism $\varphi : {}_AX \otimes_B Y_E \to \mathrm{Hom}_B({}_BX^*_A, {}_BY_E)$, defined by $x \otimes y \mapsto \varphi_{x\otimes y}$, where $(f)\varphi_{x\otimes y} = (fx)y$ for $x \in X, y \in Y$ and $f \in X^*$, is an isomorphism of A-E-bimodules.*
(2) *In the situation $({}_EP_A, {}_CX_B, {}_AU_B)$, if P_A is projective, or if X_B is projective, then ${}_EP \otimes_A \mathrm{Hom}_B({}_CX_B, {}_AU_B) \simeq \mathrm{Hom}_B({}_CX_B, {}_EP \otimes_A U_B)$ as E-C-bimodules. Dually, in the situation $({}_AP_E, {}_BX_C, {}_BU_A)$, if ${}_AP$ is projective, or if ${}_BX$ is projective, then $\mathrm{Hom}_B({}_BX_C, {}_BU_A) \otimes_A P_E \simeq \mathrm{Hom}_B({}_BX_C, {}_BU \otimes_A P_E)$ as C-E-bimodules.*

The following is a well-known result from Auslander (for example, see [2, Proposition 5.6, p. 214]).

Lemma 2.3. *Let Λ be an Artin algebra such that gl.dim$(\Lambda) \leq 2 \leq$ dom.dim(Λ). Let U be a Λ-module such that $\mathrm{add}(U)$ is the full subcategory of Λ-mod consisting of all projective-injective Λ-modules. Then*

(1) *$A := \mathrm{End}_\Lambda(U)$ is representation-finite.*
(2) *Λ is Morita equivalent to $\mathrm{End}_A(X)$, where X is an additive generator for A-mod.*

Finally, we recall the definition of $\mathcal{D}$-split sequences from [6]. For our purpose, we just restrict our attention to module categories.

Let $\mathcal{D}$ be a full subcategory of A-mod. A short exact sequence

$$0 \longrightarrow X \xrightarrow{f} M \xrightarrow{g} Y \longrightarrow 0$$

in A-mod is called a $\mathcal{D}$-split sequence if $M \in \mathcal{D}$, $\mathrm{Hom}_A(D', g)$ and $\mathrm{Hom}_A(f, D')$ are surjective for every object $D' \in \mathcal{D}$.

Note that $\mathcal{D}$-split sequences were used in [6] to construct tilting modules of projective dimension at most one.

2.2. Admissible sets and perforated orbit categories

In [7], a class of algebras, called Φ-Auslander–Yoneda algebras, were introduced, which include for example, Auslander algebras, generalized Yoneda algebras, preprojective algebras and certain trivial extensions.

Let $\mathbb{N}$ be the set of natural numbers $\{0, 1, 2, \ldots\}$. Recall that a subset Φ of $\mathbb{N}$ is said to be admissible provided that $0 \in \Phi$ and that for any $p, q, r \in \Phi$ with $p + q + r \in \Phi$ we have $p + q \in \Phi$ if and only if $q + r \in \Phi$.

As shown in [7], there are a lot of admissible subsets of $\mathbb{N}$. For example, for any $n \geq 0$, the set $\{0, 1, \ldots, n\}$ is clearly an admissible subset of $\mathbb{N}$; also, given any subset S of $\mathbb{N}$ containing 0, the set $\{x^m \mid x \in S\}$ is admissible for all $m \geq 3$ (see [7, Proposition 3.1]). But, not every subset of $\mathbb{N}$ containing zero is admissible. A counterexample is the set $\{0, 1, 2, 4\}$ or the set $\{0, 1, 2, 3, 5\}$.

Let Φ be an admissible subset of $\mathbb{N}$.

Let $\mathcal{C}$ be a k-category, and let F be an additive functor from $\mathcal{C}$ to itself. The (F, Φ)-orbit category $\mathcal{C}^{F,\Phi}$ of $\mathcal{C}$ is a category in which the objects are the same as that of $\mathcal{C}$, and the morphism set between two objects X and Y is defined to be

$$\mathrm{Hom}_{\mathcal{C}^{F,\Phi}}(X, Y) := \bigoplus_{i\in\Phi} \mathrm{Hom}_{\mathcal{C}}(X, F^iY) \in k\text{-Mod},$$

and the composition is defined in an obvious way, where k-Mod stands for the category of all vector spaces over k. Since Φ is admissible, $\mathcal{C}^{F,\Phi}$ is an additive k-category. In particular, $\mathrm{Hom}_{\mathcal{C}^{F,\Phi}}(X, X)$ is a k-algebra (which may not be finite-dimensional), and $\mathrm{Hom}_{\mathcal{C}^{F,\Phi}}(X, Y)$ is an $\mathrm{Hom}_{\mathcal{C}^{F,\Phi}}(X, X)$-$\mathrm{Hom}_{\mathcal{C}^{F,\Phi}}(Y, Y)$-bimodule. For more details, we refer the reader to [7]. In this paper, the category $\mathcal{C}^{F,\Phi}$ is simply called a perforated orbit category, and the algebra $\mathrm{Hom}_{\mathcal{C}^{F,\Phi}}(X, X)$ is called the perforated Yoneda algebra of X without mentioning F and Φ.

In case $\mathcal{C}$ is the bounded derived category $\mathscr{D}^b(A)$ with A a k-algebra, and F is the shift functor [1] of $\mathscr{D}^b(A)$, we denote simply by $\mathcal{E}_A^{\Phi}$ the (F, Φ)-orbit category $\mathcal{C}^{F,\Phi}$, by $\mathrm{E}_A^{\Phi}(X, Y)$ the set $\mathrm{Hom}_{\mathcal{E}_A^{\Phi}}(X, Y)$, and by $\mathrm{E}_A^{\Phi}(X)$ the endomorphism algebra $\mathrm{Hom}_{\mathcal{E}_A^{\Phi}}(X, X)$ of X in $\mathcal{E}_A^{\Phi}$. The latter is called Φ-Auslander–Yoneda algebra of X. Note that each element in $\mathrm{E}_A^{\Phi}(X, Y)$ can be written as $(f_i)_{i\in\Phi}$ with $f_i \in \mathrm{Hom}_{\mathscr{D}^b(A)}(X, Y[i])$. The composition of morphisms in $\mathcal{E}_A^{\Phi}$ can be interpreted as follows: for each triple (X, Y, Z) of objects in $\mathscr{D}^b(A)$,

$$\begin{aligned}&\mathrm{E}_A^{\Phi}(X, Y) \times \mathrm{E}_A^{\Phi}(Y, Z) \longrightarrow \mathrm{E}_A^{\Phi}(X, Z)\\ &\big((f_u)_{u\in\Phi}, (g_v)_{v\in\Phi}\big) \mapsto (h_i)_{i\in\Phi},\end{aligned}$$

where

$$h_i := \sum_{\substack{u,v\in\Phi\\ u+v=i}} f_u(g_v[u])$$

for each $i \in \Phi$. Clearly, if Φ is finite, then $\mathrm{E}_A^{\Phi}(X, Y)$ is finite-dimensional for all $X, Y \in A$-mod.

Now, let us state some elementary properties of the Hom-functor $\mathrm{E}_A^{\Phi}(X, -)$.

Lemma 2.4. *Suppose that A is an algebra, that X is an A-module, and that Φ is a finite admissible subset of $\mathbb{N}$.*

(1) *Let $\mathrm{add}_A^{\Phi}(X)$ stand for the full subcategory of $\mathcal{E}_A^{\Phi}$ consisting of all objects in $\mathrm{add}({}_AX)$. Then the Hom-functor $\mathrm{E}_A^{\Phi}(X, -)$: $\mathrm{add}_A^{\Phi}(X) \longrightarrow \mathrm{E}_A^{\Phi}(X)$-proj is an equivalence of categories;*
(2) *Let B be a k-algebra, and let P be a B-A-bimodule such that P_A is projective. Then there is a canonical algebra homomorphism $\alpha_P : \mathrm{E}_A^{\Phi}(X) \longrightarrow \mathrm{E}_B^{\Phi}(P \otimes_A X)$ defined by $(f_i)_{i\in\Phi} \mapsto (P \otimes_A f_i)_{i\in\Phi}$ for $(f_i)_{i\in\Phi} \in \mathrm{E}_A^{\Phi}(X)$. Thus every left (or right) $\mathrm{E}_B^{\Phi}(P \otimes_A X)$-module can be regarded as a left (or right) $\mathrm{E}_A^{\Phi}(X)$-module via α_P.*

Proof. (1) Note that the objects of $\mathrm{add}_A^{\Phi}(X)$ are the same as the objects of $\mathrm{add}(X)$. Let $\mathcal{C} := \mathrm{add}_A^{\Phi}(X)$. Then $\mathrm{Hom}_{\mathcal{C}}(Y, Z) = \mathrm{E}_A^{\Phi}(Y, Z)$ for $Y, Z \in \mathrm{add}(X)$. Hence $\mathrm{E}_A^{\Phi}(X, -) : \mathcal{C} \longrightarrow \mathrm{End}_{\mathcal{C}}(X)$-proj is the Hom-functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$. Clearly, we have $\mathrm{rad}(\mathrm{End}_{\mathcal{C}}(X)) = \mathrm{rad}(\mathrm{End}_A(X)) \oplus \bigoplus_{0\neq i\in\Phi} \mathrm{Ext}_A^i(X, X)$ and $\mathrm{Hom}_{\mathcal{C}}(X, Y \oplus Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y) \oplus \mathrm{Hom}_{\mathcal{C}}(X, Z)$ for all $Y, Z \in \mathrm{add}(X)$. Now, it is routine to verify that the functor in (1) is an equivalence of additive categories.
(2) Since the homomorphism α_P is given explicitly, we can check (2) directly. $\square$

The following homological result plays an important role in proving Theorem 1.1.

Lemma 2.5. *Suppose that A, B and C are k-algebras. Let ${}_AX$ be a module, and let ${}_AY_B$ and ${}_BP_C$ be bimodules with ${}_BP$ projective. Then, for each $i \geq 0$, we have $\mathrm{Ext}^i_A(X, Y \otimes_B P_C) \simeq \mathrm{Ext}^i_A(X, Y) \otimes_B P_C$ as C^{op}-modules. Moreover, for each admissible subset Φ of $\mathbb{N}$, we have $\mathrm{E}^{\Phi}_A(X, Y \otimes_B P_C) \simeq \mathrm{E}^{\Phi}_A(X, Y) \otimes_B P_C$ as $\mathrm{E}^{\Phi}_A(X)$-C-bimodules.*

Proof. First, let us recall the Yoneda product. Assume that U, V and W are A-modules. Fix a minimal projective resolution $P^{\bullet}_U$ of ${}_AU$:

$$\cdots \longrightarrow P^n \xrightarrow{d^n} P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} U \longrightarrow 0,$$

with all P^i projective. If $g : U \to V$ is a homomorphism, then there is a lifting of g, which is a chain map $g^{\bullet} : P^{\bullet}_U \to P^{\bullet}_V$. Then, for each $i \geq 1$, we have a short exact sequence $0 \to \Omega^i_A(U) \xrightarrow{\lambda_i} P^{i-1} \xrightarrow{\mu_i} \Omega^{i-1}_A(U) \to 0$, which gives rise to a right exact sequence of k-modules

$$\mathrm{Hom}_A(P^{i-1}, V) \xrightarrow{(\lambda_i)_*} \mathrm{Hom}_A(\Omega^i_A(U), V) \longrightarrow \mathrm{Ext}^i_A(U, V) \longrightarrow 0.$$

Hence each element of $\mathrm{Ext}^i_A(U, V)$ can be regarded as a homomorphism in $\mathrm{Hom}_A(\Omega^i_A(U), V)$ modulo the subspace of $\mathrm{Hom}_A(\Omega^i_A(U), V)$ generated by all homomorphisms that factorize through λ_i, where $i \geq 0$ and $P^{-1} := 0$. In what follows, we denote the image of $f \in \mathrm{Hom}_A(\Omega^i_A(U), V)$ by $\overline{f} \in \mathrm{Ext}^i_A(U, V)$.

Given $i, j \in \mathbb{N}$, $f_i \in \mathrm{Hom}_A(\Omega^i_A(U), V)$ and $g_j \in \mathrm{Hom}_A(\Omega^j_A(V), W)$, we know that the Yoneda product $\mu : \mathrm{Ext}^i_A(U, V) \otimes_k \mathrm{Ext}^j_A(V, W) \to \mathrm{Ext}^{i+j}_A(U, W)$ can be presented by $\overline{f_i} \otimes_k \overline{g_j} \mapsto \overline{\Omega^j_A(f_i) g_j}$, where $\Omega^j_A(f_i)$ is the j-th term of a lifting of f_i. Note that the Yoneda product is independent of the choice of a lifting of f_i.

For each ${}_AW$, there is a natural isomorphism $\theta_W : \mathrm{Hom}_A(W, Y) \otimes_B P_C \to \mathrm{Hom}_A(W, Y \otimes_B P_C)$ of C^{op}-modules, defined by $w\big(\theta_W(f \otimes p)\big) = wf \otimes p$ for $f \in \mathrm{Hom}_A(W, Y)$, $p \in P$, and $w \in W$. This is obtained by putting $X := W$, $U := Y$ in the second statement of Lemma 2.2(2). In other words, we have a natural equivalence $\theta : \mathrm{Hom}_A(-, Y) \otimes_B P_C \simeq \mathrm{Hom}_A(-, Y \otimes_B P_C)$ of functors from A-mod to C^{op}-mod. Let

$$\cdots \longrightarrow Q^i \longrightarrow Q^{i-1} \longrightarrow \cdots \longrightarrow Q^1 \longrightarrow Q^0 \longrightarrow X \longrightarrow 0$$

be a minimal projective resolution of ${}_AX$. Then, by definition, we have a right exact sequence of k-modules

$$\mathrm{Hom}_A(Q^{i-1}, Y) \longrightarrow \mathrm{Hom}_A(\Omega^i_A(X), Y) \longrightarrow \mathrm{Ext}^i_A(X, Y) \longrightarrow 0.$$

Since ${}_BP$ is projective, the following diagram is exact and commutative for $i \geq 0$:

$$\begin{tikzcd}
\mathrm{Hom}_A(Q^{i-1}, Y) \otimes_B P_C \arrow[r] \arrow[d, "\theta_{Q^{i-1}}"', "\wr"] & \mathrm{Hom}_A(\Omega^i_A(X), Y) \otimes_B P_C \arrow[r] \arrow[d, "\theta_{\Omega^i_A(X)}"', "\wr"] & \mathrm{Ext}^i_A(X, Y) \otimes_B P_C \arrow[r] \arrow[d, dashed, "\varphi_i"'] & 0 \\
\mathrm{Hom}_A(Q^{i-1}, Y \otimes_B P_C) \arrow[r] & \mathrm{Hom}_A(\Omega^i_A(X), Y \otimes_B P_C) \arrow[r] & \mathrm{Ext}^i_A(X, Y \otimes_B P_C) \arrow[r] & 0,
\end{tikzcd}$$

where we set $Q^{-1} := 0$. This induces an isomorphism $\varphi_i : \mathrm{Ext}^i_A(X, Y) \otimes_B P_C \to \mathrm{Ext}^i_A(X, Y \otimes_B P_C)$ defined by $\overline{f_i} \otimes p \mapsto \overline{\theta_{\Omega^i_A(X)}(f_i \otimes p)}$, where $f_i \in \mathrm{Hom}_A(\Omega^i_A(X), Y)$ and $p \in P$. Clearly, φ_i is a C^{op}-homomorphism for each $i \geq 0$. Thus the first part of Lemma 2.5 is proved.

Second, for each admissible subset Φ of $\mathbb{N}$, we define a map $\varphi_{\Phi} : \mathrm{E}^{\Phi}_A(X, Y) \otimes_B P_C \to \mathrm{E}^{\Phi}_A(X, Y \otimes_B P_C)$ by $(\overline{f_i}) \otimes p \mapsto (\varphi_i(\overline{f_i} \otimes p))$, where $p \in P$, and $f_i \in \mathrm{Hom}_A(\Omega^i_A(X), Y)$ with $i \in \Phi$. By the above discussion, we know that φ_{Φ} is an isomorphism of C^{op}-modules. In order to prove that φ_{Φ} is an isomorphism of $\mathrm{E}^{\Phi}_A(X)$-C-bimodules, it suffices to show that φ_{Φ} is an isomorphism of left $\mathrm{E}^{\Phi}_A(X)$-modules, or equivalently, we have to check that the following diagram commutes for $i, j \in \Phi$ with $i + j \in \Phi$:

$$\begin{tikzcd}
\mathrm{Ext}^i_A(X, X) \otimes_k \mathrm{Ext}^j_A(X, Y) \otimes_B P_C \arrow[r, "1\otimes\varphi_j"] \arrow[d, "\mu\otimes 1"] & \mathrm{Ext}^i_A(X, X) \otimes_k \mathrm{Ext}^j_A(X, Y \otimes_B P_C) \arrow[d, "\mu"] \\
\mathrm{Ext}^{i+j}_A(X, Y) \otimes_B P_C \arrow[r, "\varphi_{i+j}"] & \mathrm{Ext}^{i+j}_A(X, Y \otimes_B P_C),
\end{tikzcd}$$

where μ is the usual Yoneda product. Let $u \in \mathrm{Hom}_A(\Omega^i_A(X), X)$, $v \in \mathrm{Hom}_A(\Omega^j_A(X), Y)$ and $p \in P$. Then

$$(\mu(1 \otimes \varphi_j))\,(u \otimes v \otimes p) = \overline{\Omega^j_A(u)\,\theta_{\Omega^j_A(X)}\,(v \otimes p)} \quad \text{and} \quad (\varphi_{i+j}(\mu \otimes 1))\,(u \otimes v \otimes p) = \overline{\theta_{\Omega^{i+j}_A(X)}\,((\Omega^j_A(u)v) \otimes p)}.$$

By definition, for each $x \in \Omega^{i+j}_A(X)$, we get

$$x\big(\Omega^j_A(u)\,\theta_{\Omega^j_A(X)}\,(v \otimes p)\big) = x(\Omega^j_A(u)\,v) \otimes p = x\big(\theta_{\Omega^{i+j}_A(X)}\,((\Omega^j_A(u)v) \otimes p)\big).$$

It follows that $\mu(1 \otimes \varphi_j) = \varphi_{i+j}(\mu \otimes 1)$. This implies that φ_{Φ} is an isomorphism of $\mathrm{E}^{\Phi}_A(X)$-C-bimodules. Thus the proof is completed. $\square$

3. Inductions for stable equivalences of Morita type

In this section, we shall prove Theorem 1.1. First, we recall the definition of stable equivalences of Morita type in [3].

Definition 3.1. Let A and B be (arbitrary) k-algebras. We say that A and B are stably equivalent of Morita type if there is an A-B-bimodule ${}_AM_B$ and a B-A-bimodule ${}_BN_A$ such that

(1) M and N are projective as one-sided modules, and
(2) $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules for some projective A-A-bimodule P, and $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q.

In this case, we say that M and N define a stable equivalence of Morita type between A and B. Moreover, we have two exact functors $T_N := N \otimes_A - : A\text{-mod} \to B\text{-mod}$ and $T_M := M \otimes_B - : B\text{-mod} \to A\text{-mod}$. Similarly, the bimodules P and Q define two exact functors T_P and T_Q, respectively. Note that the images of T_P and T_Q consist of projective modules.

From now on, we assume that A, B, M, N, P and Q are fixed as in Definition 3.1, and that X is a generator for A-mod. Moreover, we fix a finite admissible subset Φ of $\mathbb{N}$, and define $\Lambda := \mathrm{E}_A^{\Phi}(X)$ and $\Gamma := \mathrm{E}_B^{\Phi}(N \otimes_A X)$.

Since the functors T_N and T_M are exact, they preserve acyclicity, and can be extended to triangle functors $T'_N : \mathscr{D}^b(A) \to \mathscr{D}^b(B)$ and $T'_M : \mathscr{D}^b(B) \to \mathscr{D}^b(A)$, respectively. Furthermore, T'_N and T'_M induce canonically two functors $F : \mathcal{E}_A^{\Phi} \to \mathcal{E}_B^{\Phi}$ and $G : \mathcal{E}_B^{\Phi} \to \mathcal{E}_A^{\Phi}$, respectively. More precisely, if $X^{\bullet} \in \mathscr{D}^b(A)$, then $F(X^{\bullet}) := (N \otimes_A X^i, N \otimes_A d_X^i)$, and if $f := (f_j)_{j \in \Phi} \in \mathrm{Hom}_{\mathcal{E}_A^{\Phi}}(X^{\bullet}, Y^{\bullet})$ with $Y^{\bullet} \in \mathscr{D}^b(A)$, then $F(f) := (N \otimes_A f_j)_{j\in\Phi} \in \mathrm{Hom}_{\mathcal{E}_B^{\Phi}}(F(X^{\bullet}), F(Y^{\bullet}))$. Similarly, we define the functor G.

The functor F gives rise to a canonical algebra homomorphism $\alpha_N : \mathrm{E}_A^{\Phi}(X^{\bullet}) \to \mathrm{E}_B^{\Phi}(F(X^{\bullet}))$ for each object $X^{\bullet} \in \mathscr{D}^b(A)$. In particular, for any $Z^{\bullet} \in \mathscr{D}^b(B)$, we can regard $\mathrm{E}_B^{\Phi}(Z^{\bullet}, F(X^{\bullet}))$ as an $\mathrm{E}_B^{\Phi}(Z^{\bullet})$-$\mathrm{E}_A^{\Phi}(X^{\bullet})$-bimodule via α_N. Note that the homomorphism α_N coincides with the one defined in Lemma 2.4, when $X^{\bullet}$ is an A-module.

Proof of Theorem 1.1. We define $U := \mathrm{E}_A^{\Phi}(X, T_M(N \otimes_A X))$ and $V := \mathrm{E}_B^{\Phi}(N \otimes_A X, T_N(X))$. In the following we shall prove that U and V define a stable equivalence of Morita type between Λ and Γ.

First, we endow U with a right Γ-module structure by $u \cdot \gamma := uG(\gamma)$ for $u \in U$ and $\gamma \in \Gamma$, where $uG(\gamma)$ denotes the composition of u with $G(\gamma)$ in the category $\mathcal{E}_A^{\Phi}$, and endow V with a right Λ-module structure by $v \cdot \lambda := vF(\lambda)$ for $v \in V$ and $\lambda \in \Lambda$. Then, U becomes a Λ-Γ-bimodule, and V becomes a Γ-Λ-bimodule.

By definition, we know $V = \Gamma$, and it is a projective left Γ-module. Since ${}_AX$ is a generator for A-mod and the image of T_P consists of projective modules, we conclude that $T_M(N \otimes_A X) = M \otimes_B (N \otimes_A X) \simeq X \oplus P \otimes_A X \in \mathrm{add}(X)$. Thus U is projective as a left Λ-module by Lemma 2.4.

(1) $U \otimes_{\Gamma} V$, as a Λ-Λ-bimodule, satisfies the condition (2) in Definition 3.1.

Indeed, we write $W := \mathrm{E}_A^{\Phi}(X, (T_MT_N)(X))$, and define a right Λ-module structure on W by $w \cdot \lambda' := w(GF)(\lambda')$ for $w \in W$ and $\lambda' \in \Lambda$. Then W becomes a Λ-Λ-bimodule. Note that there is a natural Λ-module isomorphism $\varphi : U \otimes_{\Gamma} V \to W$ defined by $x \otimes y \mapsto xG(y)$ for $x \in U$ and $y \in V$. We claim that φ is an isomorphism of Λ-Λ-bimodules. In fact, it suffices to show that φ respects the structure of right Λ-modules. However, this follows immediately from a verification: for $c \in U, d \in V$ and $a \in \Lambda$, we have

$$((c \otimes d) \cdot a)\varphi = (c \otimes (dF(a))) = cG(dF(a))\varphi = cG(d)(GF)(a) = (c \otimes d)\varphi \cdot a.$$

Combining this bimodule isomorphism φ with Lemma 2.4, we get the following isomorphisms of Λ-Λ-bimodules:

$$(*) \qquad U \otimes_{\Gamma} V \simeq \mathrm{E}_A^{\Phi}(X, (T_MT_N)(X)) \simeq \mathrm{E}_A^{\Phi}(X, X) \oplus \mathrm{E}_A^{\Phi}(X, P \otimes_A X) = \Lambda \oplus \mathrm{E}_A^{\Phi}(X, P \otimes_A X),$$

where the second isomorphism follows from $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules, and where the right Λ-module structure on $\mathrm{E}_A^{\Phi}(X, P \otimes_A X)$ is induced by the canonical algebra homomorphism $\Lambda \to \mathrm{E}_A^{\Phi}(P \otimes_A X)$, which sends $(f_i)_{i\in\Phi}$ in Λ to $(P \otimes_A f_i)_{i\in\Phi}$ (see Lemma 2.4(2)).

Now, we show that $\mathrm{E}_A^{\Phi}(X, P \otimes_A X)$ is a projective Λ-Λ-bimodule. In fact, since $P \in \mathrm{add}({}_AA \otimes_k A_A)$, we conclude that $\mathrm{E}_A^{\Phi}(X, P \otimes_A X) \in \mathrm{add}(\mathrm{E}_A^{\Phi}(X, (A \otimes_k A) \otimes_A X))$. Thus, it is sufficient to prove that $\mathrm{E}_A^{\Phi}(X, (A \otimes_k A) \otimes_A X)$ is a projective Λ-Λ-bimodule. For this purpose, we first note that the right Λ-module structure on $\mathrm{E}_A^{\Phi}(X, (A \otimes_k A) \otimes_A X)$ is induced by the canonical algebra homomorphism $\alpha_{A\otimes_k A} : \Lambda \to \mathrm{E}_A^{\Phi}((A \otimes_k A) \otimes_A X)$, which sends $g := (g_i)_{i\in\Phi}$ in Λ to $((A \otimes_k A) \otimes_A g_i)_{i\in\Phi}$. Clearly, ${}_AA \otimes_k A \otimes_A X \in \mathrm{add}({}_AA)$. It follows that $\mathrm{Ext}_A^j\big((A \otimes_k A) \otimes_A X, (A \otimes_k A) \otimes_A X\big) = 0$ for any $j > 0$, and therefore $(A \otimes_k A) \otimes_A g_i = 0$ for any $0 \neq i \in \Phi$. Thus we have $\alpha_{A\otimes_k A}(g) = (A \otimes_k A) \otimes_A g_0$. If $\pi : \Lambda \to \mathrm{End}_A(X)$ is the canonical projection and μ' is the canonical algebra homomorphism $\mathrm{End}_A(X) \to \mathrm{End}_A\big((A \otimes_k A) \otimes_A X\big)$, then $\alpha_{A\otimes_k A} = \pi\mu'$. Thus the right Λ-module structure on $\mathrm{E}_A^{\Phi}(X, (A \otimes_k A) \otimes_A X)$ is induced by $\mathrm{End}_A(X)$. Similarly, from the homomorphisms

$$\Lambda = \mathrm{E}_A^{\Phi}(X) \xrightarrow{\pi} \mathrm{End}_A(X) \xrightarrow{\mu} \mathrm{End}_A(A \otimes_k X) = \mathrm{E}_A^{\Phi}(A \otimes_k X),$$

where $\mu : \mathrm{End}_A(X) \to \mathrm{End}_A(A \otimes_k X)$ is induced by the tensor functor $A \otimes_k -$, we see that the right Λ-module structure on $\mathrm{E}_A^{\Phi}(X, A \otimes_k X)$ is also induced by $\mathrm{End}_A(X)$. Thus $\mathrm{E}_A^{\Phi}(X, (A \otimes_k A) \otimes_A X) \simeq \mathrm{E}_A^{\Phi}(X, A \otimes_k X)$ as Λ-Λ-bimodules. Moreover, it follows from Lemma 2.5 that $\mathrm{E}_A^{\Phi}(X, A \otimes_k X) \simeq \mathrm{E}_A^{\Phi}(X, A) \otimes_k X$ as Λ-$\mathrm{End}_A(X)$-bimodule. Since the A-module X can be regarded as a right Λ-module via the homomorphism π, we see that X is actually isomorphic to $\mathrm{E}_A^{\Phi}(A, X)$ as right Λ-modules. Thus $\mathrm{E}_A^{\Phi}(X, A) \otimes_k X \simeq \mathrm{E}_A^{\Phi}(X, A) \otimes_k \mathrm{E}_A^{\Phi}(A, X)$ as Λ-Λ-bimodules. Since ${}_AA \in \mathrm{add}(X)$, we know that $\mathrm{E}_A^{\Phi}(X, A)$ is a projective

Λ-module and $\mathrm{E}_A^{\Phi}(A, X)$ is a projective right Λ-module. Hence $\mathrm{E}_A^{\Phi}(X, A) \otimes_k X$ is a projective Λ-Λ-bimodule. This implies that $\mathrm{E}_A^{\Phi}(X, P \otimes_A X)$ is a projective Λ-Λ-bimodule.

(2) $V \otimes_\Lambda U$, as a Γ-Γ-bimodule, fulfills the condition (2) in Definition 3.1.

Let $Z := \mathrm{E}_B^{\Phi}(N \otimes_A X, T_N T_M(N \otimes_A X))$. Similarly, we endow Z with a right Γ-module structure defined by $z \cdot b := z(FG)(b)$ for $z \in Z$ and $b \in \Gamma$. Then Z becomes a Γ-Γ-bimodule. Observe that, for each A-module Y, there is a homomorphism $\Psi_Y : V \otimes_\Lambda \mathrm{E}_A^{\Phi}(X, Y) \to \mathrm{E}_B^{\Phi}(N \otimes_A X, T_N(Y))$ of Γ-modules, which is defined by $g \otimes h \mapsto gF(h)$ for $g \in V$ and $h \in \mathrm{E}_A^{\Phi}(X, Y)$. This homomorphism is natural in Y. In other words, $\Psi : V \otimes_\Lambda \mathrm{E}_A^{\Phi}(X, -) \to \mathrm{E}_B^{\Phi}(N \otimes_A X, T_N(-))$ is a natural transformation of functors from A-mod to Γ-mod. Clearly, Ψ_X is an isomorphism of Γ-modules. It follows from $T_M(N \otimes_A X) \in \mathrm{add}(X)$ that $\Psi_{T_M(N\otimes_A X)} : V \otimes_\Lambda U \to Z$ is a Γ-isomorphism. Similarly, we can check that $\Psi_{T_M(N\otimes_A X)}$ preserves the structure of right Γ-modules. Thus $\Psi_{T_M(N\otimes_A X)} : V \otimes_\Lambda U \to Z$ is an isomorphism of Γ-Γ-bimodules, and there are the following isomorphisms of Γ-Γ-bimodules:

$$(**) \qquad V \otimes_\Lambda U \simeq Z \simeq \Gamma \oplus \mathrm{E}_B^{\Phi}(N \otimes_A X, Q \otimes_B (N \otimes_A X)),$$

where the second isomorphism is deduced from $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules. By an argument similar to that in the proof of (1), we can show that $\mathrm{E}_B^{\Phi}(N \otimes_A X, Q \otimes_B (N \otimes_A X))$ is a projective Γ-Γ-bimodule.

It remains to show that U_Γ and V_Λ are projective. This is equivalent to showing that the tensor functors $T_U := U \otimes_\Gamma - : \Gamma\text{-mod} \to \Lambda\text{-mod}$ and $T_V := V \otimes_\Lambda - : \Lambda\text{-mod} \to \Gamma\text{-mod}$ are exact. Since tensor functors are always right exact, the exactness of T_U is equivalent to the property that T_U preserve injective homomorphisms of modules. Now, suppose that $f : C \to D$ is an injective homomorphism between Γ-modules C and D. Since $\mathrm{E}_B^{\Phi}(N \otimes_A X, Q \otimes_B (N \otimes_A X))$ is a right projective Γ-module, we know from $(**)$ that the composition functor $T_V T_U$ is exact. In particular, the homomorphism $(T_V T_U)(f) : (T_V T_U)(C) \to (T_V T_U)(D)$ is injective. Let $\mu : \mathrm{Ker}\,(T_U(f)) \to T_U(C)$ be the canonical inclusion. Clearly, we have $\mu T_U(f) = 0$, which shows $T_V(\mu T_U(f)) = T_V(\mu)(T_V T_U)(f) = 0$. It follows that $T_V(\mu) = 0$ and $(T_U T_V)(\mu) = 0$. By $(*)$, we get $\mu = 0$, which implies that the homomorphism $T_U(f)$ is injective. Hence T_U preserves injective homomorphisms. Similarly, we can show that T_V preserves injective homomorphisms, too. Consequently, U_Γ and V_Λ are projective.

Thus, the bimodules U and V define a stable equivalence of Morita type between Λ and Γ. This finishes the proof of Theorem 1.1. $\square$

Remarks. (1) If we take $\Phi = \{0\}$ in Theorem 1.1, then we get [17, Theorem 1.1]. If we assume that A is a self-injective algebra, then we get a stable equivalence of Morita type between $\mathrm{E}_A^{\Phi}(A \oplus X)$ and $\mathrm{E}_A^{\Phi}(A \oplus \Omega_A^i(X))$ for any A-module X, any finite admissible subset Φ of $\mathbb{N}$, and any integer $i \in \mathbb{Z}$. This follows from Theorem 1.1 and the fact that Ω_A provides a stable equivalence of Morita type between A and itself if A is self-injective. Thus we re-obtain the stable equivalence of [7, Corollary 3.14].

(2) For any A-module X, we denote by $\mathrm{proj.dim}({}_AX)$ the projective dimension of X. Clearly, we have $\mathrm{proj.dim}({}_AX) \leq \mathrm{proj.dim}({}_AX \oplus P \otimes_A X) = \mathrm{proj.dim}({}_AM \otimes_B N \otimes_A X) \leq \mathrm{proj.dim}({}_BN \otimes_A X) \leq \mathrm{proj.dim}({}_AX)$. This implies that the global and finitistic dimensions are invariant under stable equivalences of Morita type. In Section 4 of [17], it was shown that the dominant dimension is also invariant under stable equivalences of Morita type. Thus, Theorem 1.1 asserts actually also that these dimensions are equal for algebras $\mathrm{E}_A^{\Phi}(X)$ and $\mathrm{E}_B^{\Phi}(N \otimes_A X)$.

Many important classes of algebras are of the form $\mathrm{End}_A(A \oplus X)$ with A a self-injective algebra. Let k be a field with infinitely many elements, and let V be an n-dimensional k-space and $V^{\otimes r}$ the r-fold tensor space of V. Then the symmetric group S_r of r letters acts on the tensor space from the right hand side by permutation. Now we assume $n \geq r$. Following [5, 2.6c], the Schur algebra $S(n, r)$ is defined to be the endomorphism ring of the right $k[S_r]$-module $V^{\otimes r}$. It is well known that the Schur algebra $S(n, r)$ with $n \geq r$ is Morita equivalent to $\mathrm{End}_{k[S_n]}(k[S_n] \oplus Y)$ and has finite global dimension, where Y is the direct sum of all non-projective Young modules. From the above remarks (see also [7, Corollary 3.14]), we may get a series of algebras which are stably equivalent of Morita type to Schur algebras. For unexplained terminology in the next corollary, we refer the reader to [5].

Corollary 3.2. *Suppose that k is an algebraically closed field. Let S_n be the symmetric group of degree n. We denote by Y the direct sum of all non-projective Young modules over the group algebra $k[S_n]$ of S_n. Then,*

(1) *for every finite admissible subset Φ of $\mathbb{N}$, the algebras $\mathrm{E}_{k[S_n]}^{\Phi}(k[S_n] \oplus Y)$ and $\mathrm{E}_{k[S_n]}^{\Phi}(k[S_n] \oplus \Omega^i(Y))$ are stably equivalent of Morita type for all $i \in \mathbb{Z}$.*

(2) *All algebras $\mathrm{End}_{k[S_n]}(k[S_n] \oplus \Omega^i(Y))$ are stably equivalent of Morita type to the Schur algebra $S_k(n, n)$. In particular, $\mathrm{gl.dim}\big(\mathrm{End}_{k[S_n]}(k[S_n] \oplus \Omega^i(Y))\big) < \infty$ for all $i \in \mathbb{Z}$.*

4. Restrictions for stable equivalences of Morita type

In this section, we shall consider the general question of how to transfer stable equivalences of Morita type between algebras A and B over a field to the ones between eAe and fBf, where e and f are idempotent elements in A and B, respectively. In particular, we shall prove Theorem 1.2 in this section.

Before we start with our proof of Theorem 1.2, we state the following facts, which are essentially known in the literature. However, we would like to collect them together as a lemma for the convenience of the reader.

Lemma 4.1. *Suppose that A and B are k-algebras without semisimple direct summands. Assume that ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B, and that M and N do not have any projective bimodules as direct summands. Then,*

(1) *there are isomorphisms of bimodules: $N \simeq \mathrm{Hom}_A(M, A) \simeq \mathrm{Hom}_B(M, B)$ and $M \simeq \mathrm{Hom}_A(N, A) \simeq \mathrm{Hom}_B(N, B)$.*
(2) *Both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors.*
(3) *There are isomorphisms of bimodules: $P \simeq \mathrm{Hom}_A(P, A)$ and $Q \simeq \mathrm{Hom}_B(Q, B)$, where P and Q are the bimodules defined in Definition 3.1. Moreover, the bimodules ${}_AP_A$ and ${}_BQ_B$ are projective-injective.*
(4) *If ${}_AI$ is injective, then so is the B-module $N \otimes_A I$.*

Proof. (1) Note that, if M and N are indecomposable bimodules, then all the statements in Lemma 4.1 have been proved in [4, Theorem 2.7, Corollary 3.1, Lemma 3.2] under the hypothesis of separability on the semisimple quotient algebras $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$. One can check that they are still valid without the hypothesis of separability condition. In the following, we shall use [14, Theorem 2.2] to show Lemma 4.1 under the weaker assumption that M and N do not have any projective bimodules as direct summands.

Since A and B are stably equivalent of Morita type and do not have any semisimple direct summands, it follows from [14, Proposition 2.1] that A and B have the same number of indecomposable direct summands (of two-sided ideals). Suppose that $A = A_1 \times A_2 \times \cdots \times A_n$ and $B = B_1 \times B_2 \times \cdots \times B_n$, where all A_i and all B_i themselves are indecomposable algebras. By the proof of [14, Theorem 2.2], we know that, up to suitable reordering, for each $1 \le i \le n$, there is an A_i-B_i-bimodule M_i and a B_i-A_i-bimodule N_i such that M_i and N_i are direct summands of M and N as bimodules, respectively, and that M_i and N_i define a stable equivalence of Morita type between A_i and B_i. Set $M' := \bigoplus_{1\le i\le n} M_i$ and $N' := \bigoplus_{1\le i\le n} N_i$. Clearly, M' and N' are direct summands of M and N, respectively. Further, one can check directly that M' and N' also define a stable equivalence of Morita type between A and B. Since ${}_AM_B$ and ${}_BN_A$ do not have any projective bimodules as direct summands, it follows from [16, Lemma 4.8] that $M \simeq M'$ as A-B-bimodules and $N \simeq N'$ as B-A-bimodules. Note that A_i and B_i are indecomposable algebras, and M_i and N_i do not have any projective bimodules as direct summands. Then, by [4, Lemma 2.1], we conclude that M_i and N_i are indecomposable bimodules. This implies that Lemma 4.1 holds for the algebras A_i and B_i together with the bimodules M_i and N_i for each i. Consequently, there are isomorphisms of B-A-bimodules: $\mathrm{Hom}_A(M, A) \simeq \mathrm{Hom}_A(\bigoplus_{1\le u\le n} M_u, \bigoplus_{1\le v\le n} A_v) \simeq \bigoplus_{1\le u\le n} \mathrm{Hom}_A(M_u, A_u) \simeq \bigoplus_{1\le u\le n} N_u \simeq N$. Similarly, we can prove the other statements in (1).

(2) Note that the pair $(N \otimes_A -, M \otimes_B -)$ is an adjoint pair of functors if and only if ${}_AM_B \simeq \mathrm{Hom}_B(N, B)$ as bimodules. Thus (2) is a consequence of (1).

(3) It follows from the proof of [23, Lemma 4.5] that the first part of (3) holds true, and that P and Q are injective as one-sided modules. Furthermore, we claim that P is an injective bimodule. In fact, it suffices to show that, for any indecomposable direct summand P' of P, the bimodule ${}_AP'_A$ is injective. Since ${}_AP \in \mathrm{add}(A \otimes_k A^{op})$, there are primitive idempotents e_1 and e_2 of A such that $P' \in \mathrm{add}(Ae_1 \otimes_k e_2A)$. This implies that Ae_1 and e_2A are injective modules because P' is injective as a one-sided module. Thus P' is an injective bimodule, and so is P. Similarly, we can prove that Q is injective as a bimodule.

(4) We observe that there is an isomorphism of B-modules: $N \otimes_A I \simeq \mathrm{Hom}_A(M, I)$. Since M_B is projective and ${}_AI$ is injective, we see that $\mathrm{Hom}_A(M, I)$ is an injective B-module, and so is $N \otimes_A I$. This completes the proof of Lemma 4.1. □

By Lemma 4.1, we have the following corollary, which provides examples such that the conditions of Theorem 1.2 are satisfied. Note that the last statement in Corollary 4.2 below follows also from the proof of [23, Lemma 4.5].

Corollary 4.2. *Suppose that A and B are k-algebras. Assume that $\{e_1, \cdots, e_n\}$ and $\{f_1, \ldots, f_m\}$ are complete sets of pairwise orthogonal primitive idempotents in A and in B, respectively. Let e be the sum of all those e_i for which Ae_i is projective-injective, and let f be the sum of all those f_j for which Bf_j is projective-injective. If M and N are indecomposable bimodules that define a stable equivalence of Morita type between A and B, then $Ne \simeq N \otimes_A Ae \in \mathrm{add}(Bf)$, $Mf \simeq M \otimes_B Bf \in \mathrm{add}(Ae)$, and $Pe \in \mathrm{add}(Ae)$.*

Proof of Theorem 1.2. Let us remark that if A and B have no separable direct summands, then we may assume that M and N have no non-zero projective bimodules as direct summands. In fact, If $M = M' \oplus M''$ and $N = N' \oplus N''$ such that M' and N' have no non-zero projective bimodules as direct summands, and that M'' and N'' are projective bimodules, then it follows from [16, Lemma 4.8] that M' and N' also define a stable equivalence of Morita type between A and B.

Suppose that ${}_AM_B$ and ${}_BN_A$ do not have any non-zero projective bimodules as direct summands, and define a stable equivalence of Morita type between A and B. Then, it follows from Lemma 4.1(2) that $(M \otimes_B -, N \otimes_A -)$ and $(N \otimes_A -, M \otimes_B -)$ are adjoint pairs.

First, we note that $\mathrm{add}(Ae) = \mathrm{add}(Mf)$ and $\mathrm{add}(N \otimes_A Mf) = \mathrm{add}(Bf)$. In fact, this follows from the following equalities: $\mathrm{add}(Ae) = \mathrm{add}(M \otimes_B N \otimes_A Ae) = \mathrm{add}(M \otimes_B Bf) = \mathrm{add}(Mf)$, and the fact that $\mathrm{add}(N \otimes_A X) = \mathrm{add}(N \otimes_A \mathrm{add}(X))$ for any A-module X.

Thus, if a statement for the idempotent element e holds true, then it can be proved similarly for f, and vice versa.

Second, we shall show that the bimodules eMf and fNe satisfy the conditions of a stable equivalence of Morita type between eAe and fBf.

(1) fNe is projective as an fBf-module and as a right eAe-module, respectively.

In fact, we have $fNe \simeq \mathrm{Hom}_B(Bf, {}_BNe)$ as fBf-eAe-bimodules. Since $Ne \in \mathrm{add}(Bf)$ by the definition of f, we see that $\mathrm{Hom}_B(Bf, Ne)$ is projective as an fBf-module, that is, fNe is projective as an fBf-module. To see that fNe is a projective right eAe-module, we notice that $\mathrm{add}(Mf) = \mathrm{add}(M \otimes_B Bf) = \mathrm{add}(M \otimes_B Ne) = \mathrm{add}(Ae)$, here we use the assumption $M \otimes_B Ne \in \mathrm{add}(Ae)$. Since $(M \otimes_B -, N \otimes_A -)$ is an adjoint pair, it follows from $\mathrm{Hom}_B(Bf, {}_BN \otimes_A Ae) \simeq \mathrm{Hom}_A(M \otimes_B Bf, Ae) \simeq \mathrm{Hom}_A(Mf, Ae)$ that fNe is projective as a right eAe-module since $Mf \in \mathrm{add}(Ae)$. Thus (1) is proved.

(2) eMf is projective as an eAe-module and as a right fAf-module, respectively. The proof of (2) is similar to that of (1), we omit it here.

(3) $eMf \otimes_{fBf} fNe \simeq eAe \oplus ePe$ as bimodules.

Indeed, by the associativity of tensor products, we have the following isomorphisms of eAe-eAe-bimodules:

$$\begin{aligned} eMf \otimes_{fBf} fNe &\simeq eM \otimes_B Bf \otimes_{fBf} fB \otimes_B Ne \\ &\simeq eM \otimes_B Bf \otimes_{fBf} \mathrm{Hom}(Bf, B) \otimes_B Ne \\ &\simeq eM \otimes_B Bf \otimes_{fBf} \mathrm{Hom}(Bf, {}_BNe) \quad (\text{ by Lemma 2.2}) \\ &\simeq eM \otimes_B Ne \quad (\text{by Lemma 2.1}). \end{aligned}$$

Since M and N define the stable equivalence of Morita type between A and B, we have $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules. This implies that $eMf \otimes_{fBf} fNe \simeq eM \otimes_B Ne \simeq e(A \oplus P)e \simeq eAe \oplus ePe$ as bimodules.

(4) ePe is a projective eAe-eAe-bimodule.

In fact, it suffices to show that, for any indecomposable direct summand P' of the A-A-bimodule P, the eAe-eAe-bimodule $eP'e$ is projective. We assume $eP'e \neq 0$. Since $P \in \mathrm{add}(A \otimes_k A^{op})$, there are primitive idempotent elements e_1 and e_2 of A such that $P' \in \mathrm{add}(Ae_1 \otimes_k e_2A)$. Then ${}_AP'e \in \mathrm{add}(Ae_1 \otimes_k e_2Ae) \subseteq \mathrm{add}(Ae_1)$. This means that $P'e$ is a direct sum of copies of Ae_1. Since $P'e \in \mathrm{add}(Pe) \subseteq \mathrm{add}(Ae)$, we have $Ae_1 \in \mathrm{add}(Ae)$. Consequently, eAe_1 is a projective eAe-module. Now, we show that e_2Ae is a projective right eAe-module. Indeed, by Lemma 4.1(3), we have the following isomorphisms of A^{op}-modules: $eP \simeq \mathrm{Hom}_A(Ae, P) \simeq \mathrm{Hom}_A(Ae, \mathrm{Hom}_A(P, A)) \simeq \mathrm{Hom}_A(P \otimes_A Ae, A) \simeq \mathrm{Hom}_A(Pe, A)$. This shows that $eP \in \mathrm{add}(eA)$ since ${}_APe \in \mathrm{add}(Ae)$. Thus $eP' \in \mathrm{add}(eA)$. Since the right A-module eP' is a direct sum of copies of e_2A, it follows that $e_2A \in \mathrm{add}(eA)$ and $e_2Ae \in \mathrm{add}(eAe)$. Consequently, e_2Ae is a projective right eAe-module. Hence $eAe_1 \otimes_k e_2Ae$ is a projective eAe-eAe-bimodule, and so is its direct summand $eP'e$. This shows that ePe is a projective eAe-eAe-bimodule.

(5) Similarly, we can prove that $fNe \otimes eMf \simeq fBf \oplus fQf$ as bimodules, and that the fBf-fBf-bimodule fQf is projective.

Thus, by definition, the bimodules eMf and fNe define a stable equivalence of Morita type between eAe and fBf.

Finally, the last statement of Theorem 1.2 follows from Proposition 4.3 below, which emphasizes the view of functors.

Before we give the formulation of Proposition 4.3, we introduce here a few more notations: Set $\Lambda = \mathrm{End}_{eAe}(eA)$, $R = \mathrm{End}_{fBf}(fN)$, $\Gamma = \mathrm{End}_{fBf}(fB)$, $N' = \mathrm{Hom}_{fBf}((fB)_\Gamma, fNe \otimes_{eAe} (eA)_\Lambda)$ and $M' = \mathrm{Hom}_{eAe}((eA)_\Lambda, eMf \otimes_{fBf} (fB)_\Gamma)$.

Let $\varphi : A \to \Lambda$ be the algebra homomorphism defined by sending $a \in A$ to φ_a, where $\varphi_a : eA \to eA$, $ex \mapsto exa$ for $x \in A$. Similarly, we define an algebra homomorphism $\psi : B \to \Gamma$.

Recall that, given a diagram of functors between categories:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "H"'] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "K"] & \mathcal{D}, \end{tikzcd}$$

we say that this diagram is commutative if there is a natural isomorphism $\alpha : GF \to KH$.

Proposition 4.3. (1) *The following diagram of functors is commutative*

$$\begin{tikzcd} A\text{-mod} \arrow[r, "N\otimes_A -"] \arrow[d, "e\cdot"] & B\text{-mod} \arrow[r, "M\otimes_B -"] \arrow[d, "f\cdot"] & A\text{-mod} \arrow[d, "e\cdot"] \\ eAe\text{-mod} \arrow[r, "fNe\otimes_{eAe}-"] & fBf\text{-mod} \arrow[r, "eMf\otimes_{fBf}-"] & eAe\text{-mod}. \end{tikzcd}$$

In particular, ${}_{fBf}fNe \otimes_{eAe} eA \simeq_{fBf} fN$ *and* ${}_{eAe}eMf \otimes_{fBf} fB \simeq_{eAe} eM$.

(2) *We have the following commutative diagram of functors*

$$\begin{tikzcd} A\text{-mod} \arrow[r, "N\otimes_A -"] \arrow[d, "{}_\Lambda\Lambda\otimes_A -"] & B\text{-mod} \arrow[r, "M\otimes_B -"] \arrow[d, "{}_\Gamma\Gamma\otimes_B -"] & A\text{-mod} \arrow[d, "{}_\Lambda\Lambda\otimes_A -"] \\ \Lambda\text{-mod} \arrow[r, "{}_\Gamma N'\otimes_\Lambda -"] & \Gamma\text{-mod} \arrow[r, "{}_\Lambda M'\otimes_\Gamma -"] & \Lambda\text{-mod}, \end{tikzcd}$$

where the right A-module structure on Λ and the right B-module structure on Γ are induced by φ and ψ, respectively. Moreover, ${}_\Gamma N'_\Lambda$ *and* ${}_\Lambda M'_\Gamma$ *define a stable equivalence of Morita type between Λ and Γ.*

Proof. (1) To prove that the first square in (1) is commutative, it is sufficient to show that there is a natural transformation $\Phi : fNe \otimes_{eAe} e(-) \longrightarrow fN \otimes_A -$, which is an isomorphism. Now we define Φ to be the composition of the following two natural transformations: for each $X \in A$-mod,

$$\Phi_X : fNe \otimes_{eAe} eX \xrightarrow{\sim} fN \otimes_A Ae \otimes_{eAe} eX \xrightarrow{id_{fN}\otimes\mu} fN \otimes_A X,$$

where $\mu : Ae\otimes_{eAe}eX \to X$ is the multiplication map. Clearly, we need only to show that $id_{fN}\otimes\mu$ is a natural isomorphism, that is, for each ${}_AX$, we have to show that

$$fN \otimes_A Ae \otimes_{eAe} eX \longrightarrow fN \otimes_A X$$

is an isomorphism.

Indeed, we shall first show that if $Z \in A$-mod and $eZ = 0$, then $fN \otimes_A Z = 0$. To prove this, we observe that $fN \otimes_A Z \simeq \mathrm{Hom}_B(Bf, N \otimes_A Z) \simeq \mathrm{Hom}_A({}_AM \otimes_B Bf, Z)$, where the second isomorphism comes from the adjoint pair $(M \otimes_B -, N \otimes_A -)$. Since $\mathrm{add}(Bf) = \mathrm{add}(N \otimes_A Ae)$ and $Pe \in \mathrm{add}(Ae)$, we have $\mathrm{add}(M \otimes_B Bf) = \mathrm{add}(Ae)$. Thus $eZ = 0$ implies that $fN \otimes_A Z = 0$. Next, we consider the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\mu) \longrightarrow Ae \otimes_{eAe} eX \xrightarrow{\mu} X \longrightarrow X/AeX \longrightarrow 0.$$

Note that $e\,\mathrm{Ker}(\mu) = 0 = e(X/AeX)$ and that $fN_A \simeq fB \otimes_B N_A$ is a projective right A-module. By applying the tensor functor $fN \otimes_A -$ to the above sequence, we deduce that

$$fN \otimes_A Ae \otimes_{eAe} eX \xrightarrow{id_{fN}\otimes\mu} fN \otimes_A X$$

is an isomorphism. Thus we have proved the commutativity of the left square in (1).

Similarly, we can prove that the right square of the diagram in Proposition 4.3(1) commutes. In particular, we see that $fNe \otimes_{eAe} eA \simeq fN_A$ as fBf-A-bimodules, and $eMf \otimes_{fBf} fB \simeq eM_B$ as eAe-B-bimodules.

(2) Note that the bimodules $\mathrm{Hom}_{eAe}(eA_\Lambda, eMf \otimes_{fBf} (fN)_R)$ and $\mathrm{Hom}_{fBf}(fN_R, fNe \otimes_{eAe} (eA)_\Lambda)$ have been constructed in [17, Theorem 1.1], which induced a stable equivalence of Morita type between Λ and R. Since $\mathrm{add}(fN) = \mathrm{add}(fB)$, we see that $\mathrm{Hom}_{fBf}(fB, fN)$ and $\mathrm{Hom}_{fBf}(fN, fB)$ induce a Morita equivalence between R and Γ. As a result, N' and M' define a stable equivalence of Morita type between Λ and Γ. It can be checked directly that ${}_\Gamma N' \otimes_\Lambda \Lambda_A \simeq {}_\Gamma N'_A$ and ${}_\Lambda M' \otimes_\Gamma \Gamma_B \simeq {}_\Lambda M'_B$. So, we have

$$\begin{aligned} {}_\Gamma N' \otimes_\Lambda \Lambda_A &\simeq {}_\Gamma N'_A \simeq \mathrm{Hom}_{fBf}(fB, fN_A) \\ &\simeq \mathrm{Hom}_{fBf}(fB, fB \otimes_B N_A) \\ &\simeq {}_\Gamma \Gamma \otimes_B N_A \end{aligned}$$

and

$$\begin{aligned} {}_\Lambda M' \otimes_\Gamma \Gamma_B &\simeq {}_\Lambda M'_B = \mathrm{Hom}_{eAe}(eA, eMf \otimes_{fBf} fB_B) \\ &\simeq \mathrm{Hom}_{eAe}(eA_\Lambda, eM_B) \\ &\simeq \mathrm{Hom}_{eAe}(eA_\Lambda, eA \otimes_A M_B) \\ &\simeq {}_\Lambda \Lambda \otimes_A M_B. \end{aligned}$$

This implies that the diagram in (2) is commutative. Thus, we have proved Proposition 4.3. This also finishes the proof of Theorem 1.2. □

Remarks. (1) In Theorem 1.2, the assumption that M and N do not have any projective bimodules as direct summands is actually a very mild restriction. In fact, if $M = X' \oplus X''$ and $N = Y' \oplus Y''$ such that X' and Y' have no direct summands of projective bimodules, and that X'' and Y'' are projective bimodules, then it follows from [16, Lemma 4.8] that the bimodules X' and Y' also define a stable equivalence of Morita type between A and B. Clearly, we have $X'\otimes_B Y'e \in \mathrm{add}(Ae)$ and $\mathrm{add}(Y'e) \subseteq \mathrm{add}(Ne)$. Since ${}_AX'\otimes_B Ne$ is a direct summand of ${}_AM\otimes_B Ne$, we get $X'\otimes_B Ne \in \mathrm{add}(Ae)$, and $Y' \otimes_A X' \otimes_B Ne \in \mathrm{add}(Y' \otimes_A Ae) = \mathrm{add}(Y'e)$. This gives $Ne \in \mathrm{add}(Y'e)$. Hence $\mathrm{add}(Y'e) = \mathrm{add}(Ne)$. This means that M and N in Theorem 1.2 can be replaced by the bimodules X' and Y'.

(2) Note that $M \otimes_B Ne \in \mathrm{add}(Ae)$ is equivalent to $Pe \in \mathrm{add}(Ae)$. In Theorem 1.2, if e is an idempotent element in A such that every indecomposable projective-injective A-module is isomorphic to a summand of Ae, then $Pe \in \mathrm{add}(Ae)$. This follows immediately from Lemma 4.1(3).

(3) As pointed out in [4, Section 4], if e is an idempotent in A and if f is an idempotent in B such that $\mathrm{add}(Ae)$ and $\mathrm{add}(Bf)$ are invariant under Nakayama functors, then eAe and fBf are self-injective, and any stable equivalence of Morita type between A and B induces a stable equivalence of Morita type between eAe and fBf. Note that we may recover this result from Theorem 1.2 since the idempotents e and f satisfy the assumptions of Theorem 1.2 by [4, Lemma 4.1]. In general, however, our algebras eAe and fBf in Theorem 1.2 may not be self-injective.

Definition 4.4 (*[1]*). Let A be an algebra. A projective A-module W is called a minimal Wedderburn projective module if $\mathrm{add}(\nu_A(W)) = \mathrm{add}(I_0(A) \oplus I_1(A))$, where ν_A is the Nakayama functor of A and $0 \to A \to I_0(A) \to I_1(A)$ is the minimal injective copresentation of ${}_AA$. An idempotent element $e \in A$ is called a minimal Wedderburn idempotent element if Ae is a minimal Wedderburn projective module.

Auslander proved in [1] that, given $e^2 = e \in A$, the canonical map $\rho : A \to \mathrm{End}_{eAe}(eA)$ defined by the right multiplication map is an isomorphism if and only if $\mathrm{add}(Ae)$ contains a minimal Wedderburn projective A-module.

The following result shows that stable equivalences of Morita type preserve minimal Wedderburn projective modules or minimal Wedderburn idempotent elements.

Lemma 4.5. *Suppose that A and B are k-algebras such that A and B have no semisimple direct summands. Assume that ${}_AM_B$ and ${}_BN_A$ do not possess any projective bimodules as direct summands, and induce a stable equivalence of Morita type between A and B. Take a minimal Wedderburn idempotent $e \in A$ and a minimal Wedderburn idempotent $f \in B$. Then we have*

$$\mathrm{add}(M \otimes_B Bf) = \mathrm{add}(Ae) \quad \textit{and} \quad \mathrm{add}(N \otimes_A Ae) = \mathrm{add}(Bf).$$

Proof. We assume that $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules for some projective A-A-bimodule P, and $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q. Note that, by Lemma 4.1, the images of the functors ${}_AP\otimes_A -$ and ${}_BQ\otimes_B -$ consist of projective-injective modules.

Let $0 \to A \to I_0 \to I_1$ and $0 \to B \to J_0 \to J_1$ be minimal injective co-presentations of ${}_AA$ and ${}_BB$, respectively. We claim that

$$\mathrm{add}(M \otimes_B (J_0 \oplus J_1)) = \mathrm{add}(I_0 \oplus I_1) \quad \text{and} \quad \mathrm{add}(N \otimes_A (I_0 \oplus I_1)) = \mathrm{add}(J_0 \oplus J_1).$$

Clearly, for any A-module X, we have $\mathrm{add}(N \otimes_A \mathrm{add}(X)) = \mathrm{add}(N \otimes_A X)$. Since $0 \to A \to I_0 \to I_1$ is exact and N_A is projective, it follows that

$$0 \longrightarrow {}_BN \longrightarrow N \otimes_A I_0 \longrightarrow N \otimes_A I_1$$

is exact. Since ${}_BN \otimes_A DA$ is injective and $\mathrm{add}({}_BB) = \mathrm{add}({}_BN)$, we see that $\mathrm{add}(J_0 \oplus J_1) \subseteq \mathrm{add}(N \otimes_A (I_0 \oplus I_1))$. This implies that $\mathrm{add}(M \otimes_B (J_0 \oplus J_1)) \subseteq \mathrm{add}(M \otimes_B N \otimes_A (I_0 \oplus I_1))$. Since $P \otimes_A DA$ is projective-injective and since all indecomposable projective-injective A-modules occur in I_0, we have $\mathrm{add}(M \otimes_B N \otimes_A (I_0 \oplus I_1)) = \mathrm{add}(I_0 \oplus I_1)$. Thus, $\mathrm{add}(M \otimes_B (J_0 \oplus J_1)) \subseteq \mathrm{add}(I_0 \oplus I_1)$. Furthermore, it follows from the injectivity of the module ${}_AM \otimes_B DB$ and $\mathrm{add}({}_AA) = \mathrm{add}({}_AM)$ that $\mathrm{add}(I_0 \oplus I_1) \subseteq \mathrm{add}(M \otimes_B (J_0 \oplus J_1))$. Thus $\mathrm{add}(M \otimes_B (J_0 \oplus J_1)) = \mathrm{add}(I_0 \oplus I_1)$. Similarly, we can prove that $\mathrm{add}(N \otimes_A (I_0 \oplus I_1)) = \mathrm{add}(J_0 \oplus J_1)$. Since $e \in A$ and $f \in B$ are minimal Wedderburn idempotents, we see that $\mathrm{add}(I_0 \oplus I_1) = \mathrm{add}(\nu_A(Ae))$ and $\mathrm{add}(J_0 \oplus J_1) = \mathrm{add}(\nu_B(Bf))$. Consequently, $\mathrm{add}(N \otimes_A \nu_A(Ae)) = \mathrm{add}(\nu_B(Bf))$. It follows from $N \otimes_A \nu_A(Ae) \simeq \nu_B(N \otimes {}_AAe)$ that $\mathrm{add}(\nu_B(N \otimes {}_AAe)) = \mathrm{add}(\nu_B(Bf))$. Since the Nakayama functor ν_B is an equivalence from B-proj to B-inj, we deduce that $\mathrm{add}(N \otimes_A Ae) = \mathrm{add}(Bf)$. Similarly, we can show that $\mathrm{add}(M \otimes_B Bf) = \mathrm{add}(Ae)$. □

In the following we shall see that stable equivalences of Morita type can be transferred to "corner" algebras of Wedderburn type.

Corollary 4.6. *Suppose that A and B are k-algebras such that A and B have no semisimple direct summands. Assume that ${}_AM_B$ and ${}_BN_A$ have no projective bimodules as direct summands, and induce a stable equivalence of Morita type between A and B. Let $e \in A$ and $f \in B$ be minimal Wedderburn idempotents. Then eMf and fNe define a stable equivalence of Morita type between eAe and fBf such that $fNe \otimes_{eAe} eA \simeq fN$ and $eMf \otimes_{fBf} fB \simeq eM$ as bimodules.*

Proof. By Lemma 4.5, we see that the idempotents e and f satisfy the assumptions in Theorem 1.2. Then Corollary 4.6 follows from the first part of Theorem 1.2 together with Proposition 4.3. □

As a corollary of Corollary 4.6, we get the following result.

Corollary 4.7. *Assume that A and B are k-algebras without semisimple direct summands. Let ${}_AX$ be a generator–cogenerator for* A-mod, *and let ${}_BY$ be a generator–cogenerator for* B-mod. *If* $\mathrm{End}_A(X)$ *and* $\mathrm{End}_B(Y)$ *are stably equivalent of Morita type, then there exist bimodules ${}_AM_B$ and ${}_BN_A$ which define a stable equivalence of Morita type between A and B such that* $\mathrm{add}({}_AM\otimes_BY) = \mathrm{add}({}_AX)$ *and* $\mathrm{add}({}_BN \otimes_A X) = \mathrm{add}({}_BY)$.

Proof. Set $R = \mathrm{End}_A(X)$ and $S = \mathrm{End}_B(Y)$. First, we show that if A does not have any semisimple direct summands, then nor does R.

Suppose contrarily that R has a semisimple direct summand. Then R must have a simple projective-injective module W. Since each indecomposable projective-injective R-module is isomorphic to a direct summand of $\mathrm{Hom}_A(X, DA)$, there exists an indecomposable injective A-module I such that $W \simeq \mathrm{Hom}_A(X, I)$. Let ${}_AS$ be the socle of ${}_AI$. Then $\mathrm{Hom}_A(X, S)$ can be embedded into the simple R-module $\mathrm{Hom}_A(X, I)$, and therefore $\mathrm{Hom}_A(X, S) \simeq \mathrm{Hom}_A(X, I) \simeq W$ as R-modules. Since $A \in \mathrm{add}(X)$, we infer that $S \simeq I$. Let ${}_AP$ be the projective cover of ${}_AS$. Then it follows from $\mathrm{Hom}_R(\mathrm{Hom}_A(X, P), \mathrm{Hom}_A(X, S)) \simeq \mathrm{Hom}_A(P, S) \neq 0$ that there is a non-zero homomorphism from $\mathrm{Hom}_A(X, P)$ to the simple projective R-module $\mathrm{Hom}_A(X, S)$, which means that $\mathrm{Hom}_A(X, P) \simeq \mathrm{Hom}_A(X, S)$. Consequently, we get $P \simeq S \simeq I$. Thus A has a simple projective-injective module, and therefore it has a semisimple direct summand, which is a contradiction. This shows that R has no semisimple direct summands. Similarly, we can prove that S has no semisimple direct summands.

Note that, if X is a generator–cogenerator for A-mod, then $\mathrm{Hom}_A(X,A)$ is a minimal Wedderburn projective R-module. Similarly, $\mathrm{Hom}_B(Y,B)$ is a minimal Wedderburn projective S-module. Clearly, $\mathrm{End}_R(\mathrm{Hom}_A(X,A)) \simeq A$ and $\mathrm{End}_S(\mathrm{Hom}_B(Y,B)) \simeq B$. Note that neither R nor S has semisimple direct summands. Then, by Corollary 4.6, there exist bimodules ${}_AM_B$ and ${}_BN_A$ which define a stable equivalence of Morita type between A and B. Note that $\mathrm{Hom}_R(\mathrm{Hom}_A(X,A),R) \simeq {}_AX$ and $\mathrm{Hom}_S(\mathrm{Hom}_B(Y,B),S) \simeq {}_BY$. It follows from Corollary 4.6 that $\mathrm{add}({}_AM \otimes_B Y) = \mathrm{add}({}_AX)$ and $\mathrm{add}({}_BN \otimes_A X) = \mathrm{add}({}_BY)$. $\square$

Combining Corollary 4.7 with [17, Theorem 1.1], we have the following result on Auslander algebras.

Corollary 4.8. *Let A and B be representation-finite k-algebras. Suppose that A and B have no semisimple direct summands. Let Λ and Γ be the corresponding Auslander algebras of A and B, respectively. Then Λ and Γ are stably equivalent of Morita type if and only if so are A and B.*

For an algebra A, we denote by $[A]$ the class of all algebras B such that there is a stable equivalence of Morita type between B and A. It is known that $[A] = [A \times S]$ for any separable algebra S. Note that, if k is a perfect field, then the class of all semisimple k-algebras is the same as that of all separable k-algebras.

The following result establishes a one-to-one correspondence, up to stable equivalence of Morita type, between representation-finite algebras and Auslander algebras. This is an immediate consequence of Corollary 4.8.

Corollary 4.9. *Suppose that k is a perfect field. Let $\mathcal{F}$ be the set of equivalence classes $[A]$ of representation-finite k-algebras A under stable equivalences of Morita type, and let $\mathcal{A}$ be the set of equivalence classes $[\Lambda]$ of Auslander k-algebras Λ under stable equivalences of Morita type. Then there is a one-to-one correspondence between $\mathcal{F}$ and $\mathcal{A}$.*

Finally, we remark that Corollary 4.8 is not true for derived equivalences. Nevertheless, it was shown in [7] that if two representation-finite, self-injective algebras A and B are derived-equivalent then so are their Auslander algebras. The converse of this statement is open. For further information on constructing derived equivalences, we refer the reader to the current papers [6,7].

5. Stable equivalences of Morita type based on self-injective algebras

Of particular interest are stable equivalences of Morita type between self-injective algebras or between those related to self-injective algebras. Since derived equivalences between self-injective algebras imply stable equivalences of Morita type by a result of Rickard [20], this makes stable equivalences of Morita type closely related to the Broué abelian defect group conjecture which essentially predicates a derived equivalence between two block algebras [3], and thus also a stable equivalence of Morita type between them.

In this section, we will apply Theorems 1.1 and 1.2 to self-injective algebras. It turns out that the existence of a stable equivalence of Morita type between Φ-Auslander–Yoneda algebras of generators for one finite admissible set Φ implies the one for all finite admissible sets.

Throughout this section, we fix a finite admissible subset Φ of $\mathbb{N}$, and assume that A and B are indecomposable, non-simple, self-injective algebras. Let X be a generator for A-mod with a decomposition $X := A \oplus \bigoplus_{1\le i\le n} X_i$, where X_i is indecomposable and non-projective such that $X_i \not\simeq X_t$ for $1 \le i \ne t \le n$, and let Y be a generator for B-mod with a decomposition $Y := B \oplus \bigoplus_{1\le j\le m} Y_j$, where Y_j is indecomposable and non-projective such that $Y_j \not\simeq Y_s$ for $1 \le j \ne s \le m$.

Lemma 5.1. (1) *The full subcategory of* $\mathrm{E}_A^{\Phi}(X)$-mod *consisting of projective-injective* $\mathrm{E}_A^{\Phi}(X)$*-modules is equal to* $\mathrm{add}(\mathrm{E}_A^{\Phi}(X,A))$. *Particularly, if* $\mathrm{E}_A^{\Phi}(X) \ne \mathrm{End}_A(X)$*, then* $\mathrm{dom.dim}(\mathrm{E}_A^{\Phi}(X)) = 0$.

(2) $\mathrm{E}_A^{\Phi}(X)$ *has no semisimple direct summands.*

Proof. (1) For convenience, we set $\Lambda_0 = \mathrm{End}_A(X)$ and $\Lambda = \mathrm{E}_A^{\Phi}(X)$. Since A is self-injective, it follows from [7, Lemma 3.5] that $\nu_\Lambda(\mathrm{E}_A^{\Phi}(X,A)) \simeq \mathrm{E}_A^{\Phi}(X,\nu_A A) \simeq \mathrm{E}_A^{\Phi}(X,DA) \in \mathrm{add}\big(\mathrm{E}_A^{\Phi}(X,A)\big)$. Consequently, $\mathrm{E}_A^{\Phi}(X,A)$ is a projective-injective Λ-module. We claim that, up to isomorphism, each indecomposable projective-injective Λ-module is a direct summand of $\mathrm{E}_A^{\Phi}(X,A)$. To prove this claim, it suffices to show that $\mathrm{E}_A^{\Phi}(X,X_i)$ is not injective for all $1 \le i \le n$. We denote $\mathrm{E}_A^{\Phi}(X,X_i)$ by $\widetilde{X_i}$ for abbreviation.

First, we observe that $\mathrm{rad}(\Lambda) = \mathrm{rad}(\Lambda_0) \oplus \Lambda_+$, where $\Lambda_+ = \bigoplus_{0\ne i\in\Phi} \Lambda_i$ with $\Lambda_i = \mathrm{Ext}_A^i(X,X) = \mathrm{Hom}_{\mathscr{D}^b(A)}(X,X[i])$. Since each summand $\mathrm{Hom}_{\mathscr{D}^b(A)}(X,X[j])$ of $\widetilde{X_i}$ is a Λ_0-module and since the socle of $\widetilde{X_i}$ is the set of all elements x in $\widetilde{X_i}$ such that $\mathrm{rad}(\Lambda)x = 0$, we see that the socle of $\widetilde{X_i}$ contains $\bigoplus_{j\in\Phi}\{x \in \mathrm{soc}_{\Lambda_0}(\mathrm{Ext}_A^j(X,X_i)) \mid \Lambda_+x = 0\}$. By an argument of graded modules, we can even see that $\mathrm{soc}_\Lambda(\widetilde{X_i}) = \bigoplus_{j\in\Phi}\{x \in \mathrm{soc}_{\Lambda_0}(\mathrm{Ext}_A^j(X,X_i)) \mid \Lambda_+x = 0\}$.

Next, we shall show that $\widetilde{X_m}$ is not injective for $1 \le m \le n$. Indeed, let $f : X_m \to I$ be an injective envelope of X_m with I an injective A-module. Then $f_* : \mathrm{Hom}_A(X,X_m) \to \mathrm{Hom}_A(X,I)$ is an injective envelope of the Λ_0-module $\mathrm{Hom}_A(X,X_m)$ in Λ_0-mod. Now, we consider the following two cases:

(a) If $\widetilde{X_m} = \mathrm{Hom}_A(X,X_m)$, then $\widetilde{X_m}$ is annihilated by Λ_+. Since X_m is not injective in A-mod, we conclude that $\mathrm{Hom}_A(X,X_m)$ is not an injective Λ_0-module, which implies that $\widetilde{X_m}$ is not injective as a Λ-module.

(b) If $\widetilde{X_m} \neq \mathrm{Hom}_A(X, X_m)$, then there is a positive integer $t \in \Phi$ such that $\mathrm{Ext}^t_A(X, X_m) \neq 0$. We may assume that t is the maximal number in Φ with this property, that is, $\mathrm{Ext}^s_A(X, X_m) = 0$ for any $s \in \Phi$ with $t < s$. It follows that $\Lambda_+\mathrm{Ext}^t_A(X, X_m) = 0$, which implies that $0 \neq \mathrm{soc}_{\Lambda_0}(\mathrm{Ext}^t_A(X, X_m)) \subseteq \mathrm{soc}_\Lambda(\widetilde{X_m})$.

Now we consider $\mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, X_m))$. Since f_* is an injective envelope in Λ_0-mod, we know that $\mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, X_m)) \simeq \mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, I))$. Since $\nu_{\Lambda_0}(\mathrm{Hom}_A(X, A)) \in \mathrm{add}\big(\mathrm{Hom}_A(X, A)\big)$ and $I \in \mathrm{add}({}_AA)$, we see that $\mathrm{Hom}_{\Lambda_0}(\mathrm{Hom}_A(X, X_i), \mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, I))) = 0$ for $1 \leq i \leq n$. If e is the idempotent in Λ_0 corresponding to the direct summand A of X, then $e\,\mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, I)) = \mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, I))$. Consequently, $e\,\mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, X_m)) = \mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, X_m))$, that is, $eg = g$ whenever $g \in \mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, X_m))$, that is, g factorizes through the regular module ${}_AA$, say $g = g_1g_2$ with $g_1 : X \to {}_AA$ and $g_2 : {}_AA \to X_m$. Thus, for any element $x \in \mathrm{Hom}_{\mathscr{D}^b(A)}(X, X[i])$ with $0 \neq i \in \Phi$, we have $x \cdot g = x(g_1[i]\ g_2[i]) = (x\ g_1[i])g_2[i] = 0\,(g_2[i]) = 0$ since A is self-injective. Thus $\Lambda_+\mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, X_m)) = 0$. This implies that $\mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, X_m)) \subseteq \mathrm{soc}_\Lambda(\widetilde{X_m})$. Thus we have shown that the Λ-submodule $\mathrm{soc}_{\Lambda_0}(\mathrm{Hom}_A(X, X_m)) \oplus \mathrm{soc}_{\Lambda_0}(\mathrm{Ext}^t_A(X, X_m))$ of $\widetilde{X_m}$ is contained in the socle of $\widetilde{X_m}$. This implies that $\widetilde{X_m}$ is not injective since its socle is not simple.

Thus $\mathrm{add}(\mathrm{E}^\Phi_A(X, A))$ is just the full subcategory of $\mathrm{E}^\Phi_A(X)$-mod consisting of projective-injective modules.

Finally, we consider the dominant dimension of $\mathrm{dom.dim}(\mathrm{E}^\Phi_A(X))$. Suppose $\mathrm{E}^\Phi_A(X) \neq \mathrm{End}_A(X)$. Since A is self-injective, we have $\mathrm{E}^\Phi_A(X, A) = \mathrm{Hom}_A(X, A)$. It follows that $\mathrm{E}^\Phi_A(X, A)$ is annihilated by Λ_+. Since Λ cannot be annihilated by Λ_+, we see that Λ cannot be cogenerated by $\mathrm{E}^\Phi_A(X, A)$. This implies that $\mathrm{dom.dim}(\mathrm{E}^\Phi_A(X)) = 0$. We finish the proof.

(2) Contrarily, we suppose that the algebra $\mathrm{E}^\Phi_A(X)$ has a semisimple direct summand. Then $\mathrm{E}^\Phi_A(X)$ has a simple projective-injective module S. According to (1), we know that S must be a simple projective-injective $\mathrm{End}_A(X)$-module. Then it follows from the first part of the proof of Corollary 4.7 that A has a semisimple direct summand. Clearly, this is contrary to our initial assumption that A is indecomposable and non-simple. Thus $\mathrm{E}^\Phi_A(X)$ has no semisimple direct summands. □

Theorem 5.2. *If the algebras $\mathrm{E}^\Phi_A(X)$ and $\mathrm{E}^\Phi_B(Y)$ are stably equivalent of Morita type, then $n = m$ and there are bimodules ${}_AM_B$ and ${}_BN_A$ which define a stable equivalence of Morita type between A and B such that, up to the ordering of indices, ${}_AM \otimes_B Y_i \simeq X_i \oplus P_i$ as A-modules, where ${}_AP_i$ is projective for all i with $1 \leq i \leq n$. Moreover, for any finite admissible subset Ψ of $\mathbb{N}$, there is a stable equivalence of Morita type between $\mathrm{E}^\Psi_A(X)$ and $\mathrm{E}^\Psi_B(Y)$.*

Proof. For convenience, we set $\Lambda_0 = \mathrm{End}_A(X)$, $\Lambda = \mathrm{E}^\Phi_A(X)$, $\Gamma_0 = \mathrm{End}_B(Y)$ and $\Gamma = \mathrm{E}^\Phi_B(Y)$. By Lemma 5.1, the algebras Λ and Γ have no semisimple direct summands. Let e be the idempotent in Λ_0 corresponding to the direct summand A of X, and let f be the idempotent in Γ_0 corresponding to the direct summand B of Y. Note that $\Lambda e \simeq \mathrm{E}^\Phi_A(X, A)$ as Λ-modules and $\Gamma f \simeq \mathrm{E}^\Phi_B(Y, B)$ as Γ-modules. Clearly, $e\Lambda e \simeq A$ and $f\,\Gamma f \simeq B$ as algebras. Moreover, we see that $e\Lambda \simeq X$ as A-modules, and $f\,\Gamma \simeq Y$ as B-modules. Suppose that a stable equivalences of Morita type between Λ and Γ is given. By Corollary 4.2 and Lemma 5.1, we know that the idempotent e in Λ and the idempotent f in Γ satisfy the conditions in Theorem 1.2. It follows from Theorem 1.2 and Proposition 4.3(1) that there are bimodules ${}_AM_B$ and ${}_BN_A$ which define a stable equivalence of Morita type between A and B such that $\mathrm{add}(M \otimes_B Y) = \mathrm{add}(X)$. By the given decompositions of X and Y, we conclude that $n = m$ and, up to the ordering of direct summands, we may assume that ${}_AM \otimes_B Y_i \simeq X_i \oplus P_i$ as A-modules, where ${}_AP_i$ is projective for all i with $1 \leq i \leq n$. Now, the last statement in this corollary follows immediately from Theorem 1.1. Thus the proof is completed. □

Usually, it is difficult to decide whether an algebra is not stably equivalent of Morita type to another algebra. The next corollary, however, gives a sufficient condition to assert when two algebras are not stably equivalent of Morita type.

Corollary 5.3. *Let n be a non-negative integer. Let W be an indecomposable non-projective A-module. Suppose that $\Omega^s_A(W) \not\simeq W$ for any non-zero integer s. Set $W_n = \bigoplus_{0 \leq i \leq n} \Omega^i_A(W)$. Then, for any finite admissible subset Ψ of $\mathbb{N}$, the algebras $\mathrm{E}^\Psi_A(A \oplus W_n \oplus \Omega^l_A(W))$ and $\mathrm{E}^\Psi_A(A \oplus W_n \oplus \Omega^m_A(W))$ are not stably equivalent of Morita type whenever m and l belong to $\mathbb{N}$ with $n < m < l$.*

Proof. Suppose that there is a finite admissible subset Ψ of $\mathbb{N}$ such that $\mathrm{E}^\Psi_A(A \oplus W_n \oplus \Omega^m_A(W))$ and $\mathrm{E}^\Psi_A(A \oplus W_n \oplus \Omega^l_A(W))$ are stably equivalent of Morita type for some fixed $l, m \in \mathbb{N}$ with $n < m < l$. Set $\Phi_1 = \{0, 1, \ldots, n\} \cup \{l\}$ and $\Phi_2 = \{0, 1, \ldots, n\} \cup \{m\}$. Then, by Theorem 5.2, we know that there exist bimodules ${}_AM_A$ and ${}_AN_A$ which define a stable equivalence of Morita type between A and itself, and that there is a bijection $\sigma : \Phi_1 \to \Phi_2$ such that $M \otimes_A \Omega^j_A(W) \simeq \Omega^{\sigma(j)}_A(W) \oplus P_j$ as A-modules, where P_j is projective for each $j \in \Phi_1$. In particular, we have $M \otimes_A W \simeq \Omega^{\sigma(0)}_A(W) \oplus P_0$. Since M is projective as a one-sided module, we know that $M \otimes_A \Omega^l_A(W) \simeq \Omega^{\sigma(0)+l}_A(W) \oplus P'_l$ with $P'_l \in \mathrm{add}({}_AA)$. Note that $M \otimes_A \Omega^l_A(W) \simeq \Omega^{\sigma(l)}_A(W) \oplus P_l$. It follows that $\Omega^{\sigma(0)+l}_A(W) \simeq \Omega^{\sigma(l)}_A(W)$. Consequently, we have $\sigma(l) = \sigma(0) + l \geq l$ since W is not Ω-periodic. Hence $l \leq \sigma(l) \leq m < l$, a contradiction. This shows that $\mathrm{E}^\Psi_A(A \oplus W_n \oplus \Omega^m_A(W))$ and $\mathrm{E}^\Psi_A(A \oplus W_n \oplus \Omega^l_A(W))$ cannot be stably equivalent of Morita type whenever l and $m \in \mathbb{N}$ with $n < m < l$. □

This corollary will be used in the next section.

6. A family of derived-equivalent algebras: Application to Liu–Schulz algebras

In this section, we shall apply our results in the previous sections to solve the following problem on derived equivalences and stable equivalences of Morita type:

Problem. Is there any infinite series of finite-dimensional k-algebras such that they have the same dimension and are all derived-equivalent, but not stably equivalent of Morita type?

This problem was originally asked by Thorsten Holm about ten years ago at a workshop in Goslar, Germany.

Recall that Liu and Schulz in [13] constructed a local symmetric k-algebra A of dimension 8 and an indecomposable A-module M such that all the syzygy modules $\Omega_A^n(M)$ with $n \in \mathbb{Z}$ are 4-dimensional and pairwise non-isomorphic. This algebra A depends on a non-zero parameter $q \in k$, which is not a root of unity, and has an infinite DTr-orbit in which each module has the same dimension. A thorough investigation of Auslander–Reiten components of this algebra was carried out by Ringel in [22]. Based on this symmetric algebra and a recent result in [6] together with the results in the previous sections, we shall construct an infinite family of algebras, which provides a positive solution to the above problem.

From now on, we fix a non-zero element q in the field k, and assume that q is not a root of unity. The 8-dimensional k-algebra A defined by Liu–Schulz is an associative algebra (with identity) over k with

the generators: x_0, x_1, x_2, and

the relations: $x_i^2 = 0$, and $x_{i+1}x_i + qx_ix_{i+1} = 0$ for $i = 0, 1, 2$.

Here, and in what follows, the subscript is modulo 3.

Let n be an arbitrary but fixed natural number, and let $\Phi = \{0\}$ or $\{0, 1\}$. For $j \in \mathbb{Z}$, set $u_j := x_2 + q^j x_1$, $I_j := Au_j$, $J_j := u_jA$, $I := \bigoplus_{i=0}^n I_i$ and $\Lambda_j^\Phi := \mathrm{E}_A^\Phi(A \oplus I \oplus I_j)$.

With these notations in mind, the main result in this section can be stated as follows:

Theorem 6.1. *For any $m \geq n + 4$, we have*

(1) $\dim_k(\Lambda_m^\Phi) = \dim_k(\Lambda_{m+1}^\Phi)$.

(2) $\mathrm{gl.dim}(\Lambda_m^\Phi) = \infty$.

(3) $\mathrm{dom.dim}(\Lambda_m^\Phi) = \begin{cases} 2 & \text{if } \Phi = \{0\}, \\ 0 & \text{if } \Phi = \{0, 1\}. \end{cases}$

(4) *Λ_m^Φ and Λ_{m+1}^Φ are derived-equivalent.*

(5) *If $l > m$, then Λ_l^Φ and Λ_m^Φ are not stably equivalent of Morita type.*

An immediate consequence of Theorem 6.1 is the following corollary, which solves the above mentioned problem positively.

Corollary 6.2. *There exists an infinite series of finite-dimensional k-algebras A_i, $i \in \mathbb{N}$, such that*

(1) $\dim_k(A_i) = \dim_k(A_{i+1})$ *for all $i \in \mathbb{N}$,*

(2) *all A_i have the same global and dominant dimensions,*

(3) *all A_i are derived-equivalent, and*

(4) *A_i and A_j are not stably equivalent of Morita type for $i \neq j$.*

The proof of Theorem 6.1 will cover the rest of this section. Let us first introduce a few more notations and conventions.

Let B be an algebra and S a subset of B. Set $R(S) := \{b \in B \mid sb = 0 \text{ for all } s \in S\}$ for the right annihilator of S in B, and $L(S) := \{b \in B \mid bs = 0 \text{ for all } s \in S\}$ for the left annihilator of S in B. In case $x \in B$, we write $R(x)$ and $L(x)$ for $R(\{x\})$ and $L(\{x\})$, respectively. For $y, z \in B$, we set $B(y, z) := \{b \in B \mid L(y)bz = 0\}$, that is, $B(y, z) = \{b \in B \mid L(y)b \subseteq L(z)\}$. Note that $L(S)$ and $R(S)$ are left and right ideals in B, respectively.

Let V be a k-vector space with $y_i \in V$ for $1 \leq i \leq n \in \mathbb{N}$. We denote by $\langle y_1, \ldots, y_n\rangle$ the k-subspace of V generated by all y_i.

The following result is useful for our calculations, it may be of its own interest in describing the endomorphism rings of direct sums of cyclic left ideals.

Lemma 6.3. *Let B be a k-algebra, and let x, y and z be elements in B. Then the following statements hold:*

(1) *There is an isomorphism of k-vector spaces:*

$$\varphi_{x,y} : \mathrm{Hom}_B(Bx, By) \xrightarrow{\sim} R(L(x)) \cap By,$$

which sends f to xf for $f \in \mathrm{Hom}_B(Bx, By)$.

(2) *There is an isomorphism of k-vector spaces:*

$$\theta_{x,y} : \mathrm{Hom}_B(Bx, By) \xrightarrow{\sim} B(x, y)/L(y),$$

which sends h to $d + L(y)$ for $h \in \mathrm{Hom}_B(Bx, By)$, where $d \in B$ such that $xh = dy$.

(3) *Let $\cdot$ be the map defined by*

$$\big(B(x,y)/L(y)\big)\times\big(B(y,z)/L(z)\big)\longrightarrow B(x,z)/L(z)$$
$$\big(a+L(y),b+L(z)\big)\mapsto (a+L(y))\cdot(b+L(z)):=ab+L(z).$$

Then there is the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_B(Bx,By)\times\mathrm{Hom}_B(By,Bz) \arrow[r, "\mu_{x,y,z}"] \arrow[d, "\theta_{x,y}\times\theta_{y,z}"', "\wr"] & \mathrm{Hom}_B(Bx,Bz) \arrow[d, "\theta_{x,z}"', "\wr"] \\
(B(x,y)/L(y))\times(B(y,z)/L(z)) \arrow[r, "\cdot"] & B(x,z)/L(z),
\end{tikzcd}$$

where $\mu_{x,y,z}$ is the composition map.

(4) *Let n be a positive integer, and let x_i be elements in B for $1\le i\le n$. We define*

$$\mathrm{M}_B(x_1,x_2,\cdots,x_n):=\{(b_{i,j})_{1\le i,j\le n}\mid b_{i,j}\in B(x_i,x_j)/L(x_j)\ \text{for all}\ 1\le i,j\le n\}.$$

Then $\mathrm{M}_B(x_1,x_2,\cdots,x_n)$ becomes an associative k-algebra with the usual matrix addition and multiplication which is given by the products $\cdot$ defined in (3). Moreover, there is an algebra isomorphism $\theta:\mathrm{End}_B(\bigoplus_{1\le i\le n}Bx_i)\longrightarrow \mathrm{M}_B(x_1,x_2,\cdots,x_n)$, defined by $(f_{ij})_{1\le i,j\le n}\mapsto((f_{ij})\theta_{x_ix_j})_{1\le i,j\le n}$ for $f_{ij}\in\mathrm{Hom}_B(Bx_i,Bx_j)$.

Proof. (1) Let $f\in\mathrm{Hom}_B(Bx,By)$. Since f is a homomorphism of B-modules, we know $b(xf)=0$ whenever $b\in B$ and $bx=0$. This implies that $xf\in R(L(x))\cap By$. Thus the map $\varphi_{x,y}$ is well-defined. It is not hard to check that $\varphi_{x,y}$ is an isomorphism of k-vector spaces.

(2) For $x\in B$, we denote by ρ_x the right multiplication map from B to itself, defined by $b\mapsto bx$ for $b\in B$. Then there is a canonical exact sequence of B-modules: $\delta_x:0\to L(x)\xrightarrow{\lambda_x}B\xrightarrow{\pi_x}Bx\to 0$, where λ_x is the inclusion, and π_x is the canonical multiplication of x. Note that if μ_x denotes the inclusion of Bx into B, then $\rho_x=\pi_x\mu_x$ for all $x\in B$. By the definition of $B(x,y)$, an element w belongs to $B(x,y)$ if and only if $\lambda_x\rho_w\pi_y=0$, or equivalently, if and only if there is a unique $\alpha\in\mathrm{Hom}_B(Bx,By)$ such that $\rho_w\pi_y=\pi_x\alpha$. Clearly, $w\in L(y)$ if and only if $\rho_w\pi_y=0$. So, we have $L(y)\subseteq B(x,y)$.

First, we show that $\theta_{x,y}$ is well-defined. In fact, if $f\in\mathrm{Hom}_B(Bx,By)$, then there is an element $b\in B$, which may not be unique, such that the following diagram of left B-modules commutes:

$$\begin{tikzcd}
0 \arrow[r] & L(x) \arrow[r, "\lambda_x"] \arrow[d, dashed, "\rho_b'"'] & B \arrow[r, "\pi_x"] \arrow[d, dashed, "\rho_b"'] & Bx \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & L(y) \arrow[r, "\lambda_y"] & B \arrow[r, "\pi_y"] & By \arrow[r] & 0,
\end{tikzcd}$$

where ρ_b' is the restriction of ρ_b to $L(x)$. Hence $b\in B(x,y)$. If there is another d in B also making the above diagram commutative, then $(\rho_b-\rho_d)\pi_y=0$, and therefore $\rho_b-\rho_d$ factorizes through $L(y)$. This implies that $b-d\in L(y)$ and $b+L(y)=d+L(y)$ in $B(x,y)/L(y)$. Thus $\theta_{x,y}$ is well-defined.

Next, we shall prove that $\theta_{x,y}$ is an isomorphism of k-vector spaces. Indeed, if $(f)\theta_{x,y}=b+L(y)=0$ for some map $f\in\mathrm{Hom}_B(Bx,By)$, then $b\in L(y)$ and $\pi_xf=\rho_b\pi_y=0$. Since π_x is surjective, we get $f=0$. Thus $\theta_{x,y}$ is injective. That $\theta_{x,y}$ is surjective follows from the equivalent definitions of $B(x,y)$ discussed above.

(3) Observe that $B(x,y)B(y,z)\subseteq B(x,z)$, $L(y)B(y,z)\subseteq L(z)$ and $B(x,y)L(z)\subseteq L(z)$ for all $x,y,z\in B$. This implies that the product in (3) is well-defined. We leave the verification of the commutative diagram in (3) to the reader.

(4) It follows from the commutative diagram in (3) that $\mathrm{M}(x_1,\ldots,x_n)$ is an associative k-algebra with identity. To see that θ is an isomorphism of algebras, we first observe that θ is a k-linear isomorphism. It remains to show that θ preserves the multiplication. However, this follows straightforward from the commutative diagram in (3). $\square$

Recall that, for $i\in\mathbb{Z}$, we have defined $u_i:=x_2+q^ix_1$, $I_i:=Au_i$ and $J_i:=u_iA$. In the following lemma, we display a few properties about the Liu–Schulz algebra A.

Lemma 6.4 ([13,22]). (1) *The Liu–Schulz algebra A is an $\mathbb{N}$-graded algebra, namely, $A=\bigoplus_{i\ge 0}A_i$ with*

$$A_0=k,\quad A_1=\langle x_0,x_1,x_2\rangle,\quad A_2=\langle x_0x_1,x_1x_2,x_2x_0\rangle,\quad A_3=\langle x_0x_1x_2\rangle,\quad \text{and}\quad A_i=0\quad \text{for all } i\ge 4.$$

Moreover, A_2 is contained in the center of A. In particular, $x_0x_1x_2=x_1x_2x_0=x_2x_0x_1$ in A.

(2) *A is an 8-dimensional symmetric k-algebra.*

(3) $\dim_k(I_j)=\dim_k(J_j)=4$ *for all $j\in\mathbb{Z}$.*

(4) *$\Omega_A(I_j)=I_{j+1}$ and $\Omega_A(J_{j+1})=J_j$ for all $j\in\mathbb{Z}$.*

(5) *The A-modules I_j (respectively, A^{op}-modules J_j) are pairwise non-isomorphic for all $j\in\mathbb{Z}$.*

In the next lemma, we calculate dimensions of homomorphism groups related to the modules I_i and J_i.

Lemma 6.5. *Let i and j be integers. Then*

(1) I_j *has a basis* $\{x_2+q^jx_1, x_2x_0-q^{j-1}x_0x_1, x_1x_2, x_0x_1x_2\}$, *and* J_j *has a basis* $\{x_2+q^jx_1, x_2x_0-q^{j+1}x_0x_1, x_1x_2, x_0x_1x_2\}$.
(2) $L(u_j)=I_{j+1}$, $R(u_{j+1})=J_j$.
(3) $J_j \simeq \mathrm{Hom}_A(I_j, A)$.
(4) *As k-vector spaces,* $\mathrm{Hom}_A(I_j, I_i)\simeq J_j\cap I_i = \begin{cases}\langle x_2+q^jx_1, x_1x_2, x_0x_1x_2\rangle & \text{if } j=i,\\ \langle x_2x_0-q^{j+1}x_0x_1, x_1x_2, x_0x_1x_2\rangle & \text{if } j=i-2,\\ \langle x_1x_2, x_0x_1x_2\rangle & \text{otherwise}.\end{cases}$

In particular, $\dim_k \mathrm{Hom}_A(I_j, I_i)=\begin{cases}3 & \text{if } j=i \text{ or } i-2,\\ 2 & \text{otherwise}.\end{cases}$

(5) $\dim_k \mathrm{Ext}^1_A(I_j, I_i)=\begin{cases}1 & \text{if } j\le i\le j+3,\\ 0 & \text{otherwise}.\end{cases}$
(6) $A(1,u_i)=A$ *and* $A(u_i,1)=J_i$.
(7) $A(u_j,u_i)=\begin{cases}\langle 1, x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2\rangle & \text{if } j=i,\\ \langle x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2\rangle & \text{if } j=i-2,\\ \langle x_0, x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2\rangle & \text{otherwise}.\end{cases}$

Proof. (1) and (2). By definition, $I_j=Au_j$. One can check directly that

$$x_0u_j=(-q)(x_2x_0-q^{j-1}x_0x_1),\qquad x_2u_j=-q^{j+1}x_1x_2,\qquad x_1u_j=x_1x_2,$$
$$x_1x_2u_j=x_0x_1x_2u_j=0,\qquad x_0x_1u_j=x_0x_1x_2,\qquad x_2x_0u_k=q^jx_0x_1x_2.$$

This implies that $I_j=\langle x_2+q^jx_1, x_2x_0-q^{j-1}x_0x_1, x_1x_2, x_0x_1x_2\rangle$. Note that $0\to L(u_j)\to A\to Au_j\to 0$ is an exact sequence of A-modules. Since $u_{j+1}u_j=(x_2+q^{j+1}x_1)(x_2+q^jx_1)=0$, we have $I_{j+1}\subseteq L(u_j)$. In addition, $\dim_k I_{j+1}=\dim_k L(u_j)=4$. It follows that $L(u_j)=I_{j+1}$. Similarly, we can prove the other statements in (1) and (2) for J_j.

(3) It follows from (2) that $R(L(u_j))=R(Au_{j+1})=R(u_{j+1})=J_j$. By Lemma 6.3(1), we get an isomorphism $\varphi_{u_j,1}$: $\mathrm{Hom}_A(I_j,A)\simeq J_j$ of k-vector spaces. In fact, we can check directly that $\varphi_{u_j,1}$ is an isomorphism of A^{op}-modules. This proves (3).

(4) Note that $\mathrm{Hom}_A(I_j,I_i)=\mathrm{Hom}_A(Au_j,Au_i)\simeq u_jA\cap Au_i=J_j\cap I_i$. To prove (4), there are three cases to be considered.

Case 1: $j=i$. By (1) and (2), we conclude that $\langle x_2+q^jx_1, x_1x_2, x_0x_1x_2\rangle\subseteq I_j\cap J_j$. Since $\dim_k(I_j)=4$ and $x_2x_0-q^{j+1}x_0x_1\notin I_j$, we get $\dim_k(I_j\cap J_j)=3$. As a result, $I_j\cap J_j=\langle x_2+q^jx_1, x_1x_2, x_0x_1x_2\rangle$.

Case 2: $j=i-2$. Note that $x_2x_0-q^{j+1}x_0x_1=x_2x_0-q^{i-1}x_0x_1$. But $x_2+q^jx_1\notin I_i$. It follows that $I_i\cap J_j=\langle x_2x_0-q^{j+1}x_0x_1, x_1x_2, x_0x_1x_2\rangle$.

Case 3: $j\notin\{i,i-2\}$. We claim that $I_i\cap J_j=\langle x_1x_2, x_0x_1x_2\rangle$. Obviously, $\langle x_1x_2,\ x_0x_1x_2\rangle$ is contained in $I_i\cap J_j$. Conversely, if $\lambda\in I_i\cap J_j$, then there are elements $a_1, a_{20}, a_{21}, a_3, b_1, b_{20}, b_{21}$ and $b_3\in k$, such that $\lambda=a_1(x_2+q^jx_1)+a_{20}(x_2x_0-q^{j+1}x_0x_1)+a_{21}x_1x_2+a_3x_0x_1x_2=b_1(x_2+q^ix_1)+b_{20}(x_2x_0-q^{i-1}x_0x_1)+b_{21}x_1x_2+b_3x_0x_1x_2$. This implies that $a_1=b_1,\ a_3=b_3,\ a_{20}=b_{20},\ a_{21}=b_{21},\ a_1q^j=b_1q^i$, and $a_{20}q^{j+1}=b_{20}q^{i-1}$. Consequently, $a_1=a_{20}=0$, which means that $\lambda\in\langle x_1x_2, x_0x_1x_2\rangle$. Thus $I_i\cap J_j=\langle x_1x_2, x_0x_1x_2\rangle$.

(5) The exact sequence $0\to I_{j+1}\to A\to I_j\to 0$ of A-modules induces the following exact sequence of k-modules:

$$0\longrightarrow \mathrm{Hom}_A(I_j,I_i)\longrightarrow \mathrm{Hom}_A(A,I_i)\longrightarrow \mathrm{Hom}_A(I_{j+1},I_i)\longrightarrow \mathrm{Ext}^1_A(I_j,I_i)\longrightarrow 0.$$

By (4), we have

$$\dim_k \mathrm{Hom}_A(I_j,I_i)=\begin{cases}3 & \text{if } i\in\{j,j+2\},\\ 2 & \text{otherwise}.\end{cases}$$

Since $\dim_k(I_i)=4$, we have

$$\dim_k \mathrm{Ext}^1_A(I_j,I_i)=\begin{cases}1 & \text{if } j\le i\le j+3,\\ 0 & \text{otherwise}.\end{cases}$$

This proves (5).

(6) By definition, we know that $A(1,u_i)=A$, and $A(u_i,1)=R(u_{i+1})=J_i$.

(7) It follows from (4) and Lemma 6.3(2) that

$$\dim_k A(u_j,u_i)=\begin{cases}7 & \text{if } j=\{i-2,i\},\\ 6 & \text{otherwise}.\end{cases}$$

By definition, we know that $A(u_j,u_i)=\{a\in A\mid u_{j+1}au_i=0\}$. It is not hard to see that

$$\langle x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2\rangle\subseteq A(u_j,u_i).$$

Hence, if $j \notin \{i-2, i\}$, then $A(u_j, u_i) = \langle x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2\rangle$. If $j = i$, then $u_{j+1}u_j = 0$, and therefore $1 \in A(u_j, u_j)$. Thus $A(u_j, u_j) = \langle 1, x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2\rangle$. If $j = i-2$, then we can check that $u_{j+1}x_0u_{j+2} = 0$. Thus, $x_0 \in A(u_j, u_{j+2})$. This shows that $A(u_j, u_{j+2}) = \langle x_0, x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2\rangle$. □

For higher cohomological groups, we have the following estimation.

Lemma 6.6. *Let t be an integer and j a positive integer. Then*

(1) $\dim_k \mathrm{Ext}_A^j(I_0, I_t) = \begin{cases} 1 & \text{if } -1 \le t-j \le 2, \\ 0 & \text{otherwise}. \end{cases}$

(2) $\dim_k \mathrm{Ext}_A^j(I_t, I_0) = \begin{cases} 1 & \text{if } -2 \le t+j \le 1, \\ 0 & \text{otherwise}. \end{cases}$

(3) $\mathrm{Ext}_A^j(I_0, I_0) = 0$ *for* $j > 1$.

Proof. By Lemma 6.4, we have $\mathrm{Ext}_A^j(I_0, I_t) \simeq \mathrm{Ext}_A^1(I_0, \Omega_A^{-j+1}(I_t)) \simeq \mathrm{Ext}_A^1(I_0, I_{t-j+1})$. Now (1) follows from Lemma 6.5(5). Similarly, we can prove (2). Clearly, (3) follows from (1) and (2). □

Here and subsequently, δ_j stands for the canonical exact sequence $0 \to I_{j+1} \to A \to I_j \to 0$ in A-mod for each $j \in \mathbb{Z}$.

Lemma 6.7. *Let* $l \in \mathbb{Z}$ *and* $n \in \mathbb{N}$. *Then*

$$\{j \in \mathbb{Z} \mid \delta_j \text{ is an } \mathrm{add}(A \oplus I_l)\text{-split sequence in } A\text{-mod}\} = \{j \in \mathbb{Z} \mid j > l+2 \text{ or } j < l-3\}.$$

In particular, we have

$$\left\{j \in \mathbb{Z} \mid \delta_j \text{ is an } \mathrm{add}\left(A \oplus \bigoplus_{i=0}^{n} I_i\right)\text{-split sequence in } A\text{-mod}\right\} = \{j \in \mathbb{Z} \mid j > n+2 \text{ or } j < -3\}.$$

Proof. For any $j \in \mathbb{Z}$, we know that δ_j is an $\mathrm{add}(A \oplus I_l)$-split sequence in A-mod if and only if $\mathrm{Ext}_A^1(I_l, I_{j+1}) = \mathrm{Ext}_A^1(I_j, I_l) = 0$, which is equivalent to the condition that $j+1 \notin [l, l+3]$ and $j \notin [l-3, l]$ by Lemma 6.5(5). Thus we have (1). Clearly, (2) follows from (1) immediately. □

The following result can be directly deduced from the work in [6, Theorem 1.1] and [9, Theorem 4.1].

Lemma 6.8. *Let B be a k-algebra. Let Y and M be B-modules with M a generator for B-mod. If* $\mathrm{Ext}_B^1(M, \Omega_B(Y)) = \mathrm{Ext}_B^1(Y, M) = 0$, *then the endomorphism algebras* $\mathrm{End}_B(M \oplus Y)$ *and* $\mathrm{End}_B(M \oplus \Omega_B(Y))$ *are derived-equivalent. If, in addition,* $\mathrm{Ext}_B^2(M, \Omega_B(Y)) = \mathrm{Ext}_B^2(Y, M) = 0$, *then the* $\{0,1\}$*-Auslander–Yoneda algebras* $\mathrm{E}_B^{\{0,1\}}(M \oplus Y)$ *and* $\mathrm{E}_B^{\{0,1\}}(M \oplus \Omega_B(Y))$ *are derived-equivalent.*

Having made the previous preparations, now we can prove Theorem 6.1.

Proof of Theorem 6.1. Let $m \ge n+4$. Set $M := A \oplus I$ with $I = \bigoplus_{i=0}^n I_i$, and $V_m := M \oplus I_m$.

(1) By Lemma 6.5(5), we know that $\mathrm{Ext}_A^1(M, I_m) = \mathrm{Ext}_A^1(I_m, M) = 0$. Clearly, we have

$$\dim_k(\Lambda_m^{\{0\}}) = \dim_k \mathrm{End}_A(M) + \dim_k \mathrm{Hom}_A(M, I_m) + \dim_k \mathrm{Hom}_A(I_m, M) + \dim_k \mathrm{End}_A(I_m)$$

and

$$\dim_k(\Lambda_m^{\{0,1\}}) = \dim_k(\Lambda_m^{\{0\}}) + \dim_k \mathrm{Ext}_A^1(M, M) + \dim_k \mathrm{Ext}_A^1(I_m, I_m).$$

By Lemma 6.5, we get

$$\dim_k \mathrm{End}_A(I_m) = 3, \quad \dim_k \mathrm{Ext}_A^1(I_m, I_m) = 1, \quad \dim_k \mathrm{Hom}_A(M, I_m) = \dim_k \mathrm{Hom}_A(I_m, M) = 2n+6.$$

It follows that $\dim_k(\Lambda_m^{\Phi}) = \dim_k(\Lambda_{m+1}^{\Phi})$.

(2) We first show that $\mathrm{gl.dim}(\Lambda_m^{\{0\}}) = \infty$. By Lemma 6.5(5), we have $\mathrm{Ext}_A^1(V_m, I_j) = 0$ for any $j < 0$. Note that, for any $t < j < 0$, there is a long exact sequence

$$0 \longrightarrow I_j \longrightarrow A \longrightarrow A \longrightarrow \cdots \longrightarrow A \longrightarrow I_t \longrightarrow 0.$$

It follows that the induced sequence

$$0 \longrightarrow \mathrm{Hom}_A(V_m, I_j) \longrightarrow \mathrm{Hom}_A(V_m, A) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(V_m, A) \longrightarrow \mathrm{Hom}_A(V_m, I_t) \longrightarrow 0$$

is exact. Since $\mathrm{Hom}_A(V_m, A)$ is a projective-injective indecomposable $\Lambda_m^{\{0\}}$-module, we have $\mathrm{inj.dim}_{\Lambda_m^{\{0\}}} \mathrm{Hom}_A(V_m, I_j) = \infty$ for all $j < 0$, where inj.dim denotes the injective dimension of modules. Hence $\mathrm{gl.dim}(\Lambda_m^{\{0\}}) = \infty$. Note that there is a canonical surjective homomorphism $\pi : \Lambda_m^{\{0,1\}} \to \Lambda_m^{\{0\}}$ of algebras. Thus every $\Lambda_m^{\{0\}}$-module can be regarded as a $\Lambda_m^{\{0,1\}}$-module. In addition, $\mathrm{E}_A^{\{0,1\}}(V_m, A) = \mathrm{Hom}_A(V_m, A)$. It follows that $\mathrm{inj.dim}_{\Lambda_m^{\{0,1\}}}\mathrm{Hom}_A(V_m, I_j) = \infty$ for all $j < 0$. This yields $\mathrm{gl.dim}(\Lambda_m^{\{0,1\}}) = \infty$.

(3) Recall a classical result on dominant dimension: Let B an algebra and Y be a generator–cogenerator for B-mod. Suppose that s is a non-negative integer. Then $\mathrm{dom.dim}(\mathrm{End}_B(Y)) = s+2$ if and only if $\mathrm{Ext}_B^i(Y, Y) = 0$ for all i with $1 \le i \le s$, but $\mathrm{Ext}_B^{s+1}(Y, Y) \neq 0$. In our case, we take $Y := V_m$ and $s = 0$. By Lemma 6.5(5), we know that $\mathrm{Ext}_A^1(I_0, I_0) \neq 0$, which means that $\mathrm{Ext}_A^1(V_m, V_m) \neq 0$. Note that V_m is a generator–cogenerator for A-mod. Thus $\mathrm{dom.dim}(\Lambda_m^{\{0\}}) = 2$. By Lemma 5.1, we have $\mathrm{dom.dim}(\Lambda_m^{\{0,1\}}) = 0$.

(4) Consider the exact sequence

$$\delta_m : 0 \longrightarrow I_{m+1} \longrightarrow A \longrightarrow I_m \longrightarrow 0$$

in A-mod. Since $m \ge n+4$, it follows from Lemmas 6.5(5) and 6.4(4) that $\mathrm{Ext}_A^1(M, I_{m+1}) = \mathrm{Ext}_A^1(I_{m+1}, M) = \mathrm{Ext}_A^1(I_m, M) = \mathrm{Ext}_A^1(M, I_m) = 0$. Note that A is self-injective. By Lemma 6.8, we conclude that the algebras Λ_m^{Φ} and Λ_{m+1}^{Φ} are derived-equivalent for $\Phi = \{0\}$ or $\{0, 1\}$.

(5) It follows from Lemma 6.4 that $\Omega_A(I_j) = I_{j+1}$ for each $j \in \mathbb{Z}$ and that the A-modules I_j are pairwise non-isomorphic for all $j \in \mathbb{Z}$. Now, we define $W := I_0$ and $W_n := \oplus_{0 \le j \le n} I_j$. Then, by Corollary 5.3, the algebras Λ_l^{Φ} and Λ_m^{Φ} are not stably equivalent of Morita type if $l > m$. Thus the proof is completed. □

In the rest of this section, we consider the special case: $n = 0$ and $\Phi = 0$ in Theorem 6.1. For convenience, we set $\Lambda_m := \mathrm{End}_A(A \oplus I_0 \oplus I_m)$ for $m \in \mathbb{Z}$, and define $C := \langle 1, x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2\rangle$, $T := \langle x_1, x_2, x_0x_1, x_1x_2, x_2x_0, x_0x_1x_2\rangle$, and $S := T \oplus \langle x_0\rangle$. Note that they all are subspaces of A.

Proposition 6.9. *Let m be an integer. Then*

(1) *If $m \neq 2$, then Λ_m is isomorphic to the algebra*

$$\mathrm{M}_A(1, u_0, u_m) := \begin{pmatrix} A & A/I_1 & A/I_{m+1} \\ J_0 & C/I_1 & T/I_{m+1} \\ J_m & T/I_1 & C/I_{m+1} \end{pmatrix}.$$

(2) *Λ_2 is isomorphic to the algebra*

$$\mathrm{M}_A(1, u_0, u_2) := \begin{pmatrix} A & A/I_1 & A/I_3 \\ J_0 & C/I_1 & S/I_3 \\ J_2 & T/I_1 & C/I_3 \end{pmatrix}.$$

(3) *Suppose $m \ge 3$. Then, for any $l > m$, the algebras Λ_l and Λ_m are derived-equivalent, but not stably equivalent of Morita type.*

Proof. (1) and (2) follow easily from Lemmas 6.3(4) and 6.5, while (3) can be concluded from Lemmas 6.7, 6.8 and Corollary 5.3. □

For each positive integer $m \ge 3$, the algebra Λ_m is given by the following quiver Q with relations ρ_m:

$$Q: \begin{tikzcd} \underset{3}{\bullet} \arrow[r, shift left, "\gamma_m"] & \underset{1}{\bullet} \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] \arrow[loop above, "\alpha"] & \underset{2}{\bullet} \arrow[l, shift left, "\gamma_0"] \end{tikzcd}$$

$$\rho_m: \ \alpha^2 = \gamma_0\beta\gamma_0\alpha\beta = \gamma_m\alpha\delta\gamma_m\delta = 0;$$

$$\begin{pmatrix} \alpha\beta\gamma_0 \\ \alpha\delta\gamma_m \end{pmatrix} = \frac{1}{q - q^{m+1}} \begin{pmatrix} q^{m+2} - 1 & 1 - q^2 \\ q^{m+2} - q^m & q^m - q^2 \end{pmatrix} \begin{pmatrix} \beta\gamma_0\alpha \\ \delta\gamma_m\alpha \end{pmatrix};$$

$$\frac{\beta\gamma_0\beta}{1-q} = \frac{\delta\gamma_m\beta}{q^m - q}, \qquad \frac{\beta\gamma_0\delta}{1-q^{m+1}} = \frac{\delta\gamma_m\delta}{q^m - q^{m+1}}; \qquad \frac{\gamma_0\beta\gamma_0}{1-q} = \frac{\gamma_0\delta\gamma_m}{1-q^{m+1}}, \qquad \frac{\gamma_m\beta\gamma_0}{q^m - q} = \frac{\gamma_m\delta\gamma_m}{q^m - q^{m+1}}.$$

The Cartan matrix of Λ_m for $m \ge 3$ is

$$C = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 3 & 2 \\ 4 & 2 & 3 \end{pmatrix},$$

which is symmetric. Moreover, there is an anti-automorphism on Λ_m for ($m \ge 3$), which is given by

$$e_1 \mapsto e_1,\ e_2 \mapsto e_3,\ e_3 \mapsto e_2,\ \beta \mapsto \gamma_m,\ \gamma_m \mapsto q^m\beta,\ \alpha \mapsto \alpha,\ \delta \mapsto \gamma_0,\ \gamma_0 \mapsto \delta.$$

It follows from Proposition 6.9 that $\Lambda_t, t \geq 3$, are pairwise derived-equivalent, but not stably equivalent of Morita type.

Note that the Cartan matrix of Λ_2 is not symmetric. Thus Λ_2 is not derived-equivalent to Λ_m for $m \geq 3$ since the Cartan matrices of two derived-equivalent algebras are congruent over $\mathbb{Z}$, and therefore derived equivalences preserve the symmetry of Cartan matrices. We don't know whether Λ_1 and Λ_3 are derived-equivalent or not.

It would be interesting to show that the family of algebras in Theorem 6.1 or in Proposition 6.9 are pairwise not stably equivalent.

Acknowledgements

The research work of the corresponding author C.C. Xi is partially supported by the Fundamental Research Funds for the Central Universities (2009SD-17), while H.X. Chen is supported by the Doctor Funds of the Beijing Normal University. This revision of the first draft was partially done when C.C. Xi visited the Chern Institute of Mathematics, Tianjin, China, in July, 2010, he would like to thank Professor Chengming Bai at the Nankai University for his warm hospitality. The authors thank Yuming Liu for some helpful discussions on the subject.

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