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Explicit enumeration of 321, hexagon-avoiding permutations

Zvezdelina Stankova[a], Julian West[b]

[a] *Department of Mathematics and Computer Science, Mills College, Oakland, CA 94613, USA*
[b] *Department of Mathematics and Statistics, University of Victoria, Canada*

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Abstract

The *321*, *hexagon-avoiding* (*321-hex*) permutations were introduced and studied by Billey and Warrington (J. Alg. Comb. 13 (2001) 111–136). as a class of elements of S_n whose Kazhdan–Lusztig polynomials and the singular loci of whose Schubert varieties have certain fairly simple and explicit descriptions. This paper provides a 7-term linear recurrence relation leading to an explicit enumeration of the 321-hex permutations. A complete description of the corresponding generating tree is obtained as a by-product of enumeration techniques used in the paper, including Schensted's 321-subsequences decomposition, a 5-parameter generating function and the symmetries of the octagonal patterns avoided by the 321-hex permutations.

Keywords: 321, hexagon-avoiding permutations; Forbidden subsequences; Heaps; Linear recurrence; Schubert varieties; Kazhdan–Lusztig polynomials

1. Introduction

We start by describing the *321*, *hexagon-avoiding* permutations in two ways; first in the context of pattern-avoidance, which is the viewpoint we will be exploiting in our enumeration, and then in the context of reduced expressions, which explains the introduction of the term 321, hexagon-avoiding (for brevity, *321-hex*). Finally, we

E-mail addresses: stankova@mills.edu (Z. Stankova), westj@mala.bc.ca (J. West).
URLs: http://www.mills.edu/ACAD_INFO/mcs_stankova.html, http://www.mala.bc.ca/∼westj/wwwhome.htm

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explain the connection with Schubert varieties in [4] and how this motivates the work on the present paper.

1.1. Pattern-avoidance

From the first viewpoint, we consider permutations in S_n as bijections $w:[n]\to[n]$, and write them in one-line notation as the image of $w=[w_1,w_2,\ldots,w_n]$. In all our examples $n<10$, so we can suppress the commas without causing confusion.

Definition 1. Let $v\in S_k$ and $w\in S_n$ for some $k\leqslant n$. We say that w *contains* v if there is a sequence $1\leqslant i_1<\cdots<i_k\leqslant n$ such that the sequences $w'=[w_{i_1},w_{i_2},\ldots,w_{i_k}]$ and $[v_1,v_2,\ldots,v_k]$ obey the same pairwise relations, i.e. $w_{i_j}<w_{i_m}$ exactly when $v_j<v_m$. In such a case, we write $w'\sim v$. If w does not contain v then we say that w *avoids* v. We denote by $S_n(v)$ the set of all v-avoiding permutations of length n.

For example, the permutation $\omega=[52687431]$ avoids [2413] but does not avoid [3142] because of its subsequence [5283].

Definition 2. Two permutations σ and τ are *Wilf-equivalent* if they are equally restrictive, i.e. if $|S_n(\sigma)|=|S_n(\tau)|$ for all $n\in\mathbb{N}$.

For the past 15 years, the classification of permutations up to Wilf-equivalence has been widely researched, and as of now completed up to length 7 in [1,2,14–16,19,20]. The study of simultaneous avoidance of more than one permutation has also received certain attention in the literature. Recently, Billey and Warrington [4] have defined a class of permutations under five restrictions which is related to Schubert varieties. For a discussion and background on the topic, we direct the reader to Sections 1.3–1.4. In this paper, we enumerate the size of this class of permutations.

Definition 3. The *321-hex* permutations are those permutations which simultaneously avoid each of the following five patterns:

$$[321],\ P_1=[46718235],\ P_2=[46781235],\ P_3=[56718234],\ P_4=[56781234].$$

We denote by $\mathscr{P}$ the set of the four length-8 (*octagonal*) permutations P_1,P_2,P_3,P_4. In order to make sense of the above definition, consider the following equivalent, but perhaps more insightful, reformulation in terms of matrices.

Definition 4. Let $w\in S_n$. The *permutation matrix* M_w is the $n\times n$ matrix having a 1 in position (i,w_i) for $1\leqslant i\leqslant n$, and 0 elsewhere. (To keep the resemblance with the "shape" of w, we coordinatize M_w from the bottom left corner.) Given two permutation matrices M and N, we say that M *avoids* N if no submatrix of M is identical to N.

A permutation matrix M of length n is simply a transversal of an $n\times n$ matrix. Clearly, $w\in S_n$ contains $v\in S_k$ if and only if M_w contains M_v as a submatrix. Under this

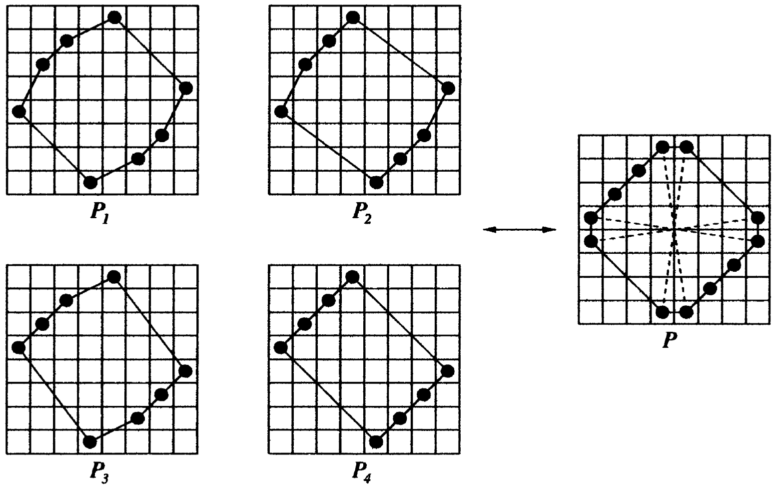

Fig. 1. Octagonal patterns.

reformulation, Fig. 1 presents the four octagonal patterns $\{P_i\}$ which must be avoided by the 321-hex permutations. The fifth pattern P does *not* come from a permutation because it is not a transversal. Yet, P is a union of the four previous permutation patterns, and it can be easily checked that a permutation matrix M_w avoids all P_i's if and only if no 8×8 permutation submatrix of M_w can be completely "covered" by P. Thus, by abuse of notation, we can say that the 321-hex permutations are defined as the permutations avoiding both [321] and P.

Our enumeration makes use of this last interpretation, exploiting the symmetries in the set $\mathscr{P}$ of octagonal patterns plus a convenient structural representation of all 321-avoiding permutations. The close relations between the four octagonal permutations are even more clearly revealed from a group-theoretical viewpoint when examining their reduced expressions. In Section 1.3, we briefly review the construction of the *heaps* of 321-avoiding permutations and their relation to the octagonal patterns P_i; for more details, see [4,8–10].

1.2. Enumeration of 321-hex permutations

Definition 5. Let $\mathscr{H}_n$ denote the set of all 321-hex permutations in S_n, and let $\alpha_n=|\mathscr{H}_n|$ be the number of such permutations.

Theorem 6. *The sequence α_n satisfies the following recursive relation*:

$$\alpha_{n+1} = 6\alpha_n - 11\alpha_{n-1} + 9\alpha_{n-2} - 4\alpha_{n-3} - 4\alpha_{n-4} + \alpha_{n-5} \quad \textit{for all } n \geqslant 6,$$

where $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 5, \alpha_4 = 14, \alpha_5 = 42, \alpha_6 = 132.$

Of the six roots of the corresponding characteristic polynomial, four are real: R_i for $i=1,2,3,4$, and two are complex conjugates: $R_5=\overline{R_6}$. This implies the same description of the six coefficients below: $c_i \in \mathbb{R}$ for $i=1,2,3,4$, and $c_5 = \overline{c_6} \in \mathbb{C}$.

Corollary 7. *The number of length-n 321, hexagon-avoiding permutations is*

$$c_1R_1^n + c_2R_2^n + c_3R_3^n + c_4R_4^n + c_5R_5^n + \overline{c_5R_5^n},$$

where the roots and coefficients are rounded off below to five digits after the decimal point:

$$R_1 \approx -0.49890, \qquad c_1 \approx -0.00328,$$
$$R_2 \approx 0.21989, \qquad c_2 \approx 0.62652,$$
$$R_3 \approx 1.95627, \qquad c_3 \approx 0.29217,$$
$$R_4 \approx 3.43526, \qquad c_4 \approx 0.07030,$$
$$R_5 \approx 0.44375 - 1.07681i, \qquad c_5 \approx 0.00714 + 0.01914i.$$

The first values of α_n are $1, 2, 5, 14, 42, 132, 429, 1426, 4806, 16329, 55740$. For further discussion of this and other results, we refer the reader to Sections 4–5. The proof of Theorem 6 follows several steps. First, we describe the nodes in the generating tree of $\mathscr{H}_n$ by using Schensted's algorithm for 321-avoiding permutations and by introducing five parameters for the generating function $h_n(x,k,l,m)$. We next observe that this function depends on fewer parameters, yielding therefore relatively few distinct values. We organize these values in five sequences, α_n, β_n, γ_n, δ_n and ε_n. Using the intrinsic symmetries of the set $\mathscr{P}$ of octagonal patterns, we deduce recursive relations expressing each sequence in terms of α_n. The latter turns out to be the number we are looking for, $|\mathscr{H}_n|$. Finally, putting together all information about the generating function and the five sequences results in the desired formula.

The remainder of the introduction can be skipped by the reader who is interested only in the pattern-avoidance interpretation.

1.3. Heaps of 321-permutations

The permutation group S_n can be regarded as generated by the set of the *adjacent transpositions* $\{s_i\}_{i=1}^{n-1}$, where $s_i = (i, i+1)$ in *cyclic* notation. In this presentation, the generators s_i and s_j commute if $|i-j| > 1$; else $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$. An *expression* is any product of generators. A *reduced* expression **a** for $w \in S_n$ is a shortest-possible expression yielding w. (It is well-known that the number of generators in **a** equals the number of inversions in w.) For example, the octagonal pattern $P_1 = [46718235]$ has a reduced expression

$$\mathbf{a} = s_3s_2s_1s_5s_4s_3s_2s_6s_5s_4s_3s_7s_6s_5.$$

Reduced expressions for the other avoided octagonal patterns are given by: $\mathbf{a} \cdot s_4$ for P_2, $s_4 \cdot \mathbf{a}$ for P_3, and $s_4 \cdot \mathbf{a} \cdot s_4$ for P_4. These expressions can be easily verified by

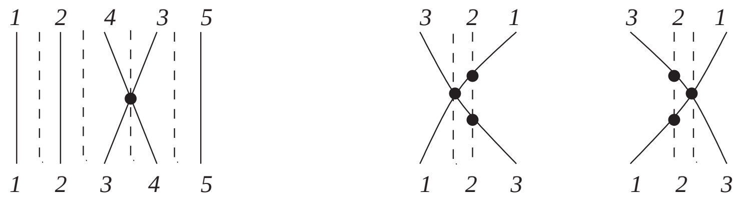

Fig. 2. String diagrams for $s_3 \in S_5$, $s_2 s_1 s_2 \in S_3$ and $s_1 s_2 s_1 \in S_3$.

considering the dashed lines in P in Fig. 1. The study of reduced expressions is a major subject of the representation theory of S_n.

After work of Viennot [18], the 321-avoiding permutations w can be represented by special ranked posets called *heaps*. The elements of Heap(w) are identified with the transpositions $\{s_{i_j}\}$ in a fixed reduced expression $\mathbf{a}$ for w. By Billey et al. [3], the 321-avoiding permutations are those in which no reduced expression contains a substring of the form $s_i s_{i\pm 1} s_i$; and by Tits [17], all reduced expressions for a 321-avoiding permutation w are equivalent up to moves $s_i s_j \to s_j s_i$ for $|i-j|>1$. Thus, the set of elements in Heap(w) and its poset structure are independent of the choice of the reduced expression $\mathbf{a}$.

We now describe the *rank function* of Heap(w), along with a Hasse diagram for the poset by embedding its elements in the integer lattice. One way to define and visualize this embedding is via *string diagrams*. To form a string diagram of $w \in S_n$, write the row of numbers $[w_1, w_2, \ldots, w_n]$ above the row $[1, 2, \ldots, n]$, thus mimicking the two-line notation for a permutation. Connect each number i on the bottom line to the corresponding number i on the top line, drawing a "string" which may change direction, but which at all times is running either due north, northwest or northeast. Strings may cross and recross, but do not run over top of one another, nor do they stray beyond the rectangular bounds formed by the two rows of numbers. For example, $s_i \in S_n$ can be realized as the crossing of two strings, as shown in Fig. 2a. In general, two or more adjacent transpositions can be applied simultaneously, provided they commute, i.e. s_i and s_j can occur on the same horizontal level of a string diagram unless $|i-j|=1$.

Thus, the crossings of a string diagram can be identified with the s_i's, labelled by the column (from 1 to $n-1$, in dashed lines on Fig. 2) in which they occur, and can be seen to form a poset in the obvious way. The linear extensions of this partial order are the expressions for w defined above. If the string diagram has the smallest possible number of crossings, then it is *minimal* and a linear extension is a reduced expression for w.

A *short braid* is a configuration obtained by applying the non-commuting transpositions s_i and s_{i+1} in the orders $s_i s_{i+1} s_i$ or $s_{i+1} s_i s_{i+1}$ (cf. Fig. 2c and b). The string diagram for a braid shows crossings at three of the four points of a small diamond, omitting either the eastern or the western point. As mentioned earlier, a permutation is 321-avoiding exactly when its minimal string diagrams avoid such configurations; such permutations are therefore also called *short-braid-avoiding* in the literature of Coxeter groups. It is now clear how to embed canonically the poset of string crossings in a

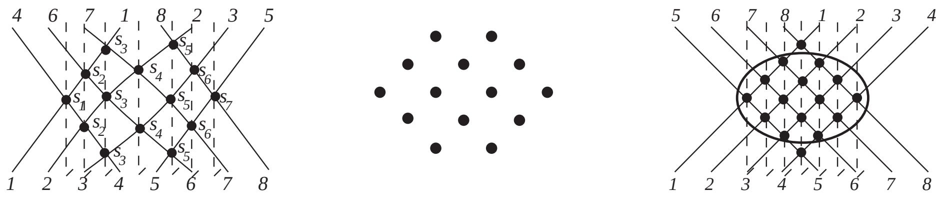

Fig. 3. String diagram for P_1, Heap(P_1), and string diagram for P_4.

minimal string diagram of w into the integer lattice. The resulting Hasse diagram is the heap of w, Heap(w). It is independent of the choice of **a** as long as w is 321-avoiding.

The heap for the octagonal pattern P_1 resembles a hexagon: Heap(P_1) has horizontal and vertical symmetries, with respectively 2,3,4,3,2 lattice points on its five ranks (see Fig. 3a and b). The string diagram of P_4 features one extra point on top and one extra point on bottom, corresponding to the crossings of the strings 1 and 8, and 4 and 5 (see Fig. 3c). The string diagrams for P_2 and P_3 have either the top or the bottom extra crossing. In all cases, Heap(P_i) contains the hexagonal Heap(P_1). (Note that any heap can be uniquely represented as a set of dots positioned on an integer lattice. Heap A is said to *contain* another heap B if heap B can be translated in the plane to overlap with some of A's dots. See the inclusion of Heap(P_1) into in Heap(P_4) in Fig. 3c, and see also Fig. 3 in [4].)

Furthermore, it can be shown that the $(321,\mathscr{P})$-avoiding permutations are exactly those 321-avoiding permutations whose minimal string diagrams avoid the hexagonal string diagram of P_1. This justifies the term *321, hexagon-avoiding* and yields the following alternative description (cf. [4]):

Definition 2′. A *321-hex permutation* is a permutation whose reduced expressions do not contain any substrings $s_j s_{j\pm 1} s_j$ for $j \geqslant 1$, or $s_{j+3}s_{j+2}s_{j+1}s_{j+5}s_{j+4}$ $s_{j+3}s_{j+2}s_{j+6}s_{j+5}s_{j+4}s_{j+3}s_{j+7}s_{j+6}s_{j+5}$ for $j \geqslant 0$.

1.4. Motivation for studying 321, hexagon-avoiding permutations

The 321-hex permutations arise in the study of Schubert varieties. The latter are defined below as subvarieties of flag manifolds.

As a set, the *complete flag manifold* $Fl(n)$ consists of complete flags

$$E. = (0 = E_0 \subset E_1 \subset \cdots \subset E_n = \mathbb{C}^n)$$

of subspaces E_i of $\mathbb{C}^n$, where $\dim E_i = i$. Fix a flag $F. = (F_0 \subset F_1 \subset \cdots \subset F_n) \in Fl(n)$. For a permutation $w \in S_n$, the *Schubert cell* $C_w \subset Fl(n)$ is

$$C_w = \{E. \in Fl(n)\colon \dim(E_j \cap F_k) = \#\{i \leqslant j\colon w(i) \leqslant k\} \text{ for } 1 \leqslant j,k \leqslant n\}.$$

Note that C_w is isomorphic to the affine space $\mathbb{C}^{l(w)}$, where $l(w)$ is the number of inversions in w, and that $Fl(n)$ is the disjoint union of Schubert cells C_w, one for each

$w \in S_n$ (cf. [7]). The Schubert cells can be viewed as the orbits of the action of the group of upper-triangular matrices $B \subset GL_n(\mathbb{C})$ on $\mathbb{C}^n$. The *Schubert variety* X_w is defined as the closure of the cell C_w, and as such, it is an irreducible closed subvariety of $Fl(n)$ of dimension $l(w)$. The action of B on the Schubert cells C_w induces an action of B on the Schubert variety X_w, making it into a union of B-orbits: $X_w = \cup_v C_v$, where the union is taken over all $v \in S_n$ with $l(v) \leqslant l(w)$.

According to Billey–Warrington, the Schubert varieties X_w corresponding to 321-hex permutations $w \in S_n$ are easily accessible: they have nice topological properties that enable us to calculate explicitly some of their topological invariants. For instance, we can determine their singular loci, derive combinatorial formulas for their Kazhdan–Lusztig polynomials, compute the Poincaré polynomial for their full intersection cohomology groups, and conclude that their Bott–Samelson resolution is small. (For definitions of these concepts and the corresponding results, see [4,5].)

Thus, the 321-hex permutations have properties that make certain algebraic and combinatorial computations easier to perform compared to arbitrary permutations. Another interesting aspect of the 321-hex permutations was examined earlier in the introduction: they are one of the few families that are known to be describable both in terms of pattern-avoidance and heap-avoidance. Yet, until now, there was no known recursive, exact or other closed form for the number of 321-hex permutations. For instance, it would be useful to know to how many permutations the results of Billey–Warrington in [4] apply; how the number of 321-hex permutations changes asymptotically, and how it compares to sizes of other well-known sets of permutations, such as $S_n(321)$, which is enumerated by the Catalan numbers.

Answers to all these questions are obtained in the present paper, where we find the 7-term linear recursive relation of Theorem 6 and derive from it the explicit exact formula of Corollary 7. With this formula at hand, answering any enumeration questions about the 321-hex permutations becomes a matter of simple observation and calculation.

2. The generating tree

2.1. What is a generating tree?

We turn now to the development of a recurrence for the 321-hex permutations. A standard tool in the enumeration of restricted permutations is the *generating tree* T introduced in [6]. Begin with an infinite tree whose nodes on level n are identified with the permutations in S_n. The node w is a child of $\hat{w} = [w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n]$ where the omitted value is $w_j = n$. Looking at this from the point of view of the parent, we can form all the children of $w \in S_n$ by *inserting* the element $n+1$ into each of the $n+1$ *sites* of w.

Then, for a given set Σ of forbidden subsequences, prune the tree by deleting all nodes containing any of the forbidden subsequences. What remains is still connected because if w does not contain any forbidden subsequence then clearly $\hat{w}$ does not either. For any node on level n of this pruned tree $T(\Sigma)$, we call a site in the corresponding permutation *active* if inserting $n+1$ at that site yields a node of the tree; conversely an *inactive* site is one where the insertion of $n+1$ creates one or more of the forbidden subsequences in Σ.

To give a complete description of a generating tree, we need to associate to each node an appropriate label, and then describe a *succession rule* for deriving the labels attached to the set of children of each node. For instance, we might characterize the original tree T generating *all* permutations as having a root labelled (2) and a succession rule $(n) \to (n+1)^n$: a permutation of length $n-1$ has label (n); each of its n children is of length n and hence has label $(n+1)$. In this instance, the label can be interpreted as revealing directly how many children each node has in the generating tree.

In the case of the 321-hex permutations, it will turn out that we need a label containing four integers. Although this is more complicated than the single integer of our motivating example, it is nevertheless a major progress to reduce the amount of information recorded at a node from a full permutation to a label of any bounded size. In particular, it is possible to apply the succession rule recursively to determine the entire downward structure of any node given only its label, regardless of whether that node is on a level corresponding to permutations on four symbols, or four thousand.

2.2. Schensted decomposition for $S_n(321)$ and active regions

Let w be any 321, hexagon-avoiding permutation on n symbols. We divide the elements $w_1, w_2, \ldots, w_n$ into two categories: the set of right-to-left minima (including w_n), and the rest. Adapting the terminology of Schensted [13], we refer to these as the *first basic subsequence* and *second basic subsequence* of w, and denote them by $\mathscr{B}_1(w)$ and $\mathscr{B}_2(w)$, respectively. For example, when $w = P_1 = [46718235]$, $\mathscr{B}_1(w) = [1,2,3,5]$ and $\mathscr{B}_2(w) = [4,6,7,8]$; and when $w = [13254768]$, $\mathscr{B}_1(w) = [1,2,4,6,8]$ and $\mathscr{B}_2(w) = [3,5,7]$. Note that both $\mathscr{B}_1(w)$ and $\mathscr{B}_2(w)$ decrease from right to left, the former—by construction, and the latter—because $w \in S_n(321)$. Now let $K = w_{i_1}$, $L = w_{i_2}$ and $M = w_{i_3}$ be the three largest (i.e. rightmost) elements in $\mathscr{B}_2(w)$ $(K < L < M)$, with each one set to zero if the corresponding element does not exist.

Since M is the largest (i.e. rightmost) element in $\mathscr{B}_2(w)$, it follows that every element to the right of M belongs to $\mathscr{B}_1(w)$. There are $x := n - i_3$ elements in this region, which we will call the *active region*. If $M = 0$ because $\mathscr{B}_2(w)$ is empty, then we consider the entire permutation to be the active region. Let k be the number of elements in the active region which are larger than K; as the elements in the active region decrease from right to left, these k elements are $w_{n-k+1}, \ldots, w_n$. Similarly, let l be the number of elements in the active region which are larger than L, and m be the number of such elements larger than M (cf. Fig. 4).

Now assign the label (x,k,l,m) to w. By construction, $x \geqslant k \geqslant l \geqslant m$. Let the X*-elements* of w be those that are counted in x, and similarly define the K-, L-, and M-elements (cf. Fig. 4). Further, let the $X \setminus K$*-elements* of w be the set of X-elements minus the set of K-elements, and similarly define the $K \setminus L$- and $L \setminus M$-elements. For example, the X-elements in P_1 are $2,3,5$, and $k = l = m = 0$.

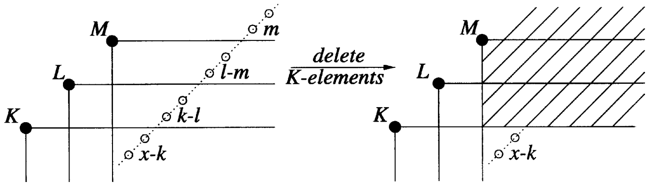

Fig. 4. Lemma 8.

2.3. The generating function for $T(321, \mathscr{P})$

Let $h_n(x,k,l,m)$ and $\mathscr{H}_n(x,k,l,m)$ be the number, resp. the set, of 321-hex permutations in S_n labelled (x,k,l,m). According to this, $h_n(0,0,0,0)$ is not defined since no permutation is labelled $(0,0,0,0)$. For convenience, denote by $h_n(0,0,0,0)$ and $\mathscr{H}_n(0,0,0,0)$ the number and the set of all 321-hex permutations in S_n which end in their largest element: $w_n = n$. It is worth noting that $\mathscr{H}_n(x,x,x,m)$ for $x > m$ corresponds to permutations w in which either L is smaller than the final "tail" of w, or L does not exist and $\mathscr{B}_2(w) = \{M\}$.

Lemma 8. *The operation of deleting all K-elements in a 321-hex permutation provides a bijection $d_K(k,l,m) : \mathscr{H}_n(x,k,l,m) \tilde{\to} \mathscr{H}_{n-k}(x-k,0,0,0)$. Hence, for all n,x,k,l,m we have $h_n(x,k,l,m) = h_{n-k}(x-k,0,0,0)$.*

Proof. Let $w \in S_n(321)$. The K-elements of w lie in $\mathscr{B}_1(w)$ and are part of the final increasing "tail" of w (since they are to the right of M). Thus, if a K-element w_i were part of an octagonal pattern P_j in w, then w_i would lie in $\mathscr{B}_1(P_j)$. But then P_j, and hence w, would contain at least three elements *larger than* and *to the left of* w_i: this contradicts the definition of a K-element, for which only L and M are *larger than* and *to the left of* it.

The above discussion shows that K-elements cannot participate in octagonal patterns in w, and hence they can be deleted without losing any relevant 321-hex information about w. (Of course, we have to rescale down appropriately M and L of w to arrive at a permutation of smaller size.) The resulting map

$$d_K(k,l,m) : \mathscr{H}_n(x,k,l,m) \to \mathscr{H}_{n-k}(x-k,0,0,0)$$

is bijective: to obtain $w \in \mathscr{H}_n(x,k,l,m)$ from its image $\tilde{w} = d_K(w) \in \mathscr{H}_{n-k}(x-k,0,0,0)$, insert the necessary number of M,L,K-elements into $\tilde{w}$ and increase appropriately L and M to fit their definitions in w. This procedure works because we can identify M as the largest element in $\tilde{w}$ (after the deletion of all K-elements in w), and then identify L as the second largest element of $\tilde{w}$, which will be necessarily to the left of M. Then insertion of the appropriate M,L,K-elements at the end of w requires rescaling M and L only (not K), and hence transforms the 321-Schensted decomposition of $\tilde{w}$ (from Section 2.2) into that of w: whether L was in $\mathscr{B}_1(\tilde{w})$ or in $\mathscr{B}_2(\tilde{w})$ does not prevent L from becoming an element of $\mathscr{B}_1(w)$ after applying $d_K^{-1}(k,l,m)$. □

2.4. The succession rule for 321-hex permutations

All children of w must avoid the subsequence [321]. This restriction by itself renders inactive all sites to the left of M, but none of the sites in the active region. Therefore when considering the children of w and their labels, we need only consider insertions taking place in the active region to the right of M. Thus, if we insert $n+1$ into a site in the active region with j elements to its right, it is easy to verify that the resulting permutation will have label:

$$\left\{\begin{array}{ll} (x+1,k+1,l+1,m+1) & \text{if } j=0 \\ (j,j,j,0) & \text{if } 0<j\leqslant m \\ (j,j,m,0) & \text{if } m<j\leqslant l \\ (j,l,m,0) & \text{if } l<j\leqslant x \end{array}\right\}. \tag{1}$$

Furthermore, the number of w's children can be computed as follows. Set

$$T := \min(k+2, \max(k+1, l+2)).$$

Then the node w has $S+1$ children, corresponding to the $S+1$ rightmost insertion sites, where

$$S = T \quad \text{if } T \leqslant x-2,$$
$$S = x \quad \text{if } T > x-2. \tag{2}$$

This allows a more compact and complete succession rule for the labels of all $S+1$ children of w:

$$(x,k,l,m) \mapsto \left\{\begin{array}{l} (x+1,k+1,l+1,m+1), \\ (i,\min(i,l),\min(i,m),0) \ \text{for } i=1,\ldots,S. \end{array}\right\}. \tag{3}$$

We shall not use directly any of the above formulas (1)–(3) in our calculations, so we leave their verification to the reader, who will find this easier after mastering the material in the rest of the paper. The importance of the above discussion is that it completely describes the structure of the generating tree $T(321,\mathscr{P})$, and hence explains in principle why the enumeration in this paper works. Why the resulting final formula for the 321-hex permutations is so simple—a *linear* recursive relation with *constant* coefficients—is a completely different matter and can be explained only by the structure of the forbidden set $\mathscr{P}$ of four octagonal patterns, as we shall see later.

2.5. Octagonal conditions

Consider a family $\mathscr{F}$ of permutations in $\mathscr{H}_n$ which are described by certain configuration conditions imposed on their basic subsequence decomposition $\mathscr{B}_1 \sqcup \mathscr{B}_2$. For example, $\mathscr{H}_n(0,0,0,0)$ is such a family defined by the condition $w_n = n \in \mathscr{B}_1(w)$.

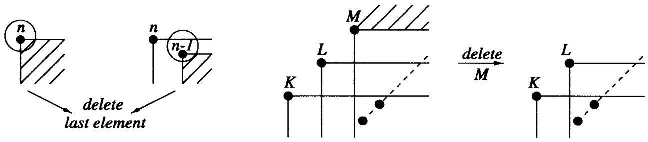

Fig. 5. (P2) and (P3).

Let a be an element of the permutations in $\mathscr{F}$ which is identified uniquely in each $w \in \mathscr{F}$ by the configuration description of $\mathscr{F}$.

Definition 9. We say that a is *pattern-free* if its deletion in each $w \in \mathscr{F}$ (and appropriate rescaling of w) results in a numerically equivalent family $\tilde{\mathscr{F}}$ of permutations in $\mathscr{H}_{n-1}$; i.e. $d_a : \mathscr{F} \to \tilde{\mathscr{F}}$ with $|\mathscr{F}| = |\tilde{\mathscr{F}}|$. For $w \in \mathscr{F}$, denote by $\tilde{w}$ the image $d_a(w) \in \tilde{\mathscr{F}}$.

For example, in $\mathscr{H}_n(0,0,0,0)$, the largest element $a = n$ is identified by being in the last position in each w, and clearly it is pattern-free (no octagonal pattern P_i has 8 in its last position):

$$d_n : \mathscr{H}_n(0,0,0,0) \xrightarrow{\sim} \mathscr{H}_{n-1}.$$

Establishing pattern-free elements and identifying the image set $\tilde{\mathscr{F}}$ is the basis of the enumeration of $\mathscr{H}_n$. The following technical lemma summarizes the pattern-free situations which will be used later in the proof of Theorem 6.

Lemma 10. *Let $\mathscr{F}$ be a family in $\mathscr{H}_n$, $a_1, \ldots, a_{x-k}$ be the $X \setminus K$-elements of $w \in \mathscr{F}$, and H be the fourth largest element in $\mathscr{B}_2(w)$, or $H = 0$ if $|\mathscr{B}_2(w)| < 4$. The set $\mathscr{P}$ imposes the following octagonal conditions on $\mathscr{F}$:*

(P1) *All K-elements are pattern-free.*
(P2) *If $w_n = n$ in $\mathscr{F}$, or $w_n = n-1$ in $\mathscr{F}$, then w_n is pattern-free.*
(P3) *If $M = n$ and $x \leqslant 2$ in $\mathscr{F}$, then M is pattern-free.*

Assume now that $x \geqslant 3$ and $k = l = m = 0$ in $\mathscr{F}$.

(P4) *If $H < a_2$ in $\mathscr{F}$, then $M, a_2, \ldots, a_x$ are pattern-free.*
(P5) *If $a_2 < H < a_3$ in $\mathscr{F}$, then $L, M, a_3, \ldots, a_x$ are pattern-free.*
(P6) *If $H > a_3$ (forcing $x = 3$) in $\mathscr{F}$, then M is pattern-free.*

Proof. (P1) follows from the proof of Lemma 8. (P2) and (P3) follow from the facts:

- Each octagonal pattern P_i has an empty 3×3 upper-right corner (see Fig. 6a).
- For (P2) (see Fig. 5a), reinserting n or $n-1$ in the last position in $\tilde{w} \in \mathscr{H}_{n-1}$ does not create 321-patterns. Similarly, for (P3) (see Fig. 5b), reinserting $M = n$ into $\tilde{w} \in \mathscr{H}_{n-1}$ as the first (rightmost) element in $\mathscr{B}_2(w)$ does not create 321-patterns (since the tail of w after M is increasing as part of $\mathscr{B}_1(\tilde{w})$).

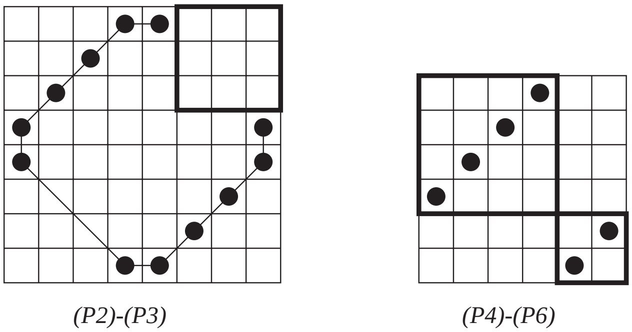

Fig. 6. Octagonal conditions.

For (P4), cf. Case 1 in Fig. 11. Start by deleting the rows and columns of the elements in the bottom row and in the rightmost column of each P_i; this leaves the permutation matrix $M[345612]$, which decomposes into a 4×4 block I_4 in the upper left corner and the fixed 2×2 block I_2 in lower right corner (corresponding to the original $a_6 = 2$ and $a_7 = 3$, see Fig. 6b).

- The fact that I_4 is to the left of and higher than I_2 proves that $a_2, \ldots, a_x$ are pattern-free: for them, only M, L, K can fit in such an I_4. In detail, if one of $a_2, \ldots, a_x$ participates in an octagonal pattern P_i, then without loss of generality we may assume that a_x is the rightmost element of P_i. This forces M, L, K to participate in P_i too as the only elements larger than a_x. Now M, being the largest element of P_i, forces at least three elements after it to participate in P_i; one is a_x, one could be a_1, hence the third one must be among $a_2, \ldots, a_{x-1}$. But none of these elements can fit into an I_2 as in Fig. 6 because only L, M, K are larger and to the left of them. Further, reinsertion of $a_2, \ldots, a_x$ into $\tilde{w}$ as the largest elements of $\mathscr{B}_1(w)$ cannot create 321-patterns. Thus, $a_2, a_3, \ldots, a_x$ are indeed pattern-free.
- After deleting $a_2, \ldots, a_x$, the largest element M lands in second to last position in $\tilde{w}$, and by (P3), it is pattern-free.

For (P5), cf. Case 2 in Fig. 11. The reasoning is similar to the case for (P4).

- First note that there can be no element a_0 of w between L and M, or else $[H, K, L, a_0, M, a_1, a_2, a_3] \sim P_1$.
- If one of $a_3, \ldots, a_x$ participates in a pattern P_i, this forces M, L, K to participate too, and in order not to run into contradiction with the $I_4 \times I_2$ argument above, we must assume that a_1 and a_2 are also in P_i, and only one among $a_3, \ldots, a_x$ is in P_i. Then the "1" in P_i will have to be between the two largest elements L and M, which was ruled earlier. Thus, $a_3, \ldots, a_x$ are pattern-free.
- Deletion of $a_3, \ldots, a_x$ leaves $M \in \mathscr{B}_2(\tilde{w})$ in third to last position, so by (P3), M is pattern-free. But L is the largest element in $\tilde{w} = d_M(w)$, with only a_1 and a_2 after it, so by (P3) again, L is pattern-free.

For (P6), cf. Case 3 in Fig. 12. Again note that there can be no element a_0 of w between L and M, or else $[H,K,L,a_0,M,a_1,a_2,a_3] \sim P_3$. If M participates in a pattern P_i, then it forces a_1,a_2,a_3 also to participate. Since the "1" in P_i cannot come from between L and M, we conclude that L does not participate in P_i. But then we can replace M by L and argue that there is a pattern P_i in $\tilde{w} = d_M(w)$, a contradiction. Hence M is pattern-free. □

As the reader may have noticed in Fig. 5b, after deletion of M in the original left-hand (LH) permutation, the "old" L and K-notation remained in the right-hand (RH) permutation. In order to keep the diagrams in Figs. 5–12 as simple to understand as possible, we have chosen to keep the M,L,K-notation for the original LH-permutation, and not to change these letters in the new RH-permutation.

3. Relations among α, β, γ, δ and ϵ

Lemma 8 shows that $h_n(x,k,l,m)$ does not depend on l or m, but rather on the differences $n-k$ and $x-k$. We shall see further that there are only five ranges for $x-k$, that completely determine the values of h: each of $x-k=0,1,2,3$ and $x-k \geqslant 4$ corresponds to exactly one of the five sequences listed in Table 1, and defined as follows:

$$\begin{cases} h_n(x,x-0,l,m) = h_{n-x+0}(0,0,0,0) =: \alpha_{n-x-1} \\ h_n(x,x-1,l,m) = h_{n-x+1}(1,0,0,0) =: \beta_{n-x+0} \\ h_n(x,x-2,l,m) = h_{n-x+2}(2,0,0,0) =: \gamma_{n-x+1} \\ h_n(x,x-3,l,m) = h_{n-x+3}(3,0,0,0) =: \delta_{n-x+2} \\ h_n(x,x-\bar{x},l,m) = h_{n-x+\bar{x}}(\bar{x},0,0,0) =: \varepsilon_{n-x+3} \end{cases} \Leftrightarrow \begin{cases} \alpha_n := h_{n+1}(0,0,0,0), \\ \beta_n := h_{n+1}(1,0,0,0), \\ \gamma_n := h_{n+1}(2,0,0,0), \\ \delta_n := h_{n+1}(3,0,0,0), \\ \varepsilon_n := h_{n+\bar{x}-3}(\bar{x},0,0,0), \end{cases}$$

where $\bar{x} \geqslant 4$. All of these definitions are justified by Lemma 8, except for the definition of ε_n, which depends only on n but not on $\bar{x}$ and whose explanation will be given later. In our search for relations between these sequences, it will be useful to define each sequence via an alternative description in the terminology of the families $\mathscr{F}$ discussed in Section 2.

Table 1
Sequences $\alpha_n, \beta_n, \gamma_n, \delta_n, \varepsilon_n$

n	1	2	3	4	5	6	7	8	9	10	11	12
α_n	1	2	5	14	42	132	429	1426	4806	16329	55740	190787
β_n	0	0	1	4	14	48	165	568	1954	6717	23082	79307
γ_n	0	0	0	1	5	20	75	271	957	3337	11559	39896
δ_n	0	0	0	0	1	6	25	93	333	1172	4083	14137
ε_n	0	0	0	0	0	1	5	19	68	240	839	2911

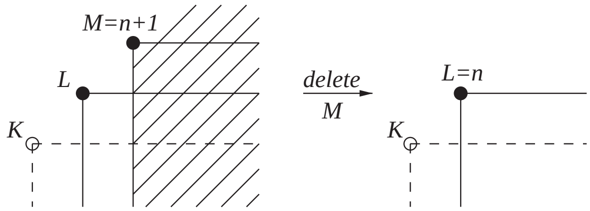

Fig. 7. Alternative description of α_n.

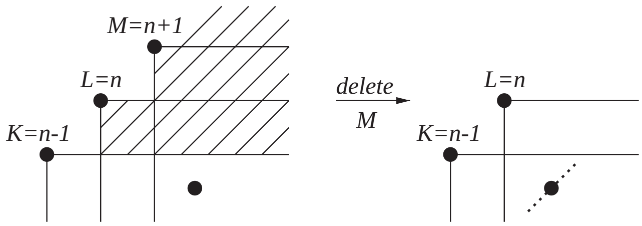

Fig. 8. Alternative description of β_n.

3.1. The sequence α

By definition, $\alpha_n = h_{n+1}(0,0,0,0)$. $M=n+1$ is the largest and the rightmost element of any $w \in \mathscr{H}_{n+1}(0,0,0,0)$, and hence it is pattern-free by (P2) (see Fig. 7). Deleting M results in a family of 321-hex permutations of length n without any further restrictions, i.e. we have a bijection

$$d_M : \mathscr{H}_{n+1}(0,0,0,0) \xrightarrow{\sim} \mathscr{H}_n = \bigcup_{x,k,l,m} \mathscr{H}_n(x,k,l,m).$$

This justifies the alternative description of α_n, stated in the introduction:

$$\alpha_n = \#\{321\text{-hex permutations in } S_n\} = |\mathscr{H}_n|. \tag{4}$$

3.2. The sequence β

By definition, $\beta_n = h_{n+1}(1,0,0,0)$. Here $M = n+1$ is second from right to left in $w \in \mathscr{H}_{n+1}(1,0,0,0)$, and by (P3) it is pattern-free (see Fig. 8). Thus, deleting M imposes only one extra condition: in the new 321-hex permutation $\tilde{w} \in \mathscr{H}_n$, the rightmost element in $\mathscr{B}_1(\tilde{w})$ is a $K \backslash L$-element or lower, i.e. the largest two elements (the original $L = n$ and $K = n-1$) belong to $\mathscr{B}_2(\tilde{w})$. This justifies the alternative description:

$$\beta_n = \#\{w \in \mathscr{H}_n \mid \{n, n-1\} \subset \mathscr{B}_2(w)\} \quad \text{for } n \geqslant 3. \tag{5}$$

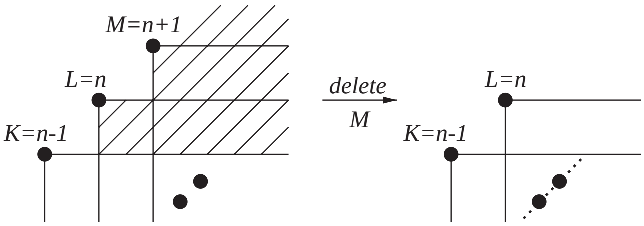

Fig. 9. Alternative description of γ_n.

In order to find a recursive description of β_n, note that each $w \in \mathscr{H}_n$ either ends in n or $n-1$, or both n and $n-1$ belong to $\mathscr{B}_2(w)$:

$$\alpha_n = \underbrace{\#\{w \in \mathscr{H}_n \mid n \text{ last}\}}_{\alpha_{n-1}} + \underbrace{\#\{w \in \mathscr{H}_n \mid n-1 \text{ last}\}}_{\alpha_{n-1}} + \beta_n. \tag{6}$$

By (P2), if n or $n-1$ is the last element in $w \in \mathscr{H}_n$, then it is pattern-free, so deleting it results in a bijection:

$$d_n : \{w \in \mathscr{H}_n \mid n \text{ last}\} \xrightarrow{\sim} \mathscr{H}_{n-1} \quad \text{and} \quad d_{n-1} : \{w \in \mathscr{H}_n \mid n-1 \text{ last}\} \xrightarrow{\sim} \mathscr{H}_{n-1}.$$

This justifies the use of $\alpha_{n-1} = |\mathscr{H}_{n-1}|$ in (6). Consequently, we have the recursive relation $\beta_n = \alpha_n - 2\alpha_{n-1}$ for $n \geqslant 3$.

3.3. *The sequence* γ

By definition, $\gamma_n = h_{n+1}(2,0,0,0)$. Here $M = n+1$ is third from right to left in $w \in \mathscr{H}_{n+1}(2,0,0,0)$, and by (P3) it is pattern-free (see Fig. 9). Deleting M imposes the following extra conditions: in the image $\tilde{w} \in \mathscr{H}_n$, the rightmost two elements in $\mathscr{B}_1(\tilde{w})$ are $K \setminus L$-elements or lower, i.e. the largest two elements (the original $L = n$ and $K = n-1$) belong to $\mathscr{B}_2(w)$ and there are at least 2 numbers *after* $L = n$. This justifies the alternative description for $n \geqslant 4$:

$$\gamma_n = \#\{w \in \mathscr{H}_n \mid \{w_s = n,\ w_t = n-1\} \subset \mathscr{B}_2(w),\ t < s \leqslant n-2\}. \tag{7}$$

In order to find a recursive description of γ_n, note that each w counted in β_n (i.e. $\{n, n-1\} \subset \mathscr{B}_2(w)$), falls into one of the following subcases: there is exactly one element *after* n (hence in $\mathscr{B}_1(w)$), or there are at least two elements *after* n (hence in $\mathscr{B}_1(w)$):

$$\beta_n = \underbrace{\#\{w \in \mathscr{H}_n \mid \{w_{n-1} = n, n-1\} \subset \mathscr{B}_2(w)\}}_{\alpha_{n-1}-\alpha_{n-2}} + \gamma_n. \tag{8}$$

The underbraced set in (8) is depicted in the LHS of Fig. 10. By (P3), deletion of $L{=}n$ results in the numerically equivalent set S in the RHS of Fig. 10. The permutations in S can be described as having their largest element $n-1 \in \mathscr{B}_2(\tilde{w})$. On the other hand,

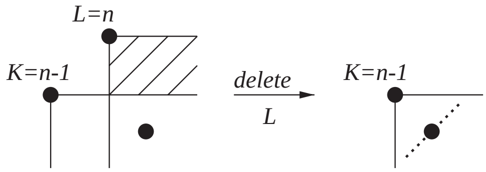

Fig. 10. Calculation of γ_n.

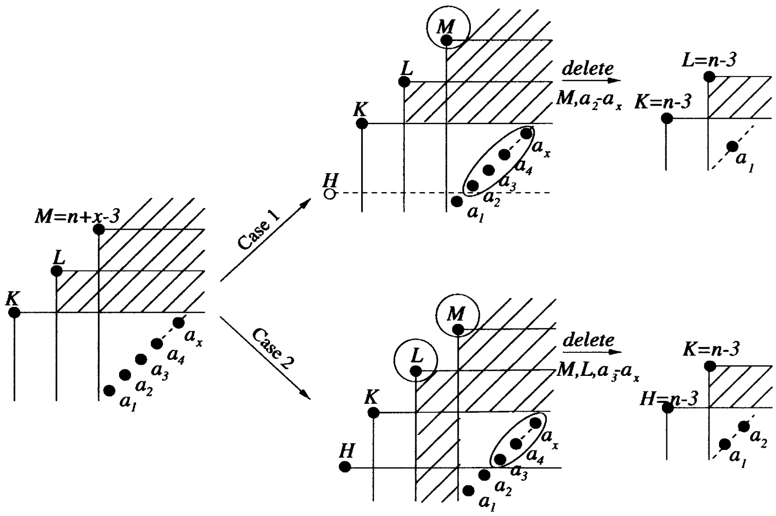

Fig. 11. Calculation of ε_n.

$\mathscr{H}_{n-1}$ breaks into two disjoint groups: group A consists of the permutations having the largest element $n-1$ in last position, and group B is the set S: $\mathscr{H}_{n-1} = A \sqcup S$. Finally, note that $d_{n-1} : A \xrightarrow{\sim} \mathscr{H}_{n-2}$, so that $|S| = \alpha_{n-1} - \alpha_{n-2}$. This justifies the underbrace notation in (8), and implies the formulas: $\gamma_n = \beta_n - (\alpha_{n-1} - \alpha_{n-2}) = \alpha_n - 3\alpha_{n-1} + \alpha_{n-2}$ for $n \geqslant 4$.

3.4. The sequence ε

By definition, $\varepsilon_n = h_{n+x-3}(x, 0, 0, 0)$ for $x \geqslant 4$. Let $a_1, \ldots, a_x$ be the $X \setminus K$-elements of $w \in \mathscr{H}_{n+x-3}(x, 0, 0, 0)$, where $x \geqslant 4$. There are two cases to consider (see Fig. 11).

Case 1: Except for M, L, K, all other elements of $\mathscr{B}_2(w)$ are smaller than a_2. Our drawing shows the next element $H \in \mathscr{B}_2(w)$ s.t. $H < a_2$ (H may not exist in w). By (P4), $a_2, a_3, a_4, \ldots, a_x$ and M are pattern-free. After deletion, the remaining configuration in $\mathscr{H}_{n-3}$ is identical to the alternative description of β_{n-3}.

Case 2: After M, L, K, the fourth element H of $\mathscr{B}_2(w)$ is between a_2 and a_3: $a_2 < H < a_3$. Recall from (P5) that there can be no element a_0 of w between the

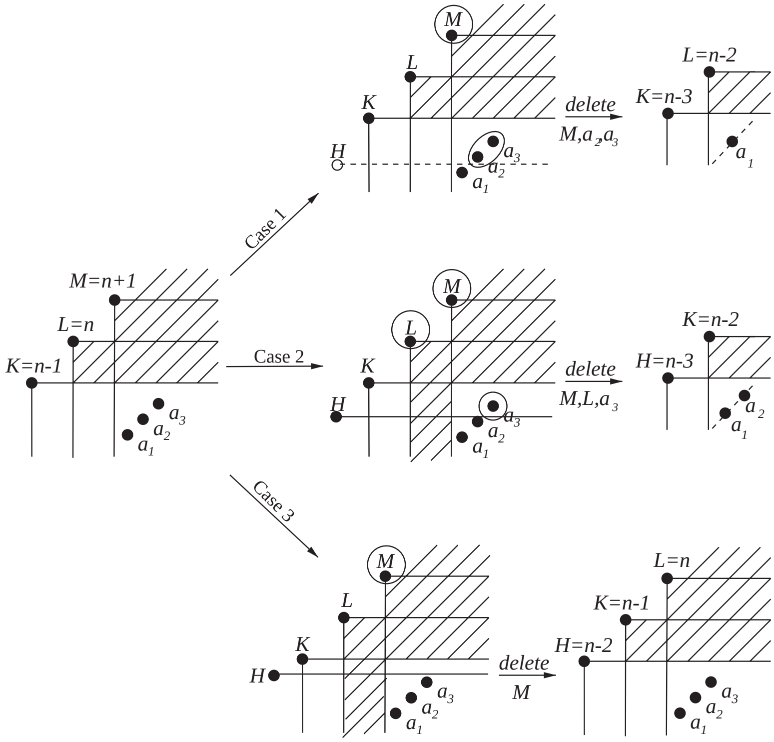

Fig. 12. Calculation of δ_n.

vertical lines of L and M. Further, $a_3, a_4, \ldots, a_x$, M and L, are pattern-free. After deletion, the remaining configuration in $\mathscr{H}_{n-3}$ is identical to the alternative description of γ_{n-3}.

Note that the case $H > a_3$ is not allowable, or else $[H, K, L, M, a_1, a_2, a_3, a_4] \sim P_2$ or $\sim P_4$ (when $a_3 < H < a_4$ or $H > a_4$, respectively).

Incidentally, the above discussion shows that $h_{n+x-3}(x, 0, 0, 0)$ does not depend on x as long as $x \geqslant 4$, and hence justifies the definition of ε_n. We conclude that $\varepsilon_n = \beta_{n-3} + \gamma_{n-3} = 2\alpha_{n-3} - 5\alpha_{n-4} + \alpha_{n-5}$ for $n \geqslant 6$.

3.5. The sequence δ

By definition, $\delta_n = h_{n+1}(3, 0, 0, 0)$. As above, denote by a_1, a_2, a_3 the $X \setminus K$-elements of $w \in \mathscr{H}_{n+1}(3, 0, 0, 0)$. There are three cases to consider (see Fig. 12).

Case 1: Except for M, L, K, all other elements of $\mathscr{B}_2(w)$ are smaller than a_2. Our drawing shows the next element $H \in \mathscr{B}_2(w)$ s.t. $H < a_2$ (H may not exist in w). By (P4), a_2, a_3 and M are pattern-free. The remaining configuration in $\mathscr{H}_{n-2}$ is identical to the alternative description of β_{n-2}.

Case 2: After M, L, K, the fourth element H of $\mathscr{B}_2(w)$ is between a_2 and a_3: $a_2 < H < a_3$. By (P5), there can be no element a_0 of w between the vertical lines of L and M; or else, $a_0 < a_1$, $a_0 \in \mathscr{B}_2(w)$, and $[H, K, L, a_0, M, a_1, a_2, a_3] \sim P_1$. Further, a_3, M and L are pattern-free. After deletion, the remaining configuration in $\mathscr{H}_{n-2}$ is identical to the alternative description of γ_{n-2}.

Case 3: In contrast to the discussion of ε, in the case of δ it is possible to have $H > a_3$ as long as there is no element a_0 of w between the vertical lines of L and M; otherwise, $[H, K, L, a_0, M, a_1, a_2, a_3] \sim P_3$. By (P6), the largest element $M = n + 1$ is pattern-free. After its deletion, the remaining configuration in $\mathscr{H}_n$ is identical to the original description of δ_{n-1}.

Therefore $\delta_n = \beta_{n-2} + \gamma_{n-2} + \delta_{n-1} = \varepsilon_{n+1} + \delta_{n-1}$ for $n \geqslant 5$ (cf. Table 1).

We summarize the results in this Section in

Lemma 11. *The sequences α, β, γ, δ and ε satisfy the recursive relations*:

$$\beta_n = \alpha_n - 2\alpha_{n-1} \qquad for\ n \geqslant 3,\ \beta_n = 0\ for\ n \leqslant 2,$$

$$\gamma_n = \alpha_n - 3\alpha_{n-1} + \alpha_{n-2} \qquad for\ n \geqslant 4,\ \gamma_n = 0\ for\ n \leqslant 3,$$

$$\delta_n = \varepsilon_{n+1} + \delta_{n-1} \qquad for\ n \geqslant 5,\ \delta_n = 0\ for\ n \leqslant 4,$$

$$\varepsilon_n = 2\alpha_{n-3} - 5\alpha_{n-4} + \alpha_{n-5} \qquad for\ n \geqslant 6,\ \varepsilon_n = 0\ for\ n \leqslant 5.$$

4. Enumeration of 321-hex permutations

We are now in a position to combine the recurrence formulas for α, β, γ, δ and ε into a single recurrence for α. We first use the interpretation of α_n as $|\mathscr{H}_n|$ and expand this in terms of the individual values of $h_n(x, k, l, m)$. For n fixed:

$$\alpha_n = \sum_{x,k,l,m} h_n(x, k, l, m),$$

where the sum is taken over $n \geqslant x \geqslant k \geqslant l \geqslant m \geqslant 0$, $x \neq 0$. Next, we break the sum into five separate sums depending on the value of $x - k$; each such sum corresponds to the definitions of α, β, γ, δ and ε, respectively. Note that the sum for α (where $x = k$) requires two extra special cases for $x = k = n - 1$ and $x = k = n$; the first case implies $l = m = n$ and the second case implies $l = n - 1$; in both cases $h_n(x, k, l, m) = 1$.

$$\Rightarrow \alpha_n = 1 + (n-1) + \sum_{\substack{n-2 \geqslant x > 0 \\ x \geqslant l \geqslant m \geqslant 0}} h_n(x, x, l, m) + \sum_{\substack{n \geqslant x \\ x-1 \geqslant l \geqslant m \geqslant 0}} h_n(x, x-1, l, m)$$

$$+ \sum_{\substack{n \geqslant x \\ x-2 \geqslant l \geqslant m \geqslant 0}} h_n(x, x-2, l, m) + \sum_{\substack{n \geqslant x \\ x-3 \geqslant l \geqslant m \geqslant 0}} h_n(x, x-3, l, m)$$

$$+ \sum_{\substack{n \geqslant x \\ x-4 \geqslant k \geqslant l \geqslant m \geqslant 0}} h_n(x, k, l, m).$$

In the next step, we replace the h_n's by the appropriate values of $\alpha, \beta, \gamma, \delta$ and ε. The case of α requires some care as we must observe the condition that not all the numerical parameters x, k, l, m can be simultaneously equal; indeed, this only happens in the special case $x=k=l=m=n$, which was broken out from the main sum earlier.

$$\Rightarrow \alpha_n = n + \sum_{\substack{n-2\geqslant x, x>m \\ x\geqslant l\geqslant m\geqslant 0}} \alpha_{n-x-1} + \sum_{\substack{n\geqslant x \\ x-1\geqslant l\geqslant m\geqslant 0}} \beta_{n-x} + \sum_{\substack{n\geqslant x \\ x-2\geqslant l\geqslant m\geqslant 0}} \gamma_{n-x+1}$$
$$+ \sum_{\substack{n\geqslant x \\ x-3\geqslant l\geqslant m\geqslant 0}} \delta_{n-x+2} + \sum_{\substack{n\geqslant x \\ x-4\geqslant k\geqslant l\geqslant m\geqslant 0}} \varepsilon_{n-x+3}.$$

We replace the indices k, l and m by binomial coefficients:

$$\alpha_n = n + \sum_{i=1}^{n-1}\left(\binom{n+1-i}{2} - 1\right)\alpha_i + \sum_{i=3}^{n-1}\binom{n+1-i}{2}\beta_i$$
$$+ \sum_{i=4}^{n-1}\binom{n+1-i}{2}\gamma_i + \sum_{i=5}^{n-1}\binom{n+1-i}{2}\delta_i + \sum_{i=6}^{n-1}\binom{n+2-i}{3}\varepsilon_i.$$

Finally, the recurrences from Section 3 yields a summation in terms of α alone:

$$\alpha_n = n + \sum_{i=1}^{n-1}\left(\binom{n+1-i}{2} - 1\right)\alpha_i + \sum_{i=3}^{n-1}\binom{n+1-i}{2}(\alpha_i - 2\alpha_{i-1})$$
$$+ \sum_{i=4}^{n-1}\binom{n+1-i}{2}(\alpha_i - 3\alpha_{i-1} + \alpha_{i-2})$$
$$+ \sum_{i=7}^{n-1}\binom{n+1-i}{2}\left(2\alpha_{i-2} - 3\alpha_{i-3} - 2\sum_{j=3}^{i-4}\alpha_j - 4\alpha_2 + \alpha_1\right)$$
$$+ \binom{n-5}{2}\delta_6 + \binom{n-4}{2}\delta_5 + \sum_{i=6}^{n-1}\binom{n+2-i}{3}(2\alpha_{i-3} - 5\alpha_{i-4} + \alpha_{i-5}). \tag{9}$$

For $i \geqslant 7$ we used the following manipulation of the δ-terms from Lemma 11:

$$\delta_i = \varepsilon_{i+1} + \delta_{i-1} = \sum_{j=6}^{i+1}\varepsilon_j + \delta_4 = \sum_{j=6}^{i+1}(2\alpha_{j-3} - 5\alpha_{j-4} + \alpha_{j-5})$$
$$= 2\sum_{j=3}^{i-2}\alpha_j - 5\sum_{j=2}^{i-3}\alpha_j + \sum_{j=1}^{i-4}\alpha_j = 2\alpha_{j-2} - 3\alpha_{j-3} - 2\sum_{j=3}^{i-4}\alpha_j - 4\alpha_2 + \alpha_1. \tag{10}$$

Note that the last expression in (10) does not work for δ_6 and δ_5 because the summation has incorrect bounds, and hence the special δ_6- and δ_5-terms in (9). In order to simplify expression (9) for α_n, we rewrite the (only) double-indexed sum in (9) as

$$\begin{aligned}-2\sum_{i=7}^{n-1}\binom{n+1-i}{2}\sum_{j=3}^{i-4}\alpha_j &= -2\sum_{j=3}^{n-5}\alpha_j\sum_{i=j+4}^{n-1}\binom{n+1-i}{2}\\ &= -2\sum_{j=3}^{n-5}\alpha_j\left(\binom{2}{2}+\binom{3}{2}+\cdots+\binom{n-j-3}{2}\right)\\ &= -2\sum_{j=3}^{n-5}\alpha_j\binom{n-j-2}{3} = -2\sum_{i=3}^{n-5}\binom{n-i-2}{3}\alpha_i.\end{aligned}$$

Further, the only term in (9) involving α_1 and α_2 equals

$$\begin{aligned}\sum_{i=7}^{n-1}\binom{n+1-i}{2}(-4\alpha_2+\alpha_1) &= -7\left(\binom{2}{2}+\binom{3}{2}+\cdots+\binom{n-6}{2}\right)\\ &= -7\binom{n-5}{3}.\end{aligned}$$

Reindexing the α-terms in (9) and grouping them together yields:

$$\begin{aligned}\alpha_n = {} & 2\alpha_{n-1}+3\alpha_{n-2}+5\alpha_{n-3}+7\alpha_{n-4}+4\alpha_{n-5}+\sum_{i=4}^{n-6}\alpha_i\left(3\binom{n+1-i}{2}\right.\\ & -5\binom{n-i}{2}+3\binom{n-1-i}{2}-3\binom{n-2-i}{2}-7\binom{n-2-i}{3}\\ & \left.+2\binom{n-1-i}{3}+\binom{n-3-i}{3}-1\right)-2\alpha_4\binom{n-5}{2}+\alpha_3\left(2\binom{n-2}{2}\right.\\ & \left.-5\binom{n-3}{2}+\binom{n-4}{2}-7\binom{n-5}{3}+2\binom{n-4}{3}+\binom{n-6}{3}-1\right)\\ & +\alpha_2\left(\binom{n-1}{2}-2\binom{n-2}{2}+\binom{n-3}{2}-5\binom{n-4}{3}\right.\\ & \left.+\binom{n-5}{3}-1\right)+\alpha_1\left(\binom{n}{2}+\binom{n-4}{3}-1\right)+n+\binom{n-4}{2}\end{aligned}$$

$$+6\binom{n-5}{2}-7\binom{n-5}{3}$$

$$=2\alpha_{n-1}+3\alpha_{n-2}+\underbrace{5\alpha_{n-3}+7\alpha_{n-4}+4\alpha_{n-5}} \tag{11}$$

$$+\sum_{i=4}^{n-6}\alpha_i\left(9-\frac{71}{6}(n-i)+\frac{11}{2}(n-i)^2-\frac{2}{3}(n-i)^3\right)+R(n), \tag{12}$$

where $R(n)=-\frac{17}{3}n^3+85n^2-\frac{1267}{3}n+704$. Here we used the initial values $\alpha_1=1,\alpha_2=2,\alpha_3=5,\alpha_4=14$. Set $Q(k)=-\frac{2}{3}k^3+\frac{11}{2}k^2-\frac{71}{6}k+9$ and note that $Q(3)=5, Q(4)=7$ and $Q(5)=4$ match the coefficients of the underbraced terms in (11), so that these terms can be pulled into the sum in (12). Thus, we obtain the following full-history, linear recurrence relation with cubic polynomial coefficients, valid for $n\geqslant 6$:

$$\alpha_n=2\alpha_{n-1}+3\alpha_{n-2}+\sum_{k=3}^{n-4}Q(k)\alpha_{n-k}+R(n).$$

Since $\deg Q(n)=\deg R(n)=3$, we need four successive history eliminations of the form $\alpha_n-\alpha_{n-1}$ (see [12] for a standard reference on techniques on linear recursive relations and characteristic polynomials). Combined with the three initial terms α_n,α_{n-1} and α_{n-2}, this produces the desired order-six constant-coefficient linear recurrence for all $n\geqslant 6$:

$$\alpha_n=6\alpha_{n-1}-11\alpha_{n-2}+9\alpha_{n-3}-4\alpha_{n-4}-4\alpha_{n-5}+\alpha_{n-6}. \tag{13}$$

Consequently, the number of the 321, hexagon-avoiding permutations of length n is given by the formula:

$$\alpha_n=c_1R_1^n+c_2R_2^n+c_3R_3^n+c_4R_4^n+c_5R_5^n+\overline{c_5R_5^n}, \tag{14}$$

where the roots and coefficients are listed in the Introduction. This completes the proof of Theorem 6.

The sixth-degree characteristic polynomial of our recurrence relation (13) is irreducible over $\mathbb{Q}$ and has Galois group S_6, as calculated by Maple. This means that there are no further algebraic relations among the roots R_i in (14), and thus we cannot hope for any better closed-form results.

However, our numerical approximations of the roots and coefficients can yield exact values for the number of 321-hex permutations for any fixed length n, exploiting the fact that the value being approximated is known to be an integer. Since the two roots of modulus less than 1 make such small contributions, they can be dropped and the following formula is exact for all n:

$$\alpha_n=[c_3R_3^n+c_4R_4^n+c_5R_5^n+\overline{c_5R_5^n}], \tag{15}$$

where the braces denote rounding to the nearest integer.

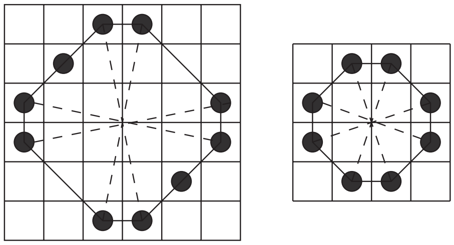

Fig. 13. 6 × 6 and 4 × 4 patterns.

5. Extensions and further discussion

The study of the octagonal patterns P_i in the present paper was motivated by their appearance in the representation theory of S_n via heap-avoidance. We refer to the enumeration of $S_n(321,\mathscr{P})$ as the "8×8 case". From a purely combinatorial viewpoint, it is natural to ask what happens in the analogous smaller 6×6 and 4×4 cases whose generalized patterns are depicted in Fig. 13.

To obtain these cases, for each $i=1,2,3,4$ shorten by one the lengths of both $\mathscr{B}_1(P_i)$ and $\mathscr{B}_2(P_i)$ by removing a chosen fixed point appearing in all P_i's. Thus, define $\mathscr{P}^6=\{[351624],[356124],[451623],[456123]\}$ and $\mathscr{P}^4=\{[2143],[3142],[2413],[3412]\}$ to be the families of avoided octagonal patterns in the 6×6 and the 4×4 cases, respectively. Both of these cases lead again to linear recursive relations with constant coefficients. The proofs below follow closely the method described in the 8×8 case, so we leave the details for verification to the reader.

5.1. The 6×6-case

Theorem 12. *Let* $\mathscr{H}_n^6=S_n(321,\mathscr{P}^6)$, *and* $\alpha_n=|\mathscr{H}_n^6|$. *Then* $\{\alpha_n\}$ *satisfies a* 6-*term linear recursive relation with constant coefficients*:

$$\alpha_{n+1}=4\alpha_n-4\alpha_{n-1}+3\alpha_{n-2}+\alpha_{n-3}-\alpha_{n-4} \quad \textit{for } n\geqslant 5, \tag{16}$$

where $\alpha_1=1,\alpha_2=2,\alpha_3=5,\alpha_4=14,\alpha_5=42$. *Consequently*, *for all* $n\geqslant 1$,

$$\alpha_n=c_1R_1^n+c_2R_2^n+c_3R_3^n+c_4R_4^n+\overline{c_4R_4^n}, \tag{17}$$

where the roots and coefficients are rounded off below to five digits after the decimal point:

$$R_1\approx -0.49569, \qquad c_1\approx -1.27509,$$

$$R_2\approx 0.51154, \qquad c_2\approx 1.03824,$$

$$R_3\approx 3.03090, \qquad c_3\approx 0.16548,$$

$$R_4\approx 0.47662-1.03635i, \qquad c_4\approx -0.15971-0.11454i.$$

The degree-5 characteristic polynomial of the recurrence is irreducible over Q and has the largest possible Galois group, S_5, as calculated by Maple. We can again drop the small roots R_1 and R_2, rounding off the remainder to the nearest integer:

$$\alpha_n = [c_3 R_3^n + c_4 R_4^n + \overline{c_4 R_4^n}] \quad \text{for all } n \geqslant 1. \tag{18}$$

The first values of α_n are: 1, 2, 5, 14, 42, 128, 389, 1179, 3572, 10825, 32810.

Proof of Theorem 12. We modify the discussion in the proof for the 8×8 case. Define the generating function $h_n(x,l,m)$ with one fewer parameter, thus taking into account only the elements M and L of $w \in \mathscr{H}_n^6$. As in Lemma 8, the L-elements are pattern-free, and therefore $h_n(x,l,m) = h_{n-l}(x-l,0,0)$.

Define the sequences $\alpha_n = h_{n+1}(0,0,0)$, $\beta_n = h_{n+1}(1,0,0)$, $\gamma_n = h_{n+1}(2,0,0)$ and $\delta_n = h_{n+x-2}(x,0,0)$ for $x \geqslant 3$. Note that $\alpha_n = h_{n+1}(0,0,0) = |\mathscr{H}_n^6|$ since α_n is the number of $w \in \mathscr{H}_{n+1}^6$ in which the largest element $n+1$ is at the end and hence it is pattern-free. Further, the relations among the sequences are:

$$\begin{cases} \beta_n = \alpha_n - \alpha_{n-1} & \text{for } n \geqslant 2, \\ \gamma_n = \alpha_{n-2} + \beta_{n-2} + \gamma_{n-1} & \text{for } n \geqslant 3, \\ \delta_n = \alpha_{n-3} + \beta_{n-3} & \text{for } n \geqslant 4. \end{cases}$$

Now we are ready to express everything in terms of $\alpha_n = \sum_{x,l,m} h_n(x,l,m)$:

$$\alpha_n = 1 + \sum h_n(x,x,m) + \sum h_n(x,x-1,m) + \sum h_n(x,x-2,m) + \sum h_n(x,l,m).$$

Here the "1" counts the identity permutation: no M-element in w; in the first sum L is smaller than all X-elements (or L does not exist); in the second sum L is larger than exactly one X-element; in the third sum L is larger than exactly two X-elements; and in the forth sum L is larger than three or more X-elements and hence $x-3 \geqslant l$. Therefore,

$$\alpha_n = 1 + \sum_{l=1}^{n-1} l\alpha_{n-l-1} + \sum_{l=0}^{n-3} (l+1)\beta_{n-l-1} + \sum_{l=0}^{n-4} (l+1)\gamma_{n-l-1} + \sum_{l=0}^{n-5} \binom{l+2}{m} \delta_{n-l-1}$$

$$\Rightarrow \alpha_{n+1} - \alpha_n = \sum_{l=1}^{n} \alpha_{n-l} + \sum_{l=0}^{n-2} \beta_{n-l} + \sum_{l=0}^{n-3} \gamma_{n-l} + \sum_{l=0}^{n-4} (l+1)\delta_{n-l}$$

$$\Rightarrow \alpha_{n+2} - 2\alpha_{n+1} + \alpha_n = \alpha_n + \beta_{n+1} + \gamma_{n+1} + \sum_{l=0}^{n-3} \delta_{n-l+1}$$

$$\Rightarrow \alpha_{n+3} - 3\alpha_{n+2} + 2\alpha_{n+1} = (\beta_{n+2} - \beta_{n+1}) + (\gamma_{n+2} - \gamma_{n+1}) + \delta_{n+2}.$$

Conveniently, each summand on the RHS can be expressed in terms of α, including $\gamma_{n+2} - \gamma_{n+1} = \alpha_n + \beta_n = 2\alpha_n - \alpha_{n-1}$:

$$\Rightarrow \alpha_{n+3} = 4\alpha_{n+2} - 4\alpha_{n+1} + 3\alpha_n + \alpha_{n-1} - \alpha_{n-2}. \qquad \square$$

5.2. The 4×4 case

Theorem 13. *Let $\mathscr{H}_n^4 = S_n(321, \mathscr{P}^4)$, and $\alpha_n = |\mathscr{H}_n^4|$. Then*

$$\alpha_n = (n-1)^2 + 1 \quad \text{for all } n \geqslant 1. \tag{19}$$

In particular, $\{\alpha_n\}$ satisfies the 4-term linear recursive relation

$$\alpha_{n+1} = 3\alpha_n - 3\alpha_{n-1} + \alpha_{n-2} \quad \text{for } n \geqslant 3,\ \alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 5. \tag{20}$$

Note that in contrast to the 6×6 and 8×8 cases, the characteristic polynomial here factors (completely) over $\mathbb{Q}$: $(x-1)^3$.

Proof. Similar to the 6×6 case. $\square$

5.3. Further discussion

Until now, there were relatively few known examples of sets of permutations whose avoidance led to linear, polynomial, or exponential formulas (see [14,20]). After the successful enumeration of the 4×4, 6×6 and the 321-hex 8×8 cases, it is tempting to generalize the recursive sequence method of this paper to the corresponding larger sets of $2k \times 2k$ patterns. At this point, it is not surprising to conjecture that all these families yield linear recursive relations with constant coefficients. In fact, when the result of the present paper was publicized, Herbert Wilf requested that many more examples of such "linear" families be found. These and other related questions are answered positively in a recent paper by Mansour and Stankova [11].

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