A Grobman–Hartman theorem for general nonuniform exponential dichotomies

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Abstract

For a nonautonomous dynamics with discrete time given by a sequence of linear operators $A_m$, we establish a version of the Grobman–Hartman theorem in Banach spaces for a very general nonuniformly hyperbolic dynamics. More precisely, we consider a sequence of linear operators whose products exhibit stable and unstable behaviors with respect to arbitrary growth rates $e^{\rho(n)}$, determined by a sequence $\rho(n)$. For all sufficiently small Lipschitz perturbations $A_m + f_m$ we construct topological conjugacies between the dynamics defined by this sequence and the dynamics defined by the operators $A_m$. We also show that all conjugacies are Hölder continuous. We note that the usual exponential behavior is included as a very special case when $\rho(n) = n$, but many other asymptotic behaviors are included such as the polynomial asymptotic behavior when $\rho(n) = \log n$.

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1. Introduction

1.1. Motivation and uniform hyperbolicity

A fundamental problem in the study of the local behavior of a map or a flow is whether the linearization along a given solution approximates well the solution itself. This problem goes...
back to the pioneering work of Poincaré. It can be interpreted as looking for an analytic change of variables, called a conjugacy, that takes the system to a linear one. Moreover, as a means to distinguish various dynamics further than in the topological category, we would like the change of variables to be as regular as possible. For example, we would like to know whether it is possible to distinguish between different types of nodes. In the case of hyperbolic fixed points, the Grobman–Hartman theorem gives a complete answer in the topological category, by constructing a topological conjugacy between the original dynamics and its linearization. The original references are Grobman [11,12] and Hartman [15,16]. Using the ideas in Moser’s proof in [21] of the structural stability of Anosov diffeomorphisms, the Grobman–Hartman theorem was extended to Banach spaces independently by Palis [23] and Pugh [26]. On the other hand, the work of Sternberg [29,30] showed that there are algebraic obstructions, expressed in terms of resonances between the eigenvalues of the linearization, that prevent the existence of conjugacies with a prescribed high regularity (see also [8,9,20,27] for further related work). In spite of this unavoidable drawback, the linearization problem still stands today as a fundamental step in the study of the local behavior of a dynamical system. Having this in mind, it is crucial to understand what is the most general class of systems with some hyperbolic behavior for which the problem can be solved. Nevertheless, there exist large classes of linear dynamics with uniform hyperbolic behavior, and the corresponding theory and its applications are widely developed. We refer to the books [10,14,17,28] for details and references related to uniform hyperbolic behavior.

1.2. Nonuniform hyperbolicity and its ubiquity

On the other hand, the classical notion of uniform hyperbolicity is very stringent for the dynamics and it is important to look for more general types of hyperbolic behavior that can be much more typical. This is precisely what happens with the notion of nonuniform hyperbolicity. Roughly speaking, a nonuniform exponential behavior includes the usual exponential contraction and expansion, but it also allows a “spoiling” of the contraction and expansion along each trajectory as the initial time increases. In other words, instead of having uniform asymptotic stability along the stable direction into the future and along the unstable direction into the past, in general we have a nonuniform asymptotic stability. This causes that at a given time, the “size” of the neighborhood in the stable and unstable directions, where respectively the exponential stability or instability of the trajectory is guaranteed, may decay with exponential rate. We refer to [1,2] for detailed expositions of large parts of the theory of nonuniform hyperbolicity, which goes back to the landmark works of Oseledets [22] and particularly Pesin [24]. As we already mentioned, the notion of nonuniform hyperbolicity (here reformulated in terms of nonuniform exponential dichotomies) is much more typical than uniform hyperbolicity. For example, almost all trajectories with nonzero Lyapunov exponents of a dynamical system preserving a finite invariant measure (such as for example any compact level set of any Hamiltonian system) are nonuniformly hyperbolic. We refer to [2,7] for a precise formulation of the results, and for related detailed discussions. Among the most important properties due to nonuniform hyperbolicity is the existence of stable and unstable manifolds, and their absolute continuity property established by Pesin in [24]. The theory also describes the ergodic properties of dynamical systems with a finite invariant measure absolutely continuous with respect to the volume [25], and it expresses the Kolmogorov–Sinai entropy in terms of the Lyapunov exponents by the Pesin entropy formula [25] (see also [19]). In another direction, combining the nonuniform hyperbolicity with the nontrivial recurrence given by the existence of a finite invariant measure, the fundamental work of Katok [18] revealed a very rich and complicated orbit structure, including an
exponential growth rate for the number of periodic points measured by the topological entropy, and an approximation of the entropy of an invariant measure by uniformly hyperbolic horseshoes. We point out that the smallness of the nonuniformity is a rather common phenomenon from the point of view of ergodic theory: namely, almost all linear variational equations obtained from a measure-preserving flow have a nonuniform exponential dichotomy with arbitrarily small nonuniformity. Nevertheless, even if arbitrarily small, in general the nonuniformity cannot be discarded a priori. In particular, it follows from work of Barreira and Schmeling in [3] that for some classes of measure-preserving transformations, the nonuniformity cannot be made arbitrarily small in a set of full topological entropy and full Hausdorff dimension. In other words, also from the topological and the dimensional points of view it is crucial to study nonuniform hyperbolicity. We also would like to mention that if an autonomous linear dynamics has a nonuniform exponential dichotomy, then in fact the dichotomy must be uniform. This is why in the context of nonuniform exponential behavior we are only interested in perturbations of a nonautonomous linear dynamics.

1.3. Brief description of our results

Our main objective is to generalize the Grobman–Hartman theorem to perturbations of a nonautonomous dynamics with discrete time

\[ z_{m+1} = A_m z_m, \quad m \in \mathbb{Z}, \]  

(1)

given by a sequence of linear operators \( A_m \) that may exhibit stable and unstable behaviors with respect to arbitrary asymptotic rates \( e^{\rho(n)} \), determined by a sequence \( \rho(n) \) (see the following paragraph for a detailed motivation for considering this general situation). Namely, for a sequence of sufficiently small Lipschitz perturbations \( A_m + f_m \) we construct topological conjugacies between the dynamics

\[ z_{m+1} = A_m z_m + f_m(z_m), \quad m \in \mathbb{Z}, \]

and the linear dynamics in (1). We emphasize that in strong contrast with the usual (exponential) stable and unstable behaviors, we allow asymptotic rates of the form \( e^{\rho(n)} \) determined by an arbitrary sequence \( \rho(n) \). The usual exponential behavior corresponds to take \( \rho(n) = n \). We point out that it is easy to construct large classes of linear dynamics as in (1) for which all or some Lyapunov exponents are infinite (either \( +\infty \) or \( -\infty \)). In this situation, one is not able to apply the existing stability theory. Nevertheless, we may still be able to distinguish between different growth rates in different directions, specified by an appropriate sequence \( \rho(n) \). It is quite reasonable, and we would even say compelling, to take advantage of such a decomposition, which allows us to develop a corresponding stability theory virtually for all linear dynamics and not only for some particular classes. From a more practical perspective, we show in [6] that for a large class of growth rates \( \rho(n) \) there exist many linear dynamics exhibiting this asymptotic behavior. We refer to that paper for more details on the notion of nonuniform exponential dichotomy in this general context, and for a discussion of its ubiquity. Moreover, we show that all conjugacies are Hölder continuous, with Hölder exponent determined by the constants \( c \) in the asymptotic rates \( e^{\rho(n)} \). We note that in the classical case of uniform exponential dichotomies, the Hölder regularity of the conjugacies seems to have been known by some experts for quite some time, although apparently, to the best of our knowledge, no correct published proof appeared in the
literature before our proof in [5] (see also [4] for the nonuniform setting). Indeed, it was claimed by some authors that the conjugacy is always Hölder continuous. Others have announced proofs of this property, but either they were never published or were not correct. We refer to [5] for detailed references to these works. It is however pointed out in [13] that the statement about the Hölder regularity is contained in a 1994 preprint of Belitski˘ı, although it remains unpublished. We follow the strategy of proof in [4], which involves three steps:

1. to show that there exist unique continuous functions \( \hat{u}_m \) satisfying
   \[
   A_m \circ \hat{u}_m = \hat{u}_{m+1} \circ (A_m + f_m)
   \]
such that the sequence \( \hat{u}_m - \Id \) has a certain boundedness property (see Theorem 1);
2. to show that there exist unique continuous functions \( \hat{v}_m \) satisfying
   \[
   \hat{v}_{m+1} \circ A_m = (A_m + f_m) \circ \hat{v}_m
   \]
such that the sequence \( \hat{v}_m - \Id \) has a certain boundedness property (see Theorem 2);
3. to verify that for each \( m \in \mathbb{Z} \) these functions satisfy
   \[
   \hat{u}_m \circ \hat{v}_m = \hat{v}_m \circ \hat{u}_m = \Id,
   \]
and thus that they are the desired conjugacies (see Corollary 1).

2. Preliminaries

Let \( X \) be a Banach space. We consider invertible linear operators \( A_m, m \in \mathbb{Z} \) such that with respect to some decomposition \( X = E \times F \) (independent of \( m \)) we can write
   \[
   A_m = \begin{pmatrix} B_m & 0 \\ 0 & C_m \end{pmatrix}, \quad m \in \mathbb{Z}.
   \]
Each sequence \( (z_m)_{m \in \mathbb{Z}} \) in \( X \) satisfying \( z_{m+1} = A_m z_m \) for every \( m \in \mathbb{Z} \) can be written in the form
   \[
   z_m = (B(m,n)x_n, C(m,n)y_n), \quad m, n \in \mathbb{Z},
   \]
where \( z_n = (x_n, y_n) \in E \times F, \) and
   \[
   B(m,n) = \begin{cases} B_{m-1} \cdots B_n, & m > n, \\ \Id, & m = n, \\ B_{m-1}^{-1} \cdots B_{n-1}^{-1}, & m < n, \end{cases}
   \]
   \[
   C(m,n) = \begin{cases} C_m \cdots C_n, & m > n, \\ \Id, & m = n, \\ C_m^{-1} \cdots C_{n-1}^{-1}, & m < n. \end{cases}
   \]
Now consider an increasing function \( \rho : \mathbb{Z} \to \mathbb{Z} \) with \( \rho(-m) = -\rho(m) \) for each \( m \in \mathbb{Z} \). We say that the sequence \( (A_m)_{m \in \mathbb{Z}} \) admits a \( \rho \)-nonuniform exponential dichotomy if there exist constants
   \[
a < 0 < b, \quad \varepsilon \geq 0 \quad \text{and} \quad D \geq 1
   \]
such that for every $m, n \in \mathbb{Z}$ with $m \geq n$ we have
\[
\|B(m,n)\| \leq De^{a\mu(m,n)+\varepsilon|\rho(n)|}, \quad \|C(m,n)^{-1}\| \leq De^{-b\mu(m,n)+\varepsilon|\rho(m)|},
\] (2)
where $\mu(m,n) = \rho(m) - \rho(n)$. Now we introduce new norms, with respect to which the nonuniform behavior in (2) becomes uniform. Choose $\sigma > 0$ such that $\sigma < \min\{-a, b\}$. For each $m \in \mathbb{Z}$ we set
\[
\|x\|_m' = \sum_{k \geq m} \|B(k,m)x\| e^{-(a+\sigma)\mu(k,m)} \quad \text{for } x \in E,
\]
\[
\|y\|_m' = \sum_{k \leq m} \|C(m,k) - 1\| y e^{(b-\sigma)\mu(m,k)} \quad \text{for } y \in F,
\]
(3)
and for each $(x, y) \in E \times F$,
\[
\|(x, y)\|_m' = \max\{\|x\|_m', \|y\|_m'\}.
\]
By (2) we have
\[
\|x\|_m' \leq D \sum_{k \geq m} e^{\varepsilon|\rho(m)|} e^{-\sigma \mu(k,m)} \|x\| \leq De^{\varepsilon|\rho(m)|} \sum_{k \geq m} e^{-\sigma \rho(k)} \|x\|
\]
\[
\leq De^{\varepsilon|\rho(m)|} e^{\sigma \rho(m)} \sum_{w \geq \rho(m)} e^{-\sigma w} \|x\| \leq \frac{D}{1 - e^{-\sigma}} e^{\varepsilon|\rho(m)|} \|x\|,
\]
with similar estimates for $\|y\|_m'$. Thus, each series in (3) converges and there exists $C > 0$ such that
\[
\frac{1}{C} \|z\| \leq \|z\|_m' \leq Ce^{\varepsilon|\rho(m)|} \|z\|
\] (4)
for every $z \in X$. Furthermore, since
\[
\mu(k, n) - \mu(k, m) = \mu(m, n)
\]
for every $m, n, k \in \mathbb{Z}$, we can easily show that
\[
\|B(m,n)\|' = \sup_{x \in E \setminus \{0\}} \frac{\|B(m,n)x\|_m'}{\|x\|_n} \leq e^{(a+\sigma)\mu(m,n)},
\]
\[
\|C(m,n)^{-1}\|' = \sup_{y \in F \setminus \{0\}} \frac{\|C(m,n)^{-1}y\|_n'}{\|y\|_m} \leq e^{-(b+\sigma)\mu(m,n)}
\]
for every $m \geq n$. This shows that with respect to the norms $\cdot \| \cdot '_m$ the sequence of operators $(A_m)_{m \in \mathbb{Z}}$ has a uniform exponential behavior.
3. Construction of topological conjugacies

We establish in this section a version of the Grobman–Hartman theorem, by constructing topological conjugacies between the sequences \( A_m \) and \( F_m = A_m + f_m \), for a large class of nonlinear perturbations \( f_m \). Namely, we consider continuous maps \( f_m : X \to X \), \( m \in \mathbb{Z} \) and a constant \( \delta > 0 \) such that for each \( m \in \mathbb{Z} \) and \( x, y \in X \):

1. \( f_m(0) = 0 \) and the map \( F_m \) is a homeomorphism;
2. \( \| f_m \|_\infty := \sup \{ \| f_m(x) \| : x \in X \} \leq \delta e^{-\epsilon |\rho(m+1)|} \); 
3. \( \| f_m(x) - f_m(y) \| \leq \delta e^{\epsilon |\rho(m+1)|} \| x - y \| \). 

(5)

We also consider the space \( \mathcal{X} \) of sequences \( u = (u_m)_{m \in \mathbb{Z}} \) of continuous functions \( u_m : X \to X \) such that

\[ \| u \|'_\infty := \sup \{ \| u_m \|'_m : m \in \mathbb{Z} \} < \infty, \]

where

\[ \| u_m \|'_m := \sup \{ \| u_m(x) \|'_m : x \in X \}. \]

One can easily verify that \( \mathcal{X} \) is a complete metric space with the norm \( \| \cdot \|'_\infty \). We start with a preliminary result.

**Theorem 1.** If the sequence \( (A_m)_{m \in \mathbb{Z}} \) admits a \( \rho \)-nonuniform exponential dichotomy, then there is a unique \( (u_m)_{m \in \mathbb{Z}} \in \mathcal{X} \) such that for every \( m \in \mathbb{Z} \) we have

\[ A_m \circ \widehat{u}_m = \widehat{u}_{m+1} \circ (A_m + f_m), \quad \text{where} \quad \widehat{u}_m = \text{Id} + u_m. \]

(7)

**Proof.** Write \( u_m = (b_m, c_m) \) and \( f_m = (g_m, h_m) \), with values in \( E \times F \). We can easily verify that (7) holds for every \( m \in \mathbb{Z} \) if and only if

\[ \bar{b}_m := (B_{m-1} \circ b_{m-1} - g_{m-1}) \circ F_{m-1}^{-1} = b_m, \]

and

\[ \bar{c}_m := C_{m-1} \circ (c_{m+1} \circ F_m + h_m) = c_m \]

for every \( m \in \mathbb{Z} \). Given \( u = (u_m)_{m \in \mathbb{Z}} = (b_m, c_m)_{m \in \mathbb{Z}} \in \mathcal{X} \), we write \( S(u) = (\bar{b}_m, \bar{c}_m)_{m \in \mathbb{Z}} \). We show that \( S(\mathcal{X}) \subset \mathcal{X} \), and that \( S \) is a contraction in the complete metric space \( \mathcal{X} \). Since each map \( F_m \) is a homeomorphism, \( (\bar{b}_m, \bar{c}_m) \) is continuous for every \( m \in \mathbb{Z} \). Furthermore, for each \( z \in X \) we have

\[ \| \bar{b}_m(z) \|'_m \leq \sum_{k \geq m} \| B(k, m) B_{m-1} b_{m-1} (F_{m-1}^{-1}(z)) \| e^{-(a+\sigma) \mu(k,m)} \]

\[ + \sum_{k \geq m} \| B(k, m) g_{m-1} (F_{m-1}^{-1}(z)) \| e^{-(a+\sigma) \mu(k,m)} \]
\[ \leq e^{(a+\sigma)\mu(m,m-1)} \sum_{k \geq m-1} \|B(k,m-1)b_{m-1}(F_{m-1}^{-1}(z))\| e^{-(a+\sigma)\mu(k,m-1)} \]

\[ + \sum_{k \geq m} \|B(k,m)\| \cdot \|g_{m-1}\| \infty e^{-(a+\sigma)\mu(k,m)} \]

\[ \leq e^{(a+\sigma)\mu(m,m-1)} \|b_{m-1}(F_{m-1}^{-1}(z))\|'_{m-1} \]

\[ + D\delta \sum_{k \geq m} e^{a\mu(k,m)+|\rho(m)|} e^{-|\rho(m)|} e^{-(a+\sigma)\mu(k,m)} \]

\[ \leq e^{(a+\sigma)\mu(m,m-1)} \|b_{m-1}(F_{m-1}^{-1}(z))\|'_{m-1} + D\delta \sum_{k \geq m} e^{-\sigma \mu(k,m)} \]

\[ \leq e^{(a+\sigma)\mu(m,m-1)} \|b_{m-1}(F_{m-1}^{-1}(z))\|'_{m-1} + \frac{D\delta}{1 - e^{-\sigma}}. \quad (9) \]

Therefore, for the sequences \(b = (b_m)_{m \in \mathbb{Z}}\) and \(\bar{b} = (\bar{b}_m)_{m \in \mathbb{Z}}\) we obtain

\[ \|\bar{b}\|'_\infty = \sup\{\|\bar{b}_m\|'_m : m \in \mathbb{Z}\} \leq \|b\|'_\infty + \frac{D\delta}{1 - e^{-\sigma}} < \infty, \]

since \(a + \sigma < 0\) and \(\rho\) is increasing. In an analogous manner, for each \(z \in X\) we have

\[ \left\|\tilde{c}_m(z)\right\|'_m \leq \sum_{k \leq m} \|C(m,k)^{-1}c_{m+1}(F_m(z))\| e^{(b-\sigma)\mu(m,k)} \]

\[ + \sum_{k \leq m} \|C(m,k)^{-1}h_m(z)\| e^{(b-\sigma)\mu(m,k)} \]

\[ \leq e^{(-b+\sigma)\mu(m+1,m)} \sum_{k \leq m+1} \|C(m+1,k)^{-1}c_{m+1}(F_m(z))\| e^{(b-\sigma)\mu(m+1,k)} \]

\[ + \sum_{k \leq m} \|C(m+1,k)^{-1}\| \cdot \|h_m\| \infty e^{(b-\sigma)\mu(m,k)} \]

\[ \leq e^{(-b+\sigma)\mu(m+1,m)} \|c_{m+1}(F_m(z))\|'_{m+1} + D\delta \sum_{k \leq m} e^{-b\mu(m+1,k)+|\rho(m+1)|} e^{-|\rho(m+1)|} e^{(b-\sigma)\mu(m,k)} \]

\[ \leq e^{(-b+\sigma)\mu(m+1,m)} \|c_{m+1}(F_m(z))\|'_{m+1} + D\delta e^{-b\mu(m+1,m)} \sum_{k \leq m} e^{\sigma \mu(k,m)} \]

\[ \leq e^{(-b+\sigma)\mu(m+1,m)} \|c_{m+1}(F_m(z))\|'_{m+1} + \frac{D}{1 - e^{-\sigma}} \delta e^{-b\mu(m+1,m)}. \quad (10) \]

Therefore, for the sequences \(c = (c_m)_{m \in \mathbb{Z}}\) and \(\tilde{c} = (\tilde{c}_m)_{m \in \mathbb{Z}}\) we obtain

\[ \|\tilde{c}\|'_\infty \leq \|c\|'_\infty + \frac{D\delta}{1 - e^{-\sigma}} < \infty, \]
since \(-b + \sigma < 0\) and \(\rho\) is increasing. This shows that \(S(\mathcal{X}) \subset \mathcal{X}\). Now we prove that \(S\) is a contraction. Given \(u_1 = (b_{1,m}, c_{1,m})_{m \in \mathbb{Z}}\) and \(u_2 = (b_{2,m}, c_{2,m})_{m \in \mathbb{Z}}\) in \(\mathcal{X}\), for each \(z \in X\) we have

\[
\left\| \tilde{b}_{1,m}(z) - \tilde{b}_{2,m}(z) \right\|_m' \leq e^{(a + \sigma)\mu(m,m-1)} \left\| b_{1,m-1}(F_{m-1}^{-1}(z)) - b_{2,m-1}(F_{m-1}^{-1}(z)) \right\|_{m-1}
\]

and

\[
\left\| \tilde{c}_{1,m}(z) - \tilde{c}_{2,m}(z) \right\|_m' \leq e^{-(b + \sigma)\mu(m,m+1)} \left\| c_{1,m+1}(F_m(z)) - c_{2,m+1}(F_m(z)) \right\|_{m+1}
\]

Thus

\[
\left\| \tilde{b}_1 - \tilde{b}_2 \right\|_\infty \leq e^{a + \sigma} \left\| b_1 - b_2 \right\|_\infty',
\]

and

\[
\left\| \tilde{c}_1 - \tilde{c}_2 \right\|_\infty \leq e^{-b + \sigma} \left\| c_1 - c_2 \right\|_\infty'.
\]

By (11) and (12) the operator \(S\) is a contraction, and there exists a unique sequence \(u \in \mathcal{X}\) such that \(S(u) = u\). This completes the proof of the theorem.

**Theorem 2.** If the sequence \((A_m)_{m \in \mathbb{Z}}\) admits a \(\rho\)-nonuniform exponential dichotomy and \(\delta\) is sufficiently small, then there is a unique \((v_m)_{m \in \mathbb{Z}} \in \mathcal{X}\) such that for every \(m \in \mathbb{Z}\) we have

\[
\hat{v}_{m+1} \circ A_m = (A_m + f_m) \circ \hat{v}_m, \quad \text{where} \quad \hat{v}_m = \text{Id} + v_m.
\]

**Proof.** Write \(v_m = (d_m, e_m)\) and \(f_m = (g_m, h_m)\), with values in \(E \times F\). We can easily verify that (13) holds for every \(m \in \mathbb{Z}\) if and only if \((d_m, e_m) = (\hat{d}_m, \hat{e}_m)\) for every \(m \in \mathbb{Z}\), where

\[
\hat{d}_m := (B_{m-1} \circ d_{m-1} + g_m \circ \hat{v}_{m-1}) \circ A_{m-1}^{-1} = d_m,
\]

and

\[
\hat{e}_m := C_m^{-1} \circ (e_{m+1} \circ A_m - h_m \circ \hat{v}_m) = e_m
\]

for every \(m \in \mathbb{Z}\). Given \(v = (v_m)_{m \in \mathbb{Z}} = (d_m, e_m)_{m \in \mathbb{Z}} \in \mathcal{X}\), we write \(T(v) = (\hat{d}_m, \hat{e}_m)_{m \in \mathbb{Z}}\).

Clearly, \((\hat{d}_m, \hat{e}_m)\) is continuous for every \(m \in \mathbb{Z}\). For each \(z \in X\) we have

\[
\left\| \hat{d}_m(z) \right\|_m' \leq \sum_{k \geq m} \left\| B(k, m) B_{m-1} d_{m-1}(A_{m-1}^{-1} z) \right\| e^{-(a + \sigma)\mu(k,m)}
\]

\[
+ \sum_{k \geq m} \left\| B(k, m) g_m \circ \hat{v}_{m-1}(A_{m-1}^{-1} z) \right\| e^{-(a + \sigma)\mu(k,m)}
\]

\[
\leq e^{(a + \sigma)\mu(m,m-1)} \sum_{k \geq m} \left\| B(k, m - 1) d_{m-1}(A_{m-1}^{-1} z) \right\| e^{-(a + \sigma)\mu(k,m-1)}
\]

\[
+ \sum_{k \geq m} \left\| B(k, m) \right\| \cdot \left\| g_m \right\|_\infty e^{(a + \sigma)\mu(k,m)}.
\]
Proceeding as in (9) this implies that $\| \tilde{d} \|'_\infty < \infty$. In an analogous manner,

$$
\left\| \tilde{e}_m(z) \right\|'_m \leq \sum_{k \leq m} \left\| \mathcal{C}(m, k)^{-1} C_m^{-1} e_{m+1}(A_m z) \right\| e^{(b-\sigma)\mu(m,k)}
+ \sum_{k \leq m} \left\| \mathcal{C}(m, k)^{-1} h_m(\tilde{v}_m(z)) \right\| e^{(b-\sigma)\mu(m,k)}
\leq e^{-(b+\sigma)\mu(m+1,m)} \sum_{k \leq m+1} \left\| \mathcal{C}(m + 1, k)^{-1} e_{m+1}(A_m z) \right\| e^{(b-\sigma)\mu(m+1,k)}
+ \sum_{k \leq m} \left\| \mathcal{C}(m, k)^{-1} \right\| \cdot \| h_m \|_\infty e^{(b-\sigma)\mu(m,k)},
$$
and proceeding as in (10) this implies that $\| \tilde{e} \|'_\infty < \infty$. Thus, $T(\mathcal{X}) \subset \mathcal{X}$. Now we prove that $T$ is a contraction. Given $v_1 = (d_{1,m}, e_{1,m})_{m \in \mathbb{Z}}$ and $v_2 = (d_{2,m}, e_{2,m})_{m \in \mathbb{Z}}$ in $\mathcal{X}$, let

$$
\tilde{v}_{i,m} = \text{Id} + v_{i,m} \quad \text{and} \quad G_{i,m} = \tilde{v}_{i,m} \circ A_{m-1}^{-1}.
$$

Proceeding as in (16), for each $z \in X$ we have

$$
\left\| \tilde{d}_{1,m}(z) - \tilde{d}_{2,m}(z) \right\|'_m
\leq e^{(a+\sigma)\mu(m,m-1)} \sum_{k \geq m-1} \| B(k, m-1)(d_{1,m-1} - d_{2,m-1})(A_{m-1}^{-1} z) \| e^{-(a+\sigma)\mu(k,m)}
+ \sum_{k \geq m} \| B(k, m) [g_{m-1}(G_{1,m-1}(z) - g_{m-1}(G_{2,m-1}(z))] \| e^{-(a+\sigma)\mu(k,m)}
\leq e^{(a+\sigma)\mu(m,m-1)} \| d_{1,m-1}(A_{m-1}^{-1} z) - d_{2,m-1}(A_{m-1}^{-1} z) \|'_m
+ \sum_{k \geq m} \| B(k, m) \| \| G_{1,m-1}(z) - G_{2,m-1}(z) \| e^{-(a+\sigma)\mu(k,m)}
\leq e^{(a+\sigma)\mu(m,m-1)} \| d_{1,m-1}(A_{m-1}^{-1} z) - d_{2,m-1}(A_{m-1}^{-1} z) \|'_m
+ \theta \| \tilde{v}_{1,m-1}(A_{m-1}^{-1} z) - \tilde{v}_{2,m-1}(A_{m-1}^{-1} z) \|
\leq e^{a+\sigma} \| d_{1,m-1} - d_{2,m-1} \|'_m + \theta C \| v_{1,m-1} - v_{2,m-1} \|'_m,
$$
where $\theta = D\delta/(1 - e^{-\sigma})$, using (4) in the last inequality. Analogously, proceeding as in (17) we have

$$
\left\| \tilde{e}_{1,m}(z) - \tilde{e}_{2,m}(z) \right\|'_m
\leq e^{-(b+\sigma)\mu(m,m+1)} \sum_{k \leq m+1} \| \mathcal{C}(m + 1, k)^{-1} e_{1,m+1}(A_m z) - e_{2,m+1}(A_m z) \| e^{(b-\sigma)\mu(m,k)}
+ \sum_{k \geq m} \| \mathcal{C}(m + 1, k)^{-1} [h_{m-1}(\tilde{v}_{1,m}(z)) - h_{m-1}(\tilde{v}_{2,m}(z)) \| \| e^{(b-\sigma)\mu(m,k)}
\leq e^{-(b+\sigma)\mu(m,m+1)} \| e_{1,m+1} - e_{2,m+1} \|'_m + \theta C \| \tilde{v}_{1,m} - \tilde{v}_{2,m} \|'_m.
$$
Thus
\[ \| \dd{d}{1,m} - \dd{d}{2,m} \|'_m \leq e^{a+\sigma} \| d_{1,m-1} - d_{2,m-1} \|'_m + \theta C \| v_{1,m-1} - v_{2,m-1} \|'_m, \]
and
\[ \| \dd{e}{1,m} - \dd{e}{2,m} \|'_m \leq e^{-b+\sigma} \| e_{1,m+1} - e_{2,m+1} \|'_m + \theta C \| v_{1,m} - v_{2,m} \|'_m. \]
This implies that
\[ \| T(v_1) - T(v_2) \|'_\infty \leq (\max\{e^{a+\sigma}, e^{-b+\sigma}\} + 2\theta C) \| v_1 - v_2 \|'_\infty. \]
Since \( \sigma < \min\{-a, b\} \), for \( \delta \) sufficiently small the operator \( T \) is a contraction, and there exists a unique sequence \( v \in \mathcal{X} \) such that \( T(v) = v \). This completes the proof of the theorem. \( \square \)

We finally obtain a version of the Grobman–Hartman theorem.

**Corollary 1.** If the sequence \((A_m)_{m \in \mathbb{Z}}\) admits a \( \rho \)-nonuniform exponential dichotomy and \( \delta \) is sufficiently small, then the maps \( \hat{u}_m = Id + u_m \) and \( \hat{v}_m = Id + v_m \) in Theorems 1 and 2 are homeomorphisms, and
\[ \hat{u}_m \circ \hat{v}_m = \hat{v}_m \circ \hat{u}_m = Id, \quad m \in \mathbb{Z}. \]

**Proof.** By (7) and (13) we obtain
\[ \hat{u}_{m+1} \circ \hat{v}_{m+1} \circ A_m = \hat{u}_{m+1} \circ F_m \circ \hat{v}_m = A_m \circ \hat{u}_m \circ \hat{v}_m \] (18)
for every \( m \in \mathbb{Z} \). Moreover, since
\[ \hat{u}_m \circ \hat{v}_m - Id = v_m + u_m \circ \hat{v}_m, \]
we have
\[ \sup\{ \| \hat{u}_m \circ \hat{v}_m - Id \|'_m : m \in \mathbb{Z} \} < \infty, \]
and \((\hat{u}_m \circ \hat{v}_m)_{m \in \mathbb{Z}} \in \mathcal{X}\). It follows from (18) and the uniqueness in Theorems 1 or 2 (for the maps \( f_m = 0 \)) that \( \hat{u}_m \circ \hat{v}_m = Id \) for every \( m \in \mathbb{Z} \). \( \square \)

The following is another consequence.

**Corollary 2.** If the sequence \((A_m)_{m \in \mathbb{Z}}\) admits a \( \rho \)-nonuniform exponential dichotomy, \( \delta \) is sufficiently small, and there exist maps \( A \) and \( f \) such that
\[ A_m = A \quad \text{and} \quad f_m = f \quad \text{for every} \quad m \in \mathbb{Z}, \] (19)
then there is a homeomorphism \( h : X \to X \) with \( \hat{u}_m = h \) for \( m \in \mathbb{Z} \).
Proof. For each continuous function \( u : X \to X \) we have
\[
p_u := \{ (u_m)_{m \in \mathbb{Z}} : u_m = u \text{ for every } m \in \mathbb{Z} \} \in X.
\]
Furthermore, when (19) holds, the contraction maps \( S \) and \( T \) in the proofs of Theorems 1 and 2 take the set \( D = \{ p_u : u \text{ is continuous} \} \) into itself. Moreover, since \( D \) is a closed nonempty subset of \( X \), the unique fixed points of the maps \( S \) and \( T \) are also in \( D \). □

4. Hölder regularity of the conjugacies

We show in this section that the topological conjugacies in Corollary 1 are always Hölder continuous.

4.1. Preliminaries

We say that the sequence \((A_m)_{m \in \mathbb{Z}}\) admits a strong \( \rho \)-nonuniform exponential dichotomy if there exist constants
\[-c \leq a < 0 < b \leq -d, \quad \varepsilon \geq 0 \quad \text{and} \quad D \geq 1\]
such that for every \( m, n \in \mathbb{Z} \) with \( m \geq n \) we have
\[
\| B(m, n) \| \leq De^{a\mu(m, n) + \varepsilon|\rho(n)|}, \quad \| C(m, n)^{-1} \| \leq De^{-b\mu(m, n) + \varepsilon|\rho(n)|},
\]
and for every \( m, n \in \mathbb{Z} \) with \( m \leq n \) we have
\[
\| B(m, n) \| \leq De^{c\mu(n, m) + \varepsilon|\rho(n)|}, \quad \| C(m, n)^{-1} \| \leq De^{-d\mu(n, m) + \varepsilon|\rho(n)|}.
\]

We also introduce new norms. Choose \( \sigma > 0 \) such that \( \sigma < \min\{-a, b\} \). For each \( m \in \mathbb{Z} \) we set
\[
\| x \|_m^* = \sum_{k \geq m} \| B(k, m) x \| e^{-(a+\sigma)\mu(k, m)} + \sum_{k < m} \| B(k, m) x \| e^{-(c+\sigma)\mu(m, k)}
\]
for \( x \in E \),
\[
\| y \|_m^* = \sum_{k \geq m} \| C(m, k)^{-1} y \| e^{(d-\sigma)\mu(k, m)} + \sum_{k < m} \| C(m, k)^{-1} y \| e^{(b-\sigma)\mu(m, k)}
\]
for \( y \in F \), and
\[
\| (x, y) \|_m^* = \max\{ \| x \|_m^*, \| y \|_m^* \}.
\]
Again there exists \( C' > 0 \) such that for every \( z \in X \),
\[
\frac{1}{C'} \| z \| \leq \| z \|_m^* \leq C' e^{\varepsilon|\rho(m)|} \| z \|.
\]
Lemma 1. For each \( m \in \mathbb{Z} \) we have

\[
\| A_m \|^* := \sup_{z \in X \setminus \{0\}} \frac{\| A_m z \|^*}{\| z \|^*} \leq e^{(-d+\sigma)\mu(m+1,m)},
\]

\[
\| A_m^{-1} \|^* := \sup_{z \in X \setminus \{0\}} \frac{\| A_m^{-1} z \|^*}{\| z \|^*} \leq e^{(c+\sigma)\mu(m,m-1)}.
\]

Proof. Setting \( z = (x, y) \in E \times F \) we have

\[
\| A_m z \|^*_{m+1} = \max \{ \| B_m x \|^*_{m+1}, \| C_m y \|^*_{m+1} \}.
\]

Furthermore

\[
\| B_m x \|^*_{m+1} = e^{(a+\sigma)\mu(m+1,m)} \sum_{k \geq m+1} \| B(k,m)x \| e^{-(a+\sigma)\mu(k,m)}
\]

\[
+ e^{-(c+\sigma)\mu(m+1,m)} \sum_{k \leq m} \| B(k,m)x \| e^{-(c+\sigma)\mu(m,k)}
\]

\[
\leq e^{(a+\sigma)\mu(m+1,m)} \| x \|^*_{m}.
\]

and

\[
\| C_m y \|^*_{m+1} = e^{(-d+\sigma)\mu(m+1,m)} \sum_{k \geq m+1} \| C(m,k)^{-1} y \| e^{(d-\sigma)\mu(k,m)}
\]

\[
+ e^{(b-\sigma)\mu(m,m+1)} \sum_{k \leq m} \| C(m,k)^{-1} y \| e^{(b-\sigma)\mu(m,k)}
\]

\[
\leq e^{(-d+\sigma)\mu(m+1,m)} \| y \|^*_{m}.
\]

Since \(-d + \sigma > a + \sigma\), we obtain

\[
\| A_m z \|^*_{m+1} \leq e^{(-d+\sigma)\mu(m+1,m)} \| z \|^*_{m}.
\]

Similarly,

\[
\| A_m^{-1} v \|^*_{m-1} = \max \{ \| B_m^{-1} x \|^*_{m-1}, \| C_m^{-1} y \|^*_{m-1} \},
\]

with

\[
\| B_m^{-1} x \|^*_{m-1} = e^{-(a+\sigma)\mu(m,m-1)} \sum_{k \geq m-1} \| B(k,m)x \| e^{-(a+\sigma)\mu(k,m)}
\]

\[
+ e^{(c+\sigma)\mu(m,m-1)} \sum_{k \leq m-1} \| B(k,m)x \| e^{-(c+\sigma)\mu(m,k)}
\]

\[
\leq e^{(c+\sigma)\mu(m,m-1)} \| x \|^*_{m}.
\]
\[
\| C_{m-1}^{-1} y \| _{m-1}^* = e^{(d+\sigma)\mu(m,m-1)} \sum_{k \geq m-1} \| C(m,k)^{-1} y \| e^{(d+\sigma)\mu(k,m)} \\
+ e^{(-b+\sigma)\mu(m,m-1)} \sum_{k < m-1} \| C(m,k)^{-1} y \| e^{(b-\sigma)\mu(m,k)} \\
\leq e^{(-b+\sigma)\mu(m,m-1)} \| y \| _m^*.
\]

Since \( c + \sigma > -b + \sigma \), we obtain
\[
\| A_{m-1}^{-1} z \| _{m-1}^* \leq e^{(c+\sigma)\mu(m,m-1)} \| z \| _m^*.
\]

This completes the proof of the lemma. \( \square \)

4.2. Hölder regularity of the conjugacies

Now we establish the Hölder regularity of the conjugacies in Corollary 1. In this section we replace condition (5) by the stronger condition
\[
\| f_m(x) - f_m(y) \| \leq \delta e^{-3\varepsilon|\rho(m+1)|} e^{(c+\sigma)\mu(m+1,m)} \| x - y \| 
\]
for every \( m \in \mathbb{Z} \) and \( x, y \in X \).

**Theorem 3.** If the sequence \( (A_m)_{m \in \mathbb{Z}} \) admits a strong \( \rho \)-nonuniform exponential dichotomy, then for each positive number

\[
\gamma < \min \left\{ \frac{-a}{c}, \frac{b}{-d} \right\},
\]

provided that \( \delta \) is sufficiently small (depending on \( \gamma \)) there exists a constant \( K = K(\gamma, \delta) > 0 \) (independent of the maps \( f_m \)) such that for every \( m \in \mathbb{Z} \) and \( x, y \in X \) with \( \| x - y \| _m^* < 1 \) we have
\[
\| v_m(x) - v_m(y) \| _m^* \leq K \| x - y \| _m^* \gamma.
\]

**Proof.** Let \( \sigma > 0 \) be so small such that
\[
\gamma < \min \left\{ \frac{-a - \sigma}{c}, \frac{b - \sigma}{-d + \sigma} \right\}.
\]

Given constants \( K > 0 \) and \( \gamma \in (0, 1) \), we consider the subset \( \mathcal{X}_\gamma \subset \mathcal{X} \) of the sequences \( (v_m)_{m \in \mathbb{Z}} \) satisfying (22) for every \( m \in \mathbb{Z} \) and \( x, y \in X \) with \( \| x - y \| _m^* < 1 \). One can easily verify that \( \mathcal{X}_\gamma \) is closed with respect to the norm \( \| \cdot \| _\infty \) in (6). Now let \( v = (v_m)_{m \in \mathbb{Z}} = (d_m, e_m)_{m \in \mathbb{Z}} \), with values in \( E \times F \), be a sequence in \( \mathcal{X}_\gamma \). By (14) we have
\[
\| \tilde{d}_m(x) - \tilde{d}_m(y) \| _m^* \leq \| B_{m-1} \delta_{m-1}(x) - B_{m-1} \delta_{m-1}(y) \| _m^* \\
+ \| g_{m-1}(\tilde{v}_{m-1}(x)) - g_{m-1}(\tilde{v}_{m-1}(y)) \| _m^*,
\]

(24)
where
\[ \delta_m = d_m \circ A_m^{-1} \quad \text{and} \quad \bar{v}_m = \hat{v}_m \circ A_m^{-1}. \]

Since \( a + \sigma > -(c + \sigma) \), the two terms in the right-hand side of (24) can be estimated respectively by

\[
\sum_{k \geq m} \left\| B(k, m - 1) \left( \delta_{m-1}(x) - \delta_{m-1}(y) \right) \right\| e^{-(a + \sigma)\mu(k, m-1)} \\
+ \sum_{k < m} \left\| B(k, m - 1) \left( \delta_{m-1}(x) - \delta_{m-1}(y) \right) \right\| e^{-(c + \sigma)\mu(m, k)} \\
= e^{(a + \sigma)\mu(m, m-1)} \sum_{k \geq m} \left\| B(k, m - 1) \left( \delta_{m-1}(x) - \delta_{m-1}(y) \right) \right\| e^{-(a + \sigma)\mu(k, m-1)} \\
+ e^{-(c + \sigma)\mu(m, m-1)} \sum_{k < m} \left\| B(k, m - 1) \left( \delta_{m-1}(x) - \delta_{m-1}(y) \right) \right\| e^{-(c + \sigma)\mu(m-1, k)} \\
\leq e^{(a + \sigma)\mu(m, m-1)} \sum_{k \geq m} \left\| B(k, m - 1) \left( \delta_{m-1}(x) - \delta_{m-1}(y) \right) \right\| e^{-(a + \sigma)\mu(k, m-1)} \\
+ e^{(a + \sigma)\mu(m, m-1)} \sum_{k < m} \left\| B(k, m - 1) \left( \delta_{m-1}(x) - \delta_{m-1}(y) \right) \right\| e^{(c + \sigma)\mu(m-1, k)} \\
\leq e^{(a + \sigma)\mu(m, m-1)} \left\| \delta_{m-1}(x) - \delta_{m-1}(y) \right\|_{m-1}^* \\
\leq e^{(a + \sigma)\mu(m, m-1)} K \left( \left\| A_{m-1}^{-1}(x - y) \right\|_{m-1}^* \right)^\gamma,
\]

and by

\[
\delta \sum_{k \geq m} \left\| B(k, m) \right\| e^{-3\varepsilon\rho(m)} e^{-(a + \sigma)\mu(m, m-2)} \left\| \bar{v}_{m-1}(x) - \bar{v}_{m-1}(y) \right\| e^{-(a + \sigma)\mu(k, m)} \\
+ \delta \sum_{k < m} \left\| B(k, m) \right\| e^{-3\varepsilon\rho(m)} e^{-(c + \sigma)\mu(m-1, m-2)} \left\| \bar{v}_{m-1}(x) - \bar{v}_{m-1}(y) \right\| e^{-(c + \sigma)\mu(m, k)} \\
\leq C' \delta e^{-(c + \sigma)\mu(m-1, m-2)} \left\| \bar{v}_{m-1}(x) - \bar{v}_{m-1}(y) \right\|_{m-1}^* \\
\leq C' \delta e^{-(c + \sigma)\mu(m-1, m-2)} (L + KL^\gamma),
\]

(25)

where
\[ L = \left\| A_{m-1}^{-1}(x - y) \right\|_{m-1}^*. \]

By Lemma 1, for \( x \neq y \) with \( \left\| x - y \right\|_{m}^* < 1 \) we obtain
\[
\frac{\|\tilde{d}_m(x) - \tilde{d}_m(y)\|_m^*}{\langle\|x - y\|_m^*,\|\rangle} \leq K\left(e^{(a+\sigma)\mu(m,m-1)} + C'\delta e^{-(c+\sigma)\mu(m-1,m-2)}\right) \\
\times e^{\nu(c+\sigma)\mu(m-1,m-2)} + C'\delta.
\]

In an analogous manner, by (15) we have
\[
\|\tilde{e}_m(x) - \tilde{e}_m(y)\|_m^* \leq \left\|C^{-1}_m \tilde{e}_{m+1}(x) - C^{-1}_m \tilde{e}_{m+1}(y)\right\|_m^* \\
+ \left\|C^{-1}_m (h_m \circ \tilde{v}_m)(x) - C^{-1}_m (h_m \circ \tilde{v}_m)(y)\right\|_m^*.
\]

where \(\tilde{e}_{m+1} = e_{m+1} \circ A_m\). The two terms in the right-hand side of (27) are respectively
\[
e^{(d-\sigma)\mu(m+1,m)} \sum_{k \geq m} \|\mathcal{C}(m+1,k)^{-1} (\tilde{e}_{m+1}(x) - \tilde{e}_{m+1}(y))\| e^{(d-\sigma)\mu(k,m+1)} \\
+ e^{(-b+\sigma)\mu(m+1,m)} \sum_{k < m} \|\mathcal{C}(m+1,k)^{-1} (\tilde{e}_{m+1}(x) - \tilde{e}_{m+1}(y))\| e^{(b-\sigma)\mu(m+1,k)} \\
\leq e^{(-b+\sigma)\mu(m+1,m)} \sum_{k \geq m+1} \|\mathcal{C}(m+1,k)^{-1} (\tilde{e}_{m+1}(x) - \tilde{e}_{m+1}(y))\| e^{(d-\sigma)\mu(k,m+1)} \\
+ e^{(-b+\sigma)\mu(m+1,m)} \sum_{k < m+1} \|\mathcal{C}(m+1,k)^{-1} (\tilde{e}_{m+1}(x) - \tilde{e}_{m+1}(y))\| e^{(b-\sigma)\mu(m+1-k)} \\
= e^{(-b+\sigma)\mu(m+1,m)} \left\|\tilde{e}_{m+1}(x) - \tilde{e}_{m+1}(y)\right\|_m^* \\
\leq e^{(-b+\sigma)\mu(m+1,m)} K \left(\|A_m(x - y)\|_m^*\right)^{\nu},
\]

and
\[
\delta \sum_{k > m} \|\mathcal{C}(m+1,k)^{-1} e^{-3e^{(d+\sigma)\mu(m+1)}} e^{-(c+\sigma)\mu(m,m-1)}\| e^{(d-\sigma)\mu(k,m)} \\
+ \delta \sum_{k \leq m} \|\mathcal{C}(m+1,k)^{-1} e^{-3e^{(d+\sigma)\mu(m+1)}} e^{-(c+\sigma)\mu(m,m-1)}\| e^{(b-\sigma)\mu(m,k)} \\
\leq \delta D e^{-(c+\sigma)\mu(m,m-1)} \left(\frac{e^{(d-\sigma)\mu(m,m+1)} + e^{(b-\sigma)\mu(m,m)}}{1 - e^{\sigma}}\right) \|\mathcal{C}(m+1,k)^{-1} e^{(d-\sigma)\mu(k,m+1)}\| e^{(d-\sigma)\mu(k,m)} \\
\leq \frac{2\delta D}{1 - e^{-\sigma}} e^{-(c+\sigma)\mu(m,m-1)} \left[\|x - y\|_m^* + K \left(\|x - y\|_m^*\right)^{\nu}\right].
\]

By Lemma 1, for \(x \neq y\) with \(\|x - y\|_m^* < 1\) we obtain
\[
\|\tilde{e}_m(x) - \tilde{e}_m(y)\|_m^* \leq K e^{(-b+\sigma)\mu(m+1,m)} e^{\nu(d+\sigma)\mu(m+1,m)} + \frac{2(1 + K)\delta D}{1 - e^{\sigma}}.
\]

It follows readily from (23) that
\[
e^{(a+\sigma)\nu(c+\sigma)} < 1 \quad \text{and} \quad e^{(-b+\sigma)\nu(-d+\sigma)} < 1.
\]
Hence, by (26) and (30), for each sufficiently small $\delta$ there exists $K > 0$ such that
\[
\max\{\|\bar{d}_m(x) - \bar{d}_m(y)\|_m, \|\bar{e}_m(x) - \bar{e}_m(y)\|_m^*\} \leq K\left(\|x - y\|_m^*\right)^\gamma
\]
for every $m \in \mathbb{Z}$ and $x, y \in X$ with $\|x - y\|_m^* < 1$. Therefore, $T(X_\gamma) \subset X_\gamma$. This completes the proof of the theorem. □

**Theorem 4.** If the sequence $(A_m)_{m \in \mathbb{Z}}$ admits a strong $\rho$-nonuniform exponential dichotomy, then for each positive $\gamma$ as in (21), provided that $\delta$ is sufficiently small (depending on $\gamma$) there exists a constant $K = K(\gamma, \delta) > 0$ (independent of the maps $f_m$) such that for every $m \in \mathbb{Z}$ and $x, y \in X$ with $\|x - y\|_m^* < 1$ we have
\[
\|u_m(x) - u_m(y)\|_m^* \leq K\left(\|x - y\|_m^*\right)^\gamma.
\]

**Proof.** Set
\[
\alpha = e^{-d + \sigma)\mu(m+1,m)} + (C')^2\delta e^{-(c+\sigma)\mu(m,m-1)} \quad \text{and} \quad \beta = (1 - C')\delta^{-1}.
\]

**Lemma 2.** For each $x, y \in X$ we have
\[
\|F_m(x) - F_m(y)\|_{m+1}^* \leq \alpha\|x - y\|_m^*,
\]
\[
\|F_m^{-1}(x) - F_m^{-1}(y)\|_m^* \leq \beta e^{(c+\sigma)\mu(m,m-1)}\|x - y\|_{m+1}^*.
\]

**Proof.** In view of Lemma 1 and (20), we have
\[
\|F_m(x) - F_m(y)\|_{m+1}^* \leq \|A_m(x - y)\|_{m+1}^* + \|f_m(x) - f_m(y)\|_{m+1}^*
\]
\[
\leq e^{-d + \sigma)\mu(m+1,m)}\|x - y\|_m^* + C'\|f_m(x) - f_m(y)\|
\]
\[
\leq e^{-d + \sigma)\mu(m+1,m)}\|x - y\|_m^* + C'\delta e^{-(c+\sigma)\mu(m,m-1)}\|x - y\|,
\]
and
\[
\|F_m(x) - F_m(y)\|_{m+1}^* \geq \|A_m(x - y)\|_{m+1}^* - \|f_m(x) - f_m(y)\|_{m+1}^*
\]
\[
\geq (e^{-(c+\sigma)\mu(m,m-1)} - (C')^2\delta e^{-(c+\sigma)\mu(m,m-1)})\|x - y\|_m^*.
\]

This yields the desired inequalities. □

Now take $x, y \in X$ with $\|x - y\|_m^* < 1$. By (8), proceeding as in (25) we obtain
\[
\|\bar{b}_m(x) - \bar{b}_m(y)\|_m^* \leq e^{(a+\sigma)\mu(m,m-1)}K\left(\|F_m^{-1}(x) - F_m^{-1}(y)\|_{m-1}^*\right)^\gamma
\]
\[
+ C'\delta e^{-(c+\sigma)\mu(m-1,m-2)}\|F_m^{-1}(x) - F_m^{-1}(y)\|_{m-1}^*,
\]
and by Lemma 2,
\[ \| \tilde{b}_m(x) - \tilde{b}_m(y) \|_m^\ast \leq e^{(a+\sigma)\mu(m,m-1)} K \beta^\gamma e^{(c+\sigma)\mu(m-1,m-2)} (\|x - y\|_m^\ast) \gamma + C' \delta \|x - y\|_m^\ast \]
\[ \leq (Ke^{(a+\sigma)\mu(m,m-1)} e^{(c+\sigma)\mu(m-1,m-2)} \beta^\gamma + C' \delta) (\|x - y\|_m^\ast)^\gamma \] (31)

(since \( \|x - y\|_m^\ast < 1 \)). Furthermore, proceeding as in (29) we obtain
\[ \| \tilde{c}_m(x) - \tilde{c}_m(y) \|_m^\ast \leq \frac{2\delta D}{1 - e^{-\sigma}} e^{-(c+\sigma)\mu(m,m-1)} \|x - y\|_m^\ast. \]

Thus, proceeding as in (28) and using Lemma 2,
\[ \| \tilde{c}_m(x) - \tilde{c}_m(y) \|_m^\ast \leq e^{(-b+\sigma)\mu(m+1,m)} K \left( \| F_m(x) - F_m(y) \|_{m+1}^\ast \right)^\gamma + \frac{2\delta D}{1 - e^{-\sigma}} e^{-(c+\sigma)\mu(m,m-1)} \|x - y\|_m^\ast \]
\[ \leq \left( Ke^{(-b+\sigma)\mu(m+1,m)} (e^{(c-d+\sigma)\mu(m+1,m)} + C' \delta) e^{-(c+\sigma)\mu(m,m-1)} \right)^\gamma \]
\[ + \frac{2\delta D}{1 - e^{-\sigma}} e^{-(c+\sigma)\mu(m,m-1)} (\|x - y\|_m^\ast)^\gamma. \]

By (31) and (32), for each sufficiently small \( \delta \) there exists \( K > 0 \) such that
\[ \max \{ \| \tilde{b}_m(x) - \tilde{b}_m(y) \|_m^\ast, \| \tilde{c}_m(x) - \tilde{c}_m(y) \|_m^\ast \} \leq K (\|x - y\|_m^\ast)^\gamma, \]

for each \( m \in \mathbb{Z} \) and \( x, y \in X \) with \( \|x - y\|_m^\ast < 1 \). This completes the proof of the theorem. \( \square \)

References