Journal of Algebra 353 (2012) 142–157

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The structure of a Laurent polynomial fibration in n variables

S.M. Bhatwadekar [a], Neena Gupta [b,*]

[a] *Bhaskaracharya Pratishthana, 56/14 Erandavane, Damle Path, Off Law College Road, Pune 411 004, India*
[b] *Stat-Math Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata 700 108, India*

ARTICLE INFO

Article history:
Received 17 March 2011
Available online 28 December 2011
Communicated by Luchezar L. Avramov

MSC:
primary 14R25, 13F20
secondary 13E15

Keywords:
Polynomial algebra
Symmetric algebra
Laurent polynomial algebra
Codimension-one
Fibre ring

ABSTRACT

Bass, Connell and Wright have proved that any finitely presented locally polynomial algebra in n variables over an integral domain R is isomorphic to the symmetric algebra of a finitely generated projective R-module of rank n. In this paper we prove a corresponding structure theorem for a ring A which is a locally Laurent polynomial algebra in n variables over an integral domain R, viz., we show that A is isomorphic to an R-algebra of the form $(\mathrm{Sym}_R(Q))[I^{-1}]$, where Q is a direct sum of n finitely generated projective R-modules of rank one and I is a suitable invertible ideal of the symmetric algebra $\mathrm{Sym}_R(Q)$. Further, we show that any faithfully flat algebra over a Noetherian normal domain R, whose generic and codimension-one fibres are Laurent polynomial algebras in n variables, is a locally Laurent polynomial algebra in n variables over R.

1. Introduction

Let R be an integral domain. Recall that an R-algebra A is called a Laurent polynomial algebra in n-variables over R if $A = R[X_1, X_1{}^{-1}, \dots, X_n, X_n{}^{-1}]$, where $X_1, X_2, \dots, X_n$ are transcendental over R. We call an R-algebra A to be a locally Laurent polynomial algebra in n variables over R if $A \otimes_R R_m$ is a Laurent polynomial algebra in n variables over the local ring R_m for every maximal ideal m of R. In this paper we explore the Laurent polynomial analogues of some results and open problems on polynomial (or $\mathbb{A}^n$) fibrations.

We shall first establish a structure theorem for locally Laurent polynomial algebras, a Laurent polynomial analogue of the famous local–global theorem of Bass, Connell and Wright [3, Theorem 4.4]

* Corresponding author.
E-mail addresses: smbhatwadekar@gmail.com (S.M. Bhatwadekar), neena_r@isical.ac.in (N. Gupta).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.11.032

which states that *any finitely presented locally polynomial algebra in n variables over a ring R is isomorphic to the symmetric algebra of a projective R-module of rank n*. While the hypothesis on finite presentation is clearly necessary in the polynomial case (consider the $\mathbb{Z}$-algebra $\mathbb{Z}[X/2, X/3, X/5, \ldots]$), our structure theorem will show that a *locally Laurent polynomial algebra* A over an integral domain R *is necessarily finitely presented* and that A is of the form $B[I^{-1}]$, where B is isomorphic to the symmetric algebra $\mathrm{Sym}_R(Q)$ of a (suitable) finitely generated projective R-module Q and I is an invertible ideal of B. Here I^{-1} denotes the B-submodule $\{a \in F \mid aI \subseteq B\}$ of the quotient field F of B and $B[I^{-1}]$ denotes the subring of F generated by B and I^{-1}. The precise statement of the structure theorem (Theorem 2.3) is given below.

Theorem A. *Let R be an integral domain and A be a locally Laurent polynomial algebra in n variables over R. Then there exist n finitely generated rank one projective R-modules L_i, $1 \leqslant i \leqslant n$, such that A is isomorphic to an R-algebra of the form*

$$\big(\mathrm{Sym}_R(Q)\big)\big[I^{-1}\big],$$

where $Q = L_1 \oplus \cdots \oplus L_n$ and I is an invertible ideal of $\mathrm{Sym}_R(Q)$ generated by the image of $L_1 \otimes \cdots \otimes L_n$. In particular, A is finitely presented over R. If $\mathrm{Pic}(R) = (0)$, then A is a Laurent polynomial algebra over R.

After describing the structure of a locally Laurent polynomial algebra, we investigate sufficient conditions for an R-algebra to be locally Laurent polynomial. Note that any locally Laurent polynomial R-algebra is faithfully flat over R. Now suppose that R is a Noetherian normal domain and A is a faithfully flat R-algebra. Under these hypotheses, we shall see that A is a locally Laurent polynomial algebra in n variables over R if $A \otimes_R R_P$ is a Laurent polynomial algebra in n variables over R_P for every prime ideal P in R of *height one* (Proposition 2.7). This result was proved in [6, Theorem 4.8] for the case $n = 1$.

Next we consider the following fibration problem:

Question. Under what (minimal) fibre conditions will a faithfully flat algebra A over a Noetherian domain R be a locally Laurent polynomial algebra?

We first investigate the case when R is a discrete valuation ring (DVR) and prove (Theorem 3.5):

Theorem B. *Let (R, t) be a discrete valuation ring with a regular parameter t, quotient field K and residue field k. Let A be an integral domain containing R such that*

(i) *$A[1/t]$ is a Laurent polynomial algebra in n variables over K.*
(ii) *A/tA is a Laurent polynomial algebra in n variables over k.*

Then A is a Laurent polynomial algebra in n variables over R.

Recall that for any $P \in \mathrm{Spec}\, R$, $k(P)$ denotes the quotient field of R/P and that $A \otimes_R k(P)$ is the fibre ring of an R-algebra A over P. Using Theorem B and Proposition 2.7, we shall show that for any faithfully flat algebra A over a Noetherian normal domain R to be locally Laurent polynomial, it is enough to ensure that the generic and codimension-one fibres of A are Laurent polynomial algebras in n variables. In fact, we prove (Theorem 3.6):

Theorem C. *Let R be a Noetherian normal domain with quotient field K and A be a faithfully flat R-algebra such that*

(i) *The generic fibre $A \otimes_R K$ is a Laurent polynomial algebra in n variables over K.*
(ii) *For each height one prime ideal P in R, $A \otimes_R k(P)$ is a Laurent polynomial algebra in n variables over $k(P)$.*

Then A is a locally Laurent polynomial algebra in n variables over R.

For the case $n = 1$, this result was proved earlier in [5, Theorem 3.11] under the additional hypothesis that A is finitely generated.

Finally we consider an arbitrary Noetherian domain. An example (see [4, Example 3.9]) of Bhatwadekar and Dutta shows that, even for $n = 1$, Theorem C cannot be extended to non-normal domains without additional hypotheses. We give the following necessary and sufficient condition for extending Theorem C to an arbitrary Noetherian domain (Theorem 4.4):

Theorem D. *Let R be a Noetherian domain with quotient field K and let A be a faithfully flat R-algebra such that*

(i) $A \otimes_R K = K[X_1, {X_1}^{-1}, \dots, X_n, {X_n}^{-1}]$, *$X_1, \dots, X_n$ are transcendental over R.*
(ii) *For each height one prime ideal P in R, $A \otimes_R k(P)$ is a Laurent polynomial algebra in n variables over $k(P)$.*
(iii) $L_i := A \cap KX_i$ *is a finitely generated projective R-module of rank one, $1 \leqslant i \leqslant n$.*

Then A is a locally Laurent polynomial algebra in n variables over R.

However, even without the hypothesis (iii), we shall show (Proposition 4.3) that A is at least finitely generated over R and that $A \otimes_R R'$ is locally Laurent polynomial over a finite birational extension R' of R.

Theorem A will be proved in Section 2, Theorems B and C in Section 3 and Theorem D in Section 4.

We recall some standard notation to be used throughout the paper. For a ring R, R^* will denote the multiplicative group of units of R. For a prime ideal P of R, and an R-algebra A, A_P denotes the ring $S^{-1}A$, where $S = R \setminus P$ and $k(P)$ denotes the residue field R_P/PR_P. The notation $A = R^{[1]}$ will mean that A is isomorphic, as an R-algebra, to a polynomial ring in one variable over R.

We also recall a few definitions (cf. [8, p. 80]). Let R be an integral domain with quotient field K. A non-zero R-submodule L of K is said to be a *fractional ideal* if there exists a non-zero element $\alpha \in R$ such that $\alpha L \subseteq R$. A fractional ideal L is said to be *invertible* if $L^{-1}L = R$, where $L^{-1} = \{\alpha \in K \mid \alpha L \subseteq R\}$.

2. On locally Laurent polynomial algebra in *n* variables

In this section we shall prove Theorem A. Throughout this section, R will denote an integral domain with quotient field K and A an integral domain containing R such that $A \cap K = R$ and

$$A \otimes_R K = K\big[X_1, {X_1}^{-1}, \dots, X_n, {X_n}^{-1}\big]$$

for some $X_1, \dots, X_n$ transcendental over R. In this set up, we shall use the following notation. For $1 \leqslant i \leqslant n$ and $j \geqslant 0$, set

$$C_{ij} := A \cap K{X_i}^j \quad \text{and} \quad D_{ij} := A \cap K{X_i}^{-j},$$

$C := \bigoplus_{(j_1,\dots,j_n)\in {\mathbb{Z}_{\geqslant 0}}^n} C_{1j_1} \cdots C_{nj_n}$, where $C_{1j_1} \cdots C_{nj_n} = \{c_1 \cdots c_n \mid c_\ell \in C_{\ell j_\ell}\}$ is an R-submodule of $A \cap K{X_1}^{j_1} \cdots K{X_n}^{j_n}$,

$$I := \text{the ideal of } C \text{ generated by } C_{11} \cdots C_{n1} \quad \text{and} \quad B := A \cap K[X_1, \dots, X_n].$$

Note that C is an R-subalgebra of B. Note also that for $g \in C_{ij}$ and $h \in D_{ij}$, $gh \in A \cap K = R$. Therefore we get an R-linear map

$$\psi_{ij} : C_{ij} \otimes_R D_{ij} \to R \quad \text{defined by } \psi_{ij}(g \otimes h) = gh.$$

Set

$$J_{ij} := \psi_{ij}(C_{ij} \otimes_R D_{ij}).$$

With the above notation, we state a few lemmas needed for the proofs. We first show that when A itself is a Laurent polynomial algebra, then C is a polynomial algebra, $C = B$ and $A = B[I^{-1}] = C[I^{-1}]$.

Lemma 2.1. *Let A be a Laurent polynomial algebra in n variables over R. Then there exist $\alpha_i \in K^*$ and $U_i \in A$ such that $U_i = \alpha_i X_i$, $1 \leqslant i \leqslant n$, $A = R[U_1, {U_1}^{-1}, \ldots, U_n, {U_n}^{-1}]$ and $B = R[U_1, \ldots, U_n]$. Further, for $1 \leqslant i \leqslant n$ and $j \geqslant 0$,*

$$C_{ij}\big(= A \cap K{X_i}^j\big) = R{U_i}^j, \qquad D_{ij}\big(= A \cap K{X_i}^{-j}\big) = R{U_i}^{-j}$$

and hence $J_{ij} = R$, $C = B = R[U_1, \ldots, U_n]$, $I = (U_1 \cdots U_n)C$, and

$$A = B\big[I^{-1}\big] = C\big[I^{-1}\big].$$

Proof. Let $A = R[Y_1, {Y_1}^{-1}, \ldots, Y_n, {Y_n}^{-1}]$. Then

$$K\big[X_1, \ldots, X_n, {X_1}^{-1}, \ldots, {X_n}^{-1}\big] = K\big[Y_1, \ldots, Y_n, {Y_1}^{-1}, \ldots, {Y_n}^{-1}\big].$$

It follows that for each i, $1 \leqslant i \leqslant n$,

$$Y_i = \lambda_i {X_1}^{a_{i1}} {X_2}^{a_{i2}} \cdots {X_n}^{a_{in}} \quad \text{and} \quad X_i = \mu_i {Y_1}^{b_{i1}} Y_2^{b_{i2}} \cdots Y_n^{b_{in}}$$

for some λ_i, $\mu_i \in K \setminus \{0\}$ and a_{ij}, $b_{ij} \in \mathbb{Z}$, $1 \leqslant j \leqslant n$, satisfying

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

For $1 \leqslant i \leqslant n$, set $\alpha_i := {\mu_i}^{-1}$ and

$$U_i := \alpha_i X_i = {Y_1}^{b_{i1}} Y_2^{b_{i2}} \cdots Y_n^{b_{in}}.$$

Then, $K[X_1, \ldots, X_n] = K[U_1, \ldots, U_n]$, and for $1 \leqslant i \leqslant n$,

$$Y_i = {U_1}^{a_{i1}} {U_2}^{a_{i2}} \cdots {U_n}^{a_{in}}.$$

Hence

$$A = R\big[U_1, {U_1}^{-1}, \ldots, U_n, {U_n}^{-1}\big], \qquad B = R[U_1, \ldots, U_n]$$

and $A \cap K{X_i}^j (= A \cap K{U_i}^j) = R{U_i}^j$ for every i, $1 \leqslant i \leqslant n$ and every $j \in \mathbb{Z}$. Thus $C = B$, $I = (U_1 \cdots U_n)C$ and $A = B[I^{-1}] = C[I^{-1}]$. □

In the general case (i.e., when A is not necessarily a Laurent polynomial algebra over R), we give below a sufficient condition for C to be the symmetric algebra of a finitely generated projective R-module of rank n and I to be an invertible ideal of C.

Lemma 2.2. *Suppose that $J_{i1} = R\ \forall\ i$, $1 \leqslant i \leqslant n$. Then for each i, $1 \leqslant i \leqslant n$, and $j \geqslant 0$,*

(I) $J_{ij} = R$.
(II) *C_{ij} and D_{ij} are finitely generated projective R-modules of rank one.*
(III) *The canonical map $\theta_{ij} : C_{i1} \otimes_R C_{i1} \otimes_R \cdots \otimes_R C_{i1}$ (j-times) $\to C_{ij}$ is an isomorphism.*
(IV) *There is a natural R-algebra isomorphism*

$$C\left(= \bigoplus_{(j_1,\dots,j_n)\in \mathbb{Z}_{\geqslant 0}{}^n} C_{1j_1}\cdots C_{nj_n}\right) \cong \operatorname{Sym}_R(C_{11} \oplus \cdots \oplus C_{n1}).$$

(V) *The ideal I of C generated by $C_{11}\cdots C_{n1}$ is an invertible ideal.*

Proof. Fix i, $1 \leqslant i \leqslant n$, and $j \geqslant 0$. Note that C_{ij} and D_{ij} are torsion-free R-modules of rank one. Moreover, if $f \in C_{i1}$ and $g \in D_{i1}$ then $f^j \in C_{ij}$, $g^j \in D_{ij}$ and $f^j g^j = (fg)^j \in R$.

(I) Since $J_{i1} = \psi_{i1}(C_{i1} \otimes_R D_{i1}) = R$, there exist $c_\ell \in C_{i1}$ and $d_\ell \in D_{i1}$, $1 \leqslant \ell \leqslant r$ for some r, such that

$$\psi_{i1}\left(\sum_\ell c_\ell \otimes d_\ell\right) = \sum_\ell c_\ell d_\ell = 1.$$

Set $a_\ell := c_\ell d_\ell$. As $c_\ell{}^j \in C_{ij}$ and $d_\ell{}^j \in D_{ij}$, we have $a_\ell{}^j = c_\ell{}^j d_\ell{}^j = \psi_{ij}(c_\ell{}^j \otimes d_\ell{}^j) \in J_{ij}$ for each ℓ. Since $\sum_\ell a_\ell = 1$, we have $(a_1{}^j,\dots,a_r{}^j)R = R$ and hence $J_{ij} = R$.

(II) Set $L := C_{ij}X_i{}^{-j}$ and $E := D_{ij}X_i{}^j$. Clearly L and E are non-zero R-submodules of K such that $LE(= C_{ij}D_{ij}) \subseteq A \cap K = R$. Thus L and E are fractional ideals. Since $R = J_{ij} = \psi_{ij}(C_{ij} \otimes_R D_{ij})$, there exist $f_s \in C_{ij}$ and $g_s \in D_{ij}$, $1 \leqslant s \leqslant t$ for some t, such that

$$1 = \sum_s f_s g_s = \sum_s \big(f_s X_i{}^{-j}\big)\big(g_s X_i{}^j\big) \in \big(C_{ij}X_i{}^{-j}\big)\big(D_{ij}X_i{}^j\big) = LE.$$

Therefore L and E are invertible ideals of K with $E = L^{-1}$ and $L = E^{-1}$ and hence C_{ij} and D_{ij} are finitely generated projective R-modules of rank one (cf. [8, p. 80]).

(III) Set $C(ij) := \theta_{ij}(C_{i1} \otimes_R C_{i1} \otimes_R \cdots \otimes_R C_{i1})$. Since $C(ij) \subseteq C_{ij}$, it is enough to show that $C(ij)_m = (C_{ij})_m$ for every maximal ideal m of R. Fix a maximal ideal m of R. By (II), $(C_{ij})_m = R_m f_{ij}$ for some $f_{ij} \in (C_{ij})_m$. Since $(J_{ij})_m = R_m$, we have $(D_{ij})_m = R_m f_{ij}{}^{-1}$ and so $f_{ij}{}^{-1} \in A_m$. Now, since $f_{i1}{}^j \in (C_{ij})_m = R_m f_{ij}$, we have $f_{i1}{}^j = \lambda_{ij} f_{ij}$ for some $\lambda_{ij} \in R_m$. Hence $\lambda_{ij}{}^{-1} = f_{i1}{}^{-j} f_{ij} \in A_m \cap K = R_m$. Thus, $f_{ij} \in R_m f_{i1}{}^j \subseteq C(ij)_m$. Hence the result follows.

(IV) follows from (II) and (III).

(V) By (II), $C_{11},\dots,C_{n1}$ are finitely generated projective R-modules and hence the ideal I of C is finitely generated and for every prime ideal p of R, I_p is a principal ideal. Thus, for any prime ideal P of C, if $p = P \cap R$, then I_P being a further localisation of I_p is principal and so I is an invertible ideal (see [8, Theorem 11.3]). $\square$

We now prove Theorem A.

Theorem 2.3. *Let R be an integral domain with quotient field K and A be a locally Laurent polynomial algebra in n variables over R. Then there exist n finitely generated rank one projective R-modules L_i, $1 \leqslant i \leqslant n$, such that A is isomorphic to an R-algebra of the form*

$$\big(\operatorname{Sym}_R(Q)\big)\big[I^{-1}\big],$$

where $Q = L_1 \oplus \cdots \oplus L_n$ and I is an invertible ideal of $\mathrm{Sym}_R(Q)$ *generated by the image of $L_1 \otimes \cdots \otimes L_n$. In fact, if $A \otimes_R K = K[X_1, X_1{}^{-1}, \ldots, X_n, X_n{}^{-1}]$ and $B = A \cap K[X_1, \ldots, X_n]$, then we may choose L_i to be $A \cap KX_i$ and* $\mathrm{Sym}_R(Q)$ *may be identified with the ring B. In particular, A is finitely presented over R. If* $\mathrm{Pic}(R) = (0)$, *then A is a Laurent polynomial algebra over R.*

Proof. As before, $C_{ij} = A \cap KX_i{}^j$, $D_{ij} = A \cap KX_i{}^{-j}$, and J_{ij} is the image of the canonical map ψ_{ij} : $C_{ij} \otimes_R D_{ij} \to R$ defined by $\psi_{ij}(g \otimes h) = gh$, $1 \leqslant i \leqslant n$, $j \geqslant 0$.

Fix i, $1 \leqslant i \leqslant n$. For any maximal ideal m of R, since A_m is a Laurent polynomial algebra in n variables over R_m, it follows from Lemma 2.1 that $(J_{i1})_m = R_m$. Thus $J_{i1} = R$. Hence, by Lemma 2.2, $L_i (= C_{i1})$ is a finitely generated projective R-module of rank one, $\mathrm{Sym}_R(L_1 \oplus \cdots \oplus L_n)$ may be identified with the subring $C(= \bigoplus_{(j_1,\ldots,j_n)\in\mathbb{Z}_{\geqslant 0}{}^n} C_{1j_1} \cdots C_{nj_n})$ of A, and the ideal I of C generated by $L_1 \cdots L_n$ is invertible.

Since $C \subseteq B$ and, by Lemma 2.1, $C_m = B_m$ for every maximal ideal m of R, we have $C = B$. Therefore, to complete the proof, we only need to show that $A = C[I^{-1}]$. Since $D_{i1} \subset A$ and $J_{i1} = R$, we have $1 \in J_{i1} \subset C_{i1}A$, i.e., $C_{i1}A = A$. Hence $IA = A$. Therefore $C[I^{-1}] \subseteq A$. Hence, it is enough to show that $A_m = C_m[I_m{}^{-1}]$ for every maximal ideal m of R. This follows from Lemma 2.1, since A_m is a Laurent polynomial algebra in n variables over R_m. □

Remark 2.4. (i) With the notation described at the beginning of this section, we have seen in Theorem 2.3 that if A is a locally Laurent polynomial, then $C = B$. This need not hold in general. Consider the faithfully flat $\mathbb{Z}$-algebra $A = \mathbb{Z}[\frac{1}{2}(X+3), X^{-1}]$. Here $\mathbb{A} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[X, X^{-1}]$ so that $B = \mathbb{Z}[\frac{1}{2}(X+3)]$, but $C = \mathbb{Z}[X] \subsetneqq B$.

(ii) Note that Theorem 2.3 is proved in two steps. The first step is to prove that each C_{i1} is a finitely generated projective R-module of rank one and $C \cong \mathrm{Sym}_R(C_{11} \oplus \cdots \oplus C_{n1})$. The second step is to show that $A = C[I^{-1}]$ where I is the invertible ideal of C generated by $C_{11} \cdots C_{n1}$. We have seen that if $J_{i1} = R$ for each i, then one achieves the first step (cf. Lemma 2.2). Moreover, in this case, since $D_{i1} \subset A$, we have $J_{i1} \subset C_{i1}A$ and hence $C_{i1}A = A$. Thus $IA = A$. Therefore $C[I^{-1}] \subseteq A$.

Now suppose that R is Noetherian or Krull and A is a faithfully flat R-algebra such that $A_P (= A \otimes_R R_P)$ is a Laurent polynomial algebra in n variables over R_P for every prime ideal P of R for which $\operatorname{depth} R_P = 1$. Under these hypotheses we will show (Proposition 2.7) that A is in fact a locally Laurent polynomial algebra in n variables over R. As in the proof of Theorem 2.3, we will first show that $J_{i1} = R$ for each i (Lemma 2.6) and then show that $A = C[I^{-1}](= B[I^{-1}])$.

We first state a lemma; the proof will follow from the argument in [7, Lemma 2.8]. Note that for a prime ideal P of a Krull domain R, $\operatorname{depth} R_P = 1$ if and only if $\operatorname{ht} P = 1$.

Lemma 2.5. *Let R be an integral domain with quotient field K which is either a Noetherian or a Krull domain and let Δ be the set of all prime ideals P of R such that* $\operatorname{depth} R_P = 1$. *For a torsion-free R-module M, the following conditions are equivalent*:

(i) *$M = \bigcap_{P\in\Delta} M_P$, where M and $M_P = M \otimes_R R_P$ are identified with their images in $M \otimes_R K$.*
(ii) *For every $a, b \in R$ such that $(aR : b) = aR$, we have $(aM : b) = aM$.*

In particular, if M is R-flat then $M = \bigcap_{P\in\Delta} M_P$.

The following is the key lemma for proving Proposition 2.7. This lemma was proved in [6, Lemma 4.2] for $n = 1$. For convenience, we give a proof in our generalised setup.

Lemma 2.6. *Let R be an integral domain which is either a Noetherian or a Krull domain and let A be a faithfully flat R-algebra such that A_P is a Laurent polynomial algebra in n variables for every prime ideal P of R such that* $\operatorname{depth} R_P = 1$. *Then C_{i1} and D_{i1} are finitely generated projective R-modules of rank one and $J_{i1} = R$ for each i, $1 \leqslant i \leqslant n$.*

Proof. We first show that the canonical map $C_{i1} \otimes_R A \to C_{i1}A$ is an isomorphism and $C_{i1}A = A$.

Since $C_{i1} \hookrightarrow KX_i$ and A is R-flat, we have $C_{i1} \otimes_R A \hookrightarrow KX_i \otimes_R A \cong K[X_1, \ldots, X_n, X_1{}^{-1}, \ldots, X_n{}^{-1}]$. Thus $C_{i1} \otimes_R A$ is a torsion-free A-module of rank one. Now if the canonical map $C_{i1} \otimes_R A \to C_{i1}A$ is not injective then the kernel of this map is a non-zero torsion-free A-submodule of $C_{i1} \otimes_R A$, which contradicts that the rank of $C_{i1} \otimes_R A$ is one. Thus, the canonical map $C_{i1} \otimes_R A \to C_{i1}A$ is injective and hence an isomorphism.

Let Δ denote the set of all prime ideals P of R such that $\operatorname{depth} R_P = 1$. For every $P \in \Delta$, since A_P is a Laurent polynomial algebra, we have, by Lemma 2.1, $(J_{i1})_P = R_P$; in particular $J_{i1} \nsubseteq P$. Choose a non-zero element $x \in J_{i1}$. Since R is either Noetherian or Krull, $\operatorname{Ass}_R(R/xR)$ is a finite subset of Δ. Therefore, by prime avoidance, we see that $J_{i1} \nsubseteq \bigcup_{P \in \operatorname{Ass}_R(R/xR)} P$. Choose $y \in J_{i1} \setminus \bigcup_{P \in \operatorname{Ass}_R(R/xR)} P$. Then $\{x, y\} \subset J_{i1}$ forms a regular sequence in R, i.e., $(xR : y) = xR$.

Since $C_{i1} = A \cap KX_i$ and A is R-flat, $C_{i1} = \bigcap_{P \in \Delta} (C_{i1})_P$ by Lemma 2.5. Therefore, again by Lemma 2.5, $(xC_{i1} : y) = xC_{i1}$, i.e., $\{x, y\}$ forms a regular sequence in C_{i1}. Since A is R-flat and $C_{i1} \otimes_R A \cong C_{i1}A$, it follows that $\{x, y\}$ forms a regular sequence in $C_{i1}A$ and hence $(xC_{i1}A : y) = xC_{i1}A$. Since $D_{i1}A \subseteq A$, we have $J_{i1}A \subseteq C_{i1}A$. Thus $x, y \in C_{i1}A$ and hence $xy \in xC_{i1}A$. Therefore $x \in (xC_{i1}A : y) = xC_{i1}A$. Thus $C_{i1}A = A$.

Since $C_{i1} \otimes_R A (\cong C_{i1}A = A)$ is a free A-module of rank one and A is faithfully flat over R, it follows that C_{i1} is a finitely presented flat and hence a projective R-module of rank one. Similarly D_{i1} is a finitely generated projective R-module of rank one. Thus $C_{i1} \otimes_R D_{i1}$ is a finitely generated projective R-module of rank one. Since $\psi_{i1}(C_{i1} \otimes_R D_{i1}) = J_{i1}$, R is a domain and $J_{i1} \neq 0$, we see that ψ_{i1} is an isomorphism and hence J_{i1} is a finitely generated projective R-module (of rank one). Therefore, by Lemma 2.5,

$$J_{i1} = \bigcap_{P \in \Delta} (J_{i1})_P = \bigcap_{P \in \Delta} R_P = R,$$

because $(J_{i1})_P = R_P$ for every $P \in \Delta$. Thus the lemma is proved. □

Proposition 2.7. *Let R be an integral domain with quotient field K which is either a Noetherian or a Krull domain and let A be a faithfully flat R-algebra such that A_P is a Laurent polynomial algebra in n variables over R_P for every prime ideal P in R such that* $\operatorname{depth} R_P = 1$. *Then A is a locally Laurent polynomial algebra in n variables over R.*

Proof. It is enough to assume that R is local. By Lemma 2.6, $C_{i1} = Rf_i$ for some $f_i \in A$ and $J_{i1} = R$ for $1 \leqslant i \leqslant n$. Therefore, by Lemma 2.2, $C = R[f_1, \ldots, f_n]$, $I = (f_1 \cdots f_n)C$ and hence $C[I^{-1}] = R[f_1, f_1{}^{-1}, \ldots, f_n, f_n{}^{-1}]$. We now show that $A = C[I^{-1}]$.

Let Δ denote the set of all prime ideals P of R such that $\operatorname{depth} R_P = 1$. Since, for every $P \in \Delta$, A_P is a Laurent polynomial algebra in n variables over R_P, we have $A_P = C_P[I_P{}^{-1}]$ by Lemma 2.1. Hence, as both A and $C[I^{-1}](= R[f_1, f_1{}^{-1}, \ldots, f_n, f_n{}^{-1}])$ are R-flat and are submodules of the quotient field of A, we have $A = C[I^{-1}]$ by Lemma 2.5. □

Remark 2.8. In contrast to Proposition 2.7, if R is a Noetherian local domain (or even a regular local ring) and B is a faithfully flat *finitely generated* R-algebra such that B_P is a polynomial algebra in n variables over R_P for every prime ideal P in R satisfying $\operatorname{depth} R_P = 1$, then B need not be a polynomial algebra. Consider

$$R = \mathbb{C}[[\pi_1, \pi_2]], \qquad B = R[X, Y, Z]/\big(\pi_2 X + \pi_1 Y + Z^2 + 1\big).$$

3. Laurent polynomial fibration over a Noetherian normal domain

In this section we shall prove Theorems B and C. We first prove Theorem B. The proof will require an auxiliary lemma. We will use the following version of the dimension inequality (cf. [8, Theorem 15.5, p. 118]).

Theorem 3.1. *Let R be a Noetherian integral domain and B an integral domain containing R. Let P be a prime ideal of B and $p = P \cap R$. Then*

$$\operatorname{ht} P + \operatorname{tr.deg}_{R/p} B/P \leqslant \operatorname{ht} p + \operatorname{tr.deg}_R B.$$

As a consequence of Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let (R, t) be a discrete valuation ring with a regular parameter t and residue field k. Let B be an integral domain containing R such that tB is a prime ideal of B. Then $\operatorname{tr.deg}_k B/tB \leqslant \operatorname{tr.deg}_R B$.*

We state below a result, the proof of which will follow from [2, Proposition 6.1 and Theorem 6.3].

Theorem 3.3. *Let (R, t) be a discrete valuation ring with a regular parameter t, quotient field K and residue field k. Let D be an integral domain containing R such that*

(i) *$D[1/t] = K^{[1]}$ and D/tD is an integral domain.*
(ii) *$\operatorname{tr.deg}_k D/tD > 0$.*

Then D is a finitely generated R-algebra and there exists a finite algebraic field extension F of k such that $D/tD = F^{[1]}$.

We now prove a lemma over discrete valuation rings which will be used in the proof of Theorem B.

Lemma 3.4. *Let (R, t) be a discrete valuation ring with a regular parameter t, quotient field K and residue field k. Let B be an integral domain containing R such that*

(i) *$B[1/t] = K[X_1, \dots, X_n]$, a polynomial ring in n variables over K.*
(ii) *B/tB is an integral domain and $\operatorname{tr.deg}_k B/tB = n$.*

Set $B_0 := R$ and for $1 \leqslant i \leqslant n$, set $B_i := B \cap K[X_1, X_2, \dots, X_i]$. Then

(I) *$B_{i+1}[1/t] = B_i[1/t][X_{i+1}]$ for $0 \leqslant i \leqslant n-1$.*
(II) *tB_i is a prime ideal of B_i of height one and $tB_i = tB_{i+1} \cap B_i$, so that*

$$k \hookrightarrow B_1/tB_1 \hookrightarrow \cdots \hookrightarrow B_n/tB_n = B/tB.$$

Let F_i denote the quotient field of B_i/tB_i.
(III) *$\operatorname{tr.deg}_k B_i/tB_i (= \operatorname{tr.deg}_k F_i) = i$, $0 \leqslant i \leqslant n$.*
(IV) *$(B_{i+1}/tB_{i+1}) \otimes_{B_i/tB_i} F_i = {K_i}^{[1]}$ for some finite algebraic field extension K_i of F_i.*

Proof. (I) follows easily from the definition of B_i's.

(II) $tB_i = tB \cap K[X_1, \dots, X_i] = tB \cap B_i$. Since by (ii) tB is prime ideal of B, we have tB_i is a prime ideal of B_i.

Since $\operatorname{tr.deg}_k B/tB = \operatorname{tr.deg}_R B = n$, and $\operatorname{ht}(tR) = 1$, from the dimension inequality (Theorem 3.1) we have $\operatorname{ht}(tB) \leqslant 1$. Therefore, as B is an integral domain, we have $\operatorname{ht}(tB) = 1$ and hence $\bigcap_{n \geqslant 0} t^n B = (0)$. Since $B_i \subset B$ for each i, $1 \leqslant i \leqslant n$, it follows that $\bigcap_{n \geqslant 0} t^n B_i = (0)$, which implies that $\operatorname{ht}(tB_i) = 1$.

Also since $B_i \subset B_{i+1}$ for each i, $0 \leqslant i \leqslant n-1$, we have $tB_i = tB \cap B_i = tB \cap B_{i+1} \cap B_i = tB_{i+1} \cap B_i$.

(III) We first note that by (II), $V_i := B_{i(tB_i)}$ is a discrete valuation ring with residue field F_i. Set $E_{i+1} := B_{i+1} \otimes_{B_i} V_i$, a localisation of B_{i+1}. Then $E_{i+1}[1/t] = V_i[1/t][X_{i+1}]$ by (I). Since tE_{i+1} is a prime ideal of E_{i+1}, we have $\operatorname{tr.deg}_{F_i}(E_{i+1}/tE_{i+1}) \leqslant 1$ by Corollary 3.2, i.e., $\operatorname{tr.deg}_{B_i/tB_i}(B_{i+1}/tB_{i+1}) \leqslant 1$. But $\operatorname{tr.deg}_k B_n/tB_n = n$ and $\operatorname{tr.deg}_k B_0/tB_0 = 0$. Hence $\operatorname{tr.deg}_k B_i/tB_i = i$ $\forall i$.

(IV) Note that $E_{i+1}[1/t] = V_i[1/t][X_{i+1}]$ by (I) and tE_{i+1} is a prime ideal of E_{i+1} by (II). Also $\text{tr.deg}_{F_i}(E_{i+1}/tE_{i+1})(= \text{tr.deg}_{B_i/tB_i}\ B_{i+1}/tB_{i+1}) = 1$. Hence by Theorem 3.3, $(B_{i+1}/tB_{i+1}) \otimes_{B_i/tB_i} F_i = E_{i+1}/tE_{i+1} = K_i^{[1]}$ for some finite algebraic field extension K_i of F_i. □

We now prove Theorem B. Over a field k, we shall call a k-algebra A to be a Laurent polynomial form in n variables if there exists an algebraic field extension F of k such that $A \otimes_k F$ is a Laurent polynomial algebra in n variables over F.

Theorem 3.5. *Let (R, t) be a discrete valuation ring with a regular parameter t, quotient field K and residue field k. Let A be an integral domain containing R such that $A[1/t]$ is a Laurent polynomial algebra in n variables over K. Then the following statements are equivalent*:

(i) *A is a Laurent polynomial algebra in n variables over R.*
(ii) *A/tA is a Laurent polynomial algebra in n variables over k.*
(iii) *A/tA is a Laurent polynomial form in n variables over k.*
(iv) *There exists a field extension F of k such that $A/tA \otimes_k F$ is an integral domain and contains a Laurent polynomial algebra in n variables over F.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are obvious. We prove (iv) ⇒ (i).

Since F is faithfully flat over k, we regard A/tA as a k-subalgebra of the integral domain $A/tA \otimes_k F$. Note that t is a prime in A. We first show that we can choose $X_1, \ldots, X_n$ in A such that

$$R\big[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}\big] \subseteq A \subset A[1/t] = K\big[X_1, X_1^{-1} \cdots, X_n, X_n^{-1}\big].$$

Choose $T_1, \ldots, T_n \in A$ such that $A[1/t] = K[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}]$. Fix an integer i, $1 \leqslant i \leqslant n$. Let $m_i \in \mathbb{Z}_{\geqslant 0}$ be the least integer such that $t^{m_i} T_i^{-1} \in A$. Let $X_i := T_i/t^{m_i}$. Then $X_i^{-1} \in A$. If $m_i = 0$, then $X_i \in A$ and we are through. If not, since $t^{m_i} = X_i^{-1} T_i \in A$ and t is a prime in A, we have either $t \mid X_i^{-1}$ or $t^{m_i} \mid T_i$ in A. If $t \mid X_i^{-1}$, then $t^{m_i - 1} T_i^{-1} \in A$, which contradicts the minimality of m_i. This shows that $t^{m_i} \mid T_i$, and hence $X_i \in A$.

We shall show that $A = R[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}]$ for the above choice of $X_1, X_2, \ldots, X_n$.

Set $B := A \cap K[X_1, \ldots, X_n]$ and $C := R[X_1, X_2, \ldots, X_n]$. We show that $B = C$. We first observe that

(1) $C \subseteq B \subset B[1/t] = K[X_1, \ldots, X_n] = C[1/t]$.
(2) $tB = tA \cap B$ and hence t is prime in B.
(3) $A = B_{X_1 \cdots X_n} (= B[(X_1 \cdots X_n)^{-1}])$.
(4) t does not divide $X_1 \cdots X_n$ in B.
(5) $\text{tr.deg}_k\, B/tB = n$.

(1) is obvious; (2) follows from the relation $tB = tA \cap K[X_1, \ldots, X_n] = tA \cap B$. To see (3), note that if $h \in A$, then there exists $\ell \in \mathbb{Z}_{\geqslant 0}$ such that $(X_1 \cdots X_n)^\ell h \in K[X_1, \ldots, X_n] \cap A (= B)$, so that $h \in B_{X_1 \cdots X_n}$. (4) follows from the fact that $B \subset A$, $X_1 \cdots X_n$ is a unit in A and t is a prime element of A. Since $\text{tr.deg}_k\, A/tA = n$ by hypothesis (iv) and Corollary 3.2, (5) follows from (3) and (4).

We now show that the map from C/tC to B/tB is one-to-one. Suppose not. Let x_i denote the image of X_i in B/tB for $1 \leqslant i \leqslant n$. Note that, by (4), x_i is a non-zero element of B/tB. As the map from C/tC to B/tB is not one-to-one, x_j is algebraic over $k[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n] (\hookrightarrow B/tB)$ for some j. Interchanging the x_i's if necessary, we assume that x_n is algebraic over $k[x_1, \ldots, x_{n-1}]$. By Lemma 3.4, if $B_{n-1} = B \cap K[X_1, \ldots, X_{n-1}]$, then $B_{n-1}/tB_{n-1} \hookrightarrow B/tB$ and

$$B/tB \hookrightarrow (B/tB) \otimes_{B_{n-1}/tB_{n-1}} F_{n-1} = K_{n-1}[Y] \big(= K_{n-1}{}^{[1]}\big),$$

where F_{n-1} is the quotient field of B_{n-1}/tB_{n-1}, K_{n-1} is a finite algebraic field extension of F_{n-1} and Y is transcendental over K_{n-1}. Since $R[X_1, \ldots, X_n] \subseteq B$, we have $R[X_1, \ldots, X_{n-1}] \subseteq B_{n-1}$, so that $x_1, x_2, \dots, x_{n-1}$ are non-zero elements of $B_{n-1}/tB_{n-1}(\hookrightarrow B/tB)$. Since by our assumption x_n is algebraic over $k[x_1, \dots, x_{n-1}](\hookrightarrow B_{n-1}/tB_{n-1})$, we have $x_n \in K_{n-1}$, and hence a unit in K_{n-1} as x_n is a non-zero element of $B/tB \hookrightarrow K_{n-1}[Y]$. Therefore $A/tA = (B/tB)_{x_1 \cdots x_{n-1} x_n}$ is contained in $K_{n-1}[Y]$.

Since F_{n-1} is the quotient field of B_{n-1}/tB_{n-1}, $(B/tB) \otimes_{B_{n-1}/tB_{n-1}} F_{n-1} (= K_{n-1}[Y])$ is a localisation of B/tB. Thus B/tB, A/tA and $K_{n-1}[Y]$ have the same quotient field, say E, and

$$B/tB \hookrightarrow A/tA \hookrightarrow K_{n-1}[Y] \hookrightarrow E.$$

Since F is k-flat, we have the following inclusions

$$A/tA \otimes_k F \hookrightarrow K_{n-1}[Y] \otimes_k F \hookrightarrow E \otimes_k F.$$

Since $A/tA \otimes_k F$ is an integral domain, and E is a localisation of A/tA, we have $E \otimes_k F$ is an integral domain. Thus $K_{n-1}[Y] \otimes_k F (= (K_{n-1} \otimes_k F)^{[1]})$ is an integral domain and hence the units of $A/tA \otimes_k F$ are contained in $K_{n-1} \otimes_k F$. It then follows from the hypothesis (iv) that $\operatorname{tr.deg}_F K_{n-1} \otimes_k F \geqslant n$. But $\operatorname{tr.deg}_F K_{n-1} \otimes_k F = \operatorname{tr.deg}_k K_{n-1} = \operatorname{tr.deg}_k F_{n-1}$ and $\operatorname{tr.deg}_k F_{n-1} = n - 1$ by Lemma 3.4. This is a contradiction. Thus the map $C/tC \to B/tB$ is one-to-one. Hence, as $C[1/t] = B[1/t]$, it follows that $B = C$. Therefore, by (3), $A = R[X_1, \dots X_n, {X_1}^{-1}, \dots, {X_n}^{-1}]$. $\square$

As a consequence of Theorem 3.5 and Proposition 2.7, we deduce Theorem C.

Theorem 3.6. *Let R be a Krull domain with quotient field K and A be a faithfully flat R-algebra such that*

(i) *The generic fibre $A \otimes_R K$ is a Laurent polynomial algebra in n variables over K.*
(ii) *For each height one prime ideal P in R, there exists a field extension $k(P)'$ of $k(P)$ such that $A \otimes_R k(P) \otimes_{k(P)} k(P)'$ is an integral domain and contains a Laurent polynomial algebra in n variables over $k(P)'$.*

Then A is a locally Laurent polynomial algebra in n variables over R.

Proof. Let Δ denote the set of all height one prime ideals of R. Since R is a Krull domain, for every $P \in \Delta$, R_P is a DVR. Thus, by Theorem 3.5, A_P is a Laurent polynomial algebra in n variables over R_P for every $P \in \Delta$. Now the result follows by Proposition 2.7. $\square$

We conclude this section with some remarks pertaining to Theorems 3.5 and 3.6.

Remark 3.7. (1) Consider a discrete valuation ring R with a regular parameter t and residue field k. Let $A = R[X, Y, Z, X^{-1}, Y^{-1}]/(tZ - XY + 1)$. Then A is generically a Laurent polynomial algebra such that the closed fibre $A/tA (= k[X, Z, X^{-1}])$ is an integral domain, k is algebraically closed in A/tA and $k^* \subsetneqq (A/tA)^*$. But A is not a Laurent polynomial algebra. This shows that the condition in (iv) of Theorem 3.5, on the existence of a Laurent polynomial algebra in n variables in a suitable extension $A/tA \otimes_k F$, is necessary. (Also see [5, Remark 3.10].)

(2) An example of Bhatwadekar and Dutta [4, Example 3.9] shows that Theorem 3.5 cannot be extended to a faithfully flat algebra A over an arbitrary Noetherian local domain R of dimension one even if the generic as well as the closed fibre is a Laurent polynomial algebra in one variable.

(3) We may contrast Theorem 3.5 with the corresponding polynomial fibration problem over a DVR. Consider the set up:

R a discrete valuation ring with a regular parameter t and B an integral domain containing R such that

(i) The generic fibre $B[1/t]$ is a polynomial algebra in n variables over $R[1/t]$.

(ii) The closed fibre B/tB is a polynomial algebra in n variables over R/tR.

Under the above hypotheses, when $n = 1$, B is a polynomial algebra in one variables over R and when $n = 2$, a theorem of Sathaye shows that B is a polynomial algebra in two variables if R contains the field of rationals $\mathbb{Q}$ (see [10, Theorem 1] and [1, Corollary 3.2]). Moreover, if ch.$(R/tR) > 0$ and $n = 2$, Asanuma has given an example to show that B need not be a polynomial algebra (see [1, Theorem 5.1]). However, for $n > 2$, it is not known whether B is a polynomial algebra even in the case $R \supseteq \mathbb{Q}$.

(4) It is not known whether a polynomial analogue of Theorem 3.6 is true for $n = 2$. For instance, even when R is a polynomial algebra in two variables over the field of complex numbers and A is a finitely generated faithfully flat R-algebra *all* of whose fibres are polynomial algebras in 2 variables, it is not known whether A is necessarily a polynomial algebra.

4. Laurent polynomial fibration over a general Noetherian domain

Let R be a Noetherian domain and A be a faithfully flat R-algebra such that all the fibre rings of A are Laurent polynomial algebras in n variables. If R is normal, we have seen (Theorem 3.6) that A is a locally Laurent polynomial algebra. However if R is not normal, then A need not be a locally Laurent polynomial algebra (Remark 3.7 (2)). In this section, we shall prove (Proposition 4.3) that at least A is finitely generated over R and that there exists a finite birational extension R' of R such that $A \otimes_R R'$ is a locally Laurent polynomial algebra in n variables over R'. We shall also prove a necessary and sufficient condition for A to be a locally Laurent polynomial algebra in n variables over R (Theorem 4.4).

The following criterion for a module M to be flat over a Noetherian ring R is known but for the lack of a proper reference, we give a proof below.

Lemma 4.1. *Let R be a Noetherian ring and M be an R-module. Then M is flat over R if and only if* $\mathrm{Tor}_1^R(M, R/P) = 0$ *for every prime ideal P of R.*

Proof. Suppose that $\mathrm{Tor}_1^R(M, R/P) = 0$ for every prime ideal P of R. To show that M is flat over R, it is enough to show that $\mathrm{Tor}_1^R(M, R/I) = 0$ for every ideal I of R (see [8, Theorem 7.8, p. 51]).

Since R is Noetherian, for every ideal I, there exist ideals $I = J_0 \subset J_1 \subset \cdots \subset J_n = R$ such that R/I has a filtration of submodules of the form

$$0 = J_0/I \subset J_1/I \subset \cdots \subset J_{n-1}/I \subset J_n/I = R/I$$

satisfying $J_{i+1}/J_i \cong R/P_i$ for some prime ideal P_i of R (see [8, Theorem 6.4, p. 39]). We prove that $\mathrm{Tor}_1^R(M, R/I) = 0$ by induction on n, the length of the filtration of R/I.

If $n = 1$, then I is a prime ideal of R and by the given hypothesis, $\mathrm{Tor}_1^R(M, R/I) = 0$.

Suppose that $n > 1$. By applying $\mathrm{Tor}_1^R(M, -)$ to the short exact sequence

$$0 \to J_1/I \to R/I \to R/J_1 \to 0$$

we get the exact sequence

$$\mathrm{Tor}_1^R(M, J_1/I) \to \mathrm{Tor}_1^R(M, R/I) \to \mathrm{Tor}_1^R(M, R/J_1).$$

Now $J_1/I \cong R/P_0$ for a prime ideal P_0 and hence $\mathrm{Tor}_1^R(M, J_1/I) = 0$. Since R/J_1 has a filtration of length $n - 1$, $\mathrm{Tor}_1^R(M, R/J_1) = 0$ by induction hypothesis. Thus $\mathrm{Tor}_1^R(M, R/I) = 0$. $\square$

We now prove an elementary result.

Lemma 4.2. *Let R be a Noetherian domain and let R' be an integral extension of R. Let D and A be flat R-algebras such that $D \subseteq A \subseteq A \otimes_R R'$ and $A \otimes_R R' = D \otimes_R R'$. Then $A = D$.*

Proof. Let $M = A/D$. Since $A \otimes_R R' = D \otimes_R R'$, it follows that $M \otimes_R R' = 0$. We will show that M is a flat R-module. It will then follow that $M \hookrightarrow M \otimes_R R' = 0$, i.e., $A = D$.

By Lemma 4.1, it is enough to show that $\mathrm{Tor}_1^R(M, R/P) = 0$ for every prime ideal P of R. Fix a prime ideal P of R. Since A is a flat R-module, we have the following exact sequence of R-modules

$$0 \to \mathrm{Tor}_1^R(M, R/P) \to D \otimes_R R/P \to A \otimes_R R/P \to M \otimes_R R/P \to 0.$$

Since R' is integral over R, there exists a prime ideal P' of R' lying over P. Since A and D are flat R-modules, we have the following injective maps

$$D \otimes_R R/P \hookrightarrow D \otimes_R R'/P' \quad \text{and} \quad A \otimes_R R/P \hookrightarrow A \otimes_R R'/P'.$$

Since the map $D \otimes_R R/P \hookrightarrow D \otimes_R R'/P'$ is a composite of the maps

$$D \otimes_R R/P \to A \otimes_R R/P \quad \text{and} \quad A \otimes_R R/P \hookrightarrow A \otimes_R R'/P' = D \otimes_R R'/P',$$

it follows that the map $D \otimes_R R/P \to A \otimes_R R/P$ is injective and hence $\mathrm{Tor}_1^R(M, R/P) = 0$. □

We now prove a result for a Laurent polynomial fibration over a Noetherian domain.

Proposition 4.3. *Let R be a Noetherian domain with quotient field K and let A be a faithfully flat R-algebra such that*

(i) *The generic fibre $A \otimes_R K$ is a Laurent polynomial algebra in n variables over K.*
(ii) *For each height one prime ideal P of R, $A \otimes_R k(P)$ is geometrically integral over $k(P)$ and there exists a field extension $k(P)'$ of $k(P)$ such that $A \otimes_R k(P) \otimes_{k(P)} k(P)'$ contains a Laurent polynomial algebra in n variables over $k(P)'$.*

Then the following statements hold:

(I) *All the fibre rings of A are Laurent polynomial forms in n variables.*
(II) *There exists a finite birational extension R' of R such that $A \otimes_R R'$ is a locally Laurent polynomial algebra in n variables over R'.*
(III) *A is finitely generated over R.*

Proof. (I) This proof is essentially the same as in [5, Theorem 3.13].

Fix any prime ideal P (need not be of height one) in R. Note that $A \otimes_R k(P) = A_P \otimes_{R_P} k(P)$. So replacing R by R_P we can assume that R is a local Noetherian domain with maximal ideal P. We prove the result by induction on the height of P.

Suppose that $\dim R = 1$. From the Krull–Akizuki theorem [9, Theorem 33.2] and the fact that R is local, it follows that the normalisation $\tilde{R}$ of R is a semilocal PID and that $k(\tilde{P})$ is a finite algebraic extension of $k(P)$ for every maximal ideal $\tilde{P}$ of $\tilde{R}$. Fix a maximal ideal $\tilde{P}$ of $\tilde{R}$, let $V = \tilde{R}_{\tilde{P}}$ and let $t \in V$ be such that $tV = \tilde{P}V$. Since R and V are birational, $A \otimes_R V$ is generically a Laurent polynomial algebra in n variables over V. Also note that, by hypothesis (ii),

$$(A \otimes_R V)/t(A \otimes_R V) = (A \otimes_R \tilde{R}) \otimes_{\tilde{R}} k(\tilde{P}) = \big(A \otimes_R k(P)\big) \otimes_{k(P)} k(\tilde{P})$$

satisfies the condition (iv) of Theorem 3.5. Hence $A \otimes_R V$ is a Laurent polynomial in n variables over V by Theorem 3.5; in particular, $(A \otimes_R k(P)) \otimes_{k(P)} k(\tilde{P}) (= A \otimes_R (V/tV))$ is a Laurent polynomial algebra in n variables over $k(\tilde{P})$. Thus $A \otimes_R k(P)$ is a Laurent polynomial form in n variables over $k(P)$.

Now suppose that $\dim R \geqslant 2$. Then by the induction hypothesis, the fibre ring $A \otimes_R k(Q)$ is a Laurent polynomial form in n variables for every prime ideal $Q \subsetneq P$. Let $\widehat{R}$ be the completion of R

with respect to the maximal ideal P and let $\widehat{P}$ denote the maximal ideal of $\widehat{R}$. Then $\dim\widehat{R} = \dim R$ and $\widehat{R}/\widehat{P} = R/P$. Let $\widehat{P_0}$ be a minimal prime ideal of $\widehat{R}$ such that $\dim\widehat{R} = \dim\widehat{R}/\widehat{P_0}$. Since $\widehat{R}$ is R-flat, by the going down theorem, for any prime ideal $\widehat{Q}$ of $\widehat{R}$, $\operatorname{ht}(\widehat{Q} \cap R) \leqslant \operatorname{ht}\widehat{Q}$. Hence, since $\operatorname{ht}\widehat{P_0} = 0$ and R is an integral domain, we have $\widehat{P_0} \cap R = (0)$. Set $\widehat{A} := A \otimes_R \widehat{R}/\widehat{P_0}$. Since $\widehat{P_0} \cap R = (0)$, $\widehat{A}$ is generically a Laurent polynomial algebra in n variables over $\widehat{R}/\widehat{P_0}$. Let $\widehat{Q}$ be a non-zero prime ideal of $\widehat{R}/\widehat{P_0}$ properly contained in $\widehat{P}$ and $Q = \widehat{Q} \cap R$. Since $\widehat{A} \otimes_{\widehat{R}} k(\widehat{Q}) = (A \otimes_R k(Q)) \otimes_{k(Q)} k(\widehat{Q})$, we have $\widehat{A} \otimes_{\widehat{R}} k(\widehat{Q})$ is a Laurent polynomial form in n variables over $k(\widehat{Q})$. Now since $A \otimes_R k(P) = \widehat{A} \otimes_{\widehat{R}} k(\widehat{P})$, we can replace R by $\widehat{R}/\widehat{P_0}$ and A by $\widehat{A}$ and assume that R is a complete Noetherian local domain. Let $\tilde{R}$ denote the normalisation of R. Then $\tilde{R}$ is a finite R-module (see [9, Theorem 32.1]) and hence a Noetherian normal local domain and for every non-zero non-maximal ideal $\tilde{Q}$ of $\tilde{R}$, $A \otimes_R \tilde{R} \otimes_{\tilde{R}} k(\tilde{Q})$ is a Laurent polynomial form in n variables. Hence, by Theorem 3.6, $A \otimes_R \tilde{R}$ is a Laurent polynomial algebra in n variables over $\tilde{R}$. This proves that the closed fibre of A is a Laurent polynomial form in n variables.

(II) Let $\tilde{R}$ be the normalisation of R. Then $\tilde{R}$ is a Krull domain (see [9, Theorem 33.10]). By (I), all the codimension-one fibres of $A \otimes_R \tilde{R}$ are Laurent polynomial forms in n variables over $\tilde{R}$. Hence, by Theorems 3.6 and 2.3, there exist n finitely generated rank one projective modules L_i, $1 \leqslant i \leqslant n$, of $\tilde{R}$ such that $A \otimes_R \tilde{R}$ is isomorphic to an $\tilde{R}$-algebra of the form $(\operatorname{Sym}_{\tilde{R}}(Q))[I^{-1}]$, where $Q = L_1 \oplus \cdots \oplus L_n$ and I is an invertible ideal of $\operatorname{Sym}_{\tilde{R}}(Q)$ generated by the image of $L_1 \otimes \cdots \otimes L_n$. Since L_i is a finitely generated projective $\tilde{R}$-module, there exists a finite extension R' of R contained in $\tilde{R}$ such that, for each i, $1 \leqslant i \leqslant n$, there exist finitely generated rank one projective R'-modules $L_i{}'$ satisfying $L_i \cong L_i{}' \otimes_{R'} \tilde{R}$. Then $(\operatorname{Sym}_{R'}(Q'))[I'^{-1}] \subseteq (\operatorname{Sym}_{\tilde{R}}(Q))[I^{-1}] = A \otimes_R \tilde{R}$, where $Q' = L_1{}' \oplus \cdots \oplus L_n{}'$ and I' is an invertible ideal of $\operatorname{Sym}_{R'}(Q')$ generated by the image of $L_1' \otimes \cdots \otimes L_n'$. Since $(\operatorname{Sym}_{R'}(Q'))[I'^{-1}](\subseteq A \otimes_R \tilde{R})$ is a finitely generated R'-algebra, by enlarging R' if necessary, we can ensure that $(\operatorname{Sym}_{R'}(Q'))[I'^{-1}] \subseteq A \otimes_R R' \subseteq A \otimes_R \tilde{R}(= (\operatorname{Sym}_{\tilde{R}}(Q))[I^{-1}])$. Since R' is a finite module over the Noetherian ring R, R' is a Noetherian ring. Hence, by Lemma 4.2, $(\operatorname{Sym}_{R'}(Q'))[I'^{-1}] = A \otimes_R R'$.

(III) Since A is flat over R, we have $R \hookrightarrow A \hookrightarrow A \otimes_R R'$. Since $A \otimes_R R'$ is a locally Laurent polynomial algebra in n variables over R', by Theorem 2.3, $A \otimes_R R'$ is a finitely generated R'-algebra. Now since R' is a finite R-module, we have $A \otimes_R R'$ is a finite A-module and $A \otimes_R R'$ is a finitely generated R-algebra. Thus, A is finitely generated over R. $\square$

We now prove Theorem D.

Theorem 4.4. *Let R be a Noetherian domain with quotient field K and let A be a faithfully flat R-algebra such that*

(i) $A \otimes_R K = K[X_1, X_1{}^{-1}, \ldots, X_n, X_n{}^{-1}]$, *for some $X_1, \ldots, X_n$ transcendental over R.*
(ii) *For each height one prime ideal P of R, $A \otimes_R k(P)$ is geometrically integral over $k(P)$ and there exists a field extension $k(P)'$ of $k(P)$ such that $A \otimes_R k(P) \otimes_{k(P)} k(P)'$ contains a Laurent polynomial algebra in n variables over $k(P)'$.*
(iii) $L_i := A \cap KX_i$ *is a finitely generated projective R-module of rank one, $1 \leqslant i \leqslant n$.*

Then A is a locally Laurent polynomial algebra in n variables over R.

Proof. We may assume that R is local. By Proposition 4.3, we can find a finite birational extension R' of R such that $A \otimes_R R'$ is a locally Laurent polynomial algebra in n variables over R'. Since R is local and R' is a finite birational extension of R, R' is a semilocal domain and hence $\operatorname{Pic} R' = (0)$. Therefore, by Theorem 2.3, $A \otimes_R R' = R'[Y_1, Y_1{}^{-1}, \ldots, Y_n, Y_n{}^{-1}]$ for some elements $Y_1, \ldots, Y_n$ which are transcendental over R' and are chosen such that $KX_i = KY_i$ for $1 \leqslant i \leqslant n$ (cf. Lemma 2.1).

Fix i, $1 \leqslant i \leqslant n$. Since R is local, by hypothesis (iii), $L_i = A \cap KX_i = Rf_i$ for some $f_i \in A(\hookrightarrow R'[Y_1, Y_1{}^{-1}, \ldots, Y_n, Y_n{}^{-1}])$. Since $f_i \in KY_i(= KX_i)$, it follows that there exists $a_i \in R'$ such that $f_i = a_i Y_i$.

We show that a_i is a unit in R'. Let $\mathcal{J}$ be the conductor ideal of R' in R. It then follows that $\mathcal{J}Y_i \subseteq A \cap KX_i = Rf_i = Ra_iY_i$. Therefore, $\mathcal{J}a_i{}^{-1} \subseteq R \subseteq R'$. It follows that $\mathcal{J}a_i{}^{-1}$ is an ideal of R' and hence $\mathcal{J}a_i{}^{-1} \subseteq \mathcal{J}$. Therefore $\mathcal{J} = a_i\mathcal{J}$. Since $\mathcal{J}$ is a non-zero finitely generated ideal of the integral domain R', it follows from NAK Lemma [8, Theorem 2.2, p. 8] that a_i is a unit in R'.

Therefore $A \otimes_R R' = R'[f_1, f_1{}^{-1}, \dots, f_n, f_n{}^{-1}]$. Since $A \otimes_R R'$ is integral over A, $f_i \in A$ and $f_i{}^{-1} \in A \otimes_R R'$, we have $f_i{}^{-1} \in A$. Thus

$$R\big[f_1, f_1{}^{-1}, \dots, f_n, f_n{}^{-1}\big] \subseteq A \subseteq A \otimes_R R' = R'\big[f_1, f_1{}^{-1}, \dots, f_n, f_n{}^{-1}\big].$$

Now by Lemma 4.2, $A = R[f_1, f_1{}^{-1}, \dots, f_n, f_n{}^{-1}]$, a Laurent polynomial algebra in n variables. □

The following lemma shows that an algebra which is stably Laurent polynomial is necessarily a Laurent polynomial algebra.

Lemma 4.5. *Let R be an integral domain and B be a Laurent polynomial algebra in n variables over R. Suppose that A is an R-algebra such that*

(i) *either $A[W_1, \dots, W_t] \cong B[Z_1, \dots, Z_t]$ as R-algebras,*
(ii) *or $A[W_1, W_1{}^{-1}, \dots, W_t, W_t{}^{-1}] \cong B[Z_1, Z_1{}^{-1}, \dots, Z_t, Z_t{}^{-1}]$ as R-algebras,*

for some W_i, Z_i, $1 \leqslant i \leqslant t$, transcendental over A and B respectively. Then $A \cong B$ as R-algebras.

Proof. Suppose that (i) holds. Let $B = R[X_1, X_1{}^{-1}, \dots, X_n, X_n{}^{-1}]$. We may assume that $A[W_1, \dots, W_t] = B[Z_1, \dots, Z_t]$. Since R is an integral domain, A is an integral domain and as $X_1, \dots, X_n$ are units in $A[W_1, \dots, W_t]$, we see that $X_1, X_1{}^{-1}, \dots, X_n, X_n{}^{-1} \in A$. Therefore $B \subseteq A$ and hence $A = B$ because B is algebraically closed in $B[Z_1, \dots, Z_t]$ and $\text{tr.deg}_R\, A = \text{tr.deg}_R\, B$.

Now suppose that (ii) holds. It is enough to consider the case when $t = 1$ and show that if $A[W, W^{-1}] = B[Z, Z^{-1}]$ as R-algebras, then $A \cong_R B$. Let $B = R[X_1, X_1{}^{-1}, \dots, X_n, X_n{}^{-1}]$. Since Z, $X_1, \dots, X_n$ are units in $A[W, W^{-1}]$, we have $Z = \lambda W^{\ell}$ and $X_i = \mu_i W^{a_i}$ for some $\lambda, \mu_i \in A^*$ and $\ell, a_i \in \mathbb{Z}$, $1 \leqslant i \leqslant n$. Again, since W is a unit in $B[Z, Z^{-1}]$, we have $W = \nu X_1{}^{b_1} \cdots X_n{}^{b_n} Z^r$, for some $\nu \in R^*$ and $r, b_i \in \mathbb{Z}$ and hence

$$W = \nu\big(\mu_1 W^{a_1}\big)^{b_1} \cdots \big(\mu_n W^{a_n}\big)^{b_n} \big(\lambda W^{\ell}\big)^r = \nu \mu_1{}^{b_1} \cdots \mu_n{}^{b_n} \lambda^r W^{a_1b_1 + \cdots + a_nb_n + \ell r}.$$

Since $\nu \mu_1{}^{b_1} \cdots \mu_n{}^{b_n} \lambda^r \in A$, we have $\sum_i a_ib_i + \ell r = 1$. Since $\mathbb{Z}$ is PID, the unimodular row $(b_1\ b_2\ \cdots\ b_n\ r)$ can be completed to an invertible $(n+1) \times (n+1)$ matrix, say $M = (b_{ij})$, such that the last row of M is $(b_1\ b_2\ \cdots\ b_n\ r)$. Set $Y_i := X_1{}^{b_{i1}} \cdots X_n{}^{b_{in}} Z^{b_{i(n+1)}}$ for $1 \leqslant i \leqslant n+1$. Then $A[W, W^{-1}] = B[Z, Z^{-1}] = R[Y_1, Y_1{}^{-1}, \dots, Y_{n+1}, Y_{n+1}{}^{-1}] = R[Y_1, Y_1{}^{-1}, \dots, Y_n, Y_n{}^{-1}, W, W^{-1}]$, since $W = \nu Y_{n+1}$ and $\nu \in R^*$. Hence $A \cong R[Y_1, Y_1{}^{-1}, \dots, Y_n, Y_n{}^{-1}]$ and so $A \cong B$. □

Remark 4.6. Let R be a Noetherian domain and B be a faithfully flat finitely generated R-algebra such that all the fibre rings $B \otimes_R k(P)$ are polynomial algebras in n variables (over $k(P)$). From a result of Asanuma [1, Theorem 3.4], it follows that if the module of 1-differential forms $\Omega_{B/R}$ is free, then B is a stably polynomial algebra over R.

Bhatwadekar and Dutta have constructed an explicit example [4, Example 3.9] of a finitely generated faithfully flat algebra A over a one dimensional Noetherian (seminormal) local domain R such that each fibre ring $A \otimes_R k(P)$ is a Laurent polynomial ring in one variable over $k(P)$, $\Omega_{A/R}$ is a free A-module of rank 1 but A is not a Laurent polynomial algebra over R. In view of Lemma 4.5, A is not a stably Laurent polynomial algebra. Thus, a Laurent polynomial analogue of Asanuma's structure theorem (see [1, Theorem 3.4]) does not seem to exist.

Acknowledgments

The authors thank Amartya K. Dutta for suggesting the Laurent polynomial analogue of the Bass–Connell–Wright theorem along the lines of Theorem 2.3 and useful discussions. The second author also thanks the CSIR for their Shyama Prasad Mukherjee Fellowship.

Appendix A. Laurent polynomial forms in *n* variables

Let k be a field and A be a k-algebra. Recall that A is said to be a Laurent polynomial form in n variables if there exists an algebraic field extension F of k such that $A \otimes_k F$ is a Laurent polynomial algebra in n variables over F, if F can be chosen to be separably algebraic, then the Laurent polynomial form A will be called separably algebraic. It was observed in [5, Proposition 2.3] that a separable Laurent polynomial form in one variable is trivial if and only if $k^* \subsetneqq A^*$. We extend the result to n variables; the proof is essentially the same as in the case $n = 1$.

Recall that a field extension F over k (not necessarily algebraic) is said to be separably generated if there exists a transcendence basis $\mathcal{B}$ of F such that F is separably algebraic over $k(\mathcal{B})$.

Proposition A.1. *Let k be a field, A a k-algebra and F a separably generated field extension of k such that $A \otimes_k F$ is a Laurent polynomial algebra in n variables over F. Suppose that A contains a Laurent polynomial algebra in n variables over k. Then A is a Laurent polynomial algebra in n variables over k.*

Proof. Let $A \otimes_k F = F[U_1, {U_1}^{-1}, \ldots, U_n, {U_n}^{-1}]$. Since $A \hookrightarrow A \otimes_k F$, we regard A as a k-subalgebra of $A \otimes_k F$. It is easy to see that there exists a finitely generated separable extension F_1 of k such that $A \otimes_k F_1 = F_1[U_1, {U_1}^{-1}, \ldots, U_n, {U_n}^{-1}]$. Thus replacing F by F_1, we can assume that F is a finitely generated separable extension of k.

We first consider the case when F is a finite separable algebraic extension of k. Replacing F by its normal extension, we can assume that F is a Galois extension of k with Galois group G. Now any $\sigma \in G$ can be extended to an A-automorphism of $A \otimes_k F$, by defining $\sigma(a \otimes \mu) = a \otimes \sigma(\mu)$. Since F over k is a Galois extension, the bilinear map

$$F \times F \to k, \quad \text{sending} \quad (x, y) \to \mathrm{Tr}(xy)$$

is non-degenerate and hence there exist $c_i \in F$ such that $\mathrm{Tr}(c_i U_i) \neq 0$, $1 \leqslant i \leqslant n$. Replacing U_i by $c_i U_i$ we can assume that $\mathrm{Tr}(U_i) \neq 0$. Let $W_i = \mathrm{Tr}(U_i)$. Then $W_i \in A$. We show that $A = k[W_1, {W_1}^{-1}, \ldots, W_n, {W_n}^{-1}]$.

Let $k[X_1, {X_1}^{-1}, \ldots, X_n, {X_n}^{-1}] \subseteq A$, where $X_1, X_2, \ldots, X_n$ are transcendental over k. Then, there exist integers a_{ij}, $1 \leqslant i, j, \leqslant n$ such that

$$X_i = \mu_i {U_1}^{a_{i1}} U_2^{a_{i2}} \cdots U_n^{a_{in}},$$

for some $\mu_i \in F^*$. Let $M = (a_{ij})$ and $M^{\mathrm{ad}} := \mathrm{Adj}(M) = (b_{ij})$ for some $b_{ij} \in \mathbb{Z}$. Since $X_1, X_2, \ldots, X_n$ are transcendental over k, $\det M \neq 0$. Set

$$Y_i := {X_1}^{b_{i1}} {X_2}^{b_{i2}} \cdots {X_n}^{b_{in}} \in A$$

for $1 \leqslant i \leqslant n$. Since $M^{\mathrm{ad}} M = (\det M) I_n$, where I_n is the identity matrix, we have

$$Y_i = \lambda_i {U_i}^{(\det M)},$$

$1 \leqslant i \leqslant n$, for some $\lambda_i \in F^*$. Fix i, $1 \leqslant i \leqslant n$. Replacing Y_i by ${Y_i}^{-1}$, we may assume that $\det M > 0$. Since for any $\sigma \in G$, $\sigma(Y_i) = Y_i$, we have $(\sigma(U_i)/U_i)^{\det M} = (\lambda_i/\sigma(\lambda_i)) \in F^*$. Therefore, $\sigma(U_i) = \nu_\sigma U_i$ for some $\nu_\sigma \in F^*$. Hence $W_i = \mathrm{Tr}(U_i) = d_i U_i$ for some $d_i \in F^*$. Thus $F[W_1, {W_1}^{-1}, \ldots, W_n, {W_n}^{-1}] =$

$F[U_1, U_1{}^{-1}, \ldots, U_n, U_n{}^{-1}]$. Since $A \otimes_k F$ is integral over A, and for $1 \leqslant i \leqslant n$, $W_i{}^{-1} \in A \otimes_k F$, we have $W_i{}^{-1} \in A$. Hence $k[W_1, W_1{}^{-1}, \ldots, W_n, W_n{}^{-1}] \subseteq A$. Since F is faithfully flat over k, we have $k[W_1, W_1{}^{-1}, \ldots, W_n, W_n{}^{-1}] = A$. Now the argument in [5, Proposition 2.3] shows that if F is an arbitrary separable extension over k then also A is a Laurent polynomial algebra. □

The following example shows that a purely inseparable Laurent polynomial form A in one variable over a field k which contains non-trivial units (i.e. $k^* \subsetneq A^*$) need not be trivial.

Example A.2. Let k be a non-perfect field of characteristic p. Let $\beta \in k$ be such that $\beta \notin k^p := \{a^p \mid a \in k\}$. Let $L = k(\alpha)$ be a purely inseparable extension of k such that $\alpha^p = \beta$. Now let $B = k[X, Y]/(X - Y^p - \beta X^p)$. It is known that B is a non-trivial inseparable A^1-form. Now

$$B \otimes_k L = \frac{L[X, Y]}{(X - Y^p - \beta X^p)} = \frac{L[X - (Y + \alpha X)^p, Y + \alpha X]}{(X - (Y + \alpha X)^p)} \cong L[Y + \alpha X].$$

Let $A = B[X^{-1}]$. Since $(Y + \alpha X)^p = X$ in $A \otimes_k L$, we have $A \otimes_k L \cong L[Y + \alpha X, (Y + \alpha X)^{-1}]$, a Laurent polynomial algebra in one variable. Also $k[X, X^{-1}] \subseteq A$. But A is not a Laurent polynomial algebra.

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