The structure of a Laurent polynomial fibration in \( n \) variables

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**Abstract**
Bass, Connell and Wright have proved that any finitely presented locally polynomial algebra in \( n \) variables over an integral domain \( R \) is isomorphic to the symmetric algebra of a finitely generated projective \( R \)-module of rank \( n \). In this paper we prove a corresponding structure theorem for a ring \( A \) which is a locally Laurent polynomial algebra in \( n \) variables over an integral domain \( R \), viz., we show that \( A \) is isomorphic to an \( R \)-algebra of the form \((\text{Sym}_R(Q))[I^{-1}])\), where \( Q \) is a direct sum of \( n \) finitely generated projective \( R \)-modules of rank one and \( I \) is a suitable invertible ideal of the symmetric algebra \( \text{Sym}_R(Q) \). Further, we show that any faithfully flat algebra over a Noetherian normal domain \( R \), whose generic and codimension-one fibres are Laurent polynomial algebras in \( n \) variables, is a locally Laurent polynomial algebra in \( n \) variables over \( R \).

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1. Introduction

Let \( R \) be an integral domain. Recall that an \( R \)-algebra \( A \) is called a Laurent polynomial algebra in \( n \)-variables over \( R \) if \( A = R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \), where \( X_1, X_2, \ldots, X_n \) are transcendental over \( R \). We call an \( R \)-algebra \( A \) to be a locally Laurent polynomial algebra in \( n \) variables over \( R \) if \( A \otimes_R R_m \) is a Laurent polynomial algebra in \( n \) variables over the local ring \( R_m \) for every maximal ideal \( m \) of \( R \).

In this paper we explore the Laurent polynomial analogues of some results and open problems on polynomial (or \( A^n \)) fibrations.

We shall first establish a structure theorem for locally Laurent polynomial algebras, a Laurent polynomial analogue of the famous local–global theorem of Bass, Connell and Wright [3, Theorem 4.4]
which states that any finitely presented locally polynomial algebra in \( n \) variables over a ring \( R \) is isomorphic to the symmetric algebra of a projective \( R \)-module of rank \( n \). While the hypothesis on finite presentation is clearly necessary in the polynomial case (consider the \( \mathbb{Z} \)-algebra \( \mathbb{Z}[X/2, X/3, X/5, \ldots] \)), our structure theorem will show that a locally Laurent polynomial algebra \( A \) over an integral domain \( R \) is necessarily finitely presented and that \( A \) is of the form \( B[I^{-1}] \), where \( B \) is isomorphic to the symmetric algebra \( \text{Sym}_R(Q) \) of a (suitable) finitely generated projective \( R \)-module \( Q \) and \( I \) is an invertible ideal of \( B \). Here \( I^{-1} \) denotes the \( B \)-submodule \( \{a \in F \mid aI \subseteq B\} \) of the quotient field \( F \) of \( B \) and \( B[I^{-1}] \) denotes the subring of \( F \) generated by \( B \) and \( I^{-1} \). The precise statement of the structure theorem (Theorem 2.3) is given below.

**Theorem A.** Let \( R \) be an integral domain and \( A \) be a locally Laurent polynomial algebra in \( n \) variables over \( R \). Then there exist \( n \) finitely generated rank one projective \( R \)-modules \( L_i \), \( 1 \leq i \leq n \), such that \( A \) is isomorphic to an \( R \)-algebra of the form

\[
(\text{Sym}_R(Q))[I^{-1}],
\]

where \( Q = L_1 \oplus \cdots \oplus L_n \) and \( I \) is an invertible ideal of \( \text{Sym}_R(Q) \) generated by the image of \( L_1 \otimes \cdots \otimes L_n \). In particular, \( A \) is finitely presented over \( R \). If \( \text{Pic}(R) = (0) \), then \( A \) is a Laurent polynomial algebra over \( R \).

After describing the structure of a locally Laurent polynomial algebra, we investigate sufficient conditions for an \( R \)-algebra to be locally Laurent polynomial. Note that any locally Laurent polynomial \( R \)-algebra is faithfully flat over \( R \). Now suppose that \( R \) is a Noetherian normal domain and \( A \) is a faithfully flat \( R \)-algebra. Under these hypotheses, we shall see that \( A \) is a locally Laurent polynomial algebra in \( n \) variables over \( R \) if \( A \otimes_R R_P \) is a Laurent polynomial algebra in \( n \) variables over \( R_P \) for every prime ideal \( P \) in \( R \) of height one (Proposition 2.7). This result was proved in [6, Theorem 4.8] for the case \( n = 1 \).

Next we consider the following fibration problem:

**Question.** Under what (minimal) fibre conditions will a faithfully flat algebra \( A \) over a Noetherian domain \( R \) be a locally Laurent polynomial algebra?

We first investigate the case when \( R \) is a discrete valuation ring (DVR) and prove (Theorem 3.5):

**Theorem B.** Let \( (R,t) \) be a discrete valuation ring with a regular parameter \( t \), quotient field \( K \) and residue field \( k \). Let \( A \) be an integral domain containing \( R \) such that

(i) \( A[1/t] \) is a Laurent polynomial algebra in \( n \) variables over \( K \).
(ii) \( A/tA \) is a Laurent polynomial algebra in \( n \) variables over \( k \).

Then \( A \) is a Laurent polynomial algebra in \( n \) variables over \( R \).

Recall that for any \( P \in \text{Spec} R \), \( k(P) \) denotes the quotient field of \( R/P \) and that \( A \otimes_R k(P) \) is the fibre ring of an \( R \)-algebra \( A \) over \( P \). Using Theorem B and Proposition 2.7, we shall show that for any faithfully flat algebra \( A \) over a Noetherian normal domain \( R \) to be locally Laurent polynomial, it is enough to ensure that the generic and codimension-one fibres of \( A \) are Laurent polynomial algebras in \( n \) variables. In fact, we prove (Theorem 3.6):

**Theorem C.** Let \( R \) be a Noetherian normal domain with quotient field \( K \) and \( A \) be a faithfully flat \( R \)-algebra such that

(i) The generic fibre \( A \otimes_R K \) is a Laurent polynomial algebra in \( n \) variables over \( K \).
(ii) For each height one prime ideal \( P \) in \( R \), \( A \otimes_R k(P) \) is a Laurent polynomial algebra in \( n \) variables over \( k(P) \).

Then \( A \) is a locally Laurent polynomial algebra in \( n \) variables over \( R \).
For the case \( n = 1 \), this result was proved earlier in [5, Theorem 3.11] under the additional hypothesis that \( A \) is finitely generated.

Finally we consider an arbitrary Noetherian domain. An example (see [4, Example 3.9]) of Bhatwadekar and Dutta shows that, even for \( n = 1 \), Theorem C cannot be extended to non-normal domains without additional hypotheses. We give the following necessary and sufficient condition for extending Theorem C to an arbitrary Noetherian domain (Theorem 4.4):

**Theorem D.** Let \( R \) be a Noetherian domain with quotient field \( K \) and let \( A \) be a faithfully flat \( R \)-algebra such that

(i) \( A \otimes_R K = K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}], X_1, \ldots, X_n \) are transcendental over \( R \).

(ii) For each height one prime ideal \( P \) in \( R \), \( A \otimes_R k(P) \) is a Laurent polynomial algebra in \( n \) variables over \( k(P) \).

(iii) \( L_i := A \cap KX_i \) is a finitely generated projective \( R \)-module of rank one, \( 1 \leq i \leq n \).

Then \( A \) is a locally Laurent polynomial algebra in \( n \) variables over \( R \).

However, even without the hypothesis (iii), we shall show (Proposition 4.3) that \( A \) is at least finitely generated over \( R \) and that \( A \otimes_R R' \) is locally Laurent polynomial over a finite birational extension \( R' \) of \( R \).

Theorem A will be proved in Section 2, Theorems B and C in Section 3 and Theorem D in Section 4.

We recall some standard notation to be used throughout the paper. For a ring \( R \), \( R^+ \) will denote the multiplicative group of units of \( R \). For a prime ideal \( P \) of \( R \), and an \( R \)-algebra \( A \), \( A_P \) denotes the ring \( S^{-1}A \), where \( S = R \setminus P \) and \( k(P) \) denotes the residue field \( R_P/PR_P \). The notation \( A = R^{[1]} \) will mean that \( A \) is isomorphic, as an \( R \)-algebra, to a polynomial ring in one variable over \( R \).

We also recall a few definitions (cf. [8, p. 80]). Let \( R \) be an integral domain with quotient field \( K \). A non-zero \( R \)-submodule \( L \) of \( K \) is said to be a *fractional ideal* if there exists a non-zero element \( \alpha \in R \) such that \( \alpha L \subseteq R \). A fractional ideal \( L \) is said to be invertible if \( L^{-1}L = R \), where \( L^{-1} = \{ \alpha \in K \mid \alpha L \subseteq R \} \).

**2. On locally Laurent polynomial algebra in \( n \) variables**

In this section we shall prove Theorem A. Throughout this section, \( R \) will denote an integral domain with quotient field \( K \) and \( A \) an integral domain containing \( R \) such that \( A \cap K = R \) and

\[
A \otimes_R K = K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]
\]

for some \( X_1, \ldots, X_n \) transcendental over \( R \). In this set up, we shall use the following notation. For \( 1 \leq i \leq n \) and \( j \geq 0 \), set

\[
C_{ij} := A \cap KX_i^j \quad \text{and} \quad D_{ij} := A \cap KX_i^{-j},
\]

\[
C := \bigoplus_{(j_1, \ldots, j_n) \in \mathbb{Z}_{\geq 0}^n} C_{1j_1} \cdots C_{nj_n}, \quad \text{where} \quad C_{1j_1} \cdots C_{nj_n} = \{ c_1 \cdots c_n \mid c_\ell \in C_{\ell j_\ell} \} \quad \text{is an} \ R\text{-submodule of} \ A \cap KX_1^{j_1} \cdots KX_n^{j_n},
\]

\[
I := \text{the ideal of} \ C \ \text{generated by} \ C_{11} \cdots C_{nn} \quad \text{and} \quad B := A \cap K[X_1, \ldots, X_n].
\]

Note that \( C \) is an \( R \)-subalgebra of \( B \). Note also that for \( g \in C_{ij} \) and \( h \in D_{ij}, gh \in A \cap K = R \). Therefore we get an \( R \)-linear map

\[
\psi_{ij} : C_{ij} \otimes_R D_{ij} \to R \quad \text{defined by} \ \psi_{ij}(g \otimes h) = gh.
\]
Let $A$ be a Laurent polynomial algebra in $n$ variables over $R$. Then there exist $\alpha_i \in K^*$ and $U_i \in A$ such that $U_i = \alpha_i X_i$, $1 \leq i \leq n$, $A = R[U_1, U_1^{-1}, \ldots, U_n, U_n^{-1}]$ and $B = R[U_1, \ldots, U_n]$. Further, for $1 \leq i \leq n$ and $j \geq 0$,

$$C_{ij}(A \cap KX_i^j) = RU_i^j, \quad D_{ij}(A \cap KX_i^{-j}) = RU_i^{-j}$$

and hence $J_{ij} = R$, $C = B = R[U_1, \ldots, U_n]$, $I = (U_1 \cdots U_n)C$, and

$$A = B[I^{-1}] = C[I^{-1}].$$

**Proof.** Let $A = R[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}]$. Then

$$K[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}] = K[Y_1, \ldots, Y_n, Y_1^{-1}, \ldots, Y_n^{-1}].$$

It follows that for each $i$, $1 \leq i \leq n$,

$$Y_i = \lambda_i X_1^{a_{i1}} X_2^{a_{i2}} \cdots X_n^{a_{in}} \quad \text{and} \quad X_i = \mu_i Y_1^{b_{i1}} Y_2^{b_{i2}} \cdots Y_n^{b_{in}}$$

for some $\lambda_i, \mu_i \in K \setminus \{0\}$ and $a_{ij}, b_{ij} \in \mathbb{Z}$, $1 \leq j \leq n$, satisfying

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$ 

For $1 \leq i \leq n$, set $\alpha_i := \mu_i^{-1}$ and

$$U_i := \alpha_i X_i = Y_1^{b_{i1}} Y_2^{b_{i2}} \cdots Y_n^{b_{in}}.$$

Then, $K[X_1, \ldots, X_n] = K[U_1, \ldots, U_n]$, and for $1 \leq i \leq n$,

$$Y_i = U_1^{a_{i1}} U_2^{a_{i2}} \cdots U_n^{a_{in}}.$$

Hence

$$A = R[U_1, U_1^{-1}, \ldots, U_n, U_n^{-1}], \quad B = R[U_1, \ldots, U_n]$$

and $A \cap KX_i^j(= A \cap KU_i^j) = RU_i^j$ for every $i$, $1 \leq i \leq n$ and every $j \in \mathbb{Z}$. Thus $C = B$, $I = (U_1 \cdots U_n)C$ and $A = B[I^{-1}] = C[I^{-1}]$. □

In the general case (i.e., when $A$ is not necessarily a Laurent polynomial algebra over $R$), we give below a sufficient condition for $C$ to be the symmetric algebra of a finitely generated projective $R$-module of rank $n$ and $I$ to be an invertible ideal of $C$. 
Lemma 2.2. Suppose that \( J_{ij} = R \) \( \forall i, 1 \leq i \leq n \). Then for each \( i, 1 \leq i \leq n \), and \( j \geq 0 \),

(I) \( J_{ij} = R \).

(II) \( C_{ij} \) and \( D_{ij} \) are finitely generated projective \( R \)-modules of rank one.

(III) The canonical map \( \theta_{ij} : C_{i1} \otimes_R C_{i1} \otimes_R \cdots \otimes_R C_{i1} \) \((j\text{-times}) \to C_{ij} \) is an isomorphism.

(IV) There is a natural \( R \)-algebra isomorphism

\[
C \left( = \bigoplus_{(j_1, \ldots, j_m) \in \mathbb{Z}^\geq 0^n} C_{ij_1} \cdots C_{ij_m} \right) \cong \text{Sym}_R(C_{i1} \oplus \cdots \oplus C_{in}).
\]

(V) The ideal \( I \) of \( C \) generated by \( C_{i1} \cdots C_{in} \) is an invertible ideal.

Proof. Fix \( i, 1 \leq i \leq n \), and \( j \geq 0 \). Note that \( C_{ij} \) and \( D_{ij} \) are torsion-free \( R \)-modules of rank one. Moreover, if \( f \in C_{i1} \) and \( g \in D_{ij} \) then \( f^j \in C_{ij}, g^j \in D_{ij} \) and \( f^j g^j = (fg)^j \in R \).

(I) Since \( J_{i1} = \psi_{i1}(C_{i1} \otimes_R D_{i1}) = R \), there exist \( c_\ell \in C_{i1} \) and \( d_\ell \in D_{i1} \), \( 1 \leq \ell \leq r \) for some \( r \), such that

\[
\psi_{i1} \left( \sum_{\ell} c_\ell \otimes d_\ell \right) = \sum_{\ell} c_\ell d_\ell = 1.
\]

Set \( a_\ell := c_\ell d_\ell \). As \( c_\ell^j \in C_{ij} \) and \( d_\ell^j \in D_{ij} \), we have \( a_\ell^j = c_\ell^j d_\ell^j = \psi_{i1}(c_\ell^j \otimes d_\ell^j) \in J_{ij} \) for each \( \ell \). Since \( \sum a_\ell = 1 \), we have \( (a_\ell^j, \ldots, a_\ell^j) R = R \) and hence \( J_{ij} = R \).

(II) Set \( L := C_{ij} X_i^{−j} \) and \( E := D_{ij} X_i^{j} \). Clearly \( L \) and \( E \) are non-zero \( R \)-submodules of \( K \) such that \( LE(= C_{ij} D_{ij}) \subseteq A \cap K = R \). Thus \( L \) and \( E \) are fractional ideals. Since \( R = J_{ij} = \psi_{ij}(C_{ij} \otimes_R D_{ij}) \), there exist \( f_s \in C_{ij} \) and \( g_s \in D_{ij} \), \( 1 \leq s \leq t \) for some \( t \), such that

\[
1 = \sum_s f_s g_s = \sum_s (f_s X_i^{−j})(g_s X_i^{j}) \in (C_{ij} X_i^{−j})(D_{ij} X_i^{j}) = LE.
\]

Therefore \( L \) and \( E \) are invertible ideals of \( K \) with \( E = L^{−1} \) and \( L = E^{−1} \) and hence \( C_{ij} \) and \( D_{ij} \) are finitely generated projective \( R \)-modules of rank one (cf. [8, p. 80]).

(III) Set \( C(ij) := \theta_{ij}(C_{ij} \otimes_R C_{ij} \otimes_R \cdots \otimes_R C_{ij}) \). Since \( C(ij) \subseteq C_{ij} \), it is enough to show that \( C(ij)_m = (C_{ij})_m \) for every maximal ideal \( m \) of \( R \). Fix a maximal ideal \( m \) of \( R \). By (II), \( (C_{ij})_m = R_m f_{ij} \) for some \( f_{ij} \in (C_{ij})_m \). Since \( (J_{ij})_m = R_m \), we have \( (D_{ij})_m = R_m f_{ij}^{-1} \) and so \( f_{ij}^{-1} \in A_m \). Now, since \( f_{i1} f_{ij} \in (C_{ij})_m = R_m f_{ij}, \) we have \( f_{i1} = \lambda_{ij} f_{ij} \) for some \( \lambda_{ij} \in R_m \). Hence \( \lambda_{ij}^{-1} = f_{i1}^{-1} f_{ij} \in A_m \cap K = R_m \).

Thus, \( f_{ij} \in R_m f_{i1} \subseteq C(ij)_m \). Hence the result follows.

(IV) follows from (II) and (III).

(V) By (II), \( C_{i1}, \ldots, C_{in} \) are finitely generated projective \( R \)-modules and hence the ideal \( I \) of \( C \) is finitely generated and for every prime ideal \( p \) of \( R \), \( I_p \) is a principal ideal. Thus, for any prime ideal \( P \) of \( C \), if \( p = P \cap R \), then \( I_p \) being a further localisation of \( I_p \) is principal and so \( I \) is an invertible ideal (see [8, Theorem 11.3]). \( \square \)

We now prove Theorem A.

Theorem 2.3. Let \( R \) be an integral domain with quotient field \( K \) and \( A \) be a locally Laurent polynomial algebra in \( n \) variables over \( R \). Then there exist \( n \) finitely generated rank one projective \( R \)-modules \( L_i, 1 \leq i \leq n \), such that \( A \) is isomorphic to an \( R \)-algebra of the form

\[
(\text{Sym}_R(Q))[I^{-1}].
\]
where \( Q = L_1 \oplus \cdots \oplus L_n \) and \( I \) is an invertible ideal of \( \text{Sym}_R(Q) \) generated by the image of \( L_1 \otimes \cdots \otimes L_n \). In fact, if \( A \otimes_R K = K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \) and \( B = A \cap K[X_1, \ldots, X_n] \), then we may choose \( I \) to be \( A \cap KX_i \) and \( \text{Sym}_R(Q) \) may be identified with the ring \( B \). In particular, \( A \) is finitely presented over \( R \).

If \( \text{Pic}(R) = 0 \), then \( A \) is a Laurent polynomial algebra over \( R \).

**Proof.** As before, \( C_{ij} = A \cap KX_{ij} \), \( D_{ij} = A \cap KX_{ij}^{-1} \), and \( j_{ij} \) is the image of the canonical map \( \psi_{ij} : C_{ij} \otimes_R D_{ij} \to R \) defined by \( \psi_{ij}(g \otimes h) = gh \), \( 1 \leq i \leq n \), \( j \geq 0 \).

Fix \( i, 1 \leq i \leq n \). For any maximal ideal \( m \) of \( R \), since \( A_m \) is a Laurent polynomial algebra in \( n \) variables over \( R_m \), it follows from Lemma 2.1 that \( (J_{1i})_m = R_m \). Hence, by Lemma 2.2, \( L_i = C_{1i} \) is a finitely generated projective \( R \)-module of rank one, \( \text{Sym}_R(L_1 \oplus \cdots \oplus L_n) \) may be identified with the subring \( \mathcal{C}(= \bigoplus_{(j_1, \ldots, j_n) \in \mathbb{Z}^n} C_{j_1} \cdots C_{j_n}) \) of \( A \), and the ideal \( I \) of \( C \) generated by \( L_1 \cdots L_n \) is invertible.

Since \( C \subseteq B \) and, by Lemma 2.1, \( C_m = B_m \) for every maximal ideal \( m \) of \( R \), we have \( C = B \). Therefore, to complete the proof, we only need to show that \( A = C[I^{-1}] \). Since \( D_{11} \subseteq A \) and \( J_{11} = R \), we have \( 1 \in J_{11} \subseteq C_{11}A \), i.e., \( C_{11}A = A \). Hence \( IA = A \). Therefore \( C[I^{-1}] \subseteq A \). Hence, it is enough to show that \( A_m = C_m[I_m^{-1}] \) for every maximal ideal \( m \) of \( R \). This follows from Lemma 2.1, since \( A_m \) is a Laurent polynomial algebra in \( n \) variables over \( R_m \). \( \Box \)

**Remark 2.4.** (i) With the notation described at the beginning of this section, we have seen in Theorem 2.3 that if \( A \) is a locally Laurent polynomial, then \( C = B \). This need not hold in general. Consider the faithfully flat \( \mathbb{Z} \)-algebra \( A = \mathbb{Z}[X, X^{-1}] \). Here \( \mathbb{A} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[X, X^{-1}] \) so that \( B = \mathbb{Z}[\frac{1}{2}(X + 3)] \), but \( C = \mathbb{Z}[X] \not\subseteq B \).

(ii) Note that Theorem 2.3 is proved in two steps. The first step is to prove that each \( C_{1i} \) is a finitely generated projective \( R \)-module of rank one and \( C \cong \text{Sym}_R(C_{11} \oplus \cdots \oplus C_{1n}) \). The second step is to show that \( A = C[I^{-1}] \) where \( I \) is the invertible ideal of \( C \) generated by \( C_{11} \cdots C_{1n} \). We have seen that if \( J_{11} = R \) for each \( i \), then one achieves the first step (cf. Lemma 2.2). Moreover, in this case, since \( D_{11} \subseteq A \), we have \( J_{11} \subset C_{11}A \) and hence \( C_{11}A = A \). Thus \( IA = A \). Therefore \( C[I^{-1}] \subseteq A \).

Now suppose that \( R \) is Noetherian or Krull and \( A \) is a faithfully flat \( \mathbb{R} \)-algebra such that \( A_P (= A \otimes_R R_P) \) is a Laurent polynomial algebra in \( n \) variables over \( R_P \) for every prime ideal \( P \) of \( R \) for which depth \( R_P = 1 \). Under these hypotheses we will show (Proposition 2.7) that \( A \) is in fact a locally Laurent polynomial algebra in \( n \) variables over \( R \). As in the proof of Theorem 2.3, we will first show that \( J_{11} = R \) for each \( i \) (Lemma 2.6) and then show that \( A = C[I^{-1}] (= B[I^{-1}] \).

We first state a lemma; the proof will follow from the argument in [7, Lemma 2.8]. Note that for a prime ideal \( P \) of a Krull domain \( R \), depth \( R_P = 1 \) if and only if \( hP = 1 \).

**Lemma 2.5.** Let \( R \) be an integral domain with quotient field \( K \) which is either a Noetherian or a Krull domain and let \( \Delta \) be the set of all prime ideals \( P \) of \( R \) such that depth \( R_P = 1 \). For a torsion-free \( R \)-module \( M \), the following conditions are equivalent:

(i) \( M = \bigcap_{P \in \Delta} M_P \), where \( M \) and \( M_P = M \otimes_R R_P \) are identified with their images in \( M \otimes_R K \).

(ii) For every \( a, b \in R \) such that \( \langle aR : b \rangle = aR \), we have \( \langle aM : b \rangle = aM \).

In particular, if \( M \) is \( R \)-flat then \( M = \bigcap_{P \in \Delta} M_P \).

The following is the key lemma for proving Proposition 2.7. This lemma was proved in [6, Lemma 4.2] for \( n = 1 \). For convenience, we give a proof in our generalised setup.

**Lemma 2.6.** Let \( R \) be an integral domain which is either a Noetherian or a Krull domain and let \( A \) be a faithfully flat \( \mathbb{R} \)-algebra such that \( A_P \) is a Laurent polynomial algebra in \( n \) variables for every prime ideal \( P \) of \( R \) such that depth \( R_P = 1 \). Then \( C_{1i} \) and \( D_{1i} \) are finitely generated projective \( R \)-modules of rank one and \( J_{1i} = R \) for each \( i, 1 \leq i \leq n \).
Proof. We first show that the canonical map $C_{11} \otimes_R A \to C_{11} A$ is an isomorphism and $C_{11} A = A$.

Since $C_{11} \to KR_1$ and $A$ is $R$-flat, we have $C_{11} \otimes_R A \to KR_1 \otimes_R A \cong K[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$. Thus $C_{11} \otimes_R A$ is a torsion-free $A$-module of rank one. Now if the canonical map $C_{11} \otimes_R A \to C_{11} A$ is not injective then the kernel of this map is a non-zero torsion-free $A$-submodule of $C_{11} \otimes_R A$, which contradicts that the rank of $C_{11} \otimes_R A$ is one. Thus, the canonical map $C_{11} \otimes_R A \to C_{11} A$ is injective and hence an isomorphism.

Let $\Delta$ denote the set of all prime ideals $P$ of $R$ such that $\text{depth}R_P = 1$. For every $P \in \Delta$, since $A_P$ is a Laurent polynomial algebra, we have, by Lemma 2.1, $(J_{11})_P = R_P$; in particular $J_{11} \not\subseteq P$. Choose a non-zero element $x \in J_{11}$. Since $R$ is either Noetherian or Krull, $\text{Ass}_R(R/xR)$ is a finite subset of $\Delta$. Therefore, by prime avoidance, we see that $J_{11} \not\subseteq \bigcup_{P \in \text{Ass}_R(R/xR)} P$. Choose $y \in J_{11} \setminus \bigcup_{P \in \text{Ass}_R(R/xR)} P$. Then $\{x, y\} \subseteq J_{11}$ forms a regular sequence in $R$, i.e., $(xR : y) = xR$.

Since $C_{11} = A \cap KR_1$ and $A$ is $R$-flat, $C_{11} = \bigcap_{P \in \Delta} (C_{11})_P$ by Lemma 2.5. Therefore, again by Lemma 2.5, $(xC_{11} : y) = xC_{11}$, i.e., $\{x, y\}$ forms a regular sequence in $C_{11}$. Since $A$ is $R$-flat and $C_{11} \otimes_R A \cong C_{11} A$, it follows that $\{x, y\}$ forms a regular sequence in $C_{11} A$ and hence $(xC_{11} A : y) = xC_{11} A$. Since $D_{11} A \subseteq A$, we have $J_{11} A \subseteq C_{11} A$. Thus $x, y \in C_{11} A$ and hence $xy \in C_{11} A$. Therefore $x \in (xC_{11} A : y) = xC_{11} A$. Thus $C_{11} A = A$.

Since $C_{11} \otimes_R A (\equiv C_{11} A = A)$ is a free $A$-module of rank one and $A$ is faithfully flat over $R$, it follows that $C_{11}$ is a finitely presented flat and hence a projective $R$-module of rank one. Similarly $D_{11}$ is a finitely generated projective $R$-module of rank one. Thus $C_{11} \otimes_R D_{11}$ is a finitely generated projective $R$-module of rank one. Since $\psi_{11}(C_{11} \otimes_R D_{11}) = J_{11}$, $R$ is a domain and $J_{11} \neq 0$, we see that $\psi_{11}$ is an isomorphism and hence $J_{11}$ is a finitely generated projective $R$-module (of rank one). Therefore, by Lemma 2.5,

$$J_{11} = \bigcap_{P \in \Delta} (J_{11})_P = \bigcap_{P \in \Delta} R_P = R,$$

because $(J_{11})_P = R_P$ for every $P \in \Delta$. Thus the lemma is proved.  

Proposition 2.7. Let $R$ be an integral domain with quotient field $K$ which is either a Noetherian or a Krull domain and let $A$ be a faithfully flat $R$-algebra such that $A_P$ is a Laurent polynomial algebra in $n$ variables over $R_P$ for every prime ideal $P$ in $R$ such that $\text{depth}R_P = 1$. Then $A$ is a locally Laurent polynomial algebra in $n$ variables over $R$.

Proof. It is enough to assume that $R$ is local. By Lemma 2.6, $C_{11} = R[f_i]$ for some $f_i \in A$ and $J_{11} = R$ for $1 \leq i \leq n$. Therefore, by Lemma 2.2, $C = R[f_1, \ldots, f_n]$, $I = (f_1 \cdots f_n)C$ and hence $C[I^{-1}] = R[f_1, f_1^{-1}, \ldots, f_n, f_n^{-1}]$. We now show that $A = C[I^{-1}]$.

Let $\Delta$ denote the set of all prime ideals $P$ of $R$ such that $\text{depth}R_P = 1$. Since, for every $P \in \Delta$, $A_P$ is a Laurent polynomial algebra in $n$ variables over $R_P$, we have $A_P = C_P[I_P^{-1}]$ by Lemma 2.1. Hence, as both $A$ and $C[I^{-1}] = (R[f_1, f_1^{-1}, \ldots, f_n, f_n^{-1}])$ are $R$-flat and are submodules of the quotient field of $A$, we have $A = C[I^{-1}]$ by Lemma 2.5.

Remark 2.8. In contrast to Proposition 2.7, if $R$ is a Noetherian local domain (or even a regular local ring) and $B$ is a faithfully flat finitely generated $R$-algebra such that $B_P$ is a polynomial algebra in $n$ variables over $R_P$ for every prime ideal $P$ in $R$ satisfying $\text{depth}R_P = 1$, then $B$ need not be a polynomial algebra. Consider

$$R = \mathbb{C}[\pi_1, \pi_2], \quad B = R[X, Y, Z]/(\pi_2X + \pi_1Y + Z^2 + 1).$$

3. Laurent polynomial fibration over a Noetherian normal domain

In this section we shall prove Theorems B and C. We first prove Theorem B. The proof will require an auxiliary lemma. We will use the following version of the dimension inequality (cf. [8, Theorem 15.5, p. 118]).
Theorem 3.1. Let $R$ be a Noetherian integral domain and $B$ an integral domain containing $R$. Let $P$ be a prime ideal of $B$ and $p = P \cap R$. Then

$$\text{ht } P + \text{tr.deg}_{R/p} B/P \leq \text{ht } p + \text{tr.deg}_{R} B.$$

As a consequence of Theorem 3.1, we have the following corollary.

Corollary 3.2. Let $(R, t)$ be a discrete valuation ring with a regular parameter $t$ and residue field $k$. Let $B$ be an integral domain containing $R$ such that $tB$ is a prime ideal of $B$. Then $\text{tr.deg}_{R} B/tB \leq \text{tr.deg}_{R} B$.

We state below a result, the proof of which will follow from [2, Proposition 6.1 and Theorem 6.3].

Theorem 3.3. Let $(R, t)$ be a discrete valuation ring with a regular parameter $t$, quotient field $K$ and residue field $k$. Let $D$ be an integral domain containing $R$ such that $\text{deg } tD = 1$. Therefore, as $D$ is a finitely generated $R$-algebra and there exists a finite algebraic field extension $F$ of $k$ such that $D/tD = K^{[1]}$.

We now prove a lemma over discrete valuation rings which will be used in the proof of Theorem B.

Lemma 3.4. Let $(R, t)$ be a discrete valuation ring with a regular parameter $t$, quotient field $K$ and residue field $k$. Let $B$ be an integral domain containing $R$. Let $P$ be a prime ideal of $B$.

(i) $B[1/t] = K[X_1, \ldots, X_{n}]$, a polynomial ring in $n$ variables over $K$.
(ii) $B/tB$ is an integral domain and $\text{tr.deg}_{R} B/tB = n$.

Set $B_0 := R$ and for $1 \leq i \leq n$, set $B_i := B \cap K[X_1, X_2, \ldots, X_i]$. Then

(i) $B_{i+1}[1/t] = B_i[1/t][X_{i+1}]$ for $0 \leq i \leq n - 1$.
(ii) $tB_i$ is a prime ideal of $B_i$ of height one and $tB_i = tB_{i+1} \cap B_i$, so that

$$k \hookrightarrow B_1/tB_1 \hookrightarrow \cdots \hookrightarrow B_n/tB_n = B/tB.$$

Let $F_i$ denote the quotient field of $B_i/tB_i$.

(iii) $\text{tr.deg}_{R} B_i/tB_i(= \text{tr.deg}_{F_i} F_i) = i, 0 \leq i \leq n$.

(iv) $(B_{i+1}/tB_{i+1}) \otimes_{B_i/tB_i} F_i = K^{[1]}$ for some finite algebraic field extension $K_i$ of $F_i$.

Proof. (i) follows easily from the definition of $B_i$'s.

(ii) $tB_i = tB \cap K[X_1, \ldots, X_i] = tB \cap B_i$. Since by (ii) $tB$ is prime ideal of $B$, we have $tB_i$ is a prime ideal of $B_i$.

Since $\text{tr.deg}_{R} B/tB = \text{tr.deg}_{R} B = n$, and $\text{ht}(R) = 1$, from the dimension inequality (Theorem 3.1) we have $\text{ht}(tB) \leq 1$. Therefore, as $B$ is an integral domain, we have $\text{ht}(tB) = 1$ and hence $\bigcap_{n \geq 0} t^n B = (0)$. Since $B_i \subset B$ for each $i, 1 \leq i \leq n$, it follows that $\bigcap_{n \geq 0} t^n B_i = (0)$, which implies that $\text{ht}(tB_i) = 1$.

Also since $B_i \subset B_{i+1}$ for each $i, 0 \leq i \leq n - 1$, we have $tB_i = tB \cap B_i = tB \cap B_{i+1} \cap B_i = tB_{i+1} \cap B_i$.

(iii) We first note that by (ii), $V_i := B_{i}(tB_i)$ is a discrete valuation ring with residue field $F_i$. Set $E_{i+1} := B_{i+1} \otimes_{B_i} V_i$, a localisation of $B_{i+1}$. Then $E_{i+1}[1/t] = V_i[1/t][X_{i+1}]$ by (i). Since $tE_{i+1}$ is a prime ideal of $E_{i+1}$, we have $\text{tr.deg}_{F_i} (E_{i+1}/tE_{i+1}) \leq 1$ by Corollary 3.2, i.e., $\text{tr.deg}_{B_i/tB_i} (B_{i+1}/tB_{i+1}) \leq 1$. But $\text{tr.deg}_{R} B_n/tB_n = n$ and $\text{tr.deg}_{R} B_0/tB_0 = 0$. Hence $\text{tr.deg}_{R} B_i/tB_i = i$ \forall i.
(IV) Note that \( E_{i+1}[1/t] = V_i[1/t][X_{i+1}] \) by (I) and \( tE_{i+1} \) is a prime ideal of \( E_{i+1} \) by (II). Also \( \text{tr.deg}(E_{i+1}/E_{i+1}) = \text{tr.deg}(B_{i+1}/B_{i+1}) = 1 \). Hence by Theorem 3.3, \( (B_{i+1}/tB_{i+1}) \otimes_{B_{i+1}/B_{i+1}} F_i = E_{i+1}/tE_{i+1} = K_i^{[1]} \) for some finite algebraic field extension \( K_i \) of \( F_i \). □

We now prove Theorem B. Over a field \( k \), we shall call a \( k \)-algebra \( A \) to be a Laurent polynomial form in \( n \) variables if there exists an algebraic field extension \( F \) of \( k \) such that \( A \otimes_k F \) is a Laurent polynomial algebra in \( n \) variables over \( F \).

**Theorem 3.5.** Let \((R, t)\) be a discrete valuation ring with a regular parameter \( t \), quotient field \( K \) and residue field \( k \). Let \( A \) be an integral domain containing \( R \) such that \( A \otimes_k F \) is a Laurent polynomial algebra in \( n \) variables over \( K \). Then the following statements are equivalent:

(i) \( A \) is a Laurent polynomial algebra in \( n \) variables over \( R \).

(ii) \( A/tA \) is a Laurent polynomial algebra in \( n \) variables over \( k \).

(iii) \( A/tA \) is a Laurent polynomial form in \( n \) variables over \( k \).

(iv) There exists a field extension \( F \) of \( k \) such that \( A/tA \otimes_k F \) is an integral domain and contains a Laurent polynomial algebra in \( n \) variables over \( F \).

**Proof.** The implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) are obvious. We prove (iv) \( \Rightarrow \) (i).

Since \( F \) is faithfully flat over \( k \), we regard \( A/tA \) as a \( k \)-subalgebra of the integral domain \( A/tA \otimes_k F \). Note that \( t \) is a prime in \( A \). We first show that we can choose \( X_1, \ldots, X_n \) in \( A \) such that

\[
R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \subseteq A \subseteq A[1/t] = K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}].
\]

Choose \( T_1, \ldots, T_n \in A \) such that \( A[1/t] = K[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}] \). Fix an integer \( i, 1 \leq i \leq n \). Let \( m_i \in \mathbb{Z}_{\geq 0} \) be the least integer such that \( t^{m_i}T_i^{-1} \in A \). Let \( X_i := T_i/t^{m_i} \). Then \( X_i^{-1} \in A \). If \( m_i = 0 \), then \( X_i \in A \) and we are through. If not, since \( t^{m_i} = X_i^{-1}T_i \in A \) and \( t \) is a prime in \( A \), we have either \( t | X_i^{-1} \) or \( t^{m_i} | T_i \) in \( A \). If \( t | X_i^{-1} \), then \( t^{m_i-1}T_i^{-1} \in A \), which contradicts the minimality of \( m_i \). This shows that \( t^{m_i} | T_i \), and hence \( X_i \in A \).

We shall show that \( A = R[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}] \) for the above choice of \( X_1, X_2, \ldots, X_n \).

Set \( B := A \cap K[X_1, \ldots, X_n] \) and \( C := R[X_1, X_2, \ldots, X_n] \). We show that \( B = C \). We first observe that

1. \( C \subseteq B \subseteq B[1/t] = K[X_1, \ldots, X_n] = C[1/t] \).
2. \( tB = tA \cap B \) and hence \( t \) is prime in \( B \).
3. \( A = B_{X_1 \cdots X_n} (= B[(X_1 \cdots X_n)^{-1}]) \).
4. \( t \) does not divide \( X_1 \cdots X_n \) in \( B \).
5. \( \text{tr.deg}_k B/tB = n \).

(1) is obvious; (2) follows from the relation \( tB = tA \cap K[X_1, \ldots, X_n] = tA \cap B \). To see (3), note that if \( h \in A \), then there exists \( \ell \in \mathbb{Z}_{\geq 0} \) such that \( (X_1 \cdots X_n)^{\ell}h \in K[X_1, \ldots, X_n] \cap A (= B) \), so that \( h \in B_{X_1 \cdots X_n} \).

(4) follows from the fact that \( B \subseteq A, X_1 \cdots X_n \) is a unit in \( A \) and \( t \) is a prime element of \( A \). Since \( \text{tr.deg}_k A/tA = n \) by hypothesis (iv) and Corollary 3.2, (5) follows from (3) and (4).

We now show that the map from \( C/tC \) to \( B/tB \) is one-to-one. Suppose not. Let \( x_i \) denote the image of \( X_i \) in \( B/tB \) for \( 1 \leq i \leq n \). Note that, by (4), \( x_i \) is a non-zero element of \( B/tB \). As the map from \( C/tC \) to \( B/tB \) is not one-to-one, \( x_j \) is algebraic over \( k[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n] \ )(\leftrightarrow B/tB) \) for some \( j \). Interchanging the \( x_i \)'s if necessary, we assume that \( x_n \) is algebraic over \( k[x_1, \ldots, x_{n-1}] \). By Lemma 3.4, if \( B_{n-1} = B \cap K[X_1, \ldots, X_{n-1}] \), then \( B_{n-1}/tB_{n-1} \leftrightarrow B/tB \) and

\[
B/tB \leftrightarrow (B/tB) \otimes_{B_{n-1}/tB_{n-1}} F_{n-1} = K_{n-1}[Y] (= K_{n-1}[1^{[1]}]),
\]

where \( F_{n-1} \) is the quotient field of \( B_{n-1}/tB_{n-1} \), \( K_{n-1} \) is a finite algebraic field extension of \( F_{n-1} \) and \( Y \) is transcendental over \( K_{n-1} \). Since \( R[X_1, \ldots, X_n] \subseteq B \), we have \( R[X_1, \ldots, X_{n-1}] \subseteq B_{n-1} \), so
that $x_1, x_2, \ldots, x_{n-1}$ are non-zero elements of $B_{n-1}/tB_{n-1} \hookrightarrow B/tB$. Since by our assumption $x_n$ is algebraic over $k[x_1, \ldots, x_{n-1}]$, we have $x_n \in K_{n-1}$, and hence a unit in $K_{n-1}$ as $x_n$ is a non-zero element of $B/tB \hookrightarrow K_{n-1}[Y]$. Therefore $A/tA = (B/tB)x_1^{-1}x_n^{-1}x_0^{-1}$ is contained in $K_{n-1}[Y]$. Since $F_{n-1}$ is the quotient field of $B_{n-1}/tB_{n-1}$, $(B/tB) \otimes_{B_{n-1}/tB_{n-1}} F_{n-1} = K_{n-1}[Y]$ is a localisation of $B/tB$. Thus $B/tB, A/tA$ and $K_{n-1}[Y]$ have the same quotient field, say $E$, and

$$B/tB \hookrightarrow A/tA \hookrightarrow K_{n-1}[Y] \hookrightarrow E.$$  
Since $F$ is $k$-flat, we have the following inclusions

$$A/tA \otimes_k F \hookrightarrow K_{n-1}[Y] \otimes_k F \hookrightarrow E \otimes_k F.$$  
Since $A/tA \otimes_k F$ is an integral domain, and $E$ is a localisation of $A/tA$, we have $E \otimes_k F$ is an integral domain. Thus $K_{n-1}[Y] \otimes_k F = (K_{n-1} \otimes_k F)^{[1]}$ is an integral domain and hence the units of $A/tA \otimes_k F$ are contained in $K_{n-1} \otimes_k F$. It then follows from the hypothesis (iv) that $\text{tr.deg}_F K_{n-1} \otimes_k F \geq n$. But $\text{tr.deg}_F K_{n-1} \otimes_k F = \text{tr.deg}_F K_{n-1} = \text{tr.deg}_F F_{n-1}$ and $\text{tr.deg}_F F_{n-1} = n - 1$ by Lemma 3.4. This is a contradiction. Thus the map $C/tC \to B/tB$ is one-to-one. Hence, as $C[1/t] = B[1/t]$, it follows that $B = C$. Therefore, by (3), $A = R[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_{n-1}].$ \hfill $\Box$

As a consequence of Theorem 3.5 and Proposition 2.7, we deduce Theorem C.

**Theorem 3.6.** Let $R$ be a Krull domain with quotient field $K$ and $A$ be a faithfully flat $R$-algebra such that

(i) The generic fibre $A \otimes_R K$ is a Laurent polynomial algebra in $n$ variables over $K$.

(ii) For each height one prime ideal $P$ in $R$, there exists a field extension $k(P)^{\prime}$ of $k(P)$ such that $A \otimes_R k(P) \otimes_{k(P)} k(P)^{\prime}$ is an integral domain and contains a Laurent polynomial algebra in $n$ variables over $k(P)^{\prime}$.

Then $A$ is a locally Laurent polynomial algebra in $n$ variables over $R$.

**Proof.** Let $\Delta$ denote the set of all height one prime ideals of $R$. Since $R$ is a Krull domain, for every $P \in \Delta$, $R_P$ is a DVR. Thus, by Theorem 3.5, $A_P$ is a Laurent polynomial algebra in $n$ variables over $R_P$ for every $P \in \Delta$. Now the result follows by Proposition 2.7. \hfill $\Box$

We conclude this section with some remarks pertaining to Theorems 3.5 and 3.6.

**Remark 3.7.** (1) Consider a discrete valuation ring $R$ with a regular parameter $t$ and residue field $k$. Let $A = R[X, Y, Z, X^{-1}, Y^{-1}]/(tZ - XY + 1)$. Then $A$ is generically a Laurent polynomial algebra such that the closed fibre $A/tA(= k[X, Z, X^{-1}])$ is an integral domain, $k$ is algebraically closed in $A/tA$ and $k^* \subset (A/tA)^*$. But $A$ is not a Laurent polynomial algebra. This shows that the condition in (iv) of Theorem 3.5, on the existence of a Laurent polynomial algebra in $n$ variables in a suitable extension $A/tA \otimes_k F$, is necessary. (Also see [5, Remark 3.10].)

(2) An example of Bhatwadekar and Dutta [4, Example 3.9] shows that Theorem 3.5 cannot be extended to a faithfully flat algebra $A$ over an arbitrary Noetherian local domain $R$ of dimension one even if the generic as well as the closed fibre is a Laurent polynomial algebra in one variable.

(3) We may contrast Theorem 3.5 with the corresponding polynomial fibration problem over a DVR. Consider the set up:

$R$ a discrete valuation ring with a regular parameter $t$ and $B$ an integral domain containing $R$ such that

(i) The generic fibre $B[1/t]$ is a polynomial algebra in $n$ variables over $R[1/t]$.

(ii) The closed fibre $B/tB$ is a polynomial algebra in $n$ variables over $R/tR$.

Under the above hypotheses, when $n = 1$, $B$ is a polynomial algebra in one variables over $R$ and when $n = 2$, a theorem of Sathaye shows that $B$ is a polynomial algebra in two variables if $R$ contains
the field of rationals $\mathbb{Q}$ (see [10, Theorem 1] and [1, Corollary 3.2]). Moreover, if $\text{ch}(R/tR) > 0$ and $n = 2$, Asanuma has given an example to show that $B$ need not be a polynomial algebra (see [1, Theorem 5.1]). However, for $n > 2$, it is not known whether $B$ is a polynomial algebra even in the case $R \supseteq \mathbb{Q}$.

(4) It is not known whether a polynomial analogue of Theorem 3.6 is true for $n = 2$. For instance, even when $K$ is a polynomial algebra in two variables over the field of complex numbers and $A$ is a finitely generated faithfully flat $R$-algebra all of whose fibres are polynomial algebras in 2 variables, it is not known whether $A$ is necessarily a polynomial algebra.

4. Laurent polynomial fibration over a general Noetherian domain

Let $R$ be a Noetherian domain and $A$ be a faithfully flat $R$-algebra such that all the fibre rings of $A$ are Laurent polynomial algebras in $n$ variables. If $R$ is normal, we have seen (Theorem 3.6) that $A$ is a locally Laurent polynomial algebra. However if $R$ is not normal, then $A$ need not be a locally Laurent polynomial algebra (Remark 3.7 (2)). In this section, we shall prove (Proposition 4.3) that at least $A$ is finitely generated over $R$ and that there exists a finite birational extension $R'$ of $R$ such that $A \otimes_R R'$ is a locally Laurent polynomial algebra in $n$ variables over $R'$. We shall also prove a necessary and sufficient condition for $A$ to be a locally Laurent polynomial algebra in $n$ variables over $R$ (Theorem 4.4).

The following criterion for a module $M$ to be flat over a Noetherian ring $R$ is known but for the lack of a proper reference, we give a proof below.

**Lemma 4.1.** Let $R$ be a Noetherian ring and $M$ be an $R$-module. Then $M$ is flat over $R$ if and only if $\text{Tor}_1^R(M, R/P) = 0$ for every prime ideal $P$ of $R$.

**Proof.** Suppose that $\text{Tor}_1^R(M, R/P) = 0$ for every prime ideal $P$ of $R$. To show that $M$ is flat over $R$, it is enough to show that $\text{Tor}_1^R(M, R/I) = 0$ for every ideal $I$ of $R$ (see [8, Theorem 7.8, p. 51]). Since $R$ is Noetherian, for every ideal $I$, there exist ideals $I = J_0 \subset J_1 \subset \cdots \subset J_n = R$ such that $R/I$ has a filtration of submodules of the form

$$0 = J_0/I \subset J_1/I \subset \cdots \subset J_{n-1}/I \subset J_n/I = R/I$$

satisfying $J_{i+1}/J_i \cong R/P_i$ for some prime ideal $P_i$ of $R$ (see [8, Theorem 6.4, p. 39]). We prove that $\text{Tor}_1^R(M, R/I) = 0$ by induction on $n$, the length of the filtration of $R/I$.

If $n = 1$, then $I$ is a prime ideal of $R$ and by the given hypothesis, $\text{Tor}_1^R(M, R/I) = 0$.

Suppose that $n > 1$. By applying $\text{Tor}_1^R(M, -)$ to the short exact sequence

$$0 \to J_1/I \to R/I \to R/J_1 \to 0$$

we get the exact sequence

$$\text{Tor}_1^R(M, J_1/I) \to \text{Tor}_1^R(M, R/I) \to \text{Tor}_1^R(M, R/J_1).$$

Now $J_1/I \cong R/P_0$ for a prime ideal $P_0$ and hence $\text{Tor}_1^R(M, J_1/I) = 0$. Since $R/J_1$ has a filtration of length $n - 1$, $\text{Tor}_1^R(M, R/J_1) = 0$ by induction hypothesis. Thus $\text{Tor}_1^R(M, R/I) = 0$. □

We now prove an elementary result.

**Lemma 4.2.** Let $R$ be a Noetherian domain and let $R'$ be an integral extension of $R$. Let $D$ and $A$ be flat $R$-algebras such that $D \subseteq A \subseteq A \otimes_R R'$ and $A \otimes_R R' = D \otimes_R R'$. Then $A = D$. 
Proof. Let $M = A/D$. Since $A \otimes_{R} R' = D \otimes_{R} R'$, it follows that $M \otimes_{R} R' = 0$. We will show that $M$ is a flat $R$-module. It will then follow that $M \otimes_{R} R' = 0$, i.e., $A = D$.

By Lemma 4.1, it is enough to show that $\text{Tor}^{R}_{1}(M, R/P) = 0$ for every prime ideal $P$ of $R$. Fix a prime ideal $P$ of $R$. Since $A$ is a flat $R$-module, we have the following exact sequence of $R$-modules

$$0 \to \text{Tor}^{R}_{1}(M, R/P) \to D \otimes_{R} R/P \to A \otimes_{R} R/P \to M \otimes_{R} R/P \to 0.$$ 

Since $R'$ is integral over $R$, there exists a prime ideal $P'$ of $R'$ lying over $P$. Since $A$ and $D$ are flat $R$-modules, we have the following injective maps

$$D \otimes_{R} R/P \hookrightarrow D \otimes_{R} R'/P' \quad \text{and} \quad A \otimes_{R} R/P \cong A \otimes_{R} R'/P'.$$

Since the map $D \otimes_{R} R/P \hookrightarrow D \otimes_{R} R'/P'$ is a composite of the maps

$$D \otimes_{R} R/P \to A \otimes_{R} R/P \quad \text{and} \quad A \otimes_{R} R/P \rightarrow A \otimes_{R} R'/P',$$

it follows that the map $D \otimes_{R} R/P \rightarrow A \otimes_{R} R/P$ is injective and hence $\text{Tor}^{R}_{1}(M, R/P) = 0$. □

We now prove a result for a Laurent polynomial fibration over a Noetherian domain.

**Proposition 4.3.** Let $R$ be a Noetherian domain with quotient field $K$ and let $A$ be a faithfully flat $R$-algebra such that

(i) The generic fibre $A \otimes_{R} K$ is a Laurent polynomial algebra in $n$ variables over $K$.

(ii) For each height one prime ideal $P$ of $R$, $A \otimes_{R} k(P)$ is geometrically integral over $k(P)$ and there exists a field extension $k(P)'$ of $k(P)$ such that $A \otimes_{R} k(P) \otimes_{k(P)} k(P)'$ contains a Laurent polynomial algebra in $n$ variables over $k(P)'$.

Then the following statements hold:

(I) All the fibre rings of $A$ are Laurent polynomial forms in $n$ variables.

(II) There exists a finite birational extension $R'$ of $R$ such that $A \otimes_{R} R'$ is a locally Laurent polynomial algebra in $n$ variables over $R'$.

(III) $A$ is finitely generated over $R$.

**Proof.** (I) This proof is essentially the same as in [5, Theorem 3.13].

Fix any prime ideal $P$ (need not be of height one) in $R$. Note that $A \otimes_{R} k(P) = A_{P} \otimes_{R_{P}} k(P)$. So replacing $R$ by $R_{P}$ we can assume that $R$ is a local Noetherian domain with maximal ideal $P$. We prove the result by induction on the height of $P$.

Suppose that $\dim R = 1$. From the Krull–Akizuki theorem [9, Theorem 33.2] and the fact that $R$ is local, it follows that the normalisation $\tilde{R}$ of $R$ is a semilocal PID and that $k(\tilde{P})$ is a finite algebraic extension of $k(P)$ for every maximal ideal $\tilde{P}$ of $\tilde{R}$. Fix a maximal ideal $\tilde{P}$ of $\tilde{R}$, let $V = \tilde{R}_{\tilde{P}}$ and let $t \in V$ be such that $tV = \tilde{P}V$. Since $R$ and $V$ are birational, $A \otimes_{R} V$ is generically a Laurent polynomial algebra in $n$ variables over $V$. Also note that, by hypothesis (ii),

$$(A \otimes_{R} V)/t(A \otimes_{R} V) = (A \otimes_{R} \tilde{R}) \otimes_{\tilde{R}} k(\tilde{P}) = (A \otimes_{R} k(P)) \otimes_{k(P)} k(\tilde{P})$$

satisfies the condition (iv) of Theorem 3.5. Hence $A \otimes_{R} V$ is a Laurent polynomial in $n$ variables over $V$ by Theorem 3.5; in particular, $(A \otimes_{R} k(P)) \otimes_{k(P)} k(\tilde{P})(= A \otimes_{R} (V/tV))$ is a Laurent polynomial algebra in $n$ variables over $k(\tilde{P})$. Thus $A \otimes_{R} k(P)$ is a Laurent polynomial form in $n$ variables over $k(P)$.

Now suppose that $\dim R \geq 2$. Then by the induction hypothesis, the fibre ring $A \otimes_{R} k(Q)$ is a Laurent polynomial form in $n$ variables for every prime ideal $Q \supsetneq P$. Let $\tilde{R}$ be the completion of $R$
with respect to the maximal ideal $P$ and let $\hat{P}$ denote the maximal ideal of $\hat{R}$. Then $\dim \hat{R} = \dim R$ and $R/\hat{P} = R/P$. Let $\hat{P}_0$ be a minimal prime ideal of $\hat{R}$ such that $\dim \hat{R} = \dim \hat{R}/\hat{P}_0$. Since $R$ is $R$-flat, by the going down theorem, for any prime ideal $Q$ of $\hat{R}$, $\text{ht}(Q \cap R) \leq \text{ht} \hat{Q}$. Hence, since $\text{ht} \hat{P}_0 = 0$ and $R$ is an integral domain, we have $\hat{P}_0 \cap R = (0)$. Set $\hat{A} := A \otimes_R \hat{R}/\hat{P}_0$. Since $\hat{P}_0 \cap R = (0)$, $\hat{A}$ is generically a Laurent polynomial algebra in $R$ variables over $R/\hat{P}_0$. Let $Q$ be a non-zero prime ideal of $\hat{R}/\hat{P}_0$ properly contained in $\hat{P}$ and $Q = \hat{Q} \cap R$. Since $A \otimes_R k(\hat{Q}) = (A \otimes_R k(Q)) \otimes_{k(\hat{Q})} k(\hat{Q})$, we have $A \otimes_R k(\hat{Q})$ is a Laurent polynomial form in $n$ variables over $k(\hat{Q})$. Now since $A \otimes_R k(\hat{P}) = \hat{A} \otimes_R \hat{P}(\hat{P})$, we can replace $R$ by $\hat{R}/\hat{P}_0$ and $A$ by $\hat{A}$ and assume that $R$ is a complete Noetherian local domain. Let $\hat{R}$ denote the normalisation of $R$. Then $\hat{R}$ is a finite $R$-module (see [9, Theorem 32.1]) and hence a Noetherian normal local domain and for every non-zero non-maximal ideal $\hat{Q}$ of $\hat{R}$, $A \otimes_R \hat{R} \otimes_R k(\hat{Q})$ is a Laurent polynomial form in $n$ variables. Hence, by Theorem 3.6, $A \otimes_R \hat{R}$ is a Laurent polynomial algebra in $n$ variables over $\hat{R}$. This proves that the closed fibre of $A$ is a Laurent polynomial form in $n$ variables.

(II) Let $\hat{R}$ be the normalisation of $R$. Then $\hat{R}$ is a Krull domain (see [9, Theorem 33.10]). By (I), all the codimension-one fibres of $A \otimes_R \hat{R}$ are Laurent polynomial forms in $n$ variables over $\hat{R}$. Hence, by Theorems 3.6 and 2.3, there exist $n$ finitely generated rank one projective modules $L_i$, $1 \leq i \leq n$, of $\hat{R}$ such that $A \otimes_R \hat{R}$ is isomorphic to an $\hat{R}$-algebra of the form $(\text{Sym}_R(Q))[I^{-1}]$, where $Q = L_1 \oplus \cdots \oplus L_n$ and $I$ is an invertible ideal of $\text{Sym}_R(Q)$ generated by the image of $L_1 \otimes \cdots \otimes L_n$. Since $L_i$ is a finitely generated projective $R$-module, there exists a finite extension $R'$ of $R$ contained in $\hat{R}$ such that, for each $i$, $1 \leq i \leq n$, there exist finitely generated rank one projective $R'$-modules $L_i'$ satisfying $L_i \cong L_i' \otimes_{R'} \hat{R}$. Then $(\text{Sym}_{R'}(Q')[I^{-1}] \subseteq (\text{Sym}_R(Q))[I^{-1}] = A \otimes_R \hat{R}$, where $Q' = L_1' \oplus \cdots \oplus L_n'$ and $I'$ is an invertible ideal of $\text{Sym}_{R'}(Q')$ generated by the image of $L_1' \otimes \cdots \otimes L_n'$. Since $(\text{Sym}_{R'}(Q'))[I^{-1}] \subseteq A \otimes_{\hat{R}} \hat{R}$ is a finitely generated $R'$-algebra, by enlarging $R'$ if necessary, we can ensure that $(\text{Sym}_{R'}(Q'))[I^{-1}] \subseteq A \otimes_{R'} \hat{R} \subseteq A \otimes_{\hat{R}} \hat{R}(= (\text{Sym}_{\hat{R}}(Q))[I^{-1}])$. Since $\hat{R}$ is a finite module over the Noetherian ring $R'$, $\hat{R}$ is a Noetherian ring. Hence, by Lemma 4.2, $(\text{Sym}_{R'}(Q'))[I^{-1}] = A \otimes_{R'} R'$.

(III) Since $A$ is flat over $R$, we have $R \hookrightarrow A \hookrightarrow A \otimes_R R'$. Since $A \otimes_R R'$ is a locally Laurent polynomial algebra in $n$ variables over $R'$, by Theorem 2.3, $A \otimes_R R'$ is a finitely generated $R'$-algebra. Now since $R'$ is a finite $R$-module, we have $A \otimes_R R'$ is a finite $A$-module and $A \otimes_R R'$ is a finitely generated $R$-algebra. Thus, $A$ is finitely generated over $R$. □

We now prove Theorem D.

**Theorem 4.4.** Let $R$ be a Noetherian domain with quotient field $K$ and let $A$ be a faithfully flat $R$-algebra such that

(i) $A \otimes_R K = K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ for some $X_1, \ldots, X_n$ transcendental over $R$.

(ii) For each height one prime ideal $P$ of $R$, $A \otimes_R k(P)$ is geometrically integral over $k(P)$ and there exists a field extension $k(P)'$ of $k(P)$ such that $A \otimes_R k(P) \otimes_{k(P)} k(P)'$ contains a Laurent polynomial algebra in $n$ variables over $k(P)'$.

(iii) $L_i := A \cap KX_i$ is a finitely generated projective $R$-module of rank one, $1 \leq i \leq n$.

Then $A$ is a locally Laurent polynomial algebra in $n$ variables over $R$.

**Proof.** We may assume that $R$ is local. By Proposition 4.3, we can find a finite birational extension $R'$ of $R$ such that $A \otimes_R R'$ is a locally Laurent polynomial algebra in $n$ variables over $R'$. Since $R$ is local and $R'$ is a finite birational extension of $R$, $R'$ is a semilocal domain and hence Pic$R' = (0)$. Therefore, by Theorem 2.3, $A \otimes_R R' = R'[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}]$ for some elements $Y_1, \ldots, Y_n$ which are transcendental over $R'$ and are chosen such that $KX_i = KY_i$ for $1 \leq i \leq n$ (cf. Lemma 2.1).

Fix $i$, $1 \leq i \leq n$. Since $R$ is local, by hypothesis (iii), $L_i = A \cap KX_i = Rf_i$ for some $f_i \in A(\hookrightarrow R'[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}])$. Since $f_i \in KY_i (= KX_i)$, it follows that there exists $a_i \in R'$ such that $f_i = a_iY_i$. 

We show that $a_i$ is a unit in $R'$. Let $\mathcal{J}$ be the conductor ideal of $R'$ in $R$. It then follows that $\mathcal{J}Y_i \subseteq A \cap KX_i = Rf_i = R\mathfrak{a}_i$. Therefore, $\mathcal{J}a_i^{-1} \subseteq R \subseteq R'$. It follows that $\mathcal{J}a_i^{-1}$ is an ideal of $R'$ and hence $\mathcal{J}a_i^{-1} \subseteq \mathcal{J}$. Therefore $\mathcal{J} = a_i\mathcal{J}$. Since $\mathcal{J}$ is a non-zero finitely generated ideal of the integral domain $R'$, it follows from NAK Lemma [8, Theorem 2.2, p. 8] that $a_i$ is a unit in $R'$.

Therefore $A \otimes_R R' = R'[f_1, f_1^{-1}, \ldots, f_n, f_n^{-1}]$. Since $A \otimes_R R'$ is integral over $A$, $f_i \in A$ and $f_i^{-1} \in A \otimes_R R'$, we have $f_i^{-1} \in A$. Thus

$$R[f_1, f_1^{-1}, \ldots, f_n, f_n^{-1}] \subseteq A \subseteq A \otimes_R R' = R'[f_1, f_1^{-1}, \ldots, f_n, f_n^{-1}]$$

Now by Lemma 4.2, $A = R[f_1, f_1^{-1}, \ldots, f_n, f_n^{-1}]$, a Laurent polynomial algebra in $n$ variables. □

The following lemma shows that an algebra which is stably Laurent polynomial is necessarily a Laurent polynomial algebra.

**Lemma 4.5.** Let $R$ be an integral domain and $B$ be a Laurent polynomial algebra in $n$ variables over $R$. Suppose that $A$ is an $R$-algebra such that

(i) either $A[W_1, \ldots, W_t] \cong B[Z_1, \ldots, Z_t]$ as $R$-algebras,

(ii) or $A[W_1, W_1^{-1}, \ldots, W_t, W_t^{-1}] \cong B[Z_1, Z_1^{-1}, \ldots, Z_t, Z_t^{-1}]$ as $R$-algebras,

for some $W_i, Z_i, 1 \leq i \leq t$, transcendental over $A$ and $B$ respectively. Then $A \cong B$ as $R$-algebras.

**Proof.** Suppose that (i) holds. Let $B = R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. We may assume that $A[W_1, \ldots, W_t] = B[Z_1, \ldots, Z_t]$. Since $R$ is an integral domain, $A$ is an integral domain and as $X_1, \ldots, X_n$ are units in $A[W_1, \ldots, W_t]$, we see that $X_1, X_1^{-1}, \ldots, X_n, X_n^{-1} \in A$. Therefore $B \subseteq A$ and hence $A = B$ because $B$ is algebraically closed in $B[Z_1, \ldots, Z_t]$ and $\text{tr.deg}_R A = \text{tr.deg}_R B$.

Now suppose that (ii) holds. It is enough to consider the case when $t = 1$ and show that if $A[W, W^{-1}] = B[Z, Z^{-1}]$ as $R$-algebras, then $A \cong B$. Let $B = R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. Since $Z, X_1, \ldots, X_n$ are units in $A[W, W^{-1}]$, we have $Z = \lambda W^\ell$ and $X_i = \mu_i W^{a_i}$ for some $\lambda, \mu_i \in \mathbb{A}^*$ and $\ell, a_i \in \mathbb{Z}$, $1 \leq i \leq n$. Again, since $W$ is a unit in $B[Z, Z^{-1}]$, we have $W = \nu X_1^{b_1} \cdots X_n^{b_n} Z^r$, for some $\nu \in \mathbb{R}^*$ and $r, b_i \in \mathbb{Z}$ and hence

$$W = \nu(\mu_1 W^{a_1})^{b_1} \cdots (\mu_n W^{a_n})^{b_n} (\lambda W^\ell)^r = \nu \mu_1^{b_1} \cdots \mu_n^{b_n} \lambda^r W^{a_1 b_1 + \cdots + a_n b_n + \ell r}.$$

Since $\nu \mu_1^{b_1} \cdots \mu_n^{b_n} \lambda^r \in A$, we have $\sum_i a_i b_i + \ell r = 1$. Since $\mathbb{Z}$ is PID, the unimodular row $(b_1, b_2, \ldots, b_n, r)$ can be completed to an invertible $(n + 1) \times (n + 1)$ matrix, say $M = (b_{ij})$, such that the last row of $M$ is $(b_1, b_2, \ldots, b_n, r)$. Set $Y_i := X_1^{b_{i1}} \cdots X_i^{b_{in}} Z^{b_{i(n+1)}}$ for $1 \leq i \leq n + 1$. Then $A[W, W^{-1}] = B[Z, Z^{-1}] = R[Y_1, Y_1^{-1}, \ldots, Y_{n+1}, Y_{n+1}^{-1}] = R[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}, W, W^{-1}]$, since $W = \nu Y_{n+1}$ and $\nu \in \mathbb{R}^*$. Hence $A \cong R[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}]$ and so $A \cong B$. □

**Remark 4.6.** Let $R$ be a Noetherian domain and $B$ be a faithfully flat finitely generated $R$-algebra such that all the fibre rings $B \otimes_R k(P)$ are polynomial algebras in $n$ variables (over $k(P)$). From a result of Asanuma [1, Theorem 3.4], it follows that if the module of 1-differential forms $\Omega_{B/R}$ is free, then $B$ is a stably polynomial algebra over $R$.

Bhatwadekar and Dutta have constructed an explicit example [4, Example 3.9] of a finitely generated faithfully flat algebra $A$ over a one dimensional Noetherian (semisegmental) local domain $R$ such that each fibre ring $A \otimes_R k(P)$ is a Laurent polynomial ring in one variable over $k(P)$, $\Omega_{A/R}$ is a free $A$-module of rank 1 but $A$ is not a Laurent polynomial algebra over $R$. In view of Lemma 4.5, $A$ is not a stably Laurent polynomial algebra. Thus, a Laurent polynomial analogue of Asanuma’s structure theorem (see [1, Theorem 3.4]) does not seem to exist.
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Appendix A. Laurent polynomial forms in $n$ variables

Let $k$ be a field and $A$ be a $k$-algebra. Recall that $A$ is said to be a Laurent polynomial form in $n$ variables if there exists an algebraic field extension $F$ of $k$ such that $A \otimes_k F$ is a Laurent polynomial algebra in $n$ variables over $F$, if $F$ can be chosen to be separably algebraic, then the Laurent polynomial form $A$ will be called separably algebraic. It was observed in [5, Proposition 2.3] that a separable Laurent polynomial form in one variable is trivial if and only if $k^* \subseteq A^*$. We extend the result to $n$ variables; the proof is essentially the same as in the case $n = 1$.

Recall that a field extension $F$ over $k$ (not necessarily algebraic) is said to be separably generated if there exists a transcendence basis $B$ of $F$ such that $F$ is separably algebraic over $k(B)$.

**Proposition A1.** Let $k$ be a field, $A$ a $k$-algebra and $F$ a separably generated field extension of $k$ such that $A \otimes_k F$ is a Laurent polynomial algebra in $n$ variables over $F$. Suppose that $A$ contains a Laurent polynomial algebra in $n$ variables over $k$. Then $A$ is a Laurent polynomial algebra in $n$ variables over $k$.

**Proof.** Let $A \otimes_k F = F[U_1, U_1^{-1}, \ldots, U_n, U_n^{-1}]$. Since $A \hookrightarrow A \otimes_k F$, we regard $A$ as a $k$-subalgebra of $A \otimes_k F$. It is easy to see that there exists a finitely generated separable extension $F_1$ of $k$ such that $A \otimes_k F_1 = F_1[U_1, U_1^{-1}, \ldots, U_n, U_n^{-1}]$. Thus replacing $F$ by $F_1$, we can assume that $F$ is a finitely generated separable extension of $k$.

We first consider the case when $F$ is a finite separable algebraic extension of $k$. Replacing $F$ by its normal extension, we can assume that $F$ is a Galois extension of $k$ with Galois group $G$. Now any $\sigma \in G$ can be extended to an $A$-automorphism of $A \otimes_k F$, by defining $\sigma(a \otimes \mu) = a \otimes \sigma(\mu)$. Since $F$ over $k$ is a Galois extension, the bilinear map

$$F \times F \to k, \quad (x, y) \mapsto \text{Tr}(xy)$$

is non-degenerate and hence there exist $c_i \in F$ such that $\text{Tr}(c_i U_i) \neq 0$, $1 \leq i \leq n$. Replacing $U_i$ by $c_i U_i$ we can assume that $\text{Tr}(U_i) \neq 0$. Let $W_i = \text{Tr}(U_i)$. Then $W_i \in A$. We show that $A = k[W_1, W_1^{-1}, \ldots, W_n, W_n^{-1}]$.

Let $k[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \subseteq A$, where $X_1, X_2, \ldots, X_n$ are transcendental over $k$. Then, there exist integers $a_{ij}$, $1 \leq i, j \leq n$ such that

$$X_i = \mu_i U_1^{a_{i1}} U_2^{a_{i2}} \cdots U_n^{a_{in}},$$

for some $\mu_i \in F^*$. Let $M = (a_{ij})$ and $M_{\text{ad}} := \text{Adj}(M) = (b_{ij})$ for some $b_{ij} \in \mathbb{Z}$. Since $X_1, X_2, \ldots, X_n$ are transcendental over $k$, $\det M \neq 0$. Set

$$Y_i := X_1^{b_{i1}} X_2^{b_{i2}} \cdots X_n^{b_{in}} \in A$$

for $1 \leq i \leq n$. Since $M_{\text{ad}} M = (\det M) I_n$, where $I_n$ is the identity matrix, we have

$$Y_i = \lambda_i U_i^{(\det M)},$$

$1 \leq i \leq n$, for some $\lambda_i \in F^*$. Fix $i$, $1 \leq i \leq n$. Replacing $Y_i$ by $Y_i^{-1}$, we may assume that $\det M > 0$.

Since for any $\sigma \in G$, $\sigma(Y_i) = Y_i$, we have $(\sigma(U_i)/U_i)^{\det M} = (\lambda_i/\sigma(\lambda_i)) \in F^*$. Therefore, $\sigma(U_i) = \nu_\sigma U_i$ for some $\nu_\sigma \in F^*$. Hence $W_i = \text{Tr}(U_i) = d_i U_i$ for some $d_i \in F^*$. Thus $F[W_1, W_1^{-1}, \ldots, W_n, W_n^{-1}] = F[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. 


\[ F[U_1, U_1^{-1}, \ldots, U_n, U_n^{-1}] \]. Since \( A \otimes_k F \) is integral over \( A \), and for \( 1 \leq i \leq n \), \( W_i^{-1} \in A \otimes_k F \), we have \( W_i^{-1} \in A \). Hence \( k[W_1, W_1^{-1}, \ldots, W_n, W_n^{-1}] \subseteq A \). Since \( F \) is faithfully flat over \( k \), we have \( k[W_1, W_1^{-1}, \ldots, W_n, W_n^{-1}] = A \). Now the argument in [5, Proposition 2.3] shows that if \( F \) is an arbitrary separable extension over \( k \) then also \( A \) is a Laurent polynomial algebra. \( \square \)

The following example shows that a purely inseparable Laurent polynomial form \( A \) in one variable over a field \( k \) which contains non-trivial units (i.e. \( k^* \not\subseteq A^* \)) need not be trivial.

**Example A.2.** Let \( k \) be a non-perfect field of characteristic \( p \). Let \( \beta \in k \) be such that \( \beta \notin k^p := \{ a^p \mid a \in k \} \). Let \( L = k(\alpha) \) be a purely inseparable extension of \( k \) such that \( \alpha^p = \beta \). Now let \( B = k[X, Y] / (X - Y^p - \beta X^p) \). It is known that \( B \) is a non-trivial inseparable \( A^1 \)-form. Now

\[
B \otimes_k L = \frac{L[X, Y]}{(X - Y^p - \beta X^p)} = \frac{L[X - (Y + \alpha X)^p, Y + \alpha X]}{(X - (Y + \alpha X)^p)} \cong L[Y + \alpha X].
\]

Let \( A = B[X^{-1}] \). Since \( (Y + \alpha X)^p = X \) in \( A \otimes_k L \), we have \( A \otimes_k L \cong L[Y + \alpha X, (Y + \alpha X)^{-1}] \), a Laurent polynomial algebra in one variable. Also \( k[X, X^{-1}] \subseteq A \). But \( A \) is not a Laurent polynomial algebra.

**References**