The Fitting–Gaschütz–Hall Relation in Certain Soluble by Finite Groups

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1. STATEMENT OF RESULTS

If $H$ is an arbitrary group $\rho_1(H), \rho(H), \phi(H)$ and $\psi(H)$ denote, respectively, the Fitting radical, the Hirsch–Plotkin radical, the Frattini subgroup and the intersection of the centralizers of the chief factors of $H$. For a finite group $H$ it is well known that

$$\rho_1(H) = \rho(H) = \psi(H) = \rho_1 \text{ mod } \phi(H)$$

and that this subgroup is nilpotent. Here $\rho_1 \text{ mod } \phi(H)$ is the subgroup $M$ defined by $M/\phi(H) = \rho_1(H/\phi(H))$. We call (1) the Fitting–Gaschütz–Hall (FGH) relation. If a group $H$ satisfies this relation and if in addition $\rho_1(H)$ is nilpotent then we say that $H$ is an FGH-group. Hence all finite groups are FGH-groups.

In [5] Hall proved that finitely generated metanilpotent by finite groups are FGH-groups. In this paper we are interested in the question of whether subgroups of groups in this class are also FGH-groups. In general, the answer to this question is in the negative: for example if $G$ is the group of Example G(i) of [7] then, in the notation of that paper, the subgroup $L = \langle K', \tau \rangle$ of $G$ has $\rho_1(L) < \rho(L)$ and so $L$ is not an FGH-group although $G$ is in fact nilpotent by Abelian. However, by restricting our attention to certain subclasses of the class of finitely generated metanilpotent by finite groups we obtain some positive results. As our first main result we shall prove

**Theorem A.** Subgroups of finitely generated metabelian by finite groups are FGH-groups.
We shall use the following letters for classes of groups:

- $\mathcal{F}$ is the class of all finite groups,
- $\mathcal{A}$ is the class of all Abelian groups,
- $\mathcal{N}$ is the class of all nilpotent groups,
- $\mathcal{G}$ is the class of all finitely generated groups.

We shall write $\mathcal{X}\mathcal{Y}$ for the class of all groups which are extensions of $\mathcal{X}$-groups by $\mathcal{Y}$-groups, and as usual $\mathcal{X}^{n+1}$ will denote $(\mathcal{X}^n)\mathcal{X}$, for $n = 1, 2, 3, \ldots$.

As an immediate corollary to Theorem A we have

**COROLLARY A.** Let $H$ be a subgroup of a $\mathcal{G} \cap (\mathcal{N}^2)\mathcal{X}$-group. Then,

1. $\phi(H)$ is nilpotent, and
2. $H' \leq \phi(H)$ if and only if $H$ is nilpotent.

Here, as usual, for a group $H$ the subgroup $H'$ is the derived subgroup generated by all the commutators $[x, y]$ with $x$ and $y$ in $H$.

Examples are known of metabelian groups with nonnilpotent Frattini subgroups (see, e.g., [5 and 6]) and these are therefore not embeddable in finitely generated metabelian groups.

Suppose $G$ is a $\mathcal{G} \cap (\mathcal{N}^2)\mathcal{X}$-group and $H \leq G$. So certainly $H$ is in $\mathcal{N}(\mathcal{G} \cap (\mathcal{N}^2)\mathcal{X})$ and therefore $\rho_1(H)$ is nilpotent since $\mathcal{G} \cap (\mathcal{N}^2)\mathcal{X}$-groups satisfy the maximal condition for subgroups. As we have seen, $\rho_1(H) < \rho(H)$ is possible. However, we do have

**LEMMA 1.** If $H$ is an $\mathcal{N}(\mathcal{G} \cap (\mathcal{N}^2)\mathcal{X})$-group then $\psi(H)$ is the unique maximal normal $ZA$-subgroup of $H$.

Here by a $ZA$-group we mean a group with an ascending central series. For any group $H$ define $\rho_\delta(H)$ to be the product of the normal $ZA$-subgroups of $H$ (see [5, p. 329]) so that $\rho_1(H) \leq \rho_\delta(H) \leq \rho(H)$. Hall in [5, p. 330] pointed out that $\rho_\delta(H) \leq \psi(H)$ and $(\rho_\delta(H))' \leq \phi(H)$. Hence combining Lemma 1 with a result of Gruenberg [2, p. 439] we have

**LEMMA 2.** If $H$ is an $\mathcal{N}(\mathcal{G} \cap (\mathcal{N}^2)\mathcal{X})$-group then

$$\rho_1(H) \leq \rho_\delta(H) - \psi(H) - \rho(H) \leq \rho_1 \mod \phi(H).$$

We have observed that the first inequality in Lemma 2 may be strict. That the second inequality may be strict is shown by Hall's example [5, p. 327] of an Abelian by cyclic group which has a Frattini subgroup which is not locally nilpotent.

Suppose now that $G$ is a $\mathcal{N}^2\mathcal{X}$-group satisfying $\text{Max-}n$, the maximal condition for normal subgroups. Then $G$ is finitely generated and by
[7, Theorem A] G is stunted, that is, there is a bound on the upper central heights of subgroups of G. Therefore if \( H \leq G \) we have that \( \rho_1(H) = \rho(H) \) is nilpotent. Combining this fact with Lemma 2 yields

**Lemma 3.** If G is an \( \mathcal{R}^{n} \)-group satisfying Max-n and H is a subgroup of G then

\[
\rho_1(H) = \rho_2(H) = \psi(H) = \rho(H)
\]

and this subgroup is nilpotent.

Lemma 3 shows that the essential problem in determining whether a subgroup H of an \( \mathcal{R}^{n} \)-group with Max-n is an FGH-group is the problem of establishing the nilpotency of \( \rho_1 \) mod \( \phi(H) \). Accordingly, it is to this problem that we now turn.

Suppose G is any group and \( H \leq G \). If we write \( f(\ ) \) for the subgroup function \( \rho \) mod \( \phi(\ ) \) then it is easy to see that for \( N \triangleleft G \) we have 

\[
f(H)N/N < f(HN/N).
\]

If, in particular, \( G/N \) is finite, then, of course, \( HN/N \) is finite and so \( f(HN/N) \) is nilpotent since finite groups are FGH-groups. Hence \( f(H)N/N \) is nilpotent. Theorem A will therefore follow from

**Lemma 4.** Let G be a \( \mathcal{G} \cap (\mathcal{R}^{n}) \)-group and suppose \( H \leq G \) such that \( HN/N \) is nilpotent for all \( N \triangleleft G \) with \( G/N \) finite. Then \( H \) is nilpotent.

In [9] Wehrfritz used an analogous result to establish similar properties in finitely generated linear groups.

However Lemma 4 cannot be generalized to the class of \( \mathcal{R}^{n} \)-groups with Max-n. In fact if \( G = D \wr C \) the (restricted) wreath product of a dihedral group D of order 8 with an infinite cyclic group C then G is an \( \mathcal{R}^{n} \)-group and G satisfies Max-n by [2, Theorem 5]. Let M be the intersection of all the normal subgroups of finite index in G. Then \( M \neq 1 \) since G is not residually finite by a theorem of Gruenberg [1]. Let t be the generator of the top group of G and set \( H = \langle M, t \rangle \). Then it is easy to see H is not nilpotent although \( HN/N \) is Abelian for all \( N \triangleleft G \) such that \( G/N \) is finite.

However, if we confine attention to the situation where G is an \( \mathcal{R}^{n} \)-group satisfying Max-n and the subgroup H is subnormal in G, that is there exists a finite series \( H = H_n \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_1 \triangleleft H_0 = G \) joining H to G, then we do have a result resembling lemma 4. In fact we shall prove

**Lemma 5.** Suppose G is an infinite \( \mathcal{G} \cap (\mathcal{R}^{n}) \)-group and H is a subnormal subgroup of G such that \( HN/N \) is nilpotent for all \( 1 \neq N \triangleleft G \) then H is nilpotent.

We remark that Lemma 5 gives an alternative way of proving the main theorem of Robinson [8] that a finitely generated hyper-(Abelian or finite)
group is nilpotent if all of its finite homomorphic images are nilpotent. For following the first part of [8] it is easy to reduce to the case where \( G \) is an infinite \( G \cap (\mathfrak{M}) \)-group with all of its proper homomorphic images nilpotent. Thus taking \( G = H \) in Lemma 5 we have \( G \) nilpotent as required.

Suppose now that \( G \) is an \( \mathfrak{R}^n \)-group satisfying \( \text{Max-}n \) and \( H \) is subnormal in \( G \). Suppose \( \rho_1 \equiv \phi(H) \) is not nilpotent. We may assume, using the \( \text{Max-}n \) property and the properties of \( \rho_1 \equiv \phi(H) \) discussed above, that \( G \) is infinite and \( \rho_1 \equiv \phi(H) / N \) is nilpotent for all \( 1 \neq N \triangleleft G \). Lemma 5 gives an immediate contradiction. Therefore \( \rho_1 \equiv \phi(H) \) is nilpotent. This result combines with Lemma 3 to prove our second main result

**Theorem B.** Subnormal subgroups of \( \mathfrak{R}^n \)-groups satisfying \( \text{Max-}n \) are \( \mathcal{FGH} \)-groups.

We note that Theorem B has as an immediate consequence

**Corollary B.** Let \( H \) be a subnormal subgroup of an \( \mathfrak{R}^n \)-group satisfying \( \text{Max-}n \). Then,

(i) \( \phi(H) \) is nilpotent, and

(ii) \( H' \leq \phi(H) \) if and only if \( H \) is nilpotent.

Hall in [5, Theorem 1(i)] gave an example of a \( G \cap \mathfrak{M} \)-group \( G \) which has \( \phi(G) \) nonnilpotent so that \( G \) is not an \( \mathcal{FGH} \)-group. This example together with Theorem B yields that the property of having all subnormal subgroups \( \mathcal{FGH} \)-groups is an \( \mathfrak{R}^n \)-property in the sense of Hall [5, p333].

Finally, it is clear from the proofs that we have as generalisations of Theorems A and B their corollaries

**Corollary A'.** Suppose for some term \( \zeta_r(G) \) of the upper central series of \( G \) we have that \( G / \zeta_r(G) \) is a \( \mathcal{Z} \cap (\mathfrak{R}^n) \)-group. Then any subgroup of \( G \) is an \( \mathcal{FGH} \)-group,

and

**Corollary B'.** Suppose \( G / \zeta_r(G) \) is an \( \mathfrak{R}^n \)-group satisfying \( \text{Max-}n \) for some \( r \geq 0 \). Then any subnormal subgroup of \( G \) is an \( \mathcal{FGH} \)-group.

2. **Proof of Lemma 1**

Suppose \( H \) is a group and \( 1 = A' \triangleleft A \triangleleft H \). Let \( c_H(A) \) denote the centralizer in \( H \) of \( A \). Then we show first of all that if \( G = H / c_H(A) \in G \cap (\mathfrak{R}^n) \) then \( A \leq \zeta_\omega(\phi(H)) \), the \( \omega \)-th term of the upper central series of \( \phi(G) \). For let \( a \in A \) and set \( C = c_H(A) \). Let \( \langle a^m \rangle \) be the normal closure of \( a \) in \( H \) and form
the split extension $L = \langle a^H \rangle \cdot G$. Then $L$ is in $\mathfrak{G} \cap (\mathfrak{H}\mathfrak{H})$ so that by Hall’s theorem $\psi(L)$ is nilpotent. But $C\psi(H) \leq \psi(L)$ since chief factors of $L$ below $\langle a^H \rangle$ are chief factors of $H$. Moreover, $a$ is in $\psi(L)$ so that $[a, n \psi(H)] = 1$ for some $n \geq 1$ and hence $a \in \zeta_n(\psi(H))$ as required. We may now prove Lemma 1 as a corollary to this as follows. Suppose that $H$ is in $\mathfrak{R}(\mathfrak{G} \cap (\mathfrak{H}\mathfrak{H}))$. We require to show that $\psi(H)$ is the unique maximal normal $ZA$-subgroup of $H$. Let $1 < K < H$ with $K \in \mathfrak{R}$ and $H/K \in \mathfrak{J} \cap (\mathfrak{H}\mathfrak{H})$. Set $A = \zeta_1(K)$. We may assume inductively that $\psi(H)/A$ is $ZA$. The previous result yields at once that $\psi(H)$ itself is $ZA$. Moreover, $\rho_a(H) \leq \psi(H)$ and Lemma 1 follows.

3. Proof of Lemma 4

Suppose $G \in \mathfrak{G} \cap (\mathfrak{H}^2\mathfrak{H})$ and let $K$ be a normal subgroup of finite index in $G$ such that $A = K'$ is Abelian. Suppose further that $G$ demonstrates that Lemma 4 is false, that is, there exists a nonnilpotent subgroup $H$ of $G$ such that $HN/N$ is nilpotent for all $N \lhd G$ with $G/N$ finite. We may assume that $G/K$ is as small as possible for this to be so. Since $G$ satisfies Max-$n$ we may also assume that $HN/N$ is nilpotent whenever $1 \neq N \lhd G$ and that $G$ is infinite. We produce a contradiction by establishing that

there exists $1 = B' < B < G$ with $\Gamma = H/c_H(B)$ finite
and $A \leq c_G(B)$. \hfill (2)

For suppose (2) is established. We separate two cases (i) $B$ is torsion free. Let $b \in B \cap H$ and let $p$ be any prime. Let $\nu$ be the natural homomorphism $H \to B^p/HB^p$, where $B^p$ is the subgroup generated by the $p$-th powers of elements of $B$. Then evidently $|bH^p| \leq p^{\nu_1}$. Since $H^p$ is nilpotent by hypothesis this gives $b^p \in \zeta_1(H^p)$. Hence $[B \cap H, H^p/H] \leq B^p$, for all $p$. Now $B$ is a finitely generated $G/A$-module and hence a finitely generated $K/A$-module. Therefore if $\pi$ is the set of all primes we have by [4, Lemma 12] that $\bigcap_{\pi \in \pi} B^p = 1$. Hence $B \cap H \leq \zeta_1(H^p)$ and so, since $H/B \cap H$ is nilpotent, $H$ is nilpotent and we have a contradiction.

(ii) $B$ is not torsion free. Let $1 \neq P$ be the periodic part of $B$ which has finite exponent $m$, say, since $G$ satisfies Max-$n$. If $b \in P \cap H$ then $|b^H| \leq m^{\nu_1}$. If $1 \neq N \lhd G$ with $G/N$ finite then we have $b^H N/N \leq \zeta_s(HN/N)$, where $s = \log_2(m^{\nu_1})$. Hence $[b_N H] \leq N$. Morover $G$ is residually finite by [4] so that $P \cap H \leq \zeta_s(H)$. However, $H/P \cap H$ is nilpotent and so $H$ is nilpotent and we again have a contradiction.

It remains to establish (2). Set $D = (A \cap H) \cap G$. If $D = 1$ then $A \cap H = 1$. Since $H$ is not nilpotent we must therefore have $A = 1$. But $G$ is infinite so that if we take $K$ for $B$ we obtain (2). Otherwise $D > 1$ and we suppose $D/E$
is a chief factor of $G$. If $E = 1$ then $D$ is finite by [4, Theorem 5.2] and we may take $D$ for $B$ to get (2). Assuming $E \neq 1$ there exists $a \in A \cap H/E$ such that $D = E\langle a^G \rangle$. Now $HE/E \in R$ so, for some $r \geq 1$, $[a, r H] \leq E$. Hence $[a, r A(K \cap H)] \leq E$. By the hypothesis on $G/K$ we clearly must have $G = KH$ so that $A(K \cap H)$ is normal in $G$. Therefore, we have $[\langle a^G \rangle, r A(K \cap H)] \leq E$ and in consequence $A(K \cap H)$ centralizes $D/E$.

It now follows from [5, Lemma 9], or directly from Hilbert's Nullstellensatz, that there exists a positive integer $n$ with $[D, n A(K \cap H)] = 1$. This application of the Nullstellensatz is the crux of the proof. Now we may take $c_D(K \cap H) = B$ to obtain (2). The proof of lemma 4 is complete.

4. PROOF OF LEMMA 5

Let $G$ be an infinite $\mathfrak{G} \cap (\mathfrak{N}_4\mathfrak{K})$-group and $H$ a subnormal subgroup of $G$ such that $HN/N$ is nilpotent for all $1 \neq N \triangleleft G$. By Hall's theorem this means that

$$H \leq \psi(G/N) \quad \text{for all } 1 \neq N \triangleleft G, \tag{3}$$

since nilpotent subnormal subgroups are contained in the Hirsch-Plotkin radical. Again by Hall's theorem $\psi(G) = \rho_1 \mod \phi(G)$ is nilpotent. If $H$ is not nilpotent then $H \leq \psi(G)$ so that there exists a chief factor $U/N$ of $G$ which $H$ does not centralize. By (3) $N = 1$. Also $\phi(G)$ must be trivial. Hence there exists a maximal subgroup $M$ of $G$ with $U \leq M$. Now it follows easily from [4, Theorem 6.1] that $M$ is of finite index in $G$ so that its normal interior $M_0$ also has finite index in $G$ since $G$ is finitely generated. Furthermore $M_0 \cap U = 1$. The nilpotency of $H$ now follows from the fact that both $H/H \cap U$ and $H/H \cap M_0$ are nilpotent.

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