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ad-nilpotent ideals of a Borel subalgebra: generators and duality ☆

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In this paper, we develop several combinatorial aspects of the theory of ad-nilpotent ideals. Let $\mathfrak{b}$ be a fixed Borel subalgebra of a complex simple Lie algebra $\mathfrak{g}$. Following [8], we say that an ideal of $\mathfrak{b}$ is ad-*nilpotent*, if it is contained in $[\mathfrak{b},\mathfrak{b}]$. Let $\mathfrak{Ad}$ or $\mathfrak{Ad}(\mathfrak{g})$ denote the set of all ad-nilpotent ideals of $\mathfrak{b}$. Any $\mathfrak{c}\in\mathfrak{Ad}$ is completely determined by the corresponding set of roots. More precisely, let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ lying in $\mathfrak{b}$ and let Δ be the root system of the pair $(\mathfrak{g},\mathfrak{t})$. Choose Δ^+, the system of positive roots, such that the roots of $\mathfrak{b}$ are positive. Then $\mathfrak{c}=\bigoplus_{\gamma\in I}\mathfrak{g}_\gamma$, where I is a suitable subset of Δ^+ and $\mathfrak{g}_\gamma$ is the root space for $\gamma\in\Delta^+$. In particular, this means that $\mathfrak{Ad}$ is finite. Abusing language, we shall say that such $I\subset\Delta^+$ is an ad-nilpotent ideal, too.

In [8], Cellini and Papi proved that there is a bijection between the ad-nilpotent $\mathfrak{b}$-ideals and the elements of the affine Weyl group $\widehat{W}$ satisfying certain property (see (1.2) below). In our paper, these elements are said to be *admissible*. Using admissible elements, Cellini and Papi established a bijection between $\mathfrak{Ad}$ and the points of the coroot lattice lying in a certain $\operatorname{rk}\mathfrak{g}$-dimensional simplex $\widetilde{D}$ with rational vertices [9]. As a consequence, they obtained a conceptual proof for the explicit formula giving the number of ad-nilpotent ideals in all simple Lie algebras.

In Section 2, we give a characterization of the generators of ad-nilpotent ideals in terms of admissible elements (Theorem 2.2). It is then shown that an ideal I has k generators if and only if the corresponding lattice point lies on the face of $\widetilde{D}$ of codimension k (Theorem 2.9). It is curious that $\widetilde{D}$ has exactly one integral vertex. We deduce this from the fact that there is only one ad-nilpotent ideal having $\operatorname{rk}\mathfrak{g}$ generators.

In Section 3, we consider the 'simple root' statistic on $\mathfrak{Ad}(\mathfrak{g})$, which assigns to any ideal the number of simple roots in it. Write $\mathfrak{Ad}(\mathfrak{g})_i$ for the set of ideals containing exactly

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i simple roots. We give recurrent formulas for these numbers and then compute them for $\mathbf{A}_p$ and the exceptional Lie algebras. It is also shown that the simple root statistic has the same distribution for $\mathbf{B}_p$ and $\mathbf{C}_p$. In case of $\mathbf{C}_p$ and $\mathbf{D}_p$, we give conjectural values for $\#\mathfrak{Ad}(\mathfrak{g})_i$, which are, no doubt, true. As a consequence of this theory, we observe some similarities between the ad-nilpotent ideals and *clusters* (see [11] for the latter). It is shown that the simple root statistic on $\mathfrak{Ad}(\mathfrak{g})$ and a certain statistic on the set of clusters have the same distribution (Theorem 3.11).

To obtain a closed formula for $\#\mathfrak{Ad}(\mathfrak{g})_0$ (Proposition 3.10), we exploit a bijection between the ad-nilpotent ideals and the regions of the Catalan arrangement lying in the dominant chamber, see [18]. We show that $I \in \mathfrak{Ad}(\mathfrak{g})_0$ if and only if the corresponding region is bounded. In turn, the number of bounded regions of any arrangement can be counted using a powerful result of Zaslavsky, once one knows the characteristic polynomial, see Proposition 3.8 for details. Having written this part, I learned that the formula for $\#\mathfrak{Ad}(\mathfrak{g})_0$ had already been obtained, in the same way, in a recent work of Athanasiadis [4]. The main result of Athanasiadis' preprint is a beautiful case-free proof of the formula for the characteristic polynomial of the Catalan arrangement.

In the last three sections, we consider the statistic that assigns to an ideal $I \in \mathfrak{Ad}(\mathfrak{g})$ the number of its generators. In case of $\mathfrak{g} = \mathfrak{sl}_n$, the ad-nilpotent ideals are identified with Dyck path of semilength n and, therefore, the generating function for this statistic is the famous *Narayana polynomial* (of degree $n-1$). For this reason, we say that the generating function for this statistic for arbitrary $\mathfrak{g}$ is a generalized Narayana polynomial. Motivated by the fact that the Narayana polynomial is palindromic, we were searching for a materialization of this property, i.e., for an involutory mapping (duality) on $\mathfrak{Ad}(\mathfrak{sl}_n)$ that takes the ideals with k generators to the ideals with $n-1-k$ generators. For $\mathfrak{sl}_n$, such a materialization does exists, and it has a number of nice properties, see Section 4. The nicety of these properties is that their formulation admits immediate generalization to all simple Lie algebras. We also show that the number of self-dual ideals in $\mathfrak{sl}_{2m+1}$ equals C_m, the mth Catalan number. In Section 5, the results concerning duality are extended to series B and C. This clearly implies that the generalized Narayana polynomials for B and C (in fact, they are equal) are palindromic. We conjecture that such a duality exists for any simple Lie algebra. At least, the generalized Narayana polynomials are always palindromic. General properties of this conjectural duality are discussed in Section 6.

After this paper has been written, there appeared preprints of E. Sommers [19] and C. Athanasiadis [5], which contain some further interesting results on ad-nilpotent ideals and admissible elements. It is worth mentioning that the generalized Narayana polynomials appear in [6, 5.2] in connection with the study of the dual braid monoid.

1. Preliminaries on ad-nilpotent ideals

1.1. Main notation

Δ is the root system of $(\mathfrak{g}, \mathfrak{t})$ and W is the usual Weyl group. For $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ is the corresponding root space in $\mathfrak{g}$.

Δ^+ is the set of positive roots and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$.

$\Pi = \{\alpha_1, \dots, \alpha_p\}$ is the set of simple roots in Δ^+.

$\mathcal{C}$ is the fundamental Weyl chamber.

We set $V := \mathfrak{t}_{\mathbb{Q}} = \bigoplus_{i=1}^p \mathbb{Q}\alpha_i$ and denote by $(\ ,\ )$ a W-invariant inner product on V. As usual, $\mu^\vee = 2\mu/(\mu, \mu)$ is the coroot for $\mu \in \Delta$.

$Q = \bigoplus_{i=1}^p \mathbb{Z}\alpha_i \subset V$ is the root lattice and $Q^\vee = \bigoplus_{i=1}^p \mathbb{Z}\alpha_i^\vee$ is the coroot lattice.

$Q^+ = \{\sum_{i=1}^p n_i\alpha_i \mid n_i \in \mathbb{N}\} \subset Q$.

Letting $\widehat{V} = V \oplus \mathbb{Q}\delta \oplus \mathbb{Q}\lambda$, we extend the inner product $(\ ,\ )$ on $\widehat{V}$ so that $(\delta, V) = (\lambda, V) = (\delta, \delta) = (\lambda, \lambda) = 0$ and $(\delta, \lambda) = 1$.

$\widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\}$ is the set of affine real roots and $\widehat{W}$ is the affine Weyl group.

Then $\widehat{\Delta}^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \geqslant 1\}$ is the set of positive affine roots and $\widehat{\Pi} = \Pi \cup \{\alpha_0\}$ is the corresponding set of affine simple roots. Here $\alpha_0 = \delta - \theta$, where θ is the highest root in Δ^+. The inner product $(\ ,\ )$ on $\widehat{V}$ is $\widehat{W}$-invariant.

For α_i ($0 \leqslant i \leqslant p$), we let s_i denote the corresponding simple reflection in $\widehat{W}$. If the index of $\alpha \in \widehat{\Pi}$ is not specified, then we merely write s_α. The length function on $\widehat{W}$ with respect to $s_0, s_1, \dots, s_p$ is denoted by l. For any $w \in \widehat{W}$, we set

$$\widehat{N}(w) = \{\alpha \in \widehat{\Delta}^+ \mid w(\alpha) \in -\widehat{\Delta}^+\}.$$

Our convention concerning $\widehat{N}(w)$ is the same as in [12,15], but opposite to that in [8,9], so that our $\widehat{N}(w)$ is $\widehat{N}(w^{-1})$ in the sense of Cellini–Papi.

1.2. ad-*nilpotent ideals*

Throughout the paper, $\mathfrak{b}$ is the Borel subalgebra of $\mathfrak{g}$ corresponding to Δ^+ and $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$. The expression "ad-nilpotent ideal" or just "ideal" always refers to a $\mathfrak{b}$-ideal lying in $\mathfrak{u}$. Let $\mathfrak{c} \subset \mathfrak{b}$ be an ad-nilpotent ideal. Then $\mathfrak{c} = \bigoplus_{\alpha \in I} \mathfrak{g}_\alpha$ for a subset $I \subset \Delta^+$, which is called the *set of roots of* $\mathfrak{c}$. As our exposition will be mostly combinatorial, an ad-nilpotent ideal will be identified with the respective set of roots. That is, I is said to be an ad-nilpotent ideal, too. Whenever we want to explicitly indicate the context, we say that $\mathfrak{c}$ is a *geometric* ad-nilpotent ideal, while I is a *combinatorial* ad-nilpotent ideal. Accordingly, being in combinatorial (respectively geometric) context, we speak about cardinality (respectively dimension) of an ideal. In the combinatorial context, the definition of an ad-nilpotent ideal can be stated as follows.

$I \subset \Delta^+$ is an ad-nilpotent ideal, if the following condition is satisfied:

$$\text{if} \quad \gamma \in I,\ \nu \in \Delta^+,\ \text{and}\ \gamma + \nu \in \Delta, \quad \text{then} \quad \gamma + \nu \in I.$$

We consider Δ^+ as poset with respect to the standard partial order '$\preceq$', i.e., $\mu \preceq \nu$ if and only if $\nu - \mu \in Q^+$. Therefore, a combinatorial ad-nilpotent ideal is nothing but a *dual order ideal* of the poset $(\Delta, \preceq)$. An element $w \in \widehat{W}$ is said to be *admissible*, if it has two properties:

(a) $w(\alpha)$ is positive for any $\alpha \in \Pi$;
(b) if $w^{-1}(\alpha)$ is negative for an $\alpha \in \widehat{\Pi}$, then $w^{-1}(\alpha) = \gamma - \delta$ for some $\gamma \in \Delta^+$.

By [8, Section 2], there is a one-to-one correspondence between the admissible elements of $\widehat{W}$ and the ad-nilpotent $\mathfrak{b}$-ideals. This correspondence is obtained as follows:

- Given $\mathfrak{c} \in \mathfrak{Ad}$, consider the members of the descending central series $\mathfrak{c} = \mathfrak{c}^1$, $\mathfrak{c}^k = [\mathfrak{c}^{k-1}, \mathfrak{c}]$ ($k \geqslant 2$) and the corresponding sets of roots I_k. Clearly, $I_k \supset I_{k+1}$ and $I_m = \varnothing$ for $m \gg 0$. Set $N_k = \{k\delta - \gamma \mid \gamma \in I_k\}$. Then $\Phi := \bigcup_{k \geqslant 1} N_k$ is a closed subset of $\widehat{\Delta}^+$ whose complement is closed as well, and therefore there is a unique $w \in \widehat{W}$ such that $\Phi = \widehat{N}(w)$. This w is the required admissible element.
- Conversely, if $w \in \widehat{W}$ is admissible, then $\widehat{N}(w) = \bigcup_{k \geqslant 1} N_k$, where $N_k = \{k\delta - \gamma \mid \gamma \in I_k\}$ and $I_k \subset \Delta^+$. Then I_1 is the set of roots of an ad-nilpotent ideal, say $\mathfrak{c}$. Furthermore, the definition of an admissible element also implies that $I_1 \supset I_2 \supset \cdots$ and I_k is the set of roots of $\mathfrak{c}^k$.

If $w \in \widehat{W}$ is admissible, then I_w (respectively $\mathfrak{c}_w$) stands for the corresponding combinatorial (respectively geometric) ad-nilpotent ideal. That is,

$$I_w = \left\{\gamma \in \Delta^+ \mid \delta - \gamma \in \widehat{N}(w)\right\} \quad \text{and} \quad \mathfrak{c}_w = \bigoplus_{\alpha \in I_w} \mathfrak{g}_\alpha.$$

Conversely, given $I \in \mathfrak{Ad}$, we write $w\langle I\rangle$ for the respective admissible element. Notice that

$$\dim \mathfrak{c}_w = \#(I_w) \quad \text{and} \quad l(w) = \sum_{k \geqslant 1} \dim(\mathfrak{c}_w)^k.$$

Throughout the paper, I or I_w stands for a combinatorial ad-nilpotent ideal. Whenever we wish to stress that $\mathfrak{Ad}$ depends on $\mathfrak{b}$ and/or $\mathfrak{g}$, we write $\mathfrak{Ad}(\mathfrak{b})$ or $\mathfrak{Ad}(\mathfrak{g})$ or even $\mathfrak{Ad}(\mathfrak{b}, \mathfrak{g})$.

2. The generators of ad-nilpotent ideals

Let I be an ad-nilpotent ideal. We say that $\gamma \in I$ is a *generator* of I, if $\gamma - \alpha \notin I$ for all $\alpha \in \Delta^+$. Obviously, this is equivalent to the fact that $I \setminus \{\gamma\}$ is still an ad-nilpotent ideal. Conversely, if $\varkappa$ is a maximal element of $\Delta^+ \setminus I$ (i.e., $(\varkappa + \Delta^+) \cap \Delta \subset I$), then $I \cup \{\varkappa\}$ is an ad-nilpotent ideal. These two procedures show that the following is true.

Proposition 2.1. *Suppose $I \subset J$ are two ad-nilpotent ideals. Then there is a chain of ideals $I = I_0 \subset I_1 \subset \cdots \subset I_m = J$ such that $\#(I_{i+1}) = \#(I_i) + 1$. In other words, $\mathfrak{Ad}$ is a ranked poset, with cardinality (dimension) of an ideal as the rank function.*

In the geometric setting, the set of generators has the following description. For an ideal $\mathfrak{c} = \bigoplus_{\gamma \in I} \mathfrak{g}_\gamma \subset \mathfrak{b}$, there is a unique $\mathfrak{t}$-stable space $\tilde{\mathfrak{c}} \subset \mathfrak{c}$ such that $\mathfrak{c} = [\mathfrak{b}, \mathfrak{c}] \oplus \tilde{\mathfrak{c}}$. Then γ is a generator of I if and only if it is a root of $\tilde{\mathfrak{c}}$. Write $\Gamma(I)$ for the set of generators of I. It is clear that a subset $\Gamma = \{\gamma_1, \ldots, \gamma_l\} \subset \Delta^+$ is the set of generators for some ideal if and only if $\gamma_i - \gamma_j \notin Q^+$ for all i, j. This means that $\Gamma \subset \Delta^+$ is the set of generators for some ad-nilpotent ideal if and only if it is an *antichain* of $(\Delta^+, \preceq)$. This is a manifestation of a general fact that, for any poset P, there is a canonical bijection between the antichains and the dual order ideals of P, see, e.g., [20, 3.1].

In what follows, we also write $I(\Gamma)$ (respectively $\mathfrak{c}(\Gamma)$) for the combinatorial (respectively geometric) ad-nilpotent ideal with the set of generators Γ. For instance, the unique maximal element of $\mathfrak{Ad}$ has the following presentation:

$$\begin{aligned} \text{Geometric}: &\quad \mathfrak{c}(\Pi)=[\mathfrak{b},\mathfrak{b}]=\mathfrak{u}; \\ \text{Combinatorial}: &\quad I(\Pi)=\Delta^+. \end{aligned}$$

It is helpful to have a description of the generators of I in terms of the respective admissible element. As usual, we write $\gamma>0$ (respectively $\gamma<0$), if $\gamma\in\widehat{\Delta}^+$ (respectively $\gamma\in-\widehat{\Delta}^+$).

Theorem 2.2. *Suppose $\gamma\in I_w$. Then γ is a generator of I_w if and only if $w(\delta-\gamma)\in-\widehat{\Pi}$.*

Proof. $\Leftarrow$. Suppose γ is not a generator of I_w, i.e., $\gamma=\bar\gamma+\nu$, where $\bar\gamma\in I_w$ and $\nu\in\Delta^+$. Then $w(\delta-\gamma)=w(\delta-\bar\gamma)-w(\nu)$ is the sum of two *negative* roots.

$\Rightarrow$. Set $w(\delta-\gamma)=-\mu<0$. If μ is not simple, then $\mu=\mu_1+\mu_2$, where both summands are positive. We have $w^{-1}(\mu_1)+w^{-1}(\mu_2)=-(\delta-\gamma)<0$. Assume for definiteness that $w^{-1}(\mu_2)<0$. Since $w^{-1}(-\mu_2)>0$ and $w(w^{-1}(-\mu_2))<0$, we have $w^{-1}(-\mu_2)\in\widehat{N}(w)$, i.e., $w^{-1}(\mu_2)=-k\delta+\nu$, where $k\geqslant 1$ and $\nu\in I_w\subset\Delta^+$.

(a) $k=1$. It follows that $w^{-1}(\mu_1)=\gamma-\nu\in\Delta$ and $w(\nu-\gamma)=-\mu_1<0$. Since w is admissible, $\nu-\gamma$ must be negative, i.e., $\gamma-\nu\in\Delta^+$. This means that γ is not a generator of I_w.
(b) $k\geqslant 2$. Let us show that there is another decomposition of μ as a sum of two positive roots such that one has $k=1$ for one of the summands. We argue by induction on k.

Since $w(k\delta-\nu)<0$, we have $\nu\in(I_w)_k$. Therefore there is a decomposition $k\delta-\nu=k'\delta-\nu'+k''\delta-\nu''$, where $k',k''>0$ and $\nu',\nu''\in I_w$. Hence $\mu_2=\mu_2'+\mu_2''$, where $w^{-1}(\mu_2')=\nu'-k'\delta$ and $w^{-1}(\mu_2'')=\nu''-k''\delta$. The following lemma shows that, in this situation, $\mu_2'+\mu_1\in\Delta^+$ or $\mu_2''+\mu_1\in\Delta^+$. If the latter holds, then $\mu=\mu_2'+(\mu_2''+\mu_1)$ is a decomposition such that $w^{-1}(\mu_2')=\nu'-k'\delta$, and $k'<k$. This completes the induction step. $\square$

Lemma 2.3. *Suppose $\mu_1,\mu_2,\mu_3\in\widehat{\Delta}^+$ and $\mu:=\mu_1+\mu_2+\mu_3\in\widehat{\Delta}^+$. Then $\mu_1+\mu_2\in\widehat{\Delta}^+$ or $\mu_1+\mu_3\in\widehat{\Delta}^+$.*

Proof. If $(\mu_2+\mu_3,\mu_1)<0$, then $(\mu_2,\mu_1)<0$ or $(\mu_3,\mu_1)<0$, and we are done. If $(\mu_2+\mu_3,\mu_1)\geqslant 0$, then $(\mu_2+\mu_3,\mu)>0$. Hence $\mu-\mu_2\in\widehat{\Delta}$ or $\mu-\mu_3\in\widehat{\Delta}$. $\square$

Corollary 2.4. *The number of generators of I_w is equal to the number of roots $\alpha\in\widehat{\Pi}$ such that $w^{-1}(\alpha)<0$.*

Proof. By the definition of an admissible element, if $w^{-1}(\alpha) < 0$, then $w^{-1}(\alpha) = \gamma - \delta$ for some $\gamma \in \Delta^+$. Hence $w(\delta - \gamma) \in -\widehat{\Pi}$ and γ is a generator of I_w. The rest is clear. $\square$

Thus, the set of generators of I_w corresponds to a certain subset of $\widehat{\Pi}$. More precisely, if $w(\gamma - \delta) = \alpha \in \widehat{\Pi}$ ($\gamma \in \Delta^+$), then we say that γ is the generator of I_w *corresponding to* α.

Recall that the *class of nilpotence* of $I \in \mathfrak{A}\mathfrak{d}$, denoted $\mathrm{cl}(I)$, is the maximal k such that $I_k \neq \varnothing$. Making use of the admissible element w defining the ad-nilpotent ideal I_w, one can readily determine $\mathrm{cl}(I_w)$.

Proposition 2.5. $\mathrm{cl}(I_w) = k$ *if and only if* $w(\alpha_0) + k\delta \in \Delta^+ \cup (\delta - \Delta^+)$.

Proof. Since each $(I_w)_m$ is an ad-nilpotent ideal, we have $(I_w)_m \neq \varnothing$ if and only if $\theta \in (I_w)_m$. Therefore, the very definition of the admissible element corresponding to an ad-nilpotent ideal (see (1.2)) implies that $\mathrm{cl}(I_w) = k$ if and only if $w(k\delta - \theta) < 0$ and $w((k+1)\delta - \theta) > 0$. In other words, $w(\alpha_0) + (k-1)\delta < 0$ and $w(\alpha_0) + k\delta > 0$. Hence the conclusion. $\square$

Remark. If I is a non-trivial Abelian ideal, then $\mathrm{cl}(I) = 1$ and the proposition asserts that $w(\alpha_0) + \delta \in \Delta^+ \cup (\delta - \Delta^+)$. However, Proposition 2.4 in [15] says that only the first possibility actually realizes, i.e., $w(\alpha_0) + \delta \in \Delta^+$. But, it can be shown that in case $k = \mathrm{cl}(I) > 1$ we do have both possibilities for $w(\alpha_0) + k\delta$.

Example 2.6. Take $w = s_\theta s_0 \in \widehat{W}$, where $s_\theta \in W$ is the reflection with respect to θ. Then

$$s_\theta s_0(\alpha) = \begin{cases} \alpha + \delta, & (\alpha, \theta) \neq 0, \\ \alpha, & (\alpha, \theta) = 0, \end{cases} \quad \text{for } \alpha \in \Pi.$$

We also have

$$w^{-1} : \begin{cases} \alpha_0 \mapsto \alpha_0 + 2\delta, & \\ \alpha_i \mapsto \alpha_i & \text{if } (\alpha_i, \theta) = 0,\ i \neq 0, \\ \alpha_i \mapsto \alpha_i - \delta & \text{if } (\alpha_i, \theta) \neq 0,\ i \neq 0. \end{cases}$$

Hence w is admissible. The corresponding combinatorial ad-nilpotent ideal is $\mathcal{H} = \{\gamma \in \Delta^+ \mid (\gamma, \theta) > 0\}$ and the set of generators is $\Gamma(\mathcal{H}) = \mathcal{H} \cap \Pi$. The (geometric) ideal $\mathfrak{c} = \bigoplus_{\gamma \in \mathcal{H}} \mathfrak{g}_\gamma$ is the standard Heisenberg subalgebra of $\mathfrak{g}$. Obviously, $\mathrm{cl}(\mathcal{H}) = 2$, and we have $s_\theta s_0(\alpha_0) + 2\delta = \delta - \theta$.

The work of Cellini and Papi [9] establishes a bijection between the ad-nilpotent ideals of $\mathfrak{b}$ and the points of certain simplex in V lying in $Q^\vee$, the coroot lattice. This was used for giving a uniform proof of the formula for the number of ad-nilpotent ideals. Below, we describe that bijection in a form adapted to our notation, and show that this can also be used for determining the number of generators of an ideal.

As is well known, $\widehat{W}$ is isomorphic to a semi-direct product of W and $Q^\vee$. Given $w \in \widehat{W}$, there is a unique decomposition

$$w = v_w \cdot t_{r_w}, \tag{2.7}$$

where $v_w \in W$ and t_{r_w} is the translation corresponding to $r_w \in Q^\vee$. The word "translation" means the following. The group $\widehat{W}$ has two natural actions:

(a) the linear action on $\widehat{V} = V \oplus \mathbb{Q}\delta \oplus \mathbb{Q}\lambda$;
(b) the affine-linear action on V.

For $r \in Q^\vee$, the linear action of $t_r \in \widehat{W}$ on $V \oplus \mathbb{Q}\delta$ is given by $t_r(x) = x - (x,r)\delta$ (we do not need the formulas for the whole of $\widehat{V}$), while the affine-linear action on V is given by $t_r \circ y = y + r$. So that t_r is a true translation for this action on V. For instance, the formulae of Example 2.6 show that $s_\theta s_0 = t_{-\theta^\vee}$.

There is a simple procedure for obtaining the affine-linear action on V from the linear action on $\widehat{V}$, which is explained in [9], but we do not need this.

Using the decomposition (2.7), one can define the mapping $\widehat{W} \to Q^\vee$ by $w \mapsto v_w(r_w) =: d_w$. One of the main results of [9] is that the set $\{d_w\}$, where w ranges over all admissible elements of $\widehat{W}$, provides a nice parametrization of ad-nilpotent ideals. Namely, set

$$\widetilde{D} = \bigl\{\tau \in V \mid (\tau,\alpha) \geqslant -1\ \forall\alpha \in \Pi \text{ and } (\tau,\theta) \leqslant 2\bigr\}.$$

It is a simplex in V. The following is Proposition 3 in [9].

Theorem 2.8 (Cellini–Papi). *The mapping $\mathfrak{Ad} \to Q^\vee$, defined by $I \mapsto w\langle I\rangle \mapsto d_{w\langle I\rangle} =: d_I$, sets up a bijection between $\mathfrak{Ad}$ and $\widetilde{D} \cap Q^\vee$.*

Remark. Our $\widetilde{D} \cap Q^\vee$ is $-D$ in the notation of [9].

Now, we provide a link between the number of generators of I and the position of d_I inside of $\widetilde{D}$.

Theorem 2.9. *The number of generators of I equals the codimension* (*in V*) *of the minimal face of $\widetilde{D}$ containing d_I.*

Proof. We have $w = w\langle I\rangle$ is an admissible element of $\widehat{W}$. Let us realise how the vector $d_I = v_w(r_w)$ can be determined by the linear action of w. If $w = v_w \cdot t_{r_w}$, then $w^{-1} = v_w^{-1} \cdot t_{-v_w(r_w)}$. In the following computations, we repeatedly use the facts that δ is isotropic and $w(\delta) = \delta$ for all $w \in \widehat{W}$. If $x \in V \oplus \mathbb{Q}\delta$, then

$$\begin{aligned} w^{-1}(x) &= v_w^{-1}\bigl(t_{-v_w(r_w)}(x)\bigr) = v_w^{-1}\bigl(x + \bigl(x, v_w(r_w)\bigr)\delta\bigr) = v_w^{-1}(x) + \bigl(x, v_w(r_w)\bigr)\delta \\ &= v_w^{-1}(x) + (x, d_I)\delta. \end{aligned}$$

In particular, we have

$$w^{-1}(\alpha_i)=v_w^{-1}(\alpha_i)+(\alpha_i,d_I)\delta,\quad i\geqslant 1,$$

and

$$w^{-1}(\alpha_0)=v_w^{-1}(\alpha_0)+(\alpha_0,d_I)\delta=-v_w^{-1}(\theta)+\bigl(1-(\theta,d_I)\bigr)\delta.$$

Note that $v_w^{-1}(\alpha_i)$ and $-v_w^{-1}(\theta)$ are in Δ. Therefore, by the very definition of an admissible element, we have $(\alpha_i,d_I)\geqslant -1$ $(i\geqslant 1)$ and $1-(\theta,d_I)\geqslant -1$, i.e., $(\theta,d_i)\leqslant 2$. (In particular, we have recovered the fact that $d_I\in\widetilde{D}$.) Set $k_i=(\alpha_i,d_I)$ and $k_0=1-(\theta,d_I)$. By Theorem 2.2, we have $k_i=-1$ if and only if $v_w^{-1}(\alpha_i)$ is a generator of I; that is, I has a generator corresponding to α_i. Similarly, $k_0=-1$ if and only if I has a generator corresponding to α_0. It remains to observe that $k_i=-1$ $(i=0,1,\ldots,p)$ are precisely the equations of facets of $\widetilde{D}$. $\square$

It follows that an ad-nilpotent ideal has at most n generators, and the ideals having exactly n generators correspond to the *integral* (i.e., lying in $Q^\vee$) vertices of $\widetilde{D}$. Next, we give an elementary proof for the first observation and show that $\widetilde{D}$ always has a unique integral vertex.

Proposition 2.10. *Let $\Gamma\subset\Delta^+$ be an antichain. Then*

(i) *The elements of Γ are linearly independent and hence* $\#(\Gamma)\leqslant \operatorname{rk}\mathfrak{g}$;
(ii) *If* $\#(\Gamma)=\operatorname{rk}\mathfrak{g}$, *then* $\Gamma=\Pi$.

Proof. (i) Suppose $\Gamma=\{\gamma_1,\ldots,\gamma_t\}$. Since $\gamma_i-\gamma_j\notin\Delta$, the angle between any pair of elements of Γ is non-acute. Because all γ_i's lie in open half-space of V, they are linearly independent.

(ii) Suppose $\Gamma=\{\gamma_1,\ldots,\gamma_p\}$, and let $w\in\widehat{W}$ be the corresponding admissible element.

We argue by induction on p. The case $p=1$ being obvious, we assume that $p\geqslant 2$. If $\Gamma\cap\Pi\neq\varnothing$, say $\gamma_1\in\Pi$, then $\{\gamma_2,\ldots,\gamma_p\}$ is an antichain in a root system whose rank is $p-1$. Hence $\Gamma=\Pi$ by the induction assumption. So, we have only to prove that the case $\Gamma\cap\Pi=\varnothing$ is impossible. Assume not, i.e., $\operatorname{ht}(\gamma_i)\geqslant 2$ for all i. By Theorem 2.2,

$$w(\gamma_i)-\delta=\alpha_{l_i}\in\widehat{\Pi}.\tag{2.11}$$

Since $p\geqslant 2$, we may choose i such that α_{l_i} lies in Π. Without loss of generality, we may assume that $i=1$. Choose also roots $\mu,\bar\mu\in\Delta^+$ such that $\gamma_1=\mu+\bar\mu$. Obviously, then $\mu,\bar\mu\notin I=I(\Gamma)$. By part (i), Γ is a basis for V. Hence,

$$\mu=\sum_{j\in J}d_j\gamma_j-\sum_{k\in K}c_k\gamma_k,$$

where J, K are disjoint subsets of $\{1, 2, \ldots, p\}$ and $c_k, d_j > 0$. Therefore,

$$\bar{\mu} = -\sum_{j \in J} d_j \gamma_j + \sum_{k \in K} c_k \gamma_k + \gamma_1.$$

Given $\nu \in \widehat{\Delta}$, we say that the *level* of ν, denoted $\mathrm{lev}(\nu)$, is $m \in \mathbb{Z}$, if $\nu - m\delta \in \Delta$. Consider the roots $w(\mu), w(\bar{\mu}) \in \widehat{\Delta}$. Since $w(\delta - \mu) > 0$ and $w(\mu) > 0$, we have $\mathrm{lev}(w(\mu))$ is either 1 or 0, and likewise for $\bar{\mu}$. As $w(\mu + \bar{\mu}) = \delta + \gamma_1$ has level 1, we may assume without loss that $\mathrm{lev}(w(\mu)) = 0$ and $\mathrm{lev}(w(\bar{\mu})) = 1$. Using Eq. (2.11) for the $w(\gamma_i)$'s, we obtain

$$w(\mu) = \left(\sum_j d_j - \sum_k c_k\right)\delta + \sum_{j \in J} d_j \alpha_{l_j} - \sum_{k \in K} c_k \alpha_{l_k},$$

and

$$w(\bar{\mu}) = \left(1 - \sum_j d_j + \sum_k c_k\right)\delta - \sum_{j \in J} d_j \alpha_{l_j} + \sum_{k \in K} c_k \alpha_{l_k} + \alpha_{l_1}.$$

If one of the roots α_{l_i}, $i \in J \cup K$, is equal to $\alpha_0 = \delta - \theta$, then the equality $\mathrm{lev}(w(\bar{\mu})) - \mathrm{lev}(w(\mu)) = 1$ cannot be satisfied. Hence all these roots lie in Π and hence $\sum_j d_j - \sum_k c_k = \mathrm{lev}(w(\mu)) = 0$. But the equality $w(\mu) = \sum_{j \in J} d_j \alpha_{l_j} - \sum_{k \in K} c_k \alpha_{l_k} \in \Delta$ contradicts the fact that $w(\mu)$ is positive. $\square$

Corollary 2.12. *The simplex $\widetilde{D}$ has a unique integral vertex, corresponding to the unique maximal* ad*-nilpotent ideal.*

The vertices of $\widetilde{D}$ can explicitly be described, see [9]. Indeed, let $\{\pi_i\}$ be the basis for V dual to $\{\alpha_i\}$, $1 \leqslant i \leqslant p$, and h the Coxeter number for Δ. If $\theta = \sum_{i=1}^{p} m_i \alpha_i$ and $\rho^\vee$ is the half-sum of all positive coroots, then the vertices of $\widetilde{D}$ are $-\rho^\vee$ and $-\rho^\vee + \frac{h+1}{m_i}\pi_i$, $1 \leqslant i \leqslant p$. However, it is not immediately clear from this that exactly one vertex lies in $Q^\vee$.

3. A combinatorial statistic on $\mathfrak{Ad}(\mathfrak{g})$, Catalan arrangements, and clusters

By [9], the cardinality of $\mathfrak{Ad}(\mathfrak{g})$ is equal to $\prod_{i=1}^{p} \frac{h+e_i+1}{e_i+1}$, where the e_i's are the exponents and h is the Coxeter number of $\mathfrak{g}$. In this section, we consider the *simple root statistic* on $\mathfrak{Ad}(\mathfrak{g})$. It is given by

$$\mathrm{sim}(I) = \#(I \cap \Pi), \quad I \in \mathfrak{Ad}(\mathfrak{g}).$$

Accordingly, we set

$$\mathfrak{Ad}(\mathfrak{g})_i = \bigl\{I \in \mathfrak{Ad}(\mathfrak{g}) \mid \#(I \cap \Pi) = i\bigr\}, \quad i = 0, 1, \ldots, p.$$

Because we know the number of ad-nilpotent ideals for all simple $\mathfrak{g}$, the number $\mathfrak{Ad}(\mathfrak{g})_0$ can be counted via the inclusion–exclusion principle. Indeed, the ideals containing $\alpha_i \in \Pi$ can be identified with the ideals of the semisimple subalgebra of $\mathfrak{g}$ whose simple roots are $\Pi \setminus \{\alpha_i\}$. Write $\mathfrak{g}(J)$ for the semisimple subalgebra of $\mathfrak{g}$ whose set of simple roots is $J \subset \Pi$. Then

$$\#\mathfrak{Ad}(\mathfrak{g})_0 = \sum_{J \subset \Pi} (-1)^{p-\#J} \#\mathfrak{Ad}\bigl(\mathfrak{g}(J)\bigr). \tag{3.1}$$

In turn, the numbers $\mathfrak{Ad}(\mathfrak{g})_i$ $(i > 0)$ are easily computed, once one knows $\mathfrak{Ad}(\mathfrak{g})_0$. For instance, the number of ideals containing exactly one simple root, say α_i, is equal to the number of all ideals in $\mathfrak{g}(\Pi \setminus \{\alpha_i\})$ that do not contain simple roots. Hence

$$\#\mathfrak{Ad}(\mathfrak{g})_1 = \sum_{\#J = p-1} \#\mathfrak{Ad}\bigl(\mathfrak{g}(J)\bigr)_0.$$

Similarly, one obtains the general formula:

$$\#\mathfrak{Ad}(\mathfrak{g})_i = \sum_{\#J = p-i} \#\mathfrak{Ad}\bigl(\mathfrak{g}(J)\bigr)_0. \tag{3.2}$$

Of course, applying Eqs. (3.1) and (3.2), one should use the relation that if $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, then $\#\mathfrak{Ad}(\mathfrak{h}) = \#\mathfrak{Ad}(\mathfrak{h}_1) \cdot \#\mathfrak{Ad}(\mathfrak{h}_2)$, and likewise for $\mathfrak{Ad}(\mathfrak{h})_0$.

The distribution of the simple root statistic over $\mathfrak{Ad}(\mathfrak{g})$ yields the polynomial

$$\mathcal{S}_{\mathfrak{g}}(q) = \sum_{i=0}^{p} \#\bigl(\mathfrak{Ad}(\mathfrak{g})_i\bigr) q^i,$$

which is not hard to compute. For instance, Table 1 contains the relevant data for exceptional Lie algebras.

It immediately follows from Eq. (3.2) that $\#\mathfrak{Ad}(\mathfrak{g})_p = 1$ and $\#\mathfrak{Ad}(\mathfrak{g})_{p-1} = p$. A bit longer analysis yields

Proposition 3.3. *If $\mathfrak{g}$ is simply-laced, then $\#\mathfrak{Ad}(\mathfrak{g})_{p-2} = (p-1)(p+2)/2$; If $\mathfrak{g} \in \{B, C, F\}$, then $\#\mathfrak{Ad}(\mathfrak{g})_{p-2} = p(p+1)/2$.*

Table 1

i		0	1	2	3	4	5	6	7	8
$\#\mathfrak{Ad}(\mathfrak{g})_i$	$\mathbf{F}_4$	66	24	10	4	1				
	$\mathbf{E}_6$	418	228	110	50	20	6	1		
	$\mathbf{E}_7$	2431	1001	429	187	77	27	7	1	
	$\mathbf{E}_8$	17342	4784	1771	728	299	112	35	8	1

Proof. There are $p(p-1)$ subalgebras of the form $\mathfrak{g}(J)$ with $\#J = 2$. Of these subalgebras, we have

- $p-1$ subalgebra of type $\mathbf{A}_2$ and $(p-1)(p-2)/2$ subalgebras of type $\mathbf{A}_1 \times \mathbf{A}_1$, if $\mathfrak{g}$ is simply-laced;
- one subalgebra of type $\mathbf{C}_2$, $p-2$ subalgebras of type $\mathbf{A}_2$ and $(p-1)(p-2)/2$ subalgebras of type $\mathbf{A}_1 \times \mathbf{A}_1$, if $\mathfrak{g}$ is doubly-laced. $\square$

Our results and conjectures for the classical series are as follows.

Theorem 3.4.

$$\#\mathfrak{Ad}(\mathbf{A}_p)_i = \frac{i+1}{p+1}\binom{2p-i}{p}, \quad i = 0, 1, \ldots, p.$$

We defer the proof to Section 4. Arguing by induction on p and using Eq. (3.2), one obtains $\mathcal{S}_{\mathfrak{g}}(q) = \mathcal{S}_{\mathfrak{g}^\vee}(q)$, where $\mathfrak{g}^\vee$ is the Langlands dual Lie algebra. The only practical output of this equality is that the simple root statistic has the same distribution for $\mathbf{B}_p$ and $\mathbf{C}_p$. However, we have only conjectural values for $\mathbf{C}_p$ and $\mathbf{D}_p$, which are verified for $p \leqslant 8$.

Conjecture 3.5.

$$\#\mathfrak{Ad}(\mathbf{C}_p)_i = \binom{2p-1-i}{p-1}, \quad i = 0, 1, \ldots, p.$$

$$\#\mathfrak{Ad}(\mathbf{D}_p)_i = \binom{2p-2-i}{p-2} + \binom{2p-3-i}{p-2}, \quad i = 1, 2, \ldots, p.$$

Notice that the conjecture does not give an expression for $\#\mathfrak{Ad}(\mathbf{D}_p)_0$. As we will see below, the right value for $\#\mathfrak{Ad}(\mathbf{D}_p)_0$ is $\binom{2p-2}{p-2} + \binom{2p-3}{p-3}$.

Using Eq. (3.1), it is easy to compute $\#\mathfrak{Ad}(\mathfrak{g})_0$ for any simple Lie algebra. However, obtaining a closed expression in the classical case requires some work. In order to obtain a more conceptual explanation and the closed formula valid for all $\mathfrak{g}$, we use the theory of arrangements.

Remark 3.6. Remark Having written up Propositions 3.7 and 3.10 below, I found that exactly the same results are obtained in the recent preprint of C. Athanasiadis [4]. In this preprint, he gave a conceptual proof of the formula (3.9) for the characteristic polynomial of the Catalan arrangement. In fact, Eq. (3.9) was known for all simple Lie algebras via case-by-case verification, and this was used in my original argument.

Recall a bijection between the ad-nilpotent ideals and the regions of the Catalan arrangement that are contained in the fundamental Weyl chamber. This bijection is due to Shi [18, Theorem 1.4], see also [9, §4]. The *Catalan arrangement* $\mathrm{Cat}(\Delta)$ is the set of hyperplanes in V having the equations

$$(x,\mu)=1, \quad (x,\mu)=0, \quad (x,\mu)=-1 \quad \bigl(\mu\in\Delta^+\bigr).$$

The *regions* of an arrangement are the connected components of the complement in V of the union of all its hyperplanes. Clearly, $\mathcal{C}$ is a union of regions of $\mathrm{Cat}(\Delta)$. Any region lying in $\mathcal{C}$ is said to be *dominant*. The bijection takes an ideal $I\subset\Delta^+$ to the region

$$R_I=\bigl\{x\in\mathcal{C}\mid (x,\gamma)>1, \text{if } \gamma\in I \text{ and } (x,\gamma)<1, \text{ if } \gamma\notin I\bigr\}.$$

Obviously, the dominant regions of $\mathrm{Cat}(\Delta)$ are the same as those for the *Shi arrangement* $\mathrm{Shi}(\Delta)$. Here $\mathrm{Shi}(\Delta)$ is the set of hyperplanes in V having the equations

$$(x,\mu)=1, \quad (x,\mu)=0 \quad \bigl(\mu\in\Delta^+\bigr).$$

It will be more convenient for us to deal with the arrangement $\mathrm{Cat}(\Delta)$, since it is W-invariant. A region (of an arrangement) is called *bounded*, if it is contained in a sphere about the origin.

Proposition 3.7. *$I\in\mathfrak{Ad}(\mathfrak{g})_0$ if and only if the region R_I is bounded.*

Proof. (1) Suppose $I\cap\Pi=\varnothing$. Then the definition of R_I shows that it is contained in the bounded domain in $\mathcal{C}$ given by the inequalities $(\alpha,x)<1$, $\alpha\in\Pi$.

(2) Suppose $\alpha_i\in I\cap\Pi$. Then I also contains all positive roots whose coefficient of α_i in the expression through the simple roots is positive. Hence for all roots γ such that $(\gamma,\varphi_i)>0$ we have the constraints $(x,\gamma)>1$. This means that if $x\in R_I$, then all constraints are satisfied for $x+a\varphi_i$ with any $a\in\mathbb{R}_{\geqslant 0}$. Thus, R_I is unbounded. □

The number of regions and bounded regions of any hyperplane arrangement can be counted through the use of a striking result of T. Zaslavsky. Let $\chi(\mathcal{A},t)$ denote the characteristic polynomial of a hyperplane arrangement $\mathcal{A}$ in V (see, e.g., [2,17] for precise definitions).

Theorem 3.8 (Zaslavsky [23, Section 2]). (1) *The number of regions into which $\mathcal{A}$ dissects V equals $r(\mathcal{A})=(-1)^p\chi(\mathcal{A},-1)$.*

(2) *The number of bounded regions into which $\mathcal{A}$ dissects V equals $b(\mathcal{A})=|\chi(\mathcal{A},1)|$.*

Recently, Athanasiadis [4] found a rather simple case-free proof of the following formula for the characteristic polynomial of the Catalan arrangement:

$$\chi\bigl(\mathrm{Cat}(\Delta),t\bigr)=\prod_{i=1}^{p}(t-h-e_i). \tag{3.9}$$

(For the classical series, it was computed earlier in [2].) Now, combining the preceding results, we arrive at our goal.

Proposition 3.10.

$$\#\mathfrak{Ad}(\mathfrak{g})_0 = \prod_{i=1}^{p} \frac{h+e_i-1}{e_i+1}.$$

Proof. Since the arrangement $\mathrm{Cat}(\Delta)$ is W-invariant, the number of its bounded regions lying in $\mathcal{C}$ is equal to $\frac{1}{\#W}|\chi(\mathrm{Cat}(\Delta),1)|$. It remains to observe that $\#W = \prod_i (e_i+1)$. □

Similarly, using the value $\chi(\mathrm{Cat}(\Delta),-1)$, as Athanasiadis also did in [4], one obtains the formula for the number of all ad-nilpotent ideals stated at the beginning of this section. This proof is not so elementary as the proof of Cellini–Papi [9], for it requires some deep results from the theory of arrangements.

It is quite interesting that the numbers $\prod_{i=1}^{p} \frac{h+e_i+1}{e_i+1}$ and $\prod_{i=1}^{p} \frac{h+e_i-1}{e_i+1}$ also appear in [11, Theorem 1.9 and Proposition 3.9] as the numbers of all and positive clusters, respectively. We are not going to discuss the theory of clusters related to the root systems, referring to that paper for all relevant definitions. For our current purposes, it suffices to know that clusters are certain subsets of $\Delta^+ \cup (-\Pi)$. Each cluster is a linearly independent subset of V having exactly p elements. A cluster is called *positive*, if all its elements are positive roots.

A close relationship between clusters and ad-nilpotent ideals is seen in the following curious fact. Let $\mathfrak{Clus}(\mathfrak{g})_i$ denote the set of clusters having exactly i elements from $-\Pi$.

Theorem 3.11. *One always has the equality* $\#\mathfrak{Ad}(\mathfrak{g})_i = \#\mathfrak{Clus}(\mathfrak{g})_i$.

Proof. From Proposition 3.6 in [11], it follows that the numbers $\mathfrak{Clus}(\mathfrak{g})_i$, $i = 0,1,\dots,p$, also satisfy the recurrent relations Eqs. (3.1) and (3.2). □

It is not too brave to suggest that there exists a natural bijection between clusters and ad-nilpotent ideals that takes $\mathfrak{Clus}(\mathfrak{g})_i$ to $\mathfrak{Ad}(\mathfrak{g})_i$ for all i.

4. On ad-nilpotent ideals for $\mathfrak{g} = \mathfrak{sl}_n$

For the rest of the paper, we are going to study another combinatorial statistic on the set of ad-nilpotent ideals, which is related to the theory developed in Section 2. We first consider the classical series in Sections 4 and 5, and then move to the general case in Section 6.

At the rest of this section, $\mathfrak{g} = \mathfrak{sl}_n$ and hence $p = n-1$. We assume that $\mathfrak{b}$ (respectively $\mathfrak{t}$) is standard, i.e., it is the space of upper-triangular (respectively diagonal) matrices. Then the positive roots are identified with the pairs (i,j), where $1 \leqslant i < j \leqslant n$. For instance, $\alpha_i = (i,i+1)$ and $\theta = (1,n)$. An ad-nilpotent $\mathfrak{b}$-ideal is represented by a right-justified Ferrers diagram with at most $n-1$ rows, where the length of ith row is at most $n-i$. If a box of a Ferrers diagram corresponds to a positive root (i,j), then we say that this box has the coordinates $(i.j)$. The unique northeast corner of the diagram corresponds to θ and the southwest corners give rise to the generators of the corresponding ideal, see Fig. 1.

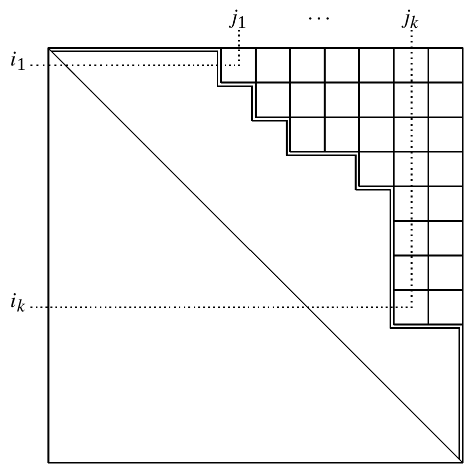

Fig. 1. An ad-nilpotent ideal in $\mathfrak{sl}_n$.

Such a diagram (ideal) I is completely determined by the coordinates of boxes that contain the southwest corners of the diagram, say $(i_1, j_1), \ldots, (i_k, j_k)$. Then we obviously have $\Gamma(I) = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ and

$$1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n-1, \qquad 2 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n.$$

Various enumerative results for ad-nilpotent ideals in $\mathfrak{sl}_n$ are obtained in [1,8,14]. In particular, the total number of ad-nilpotent ideals equals $C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. There is a host of combinatorial objects that are counted by Catalan numbers, see [21, Chapter 6, Example 6.19] and the "Catalan addendum" at www-math.mit.edu/~rstan/ec. We shall use the fact that C_n is equal to

(a) the number of all sequences $v = v_1 v_2 \cdots v_{2n}$ of n 1's and n -1's with all partial sums nonnegative, or
(b) the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0)$ and $(0,1)$, always staying in the domain $x \leqslant y$, i.e., the number of Dyck paths of semilength n.

In our matrix interpretation, we are forced to assume that the x-axis is vertical and directed downwards, while the y-axis is horizontal. Therefore $(0,0)$ is the upper-left corner and (n,n) is the lower-right corner of the matrix. The Dyck path corresponding to an ad-nilpotent ideal is the double path in Fig. 1. It has $2n$ steps. The corresponding sequence v is obtained as follows. We start from $(0,0)$ and attach $+1$ to the horizontal step (i.e., $(0,1)$) and -1 to the vertical step (i.e., $(1,0)$).

Remark. Coordinates of boxes of Ferrers diagrams and lattice points considered above are compatible in the sense that the coordinates of a box are equal to the coordinates of its southeast corner.

Proof of Theorem 3.4. Once the relationship between the ad-nilpotent ideals in $\mathfrak{sl}_n$ and Dyck paths is established, one may appeal to huge combinatorial literature on the latter. It is clear that $I \in \mathfrak{Ad}(\mathfrak{sl}_n)$ contains a simple root if and only if the corresponding Dyck path touches the diagonal somewhere except the points $(0,0)$ and (n,n). In other words, the number of simple roots in I equals the number of (intermediate) *returns* of the Dyck path. The distribution of this statistic is well-known, see, e.g., [10, 6.6]. □

Let $\mathfrak{Ad}_n$ denote the set of all ad-nilpotent ideals for $\mathfrak{sl}_n$. From now on, we stick to considering the statistic $\mathrm{gen}: \mathfrak{Ad}_n \to \mathbb{N}$, which assigns to an ideal the number of its generators. Let $\mathfrak{Ad}_n^k$, $0 \leqslant k \leqslant n-1$, be the set of ideals with k generators, i.e., the set of Ferrers diagrams, as above, with exactly k southwest corners.

Proposition 4.1.

$$\#\bigl(\mathfrak{Ad}_n^k\bigr) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

Proof. The numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, $k = 1,\dots,n$, are called the *Narayana numbers*, so that we are to show that $\#(\mathfrak{Ad}_n^{k-1}) = N(n,k)$. It is known that the Narayana numbers have the following combinatorial interpretation, see [21, Chapter 6, Example 36(a)]. Let X_{nk} be the set of all sequences $v = v_1 \cdots v_{2n}$ as in (a) above, such that

$$k = \#\{j \mid v_j = 1,\ v_{j+1} = -1\}.$$

Then $\#(X_{nk}) = N(n,k)$. For $v \in X_{nk}$, it is easily seen that

$$k-1 = \#\{j \mid v_j = -1,\ v_{j+1} = 1\}.$$

The change of sign from 1 to -1 (respectively from -1 to 1) in v corresponds to the turn of the type "horizontal followed by vertical" (respectively "vertical followed by horizontal") step in the respective lattice path. Geometrically, the steps of second type correspond to the southwest corners of our Ferrers diagram. It follows that the sequences $v \in X_{nk}$ are in bijection with the Ferrers diagrams with $k-1$ southwest corners, and we are done. □

Since $N(n,k) = N(n, n-k+1)$, one may suggest that there is a bijective interpretation of this equality. This is really the case.

Theorem 4.2. *There is a natural bijection between $\mathfrak{Ad}_n^k$ and $\mathfrak{Ad}_n^{n-k-1}$.*

Proof. Let $(i_1, j_1), \dots, (i_k, j_k)$ be the generators of an ad-nilpotent ideal $I \in \mathfrak{Ad}_n^k$. Consider separately the ordered sets of the first and second coordinates for these generators, i.e. put $X(I) = \{i_1, \dots, i_k\}$ and $Y(I) = \{j_i, \dots, j_k\}$. We wish to construct two other ordered sets that will form the first and the second coordinates of the generators for the dual ideal. To this end, put

$$X\bigl(I^*\bigr) = \{1, \dots, n-1\} \setminus \{j_1 - 1, \dots, j_k - 1\}.$$

$$Y(I^*) = \{2, \dots, n\} \setminus \{i_1 + 1, \dots, i_k + 1\}.$$

For $A = \{a_1, \dots, a_m\}$, it is convenient to introduce notation $A[a] = \{a_1 + a, \dots, a_m + a\}$. Then the previous formulas can be written as

$$\begin{aligned} X(I^*) &= \bigl(\{2, \dots, n\} \setminus Y(I)\bigr)[-1], \\ Y(I^*) &= \bigl(\{1, \dots, n-1\} \setminus X(I)\bigr)[1]. \end{aligned} \tag{4.3}$$

It is then easily seen that the square of this transformation is the identity on $\mathfrak{Ad}_n^k$. Therefore one has only to prove that the ordered sets $X(I^*)$, $Y(I^*)$ determine an ad-nilpotent ideal. The latter means that if $X(I^*) = \{i_1^*, \dots, i_{n-k-1}^*\}$ and $Y(I^*) = \{j_1^*, \dots, j_{n-k-1}^*\}$, then $i_q^* < j_q^*$ for all q. (Of course, $i_1^* < i_2^* < \cdots$ and likewise for j_l^*.)

(a) Given $q \in \{1, \dots, n-k-1\}$, suppose there is m such that $i_m > m + q - 1$. Assume also that m is the minimal number with this property. Then $i_m \geqslant m + q$ and $i_{m-1} < m - 1 + q$. Therefore the qth element of $\{1, \dots, n-1\} \setminus X(I)$ is $m - 1 + q$ and hence $j_q^* = m + q$. Since $j_m > i_m = m + q$, we can find the minimal number l such that $j_l > l + q$. Then $l \leqslant m$ and the qth element of $\{2, \dots, n\} \setminus Y(I)$ is $l + q$. Thus, $i_q^* = l + q - 1 < m + q = j_q^*$.

(b) Suppose $i_m \leqslant m + q - 1$ for all $m \in \{1, \dots, k\}$, that is, $i_k \leqslant k + q - 1$. Then the qth element of $\{1, \dots, n-1\} \setminus X(I)$ is $k + q$ and hence $j_q^* = k + q + 1$. On the other hand, the inequalities $i_q^* < i_{q+1}^* < \cdots < i_{n-k-1}^* \leqslant n - 1$ show that $i_q^* \leqslant (n-1) - ((n-k-1) - q) = q + k$.

Thus, $X(I^*)$ and $Y(I^*)$ determine an element of $\mathfrak{Ad}_n^{n-k-1}$, which we denote by I^*. □

For all $k \in \{0, 1, \dots, n-1\}$, we have constructed bijections

$$\mathfrak{Ad}_n^k \to \mathfrak{Ad}_n^{n-k-1}, \quad I \mapsto I^*.$$

which give rise to an involutory transformation $*: \mathfrak{Ad}_n \to \mathfrak{Ad}_n$. Although this transformation is not order-reversing with respect to the inclusion of ideals, it has interesting properties. The formulation of these properties is "universal," i.e., it makes sense for any (semi)simple Lie algebra:

Lemma 4.4. *Suppose $A \subset \Pi$ is an arbitrary subset, and $I = I(A)$. Then $I^* = I(\Pi \setminus A)$.*

Proof. Straightforward. Use formulae (4.3). □

To state one more property, we need some notation. As usual, the height of a root $\gamma \in \Delta^+$ is denoted by $\mathrm{ht}(\gamma)$. Recall that $h = \mathrm{ht}(\theta) + 1$ is the Coxeter number of $\mathfrak{g}$. Set $\Delta^+(k) = \{\gamma \in \Delta^+ \mid \mathrm{ht}(\gamma) = k\}$ and $\Delta_k^+ = \{\gamma \in \Delta^+ \mid \mathrm{ht}(\gamma) \geqslant k\}$. It is clear that Δ_k^+ is a combinatorial ad-nilpotent ideal and $\Gamma(\Delta_k^+) = \Delta^+(k)$.

For $\mathfrak{sl}_n$, we have $\mathrm{ht}(i, j) = j - i$ and the Coxeter number is n.

Lemma 4.5. *In case of* $\mathfrak{sl}_n$, *we have* $(\Delta_k^+)^* = \Delta_{h+1-k}^+ = \Delta_{n+1-k}^+$.

Proof. Set $I = \Delta_k^+$. In our notation, the roots in $\Delta^+(k)$ are $(1, k+1), (2, k+2), \dots, (n-k, n)$. Hence $X(I) = \{1, 2, \dots, n-k\}$ and $Y(I) = \{k+1, k+2, \dots, n\}$. Therefore $X(I^*) = \{1, 2, \dots, k-1\}$ and $Y(I^*) = \{n-k+2, \dots, n\}$. This means that I^* is generated by the roots $(1, n-k+2), \dots, (k-1, n)$, i.e., all roots of height $n-k+1$. $\square$

Examples. In the geometric context, taking $k = 1$, we obtain $\mathfrak{u}^* = \{0\}$. For $k = 2$, we have $[\mathfrak{u}, \mathfrak{u}]^* = \mathfrak{g}_\theta$, because θ is the only root of height $h-1$.

It is curious that our definition of the dual ad-nilpotent ideal for $\mathfrak{sl}_n$ leads to another occurrence of Catalan numbers. Namely, let us try to describe and enumerate the self-dual ideals. For $I \in \mathfrak{Ad}_n^m$, the necessary condition of self-duality is $m = n - m - 1$. That is, $n = 2m + 1$.

Theorem 4.6. *There are no self-dual* ad*-nilpotent ideals for* $\mathfrak{sl}_{2m}$. *For* $\mathfrak{sl}_{2m+1}$, *the number of self-dual* ad*-nilpotent* $\mathfrak{b}$*-ideals is equal to* $\frac{1}{m+1}\binom{2m}{m}$.

Proof. We use the notation introduced in Theorem 4.2. Suppose $I \in \mathfrak{Ad}_{2m+1}^m$ and $X = X(I) = \{i_1, i_2, \dots, i_m\}$, $Y = Y(I) = \{j_1, j_2, \dots, j_m\}$. The condition $I = I^*$ means that $X = X^* = (\{2, 3, \dots, 2m+1\} \setminus Y)[-1]$ and $Y = Y^* = (\{1, 2, \dots, 2m\} \setminus X)[1]$. Clearly, all these equalities are equivalent to the following

$$\{1, 2, \dots, 2m\} = \{i_1, i_2, \dots, i_m\} \sqcup \{j_1 - 1, j_2 - 1, \dots, j_m - 1\} = X \sqcup Y[-1].$$

Therefore Y is determined by X and vice versa. However, X cannot be an arbitrary m-element subset of $\{1, 2, \dots, 2m\}$, since the conditions $i_k < j_k$, $k = 1, \dots, m$, must also be satisfied. Given $X \subset \{1, 2, \dots, 2m\}$ with $\#(X) = m$, define the sequence $v = v_1 v_2 \dots v_{2m}$ by the following rule:

$$v_i = \begin{cases} 1, & \text{if } i \in X, \\ -1, & \text{if } i \notin X. \end{cases}$$

Then the pair $(X, \{1, 2, \dots, 2m\} \setminus X\} = Y[-1])$ determines an ad-nilpotent ideal if and only if all partial sums of v are nonnegative. Indeed, $\sum_{i=1}^{2k-1} v_i < 0$ if and only if $i_k \geqslant j_k$. As was mentioned above, the number of such sequences is the mth Catalan number. $\square$

To illustrate Theorem 4.6, we list the generators of all self-dual ideals for $\mathfrak{sl}_7$:

$$\Gamma_1 = \{(1,5), (2,6), (3,7)\}, \qquad \Gamma_2 = \{(1,4), (2,6), (4,7)\},$$
$$\Gamma_3 = \{(1,4), (2,5), (5,7)\}, \qquad \Gamma_4 = \{(1,3), (3,6), (4,7)\},$$
$$\Gamma_5 = \{(1,3), (3,5), (5,7)\}.$$

Remark 4.7. The equality of Theorem 4.6 is (almost) an instance of the so-called "$q = -1$ phenomenon" studied by J. Stembridge [22]. The distribution of the statistic "number of generators" yields the polynomial

$$N_n(q) = \sum_{k=0}^{n-1} \#\big(\mathfrak{Ad}_n^k\big) q^k = \sum_{k=0}^{n-1} \frac{1}{n}\binom{n}{k}\binom{n}{k+1} q^k,$$

which is often called the *Narayana polynomial*. The $q = -1$ phenomenon is said to occur if $N_n(-1)$ counts the number of fixed points of some natural involution on $\mathfrak{Ad}_n$. We already have the involution '$*$' and know the number of its fixed points. On the other hand, it follows from [7, Proposition 2.2] that

$$N_n(-1) = \begin{cases} 0, & \text{if } n \text{ is even}, \\ (-1)^{(n-1)/2} C_{n-1/2}, & \text{if } n \text{ is odd}. \end{cases} \tag{4.8}$$

(Actually, the authors of [7] deal with the polynomial $d_n(q) = (1+q)N_n(q+1)$. However, the sign given there for the value $d_n(-2)$ should be opposite.) Thus, we see that the $q = -1$ phenomenon occurs up to sign. It is interesting that Eq. (4.8) appears also in [13, p. 276] in connection with a discussion of the Charney–Davis conjecture and properties of the Coxeter zonotope of type A.

The involution on $\mathfrak{Ad}_n$ (and hence on the set of Dyck paths of semilength n) described in Theorem 4.2 seems to be new.

5. ad-nilpotent $\mathfrak{b}$-ideals for orthogonal and symplectic Lie algebras

A possible idea for constructing an involutory mapping $* : \mathfrak{Ad}(\mathfrak{g}) \to \mathfrak{Ad}(\mathfrak{g})$ for the other classical Lie algebras can be the following:

Consider the standard embedding $\mathfrak{g} \subset \mathfrak{sl}_N$, and choose a Borel subalgebra $\bar{\mathfrak{b}} \subset \mathfrak{sl}_N$ such that $\bar{\mathfrak{b}} \cap \mathfrak{g} = \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. Making use of the embedding $\mathfrak{b} \subset \bar{\mathfrak{b}}$, one can regard $\mathfrak{Ad}(\mathfrak{b}, \mathfrak{g})$ as a subset of $\mathfrak{Ad}(\bar{\mathfrak{b}}, \mathfrak{sl}_N)$ consisting of ideals satisfying a symmetry condition. Then we apply to $\mathfrak{Ad}(\bar{\mathfrak{b}}, \mathfrak{sl}_N)$ the duality procedure described in the previous section. The last step should be to interpret the resulting ideal in $\mathfrak{sl}_N$ as an element of $\mathfrak{Ad}(\mathfrak{b}, \mathfrak{g})$.

It turns out that this recipe yields "expected" results for $\mathfrak{sp}_{2p}$, but not immediately for $\mathfrak{so}_p$. The obstacle is that the last step in the above program cannot always be fulfilled in the orthogonal case. Still, one can modify this procedure, so that to get a suitable result for $\mathfrak{so}_{2p+1}$. However, I do not know how to deal with the case of $\mathfrak{so}_{2p}$.

5.1. *The symplectic case*

Choose a basis for a $2p$-dimensional symplectic $\Bbbk$-vector space $\mathbb{V}$ so that the skew-symmetric non-degenerate bilinear form has the matrix $\begin{pmatrix} 0 & \Upsilon_p \\ -\Upsilon_p & 0 \end{pmatrix}$, where Υ_p is the $p \times p$ matrix whose only nonzero entries are 1's along the antidiagonal.

For any $A \in \mathrm{Mat}_p(\Bbbk)$, let $\widehat{A}$ denote the matrix $\Upsilon_p(A^t)\Upsilon_p$, where A^t is the usual transpose of A. The transformation $A \mapsto \widehat{A}$ is the transpose relative to the antidiagonal. In the above basis for $\mathbb{V}$, the algebra $\mathfrak{sp}_{2p}$ has the following block form:

$$\mathfrak{sp}_{2p} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| B = \widehat{B},\ C = \widehat{C},\ D = -\widehat{A} \right\},$$

where A, B, C, D are $p \times p$ matrices. If $\bar{\mathfrak{b}}$ is the standard Borel subalgebra of $\mathfrak{sl}_{2p}$, then $\mathfrak{b} := \bar{\mathfrak{b}} \cap \mathfrak{sp}_{2p}$ is a Borel subalgebra of $\mathfrak{sp}_{2p}$. It follows that $\mathfrak{Ad}(\mathfrak{sp}_{2p})$ can be identified with *the* subset of $\mathfrak{Ad}(\mathfrak{sl}_{2p})$ consisting of all Ferrers diagram that are symmetric relative to the antidiagonal.

Let us say that $\bar{I} \in \mathfrak{Ad}(\mathfrak{sl}_{2p})$ is *self-conjugate*, if the corresponding Ferrers diagram is symmetric with respect to the antidiagonal. It is easily seen that if $\bar{I} \in \mathfrak{Ad}(\mathfrak{sl}_{2p})$ is self-conjugate, then $\bar{I}^*$ is self-conjugate as well, see below. This induces the desired involution on $\mathfrak{Ad}(\mathfrak{sp}_{2p})$, and a straightforward verification shows that this involution satisfies properties (4.4) and (4.5).

Since the Ferrers diagram corresponding to an ad-nilpotent $\mathfrak{b}$-ideal has a symmetry property, we may cancel out its part which is below the antidiagonal. What we obtain is a shifted Ferrers diagram.

Example 5.1.1. $\mathfrak{g} = \mathfrak{sp}_8$. In our matrix interpretation, the array of positive roots is

$$\begin{array}{ccccccc} 1000 & 1100 & 1110 & 1111 & 1121 & 1221 & 2221 \\ & 0100 & 0110 & 0111 & 0121 & 0221 & \\ & & 0010 & 0011 & 0021 & & \\ & & & 0001 & & & \end{array}$$

where the quadruple $c_1c_2c_3c_4$ stands for the root $\sum c_i\alpha_i$. Consider the ad-nilpotent ideal I whose generators are $\alpha_1, \alpha_2+\alpha_3, 2\alpha_3+\alpha_4$. The corresponding shifted Ferrers diagram is depicted on the left hand side in Fig. 2.

The dotted lines demonstrate the positive roots that are not in I, and the whole array corresponds to Δ^+ (or $\mathfrak{u}$). The boxes marked with '$\circ$' represent the generators. The corresponding self-conjugate ideal $\bar{I} \in \mathfrak{Ad}(\mathfrak{sl}_8)$ is depicted in Fig. 3, where the dotted line is the antidiagonal.

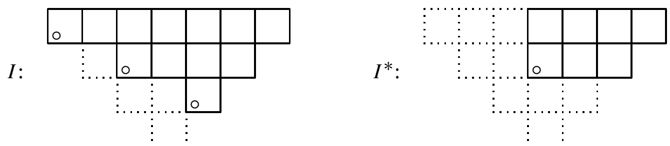

Fig. 2. An ad-nilpotent ideal in $\mathfrak{Ad}(\mathfrak{sp}_8)$ and its dual.

$\bar{I}$:

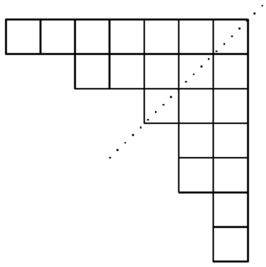

Fig. 3. The self-conjugate ad-nilpotent ideal $\bar{I}$ in $\mathfrak{Ad}(\mathfrak{sl}_8)$.

From the picture representing $\bar{I}$, we find that $X(\bar{I}) = \{1, 2, 3, 5, 7\}$ and $Y(\bar{I}) = \{2, 4, 6, 7, 8\}$. Therefore $X(\bar{I}^*) = \{2, 4\}$ and $Y(\bar{I}^*) = \{5, 7\}$. This leads to the diagram depicted on the right hand side in Fig. 2. The sole generator of the ideal I^* is $\alpha_2 + \alpha_3 + \alpha_4$.

Formally, our recipe for constructing the dual ad-nilpotent ideal in $\mathfrak{Ad}(\mathfrak{sp}_{2p})$ is as follows. We use the same coordinate system as in the $\mathfrak{sl}_n$-case. The shifted Ferrers diagram (as in Fig. 2) is determined by the coordinates of the boxes that contain its southwest corners, and these boxes give rise to the generators of the respective ad-nilpotent ideal. Suppose $\Gamma = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ is the set of generators of $I \in \mathfrak{Ad}(\mathfrak{sp}_{2p})$, and $i_1 < i_2 < \cdots < i_k$. Then $i_l < j_l$ for all l, $j_1 < j_2 < \cdots < j_k$, and $i_k + j_k \leqslant 2p + 1$. Conversely, if a set Γ satisfies all these inequalities, then it is the set of generators of an ad-nilpotent ideal. Denoting by $\bar{I}$ the corresponding self-conjugate ideal in $\mathfrak{Ad}(\mathfrak{sl}_{2p})$, we obtain

$$X(\bar{I}) = (i_1, \ldots, i_k, 2p + 1 - j_k, \ldots, 2p + 1 - j_1),$$
$$Y(\bar{I}) = (j_1, \ldots, j_k, 2p + 1 - i_k, \ldots, 2p + 1 - i_1).$$

[If $i_k + j_k = 2p + 1$, then one should cancel out the repetition in the middle.] The coordinates of vectors $X(\bar{I}), Y(\bar{I})$ can be paired so that the sum in each pair is equal to $2p + 1$. Therefore the same property holds for the shifted complements $X(\bar{I}^*), Y(\bar{I}^*)$. That is, $\bar{I}^*$ is again a self-conjugate ideal in $\mathfrak{Ad}(\mathfrak{sl}_{2p})$, and we can define the ideal $I^* \in \mathfrak{Ad}(\mathfrak{sp}_{2p})$.

Notice that

$$\#\Gamma(I) + \#\Gamma(I^*) = p$$

and the multiset $\{\Gamma(I), \Gamma(I^*)\}$ contains a unique long root, i.e., the distribution of long and short roots is always the same as in Π. (A long root corresponds to the generator (i_k, j_k) with $i_k + j_k = 2p + 1$.) In particular, the equality $I = I^*$ is impossible, i.e., there are *no* self-dual ad-nilpotent ideals.

Example 5.1.2. $\mathfrak{g} = \mathfrak{sp}_6$. In Table 2, we list all pairs of dual ad-nilpotent ideals including the ideals with one and two generators. The column with I (respectively I^*) contains all

Table 2
Pairs of dual ad-nilpotent ideals in $\mathfrak{sp}_6$

No.	$\Gamma(I)$	$\Gamma(I^*)$
1–3	α_i	$\Pi \setminus \{\alpha_i\}$
4	$\alpha_1+\alpha_2$	$\alpha_1+\alpha_2, \alpha_3$
5	$\alpha_2+\alpha_3$	$2\alpha_2+\alpha_3, \alpha_1$
6	$2\alpha_2+\alpha_3$	$\alpha_2+\alpha_3, \alpha_1$
7	$\alpha_1+\alpha_2+\alpha_3$	$\alpha_1+\alpha_2, 2\alpha_2+\alpha_3$
8	$\alpha_1+2\alpha_2+\alpha_3$	$\alpha_1+\alpha_2+\alpha_3, 2\alpha_2+\alpha_3$
9	$2\alpha_1+2\alpha_2+\alpha_3$	$\alpha_1+\alpha_2, \alpha_2+\alpha_3$

ideals with one (respectively) two generators. The numeration of simple roots is standard: $\alpha_1=\varepsilon_1-\varepsilon_2,\ \alpha_2=\varepsilon_2-\varepsilon_3,\ \alpha_3=2\varepsilon_3$.

It is clearly seen that properties of Lemmas 4.4 and 4.5 are satisfied here.

5.2. *The orthogonal case*

Choose a basis for an n-dimensional orthogonal $\Bbbk$-vector space $\mathbb{V}$ so that the symmetric non-degenerate bilinear form has the matrix Υ_n. In the above basis for $\mathbb{V}$, we have:

$$\mathfrak{so}_n=\left\{A \mid A=-\widehat{A}\right\}.$$

Here we also have $\mathfrak{b}:=\bar{\mathfrak{b}}\cap\mathfrak{so}_n$ is a Borel subalgebra. This means that to any ad-nilpotent $\mathfrak{b}$-ideal in $\mathfrak{so}_n$, one can again attach a self-conjugate ad-nilpotent $\bar{\mathfrak{b}}$-ideal in $\mathfrak{sl}_n$. But unlike the symplectic case this mapping is not onto. The reason is that the orthogonal matrices have zero antidiagonal entries. Therefore a self-conjugate ad-nilpotent ideal in $\mathfrak{sl}_n$ having a generator on the antidiagonal cannot correspond to a $\mathfrak{b}$-ideal in $\mathfrak{so}_n$. It may happen that, for $I\in\mathfrak{Ad}(\mathfrak{so}_n)$, the last element in the sequence $I\to\bar{I}\to\bar{I}^*$ cannot be interpreted as an ideal in $\mathfrak{so}_n$. So, a naive attempt to repeat the "symplectic" procedure fails.

In the odd-dimensional case, this difficulty can be circumvented by associating to a $\mathfrak{b}$-ideal in $\mathfrak{so}_{2p+1}$ the ideal in $\mathfrak{sp}_{2p}$ having the same shape (shifted Ferrers diagram). This is achieved by cancelling out from a symmetric Ferrers diagram both the antidiagonal (which corresponds to zero entries in the matrix) and the part below the antidiagonal. This leads to a satisfactory procedure.

Example 5.2.1. $\mathfrak{g}=\mathfrak{so}_7$. In Table 3, we list all pairs of dual ad-nilpotent ideals including the ideals with one and two generators. The column with I (respectively I^*) contains all ideals with one (respectively) two generators. The numeration of simple roots is standard: $\alpha_1=\varepsilon_1-\varepsilon_2,\ \alpha_2=\varepsilon_2-\varepsilon_3,\ \alpha_3=\varepsilon_3$. One can see some small distinctions from Table 2.

Again, the properties of Lemmas 4.4 and 4.5 are satisfied here. In the following section, we also summarize some other properties of the duality mapping that are inspired by our computations in classical cases.

Table 3
Pairs of dual ad-nilpotent ideals in $\mathfrak{so}_7$

No.	$\Gamma(I)$	$\Gamma(I^*)$
1–3	α_i	$\Pi \setminus \{\alpha_i\}$
4	$\alpha_1+\alpha_2$	$\alpha_1+\alpha_2, \alpha_3$
5	$\alpha_2+\alpha_3$	$\alpha_2+2\alpha_3, \alpha_1$
6	$\alpha_2+2\alpha_3$	$\alpha_2+\alpha_3, \alpha_1$
7	$\alpha_1+\alpha_2+\alpha_3$	$\alpha_1+\alpha_2, \alpha_2+2\alpha_3$
8	$\alpha_1+\alpha_2+2\alpha_3$	$\alpha_1+\alpha_2+\alpha_3, \alpha_2+2\alpha_3$
9	$\alpha_1+2\alpha_2+2\alpha_3$	$\alpha_1+\alpha_2, \alpha_2+\alpha_3$

6. Towards the general case

In view of Theorem 4.2, it is natural to ask whether there is a natural involutory mapping $*: \mathfrak{Ad}(\mathfrak{g}) \to \mathfrak{Ad}(\mathfrak{g})$ for any simple Lie algebra $\mathfrak{g}$ such that

$$\#\bigl(\Gamma(I)\bigr) + \#\bigl(\Gamma\bigl(I^*\bigr)\bigr) = \operatorname{rk}\mathfrak{g}$$

and the two properties of Lemmas 4.4 and 4.5 are also satisfied?

It is plausible that a conjectural definition of duality should exploit somehow admissible elements of $\widehat{W}$ and the simplex $\widetilde{D}$. Although my attempts to define such a mapping in a uniform way were unsuccessful, I believe that such a mapping does exist.

Since an ad-nilpotent ideal $I \in \mathfrak{Ad}(\mathfrak{g})$ is completely determined by the corresponding antichain $\Gamma = \Gamma(I) \subset \Delta^+$, properties of the conjectural duality on $\mathfrak{Ad}(\mathfrak{g})$ can be restated in terms of antichains in Δ^+. Let $\mathfrak{An}(\Delta^+)$ denote the set of all antichains in Δ^+. For a moment, we assume that Δ is not necessarily irreducible, and $\Delta = \bigsqcup_i \Delta_i$, where each Δ_i is an irreducible root system and the rank of Δ_i is p_i.

Conjecture 6.1. *There exists a natural involutory mapping*

$$*: \mathfrak{An}\bigl(\Delta^+\bigr) \to \mathfrak{An}\bigl(\Delta^+\bigr)$$

such that the following hol ds for $\Gamma \in \mathfrak{An}(\Delta^+)$:

(i) $\Gamma^* = \bigsqcup(\Gamma \cap \Delta_i)^*$ *and* $(\Gamma \cap \Delta_i)^*$ *depends only on* $\Gamma \cap \Delta_i$;
(ii) $\#(\Gamma \cap \Delta_i) + \#(\Gamma^* \cap \Delta_i) = p_i$ *for all* i;
(iii) *Suppose* Γ *contains a simple root* α. *Write* $\Delta(\Pi \setminus \{\alpha\})$ *for the root subsystem spanned by the set of simple roots* $\Pi \setminus \{\alpha\}$. *Then* $\Gamma^* \subset \Delta(\Pi \setminus \{\alpha\})^+$ *and moreover,* $\Gamma^* = (\Gamma \setminus \{\alpha\})^*$, *where* $\Gamma \setminus \{\alpha\}$ *is regarded as antichain in* $\Delta(\Pi \setminus \{\alpha\})^+$;
(iv) (*Approximately a converse to the previous property.*) *If* $\Gamma \subset \Delta(\Pi \setminus \{\alpha\})^+$, *then* $\Gamma^* = \{\alpha\} \cup \{$*the dual of* Γ *taken in* $\Delta(\Pi \setminus \{\alpha\})^+\}$;
(v) *If* Δ *is irreducible, then* $(\Delta^+(k))^* = \Delta^+(h+1-k)$, *where* h *is the Coxeter number of* Δ (*cf. Lemma* 4.5);
(vi) *the distribution of long and short roots in the multiset* $\{\Gamma, \Gamma^*\}$ *is the same as in* Π. (*This condition is vacuous in the simply-laced case.*)

It is easy to see that the duality defined for the root systems of type $\mathbf{A}_p,\ \mathbf{B}_p,\ \mathbf{C}_p$ satisfies all these properties. Also, it is immediate that '$*$' can uniquely be defined for $\mathbf{G}_2$.

Now, we again assume that Δ is irreducible. Clearly, a necessary condition for such a duality to exist is that the number of antichains of cardinality k ought to be equal to the number of antichains of cardinality $p-k$. This holds in all cases, where the corresponding values are known, see below. If $k=0$, then the assertion follows from Proposition 2.10. In case $k=1$, one should be able to prove that the number of positive roots is equal to the number of antichains of cardinality $p-1$. Unfortunately, the only proof I know amounts to a case-by-case verification.

For each simple Lie algebra $\mathfrak{g}$, we define an analogue of Narayana polynomial as follows. Let $d_k(\mathfrak{g})$ be the number of all ad-nilpotent ideals with k generators or, equivalently, the number of all k-element antichains in Δ^+. Then

$$\mathcal{N}_{\mathfrak{g}}(q)=\sum_{i=0}^{p} d_k(\mathfrak{g})q^k \tag{6.2}$$

is said to be the *Narayana polynomial of type* $\mathfrak{g}$ (or, a generalized Narayana polynomial). Clearly, $d_0(\mathfrak{g})=d_p(\mathfrak{g})=1$ and $d_1(\mathfrak{g})=\#\Delta^+$. By Theorem 2.9, $d_{p-1}(\mathfrak{g})$ equals the number of integral points lying on the edges of the simplex $\widetilde{D}$ (except of the unique integral vertex). Below, we list all generalized Narayana polynomials:

$$\mathcal{N}_{A_p}(q)=\sum_{k=0}^{p}\frac{1}{p+1}\binom{p+1}{k}\binom{p+1}{k+1}q^k;$$

$$\mathcal{N}_{B_p}(q)=\mathcal{N}_{C_p}(q)=\sum_{k=0}^{p}\binom{p}{k}^2 q^k;$$

$$\mathcal{N}_{D_p}(q)=\sum_{k=0}^{p}\left(\binom{p}{k}^2-\frac{p}{p-1}\binom{p-1}{k}\binom{p-1}{k-1}\right)q^k.$$

$$\mathcal{N}_{G_2}(q)=1+6q+q^2;$$

$$\mathcal{N}_{F_4}(q)=1+24q+55q^2+24q^3+q^4;$$

$$\mathcal{N}_{E_6}(q)=1+36q+204q^2+351q^3+204^4+36q^5+q^6;$$

$$\mathcal{N}_{E_7}(q)=1+63q+546q^2+1470q^3+1470^4+546q^5+63q^6+q^7;$$

$$\mathcal{N}_{E_8}(q)=1+120q+1540q^2+6120q^3+9518^4+6120q^5+1540q^6+120q^7+q^8.$$

In type A, it is the usual Narayana polynomial (cf. Remark 4.7). The result for types B and C follows from [3, Corollary 5.8]. In that place, Athanasiadis computes the number of non-nesting partitions on $\mathbf{B}_p$ or $\mathbf{C}_p$ whose 'type' has k parts. However, it follows from his previous exposition that a non-nesting partition whose type has k parts is exactly an antichain of cardinality $p-k$. The case of $\mathbf{D}_p$ is dealt with in [5]. Here one also has

$\binom{p}{k}^2 - \frac{p}{p-1}\binom{p-1}{k}\binom{p-1}{k-1}$ is the number of *non-crossing* partitions on $\mathbf{D}_p$ whose type has k parts [16, Section 4]. The case of $\mathbf{G}_2$ is trivial and that of $\mathbf{F}_4$ is relatively easy.

The case of $\mathbf{E}_n$ requires more work. The result can be obtained through the counting of all integral points in $\widetilde{D}$ and use of Theorem 2.9.

Thus, all generalized Narayana polynomials are palindromic.

By [9], we have $\mathcal{N}_{\mathfrak{g}}(1) = \#\mathfrak{Ad}(\mathfrak{g}) = \prod_{i=1}^{p} \frac{h+e_i+1}{e_i+1}$. It would be interesting to find a uniform expression for the coefficients of the generalized Narayana polynomials.

Another intriguing feature is that there are nice formulae for the values $\mathcal{N}_{\mathfrak{g}}(-1)$. For $\mathbf{A}_p$, we refer again to Remark 4.7. The $\mathbf{B}_p$- or $\mathbf{C}_p$-case amounts to a well-known combinatorial identity:

$$\sum_{k=0}^{p}(-1)^k \binom{p}{k}^2 = \begin{cases} 0, & \text{if } p \text{ is odd,} \\ (-1)^{p/2}\binom{p}{p/2}, & \text{if } p \text{ is even.} \end{cases}$$

Combining the expressions for $\mathbf{A}_p$ and $\mathbf{B}_p$ cases, we obtain

$$\mathcal{N}_{D_p}(-1) = \begin{cases} 0, & \text{if } p \text{ is odd,} \\ (-1)^{p/2}\left[\binom{p}{p/2} - 2\binom{p-2}{p/2-1}\right] = (-1)^{p/2} 2\binom{p-2}{p/2}, & \text{if } p \text{ is even.} \end{cases}$$

One may also observe that if p is even, then $(-1)^{p/2}\mathcal{N}_{\mathfrak{g}}(-1)$ is positive for all simple Lie algebras $\mathfrak{g}$.

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