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On Linear Combinations of Special Operators*

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1. INTRODUCTION

In this note we continue the study begun in ([*4*], [*7*], [*8*], [*9*]) of the linear span of sets of special operators in an operator algebra.

First we apply a result of Herstein to simple C^*-algebras which arise as quotients of properly infinite von Neumann algebras. If $\mathscr{A}$ is a properly infinite von Neumann algebra, $\mathscr{S}$ is a linear subspace of $\mathscr{A}$, invariant under certain inner $*$-automorphisms, and $\mathscr{M}$ is a maximal ideal of $\mathscr{A}$, then either $\mathscr{S} \subset \{\lambda I\} + \mathscr{M}$ or else $\mathscr{A} = \mathscr{S} + \mathscr{M}$. Specializing this to $\mathscr{A} = \mathscr{B}(\mathscr{H})$, the ring of all bounded operators on the Hilbert space $\mathscr{H}$, we conclude that if S is an operator which is not congruent to a scalar, modulo the ideal of compact operators, then *every* operator is a finite linear combination of operators, each unitarily equivalent to S, plus a compact operator.

This last result has a curious application to the invariant subspace problem. A *transitive* operator is one having no closed invariant subspaces except the trivial two (Arveson and Halmos are responsible for the term). Of course, the existence of a transitive operator on separable Hilbert space is a long-standing problem, now generally acceded to have an affirmative answer. Our result shows that *if* there is at least one transitive operator, then every operator can be written as a finite sum of transitives and a compact.

Our next observation is a sharpening of the Herstein Theorem in the case of simple von Neumann algebras (factors of type II_1, and type III factors acting separably). This strengthened form is probably valid for any simple C^*-algebra, but we have not been able to settle this; the existence of a non-trivial idempotent appears to be an annoying necessity.

Finally, another contribution is made toward solving the problem of which von Neumann algebras are linearly spanned by their projections. In [*7*], Pearcy and the author showed that any properly infinite algebra is so spanned, extending the result of Fillmore [*3*] for $\mathscr{B}(\mathscr{H})$. Pearcy and the author have recently shown [*8*] that certain II_1-factors discovered by Wright are linearly spanned by their projections E satisfying $E \sim I - E$. In [*4*], Fillmore

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and the author noted that a type I_2 algebra with infinite-dimensional center is not spanned by its projections. Thereupon Kaplansky suggested to the author that one should consider the *central span* of the projections (i.e., finite linear combinations of projections with coefficients in the center of the algebra). A glance at the techniques employed in [*4*] makes it immediately clear that any homogeneous algebra of finite type (type I_n) is the central span of its projections, but this does not obviously imply that the same is true for any finite algebra of type I. We prove that it is in Section 3.

Thus it now appears that the general situation is as follows.

CONJECTURE. *Any von Neumann algebra is the central span of its projections.*

An affirmative answer to this conjecture would bear importantly on the study of II_1-algebras ([*8*]; Section 5). We comment briefly on this in the last section.

We are indebted to Arlen Brown for a remark which simplified the proof of Theorem 1, and to Peter Fillmore for shortening the proof of Lemma 2.

2. UNITARILY INVARIANT SUBSPACES

The following theorem underlies the results of this section. For a C^*-algebra $\mathscr{A}$ and subsets $\mathscr{S}, \mathscr{T} \subset \mathscr{A}$ we write $[\mathscr{S}, \mathscr{T}]$ for the (unclosed) complex linear span of all operators $ST - TS$, where $S \in \mathscr{S}$ and $T \in \mathscr{T}$.

HERSTEIN THEOREM (C^*-version). *Let a be $\mathscr{A}$ simple C^*-algebra with identity I, and let $\mathscr{S}$ be a linear subspace (not necessarily closed) of $\mathscr{A}$ such that $[\mathscr{S}, [\mathscr{A}, \mathscr{A}]] \subset \mathscr{S}$. Then either $\mathscr{S} \subset \{\lambda I\}$ (the scalar multiples of I) or else $\mathscr{S} \supset [\mathscr{A}, \mathscr{A}]$.*

For the proof, we refer the reader to [*5*].

THEOREM 1. *Let $\mathscr{A}$ be a properly infinite von Neumann algebra, and let $\mathscr{M}$ be a maximal ideal of $\mathscr{A}$. Suppose that $\mathscr{S}$ is a linear subspace of $\mathscr{A}$ such that $U^*\mathscr{S}U \subset \mathscr{S}$ for each unitary operator U of the form $U = E \pm i(I - E)$, E a projection in $\mathscr{A}$. Then either $\mathscr{S} \subset \{\lambda I\} + \mathscr{M}$ or else $\mathscr{A} = \mathscr{S} + \mathscr{M}$.*

Proof. Set $\tilde{\mathscr{A}} = \mathscr{A}/\mathscr{M}$. It is known ([7]; Theorem 6, p. 461) that $\mathscr{A}$ is the linear span of its projections, and also ([7]; Lemma 3.1, p. 458) that $\mathscr{A} = [\mathscr{A}, \mathscr{A}]$. Evidently the two properties persist on passing to the simple C^*-algebra $\tilde{\mathscr{A}}$.

Now let $S \in \mathscr{S}$ and take any projection $E \in \mathscr{A}$. Then

$$(*) \qquad 2i(ES - SE) = U^*SU - USU^* \in \mathscr{S},$$

where $U = E + i(I - E)$.

The two facts mentioned earlier imply that $[\mathscr{S}, [\mathscr{A}, \mathscr{A}]] \subset \mathscr{S}$ and hence that $[\bar{\mathscr{S}}, [\bar{\mathscr{A}}, \bar{\mathscr{A}}]] \subset \bar{\mathscr{S}}$. By the Herstein Theorem, either $\bar{\mathscr{S}} \subset \{\lambda \bar{I}\}$ or else $\bar{\mathscr{S}} \supset [\bar{\mathscr{A}}, \bar{\mathscr{A}}] = \bar{\mathscr{A}}$. The result follows on reinterpreting these inclusions back in $\mathscr{A}$.

Remark. A careful reading of ([7]; Theorem 3) reveals that one can restrict the projections E in Theorem 1 to satisfy $E \sim I - E$.

COROLLARY 1. *Let $\mathscr{A} = \mathscr{B}(\mathscr{H})$, with $\mathscr{H}$ separable, and let $\mathscr{M}$ be the (maximal) ideal of compact operators. Suppose that S is an operator such that $S \not\equiv \lambda I (\mathrm{mod}\, \mathscr{M})$, for every scalar λ, and let $\mathscr{S}$ be the linear span of all operators U^*SU, where $U = E \pm i(I - E)$, and E is a projection with $E \sim I - E$. Then $\mathscr{A} = \mathscr{S} + \mathscr{M}$.*

Proof. The assumptions imply that $\mathscr{S} \not\subset \{\lambda I\} + \mathscr{M}$, forcing the other alternative in Theorem 1.

COROLLARY 2. *If a transitive operator T exists, then any operator is the finite sum of transitive operators, each unitarily equivalent to T, and a compact operator.*

Proof. A well-known theorem of von Neumann, Aronszajn and Smith [1] states that no operator of the form $\lambda I + C$, λ complex and C a compact operator, is transitive. Corollary 1 now applies, completing the proof.

THEOREM 2. *Let $\mathscr{A}$ be a von Neumann factor which is either of type* II_1 *or else is type* III *and acts separably. Suppose that $\mathscr{S}$ is a linear subspace of $\mathscr{A}$ such that $U^*\mathscr{S}U \subset \mathscr{S}$ for each unitary operator U of the form $U = E \pm i(I - E)$, E a projection in $\mathscr{A}$ with $E \sim I - E$. Then either $\mathscr{S} \subset \{\lambda I\}$ or else $\mathscr{S} \supset [\mathscr{A}, \mathscr{A}]$.*

Remark. If $\mathscr{A}$ is type III, the conclusion reads: "either $\mathscr{S} \subset \{\lambda I\}$ or else $\mathscr{S} = \mathscr{A}$."

Proof. The factors in question are simple algebras ([2]; Corollary 3, p. 275; Exercice 1(c), p. 323). Pearcy and the author have shown ([8]; Theorem 3) that projections $E \in \mathscr{A}$ with $E \sim I - E$ span $[\mathscr{A}, \mathscr{A}]$ in case $\mathscr{A}$ is an algebra of type II_1. As noted in the proof of Theorem 1, $\mathscr{A} = [\mathscr{A}, \mathscr{A}]$ is spanned by such projections if $\mathscr{A}$ is of type III. Using equation (*), we can then conclude that $[\mathscr{S}, [\mathscr{A}, \mathscr{A}]] \subset \mathscr{S}$, and the conclusion then follows from the Herstein Theorem.

We have ignored factors of type I_n, as these are relatively uninteresting; Theorem 2 is easily seen to hold for such factors.

3. Finite Type I Algebras

Let $\mathscr{A}$ be an abelian C^*-algebra containing the identity operator I and let $\mathscr{M}_n(\mathscr{A})$ be the algebra of all $n \times n$ matrices ($n \geqslant 2$) with entries from $\mathscr{A}$. Denote by $M_{ij}(A)$ the matrix with $A \in \mathscr{A}$ in the (i, j) position and zeros elsewhere. The center $\mathscr{Z}$ of $\mathscr{M}_n(\mathscr{A})$ then consists of all matrices of the form $\sum_{i=1}^n M_{ii}(A)$.

Lemma 1. *Let $\mathscr{A}$ be an abelian C^*-algebra with $I \in \mathscr{A}$. Then $\mathscr{M}_n(\mathscr{A})$ is the central span of its projections.*

Proof. As noted in ([4]; p. 333), any matrix of the form $M_{ij}(A)$ with $i \neq j$ is a (scalar) linear combination of projections, so it is enough to treat diagonal matrices of the form $M_{ii}(A)$. But

$$M_{ii}(A) = \left(\sum_{k=1}^{n} M_{kk}(A)\right) \cdot M_{ii}(I)$$

is a central multiple of a projection, and the lemma is proved.

Corollary 3. *Any finite homogeneous von Neumann algebra (type I_n) is the central span of its projections.*

Now the most general type I finite algebra is a (perhaps infinite) direct sum of homogeneous algebras ([2]; Proposition 2, p. 252) so the last corollary is not directly applicable. We require a lemma.

Lemma 2. *Let $\mathscr{A}$ be a finite type* I *von Neumann algebra which contains a projection R such that $R \sim I - R$. Then $\mathscr{A}$ is the central span of such projections.*

Proof. The analysis of $[\mathscr{A}, \mathscr{A}]$ carried out in ([8]; Lemmas 4.1, 4.2, 4.3, and Theorem 3) applies almost verbatim, and shows that $[\mathscr{A}, \mathscr{A}]$ is the set of linear combinations of the form $\sum_{i=1}^n \alpha_i E_i$ where $\sum_{i=1}^n \alpha_i = 0$ and each $E_i \in \mathscr{A}$ is a projection with $E_i \sim I - E_i$. This is achieved by writing $\mathscr{A}$ as a 2×2 matrix algebra in the standard fashion; we refer to [8] for details.

One point in the argument ([8]; Lemma 4.2) needs to be modified as follows. It must be shown that if $T \in \mathscr{A}$ satisfies $T^2 = 0$, then there is a projection $E \in \mathscr{A}$ with $ET = T$, $TE = 0$ and $E \sim I - E$. To see this, let V be the partial isometry appearing in the canonical polar decomposition of T.

As noted in ([8]; proof of Lemma 4.2), the projections $F = V^*V$ and $G = VV^*$ are orthogonal and $GT = T$, while $I - F - G$ left and right annihilates T. Our task is to split $I - F - G$ into two orthogonal equivalent projections.

A simple comparison argument like that used in ([7]; Lemma 3.2, p. 460) shows that there are projections $M, N \in \mathscr{A}$ with $F \sim M \leqslant R$ and $G \sim N \leqslant I - R$. Applying the Cancellation Law for Finite Projections ([6]; Lemma 2.5, p. 15) first to the equivalence $M + (R - M) \sim N + (I - R - N)$, and then to the equation $F + G + (I - F - G) = M + N + (R - M) + (I - R - N,)$ we obtain the desired representation $I - F - G \sim (R - M) + (I - R - N)$, with $R - M \sim I - R - N$. Thus we can split $I - F - G$ into two orthogonal equivalent projections, either of which added to G yields a projection E with the desired properties.

Finally, it is shown in ([8]; Theorem 1) that $[\mathscr{A}, \mathscr{A}]$ is precisely the set of operators in $\mathscr{A}$ having central trace zero. Thus if $\mathscr{Z}$ is the center of $\mathscr{A}$, $\mathscr{A} = \mathscr{Z} + [\mathscr{A}, \mathscr{A}]$. Since each operator $Z \in \mathscr{Z}$ admits the representation $Z = ZR + Z(I - R)$, the lemma follows.

The main result of this section is now accessible.

THEOREM 3. *Every finite type* I *von Neumann algebra is the central span of its projections.*

Proof. It will suffice to show that each self-adjoint operator S in a finite type I algebra $\mathscr{A}$ is a finite linear combination, with central coefficients, of projections in $\mathscr{A}$.

As in ([4]; proof of Lemma], we can find orthogonal projections P, Q and R commuting with S, with sum I, such that $P \sim Q$ and R is abelian. Lemma 2 then applies to the subalgebra $(P + Q)\mathscr{A}(P + Q)$, and enables us to write $S(P + Q)$ as a central linear combination of projections. If $\mathscr{Z}$ is the center of $\mathscr{A}$, then $R\mathscr{A}R = \mathscr{Z}R$, since R is an abelian projection ([2]; Corollary, p. 19). Hence SR is a central multiple of R, and the proof is complete.

4. CONCLUDING REMARKS

We observed in [8] that if a II_1-algebra $\mathscr{A}$ is centrally spanned by its projections E with $E \sim I - E$, then $[\mathscr{A}, \mathscr{A}]$ is automatically norm closed. This important property was shown to hold for at least one class of II_1-factors.

A presumably easier question is whether a II_1-factor is the linear span of *all* of its projections. We simplify this question somewhat by noting that the span of the projections of dimension one-half coincides with the span of all projections with *rational* dimension.

First assume that E is a projection in a II_1-factor $\mathscr{A}$ with $\dim(E) = n^{-1}$ $(n \geqslant 3)$. Put $E_1 = E$ and choose orthogonal projections $E_2, \ldots, E_n$, each orthogonal to E with $E_i \sim E$ $(i = 2, \ldots, n)$ and $E_1 + E_2 + \cdots + E_n = I$.

Let $U_i \in \mathscr{A}$ be a partial isometry from E to E_i and let $S_i = U_i + U_i^*$. It is easy to verify that

$$E - E_i = \tfrac{1}{2}[S_i(S_i(E - E_i)) - (S_i(E - E_i))S_i]$$

so that $E - E_i \in [\mathscr{A}, \mathscr{A}]$. Denoting by $\mathscr{S}$ the span of all projections $P \in \mathscr{A}$ with $P \sim I - P$, we see from ([8]; Theorem 3) that $\mathscr{S} = \{\lambda I\} + [\mathscr{A}, \mathscr{A}]$. Now it is clear that

$$E = n^{-1}\left(I + \sum_{i=2}^{n} (E - E_i)\right) \in \mathscr{S}.$$

Finally, suppose that $\dim(E) = mn^{-1}$, where $0 < mn^{-1} < 1$. Write E as the orthogonal sum $E = E_1 + E_2 + \cdots + E_m$, where $\dim(E_i) = n^{-1}$. By the above reasoning, each E_i belongs to $\mathscr{S}$, as therefore does E.

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