Stability radii for linear time-varying
differential–algebraic equations with respect
to dynamic perturbations

Nguyen Huu Du, Vu Hoang Linh *

Faculty of Mathematics, Mechanics and Informatics, University of Natural Sciences, Vietnam National University,
334 Nguyen Trai Str., Thanh Xuan, Hanoi, Viet Nam
Received 8 December 2005; revised 30 June 2006
Available online 4 August 2006

Abstract

This paper is concerned with the robust stability for linear time-varying differential–algebraic equations. We consider the systems under the effect of uncertain dynamic perturbations. A formula of the structured stability radius is obtained. The result is an extension of a previous result for time-varying ordinary differential equations proven by Birgit Jacob [B. Jacob, A formula for the stability radius of time-varying systems, J. Differential Equations 142 (1998) 167–187].

Keywords: Robust stability; Linear time-varying system; Differential–algebraic equation; Input–output operator

1. Introduction

In lots of applications there is a frequently arising question, namely, how robust is a characteristic qualitative property of a system (e.g., the stability) when the system comes under the effect of uncertain perturbations. This is the subject of the robust stability analysis which has attracted serious attention of researchers recently. This paper is concerned with time-varying systems of differential–algebraic equations (DAEs) of the form

\[ E(t)x'(t) = A(t)x(t), \quad t \geq 0, \]  

(1.1)

* Corresponding author.
E-mail address: vhlinh@hn.vnn.vn (V.H. Linh).
where \( E(\cdot) \in L^{\text{loc}}_{\infty}(0, \infty; \mathbb{K}^{n \times n}) \), \( A(\cdot) \in L^{\text{loc}}_{\infty}(0, \infty; \mathbb{K}^{n \times n}) \), \( \mathbb{K} = \{ \mathbb{C}, \mathbb{R} \} \). We assume that the leading term \( E(t) \) is singular for almost all \( t \geq 0 \) and \( \ker E(\cdot) \) is absolutely continuous. In addition, we suppose that (1.1) generates an exponentially stable evolution operator \( \Phi = \{ \Phi(t,s) \}_{t,s \geq 0} \), i.e., there exist positive constants \( M \) and \( \omega \) such that

\[
\| \Phi(t,s) \|_{\mathbb{K}^{n \times n}} \leq Me^{-\omega(t-s)}, \quad t \geq s \geq 0. \tag{1.2}
\]

We consider system (1.1) subjected to structured perturbation of the form

\[
E(t)x'(t) = A(t)x(t) + B(t)\Delta(C(\cdot)x(\cdot))(t), \quad t \geq 0, \tag{1.3}
\]

where \( B(\cdot) \in L_{\infty}(0, \infty; \mathbb{K}^{n \times m}) \) and \( C(\cdot) \in L_{\infty}(0, \infty; \mathbb{K}^{q \times n}) \) are given matrices defining the structure of the perturbation and \( \Delta : L_p(0, \infty; \mathbb{K}^{m}) \rightarrow L_p(0, \infty; \mathbb{K}^{q}) \) is an unknown disturbance operator which is supposed to be linear, dynamic, and causal. Thus, system (1.3) represents a large class of linear functional differential equations including, e.g., delay equations, integro-differential equations, etc. In applications, the nominal system (1.1) plays the role of a simplified model problem, while the perturbed system (1.3) can be considered as a real-life problem.

The so-called stability radius is defined by the largest bound \( r \) such that the stability is preserved for all perturbations \( \Delta \) of norm strictly less than \( r \). This measure of the robust stability was introduced by Hinrichsen and Pritchard [10] for linear time-invariant systems of ordinary differential equations (ODEs) with respect to time- and output-invariant, i.e., static perturbations. Formulae of the structured stability radii were obtained in [10,13]. For further considerations in abstract spaces, see [5] and the references therein. In lots of problems, uncertain perturbations may depend on the output feedback, as well. In [9], explicit time-invariant systems with respect to dynamic perturbations were considered and a formula of the stability radius was given in terms of the norm of a certain input–output operator. Earlier results for time-varying systems can be found, e.g., in [7,8]. The most successful attempt for finding a formula of the stability radius was an elegant result given by Jacob [7]. In that paper, the author considered the explicit system, that is the special case of (1.1) with the leading term \( E = I \), and succeeded in proving that the stability radius is equal to

\[
\sup_{t_0 \geq 0} \left\{ \| L_{t_0} \|_{L(L_p(t_0, \infty; \mathbb{K}^m), L_p(t_0, \infty; \mathbb{K}^q))}^{-1} \right\} \quad \forall t \geq t_0, \quad \forall t \geq t_0,
\]

where \( L_{t_0} := C(t) \int_{t_0}^{t} \Phi(t,s)B(s)u(s) \, ds \).

On the other hand, systems occurring in various applications, such as optimal control, electronic circuit simulation, multibody mechanics, etc. are described by differential–algebraic systems, see [1,2]. Therefore, it is natural to extend the notion of the stability radius to differential–algebraic equations. This problem has been solved for linear time-invariant DAEs, see [1,3,4,14]. It is worth mentioning that the index notion, which plays a key role in the qualitative theory and in the numerical analysis of DAEs, should be taken into consideration in the robust stability analysis, too. The aim of this paper is to extend Jacob’s result to time-varying systems (1.1) with index-1. In this paper we follow the tractability index approach proposed by März et al., see [6,12].

The paper is organized as follows. In the next section we recall some basic notions and preliminary results on the theory of linear DAEs. Section 3 deals with the existence and uniqueness of the mild solution, and the stability concepts for (1.1). In particular, we call the attention to some differences between DAEs and ODEs. In Section 4, a definition of the structured stability
radii for DAEs is given. It is slightly different from the case of ODEs that not only the stability, but also the index-1 property are required to be preserved. Then, we propose a formula of the stability radius for (1.1) subjected to (1.3) which is a little bit different from and more complicated than (1.4). In the last section, some special cases are analyzed. In particular, the result obtained for time-invariant systems is compared to those appeared in earlier literature.

2. Preliminary

2.1. Notations

Throughout the paper we use the following standard notations as in [7]. Let $K \in \{\mathbb{R}, \mathbb{C}\}$, let $X, Y$ be finite-dimensional vector spaces and let $t_0 \geq 0$. For every $p, 1 \leq p < \infty$, we denote by $L^p(s,t; X)$ the space of measurable function $f$ with

$$\|f\|_p := \left( \int_s^t \|f(\rho)\|^p d\rho \right)^{1/p} < \infty$$

and by $L_\infty(s,t; X)$ the space of measurable and essentially bounded functions $f$ with $\|f\|_\infty := \text{ess sup}_{\rho \in [s,t]} \|f(\rho)\|$, where $t_0 \leq s < t \leq \infty$. We also consider the spaces $L^p_{\text{loc}}(t_0, \infty; X)$ and $L^\infty_\text{loc}(t_0, \infty; X)$, which contain all functions $f$ satisfying $f \in L^p(s,t; X)$ and $f \in L^\infty(s,t; X)$, respectively, for every $s, t$, $t_0 \leq s < t < \infty$. For $k \geq 0$ the operator of truncation $\pi_k$ at $k$ on $L^p(0, \infty; X)$ is defined by

$$\pi_k(u)(t) := \begin{cases} u(t), & t \in [0,k], \\ 0, & t > k. \end{cases}$$

We use the conventional notation $\mathcal{L}(L^p(t_0, \infty; X), L^p(t_0, \infty; Y))$ to denote the Banach space of linear bounded operators $\mathbb{P}$ from $L^p(t_0, \infty; X)$ to $L^p(t_0, \infty; Y)$ supplied with the norm

$$\|\mathbb{P}\| := \sup_{x \in L^p(t_0, \infty; X), \|x\| = 1} \|\mathbb{P}x\|_{L^p(t_0, \infty; Y)}.$$

An operator $\mathbb{P} \in \mathcal{L}(L^p(0, \infty; X), L^p(0, \infty; Y))$ is called to be causal, if $\pi_t \mathbb{P} \pi_t = \pi_t \mathbb{P}$ for every $t \geq 0$. For $k \geq 0$, $S_k$ denotes the operator of left shift by $k$ on $L^p(0, \infty; X)$: $S_k(u)(t) = u(t+k)$. In the whole paper, we omit for brevity the time variable $t$, where it does not cause misunderstanding.

2.2. Linear differential–algebraic equations

We consider the linear differential–algebraic system

$$E(t)x'(t) = A(t)x(t) + q(t), \quad t \geq 0, \quad (2.1)$$

where $E, A$ are supposed as in Section 1, $q \in L^\infty_\text{loc}(0, \infty; \mathbb{K}^n)$. Let $N(t)$ denote $\ker E(t)$ for all $t$. Then due to the assumption on $\ker E(\cdot)$ in Section 1, there exists an absolutely continuous projector $Q(t)$ onto $N(t)$, i.e., $Q \in C(0, \infty; \mathbb{K}^{n \times n})$, $Q$ is differentiable almost everywhere,
\( Q^2 = Q \), and \( \text{Im } Q(t) = N(t) \) for all \( t \geq 0 \). We assume in addition that \( Q' \in L^\text{loc}_\infty(0, \infty; \mathbb{K}^{n \times n}) \).

Set \( P = I - Q \), then \( P(t) \) is a projector along \( N(t) \). System (2.1) is rewritten into the form

\[
E(t)(Px)'(t) = \overline{A}(t)x(t) + q(t),
\]

where \( \overline{A} := A + EP' \in L^\text{loc}_\infty(0, \infty; \mathbb{K}^{n \times n}) \). We define \( G := E - \overline{A}Q \).

**Definition 1.** (See also [6, Section 1.2].) The DAE (2.1) is said to be index-1 tractable if \( G(t) \) is invertible for almost every \( t \in [0, \infty) \) and \( G^{-1} \in L^\text{loc}_\infty(0, \infty; \mathbb{K}^{n \times n}) \).

Now let (2.1) be index-1. Note that the index-1 property does not depend on the choice of projectors \( P(Q) \), see [6,12]. We consider the homogeneous case \( q(t) = 0 \) and construct the Cauchy operator generated by (2.1). Taking into account the equalities

\[
G^{-1}E = P, \quad G^{-1}\overline{A} = -Q + G^{-1}\overline{AP},
\]

and multiplying both sides of (2.2) with \( PG^{-1}, QG^{-1} \), we obtain

\[
\begin{aligned}
(Px)' &= (P' + PG^{-1}\overline{A})Px, \\
Qx &= QG^{-1}\overline{AP}x.
\end{aligned}
\]

Thus, the system is decomposed into two parts: a differential part and an algebraic one. Hence, it is clear that we need to address the initial value condition to the differential components, only. Denote \( u = Px \), the differential part becomes

\[
u' = (P' + PG^{-1}\overline{A})u.
\]

This equation is called the inherent ordinary differential equation (INHODE) of (2.1). Multiplying both sides of (2.3) with \( Q \) yields

\[
(Qu)' = Q'Qu.
\]

Hence, the INHODE (2.3) has the invariant property that every solution starting in \( \text{Im}(P(t_0)) \) remains in \( \text{Im}(P(t)) \) for all \( t \). Let \( \Phi_0(t, s) \) denote the Cauchy operator generated by the INHODE (2.3), i.e.,

\[
\begin{aligned}
\frac{d}{dt} \Phi_0(t, s) &= (P' + PG^{-1}\overline{A})\Phi_0(t, s), \\
\Phi_0(s, s) &= I.
\end{aligned}
\]

Then, the Cauchy operator generated by system (2.1) is defined by

\[
\begin{aligned}
E \frac{d}{dt} \Phi(t, s) &= A\Phi(t, s), \\
P(s)(\Phi(s, s) - I) &= 0,
\end{aligned}
\]

and can be given as follows

\[
\Phi(t, s) = (I + QG^{-1}\overline{A}(t))\Phi_0(t, s)P(s).
\]
By the arguments used in [6, Section 1.2], [12], the unique solution of the initial value problem (IVP) for (2.1) with the initial condition

\[ P(t_0)(x(t_0) - x_0) = 0, \quad t_0 \geq 0, \] (2.4)

can be given by the constant-variation formula

\[ x(t) = \Phi(t, t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi(t, \rho)PG^{-1}q(\rho) d\rho + QG^{-1}q(t). \]

**Remark 1.** In general, the equality \( x(t_0) = x_0 \) for a given \( x_0 \in \mathbb{K}^n \) cannot be expected as in an initial value problem for ODEs. However, the so-called fully consistent initial value related to (2.1), (2.4) can be given as follows

\[ x(t_0) = (I + QG^{-1}\tilde{A}(t_0))P(t_0)x_0 + QG^{-1}q(t_0). \]

Finally, we remark that, due to very mild conditions on the data of (2.1), only the differential part \( P(t)x(t) \) can be expected to be smooth.

3. Mild solution and stability notions

From now, let the following assumptions hold.

**Assumption A1.** System (1.1) is index-1 and there exist \( M > 0, \omega > 0 \) such that

\[ \| \Phi_0(t, s)P(s) \| \leq Me^{-\omega(t-s)}, \quad t \geq s \geq 0. \]

**Assumption A2.** \( PG^{-1}, QG^{-1} \) and \( Q_s := -QG^{-1}\tilde{A} \) are essentially bounded on \([0, \infty)\).

**Remark 2.** We note that the above assumptions imply immediately the estimate

\[ \| \Phi(t, s) \| = \| (I - Q_s(t))\Phi_0(t, s)P(s) \| \leq \left( 1 + \text{ess sup}_{t \geq 0} \| Q_s(t) \| \right)Me^{-\omega(t-s)}, \]

that is, (1.2) holds for almost all \( t \geq s \geq 0 \) with \( M := (1 + \text{ess sup}_{t \geq 0} \| Q_s(t) \|)\tilde{M} \). Furthermore, due to the invariant property of the solutions of the INHODE (2.3), we have

\[ P(t)\Phi(t, s) = P(t)\Phi_0(t, s)P(s) = \Phi_0(t, s)P(s). \]

It is also remarkable that the terms \( QG^{-1}, Q_s \) do not depend on the choice of projector \( Q \) (see [6,12]). We will see later that the restriction on the boundedness of \( PG^{-1}, QG^{-1} \) might be relaxed somewhat.

First, the index notion is extended to the perturbed system (1.3), where the disturbance operator \( \Delta \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^q), L_p(0, \infty; \mathbb{K}^m)) \) is supposed to be causal. Let the linear operator \( \tilde{G} \in \mathcal{L}(L_p^{loc}(0, \infty; \mathbb{K}^q), L_p^{loc}(0, \infty; \mathbb{K}^m)) \) be defined as follows

\[ (\tilde{G}u)(t) = (E - \tilde{A}Q)u(t) - B\Delta(\tilde{C}Q(\cdot)u(\cdot))(t), \quad t \geq 0. \]
Writing formally, we have
\[ \tilde{G} = (I - B \Delta C Q G^{-1}) G. \] (3.1)

**Definition 2.** The functional differential–algebraic system (1.3) is said to be index-1 (in the generalized sense) if for every \( T > 0 \), the operator \( \tilde{G} \) restricted to \( L_p(0, T; \mathbb{K}^n) \) is invertible and the inverse operator \( \tilde{G}^{-1} \) is bounded.

**Definition 3.** We say that the IVP for the perturbed system (1.3) with (2.4) admits a mild solution if there exists \( x \in L^{\text{loc}}_p(t_0, \infty; \mathbb{K}^n) \) satisfying
\[
x(t) = \Phi(t, t_0) P(t_0) x_0 + \int_{t_0}^{t} \Phi(t, \rho) P G^{-1} B \Delta \left( \left[ C x(\cdot) \right]_{t_0} \right)(\rho) \, d\rho + Q G^{-1} B \Delta \left( \left[ C x(\cdot) \right]_{t_0} \right)(t)
\] (3.2)
for \( t \geq t_0 \), where
\[
\left[ C x(\cdot) \right]_{t_0} = \begin{cases} 0, & t \in [0, t_0), \\ C(t) x(t), & t \in [t_0, \infty). \end{cases}
\]

**Definition 4.** Let \( X, Y \) be Banach spaces and \( M : X \rightarrow Y \) be a linear bounded operator. We say that \( M \) is stable if it is boundedly invertible, i.e., \( M \) is invertible and its inverse is bounded.

**Lemma 1.** Suppose that the bounded linear operator triplet: \( \mathbb{M} : X \rightarrow Y, \; \mathbb{P} : Y \rightarrow Z, \; \mathbb{N} : Z \rightarrow X \) is given, where \( X, Y, Z \) are Banach spaces. Then the operator \( I - \mathbb{M} \mathbb{P} \mathbb{N} \) is invertible if and only if \( I - \mathbb{P} \mathbb{N} \mathbb{M} \) is invertible. Furthermore, if
\[
\|\mathbb{P}\| < \|\mathbb{N} \mathbb{M}\|^{-1}
\]
is provided, both the operators \( I - \mathbb{M} \mathbb{P} \mathbb{N} \) and \( I - \mathbb{P} \mathbb{N} \mathbb{M} \) are stable.

**Proof.** First suppose that \( I - \mathbb{M} \mathbb{P} \mathbb{N} \) is invertible. By direct calculation, it is easy to verify that
\[
(I - \mathbb{P} \mathbb{N} \mathbb{M})^{-1} = I + \mathbb{P} \mathbb{N} (I - \mathbb{M} \mathbb{P} \mathbb{N})^{-1} \mathbb{M}.
\]
That is \( I - \mathbb{P} \mathbb{N} \mathbb{M} \) is invertible, too. Furthermore, if \( (I - \mathbb{M} \mathbb{P} \mathbb{N})^{-1} \) is bounded then so is \( (I - \mathbb{P} \mathbb{N} \mathbb{M})^{-1} \). To verify the inverse direction of the statement, we proceed analogously. The second statement is a simple consequence of a well-known theorem of functional analysis (e.g., see [11, pp. 231–232]).

Applying the lemma with \( \mathbb{M} = B, \; \mathbb{P} = \Delta \) and \( \mathbb{N} = C Q G^{-1} \), we obtain that \( \tilde{G} \) is invertible if and only if \( I - \Delta C Q G^{-1} B \) and \( I - C Q G^{-1} B \Delta \) are invertible.
Theorem 1. Consider the IVP (1.3), (2.4). If (1.3) is index-1, then it admits a unique mild solution \( x \in L^p_{\text{loc}}(t_0, \infty; \mathbb{K}^n) \) with absolutely continuous \( Px \) for all \( t_0 \geq 0, x_0 \in \mathbb{K}^n \). Furthermore, for an arbitrary \( T > 0 \), there exists a constant \( M_1 \) such that
\[
\| P(t)x(t) \| \leq M_1 \| P(t_0)x_0 \| \quad \text{for all } t \in [t_0, T].
\]

Proof. Fix an arbitrary \( T > t_0 \) and consider the perturbed system (1.3) on \([t_0, T] \). It can be rewritten as follows
\[
\begin{align*}
(Px)' &= (P' + PG^{-1}A)Px + PG^{-1}B \Delta(Cx), \\
Qx &= QG^{-1}APx + QG^{-1}B \Delta(Cx).
\end{align*}
\]

We define \( u := Px, v := Qx \). Multiplying the algebraic equation with \( C \), we obtain
\[
(I - CQG^{-1}B \Delta)(Cv) = CQG^{-1}(A u + B \Delta(Cu)).
\]

Due to the index-1 assumption and Lemma 1, it is clear that the operator \( I - CQG^{-1}B \Delta \) is boundedly invertible. Let us define
\[
\forall u := (I - CQG^{-1}B \Delta)^{-1}(CQG^{-1}(A u + B \Delta(Cu))).
\]

It is clear that \( \forall \) is linear, bounded and causal. By substituting \( Cv = \forall u \) into the differential part, the INHODE becomes
\[
u' = (P' + PG^{-1}A)u + PG^{-1}B \Delta((C + \forall)u).
\]

By invoking [7, Proposition 3.2], the INHODE has a unique mild solution and this solution can be given by the constant-variation formula. By setting \( x = Px + Qx = u + v \), we obtain the unique mild solution to (1.3). It is easy to see that this unique solution can be given by the “constant-variation formula” (3.2) and the differential part \( Px \) is absolutely continuous.

To verify the remainder part, define an operator \( \mathbb{W}: L_p(t_0, T; \mathbb{K}^n) \to L_p(t_0, T; \mathbb{K}^n) \)
\[
\mathbb{W}u := (P' + PG^{-1}A)u + PG^{-1}B \Delta((C + \forall)u).
\]

It is obvious that \( \mathbb{W} \) is linear, bounded and causal. The INHODE is equivalent to the integral equation
\[
u(t) = \nu(t_0) + \int_{t_0}^t \mathbb{W}u(\rho) \, d\rho.
\]

Taking norm on \( L_p(t_0, t; \mathbb{K}^n) \), we have
\[
\| \nu(t) \|_{L_p(t_0, t; \mathbb{K}^n)} \leq \| \nu(t_0) \| + \left\| \int_{t_0}^t \mathbb{W}u(\rho) \, d\rho \right\|_{L_p(t_0, t; \mathbb{K}^n)}
\]
\[
\leq \| \nu(t_0) \| + \left( \int_{t_0}^t \left\| \mathbb{W}u(\rho) \right\|_{L_p(t_0, \rho; \mathbb{K}^n)}^p d\rho \right)^{1/p} \quad \text{(by Minkowski’s inequality)}.
\]
\[ \|u\| + \int_{t_0}^{t} \left( \int_{t_0}^{s} \|W(u(\rho))\|_p d\rho \right)^{1/p} ds \]
\[ \leq \|u\| + \|W\| \int_{t_0}^{t} \|u(s)\|_{L_p(t_0,s;\mathbb{R}^n)} ds. \]

It follows from the Gronwall–Bellman inequality that
\[ \|u(s)\|_{L_p(t_0,T;\mathbb{R}^n)} \leq \|u\| e^{\|W\|(T-t_0)} \leq \|u\| e^{\|W\|T} \]
for all \(0 \leq t_0 < T < +\infty\). Taking the vector norm of both sides of the integral equation for \(u\) and applying Hölder’s inequality, we have
\[ \|u(t)\| \leq \|u\| + (t-t_0)^{1/q} \left( \int_{t_0}^{t} \|W(u(\rho))\|_p d\rho \right)^{1/p} \]
\[ \leq \|u\| + T^{1/q} \|W(u)\|_{L_p(t_0,T;\mathbb{R}^n)} \leq \|u\| + T^{1/q} \|W\| \|u\| e^{\|W\|T}. \]

Here \(q\) is such a number that \(1/p + 1/q = 1\). By setting \(M_1 = 1 + T^{1/q} \|W\| e^{\|W\|T}\) the proof is complete. \(\square\)

**Remark 3.** We call the attention to the fact that for functional DAEs (1.3), with respect to very mild conditions on its coefficients, only the differential components of the solution are expected to be continuously dependent on the initial value.

Now let the unique mild solution to the initial value problem for (1.3) with initial value condition (2.4) denote by \(x(t; t_0, x_0) = x(t; t_0, P(t_0)x_0)\). It is obvious that for \(t > T\) the following representation holds
\[ x(t; t_0, x_0) = \Phi(t, T)P(T)x(T; t_0, x_0) + \int_{T}^{t} \Phi(t, \rho)PG^{-1}B\Delta([\pi_T(Cx(\cdot; t_0, x_0))]_{t_0})_{T}(\rho) d\rho 
+ QG^{-1}B\Delta([\pi_T(Cx(\cdot; t_0, x_0))]_{t_0})_{T}(t) 
+ \int_{T}^{t} \Phi(t, \rho)PG^{-1}B\Delta([Cx(\cdot; t_0, x_0)]_{T})(\rho) d\rho 
+ QG^{-1}B\Delta([Cx(\cdot; t_0, x_0)]_{T})(t). \] (3.3)

We define the following operators
\[ (\mathbb{L}_u)(t) = C(t) \int_{t_0}^{t} \Phi(t, \rho)PG^{-1}B(\rho)u(\rho) d\rho + CQG^{-1}B(t)u(t), \]
\[
(\hat{L}_{t_0} u)(t) = C(t) \int_{t_0}^{t} \Phi(t, \rho) P G^{-1} B(\rho) u(\rho) \, d\rho, \quad (\check{L}_{t_0} u)(t) = C Q G^{-1} B(t) u(t),
\]

\[
(M_{t_0} u)(t) = t \int_{t_0}^{t} \Phi(t, \rho) P G^{-1} B(\rho) u(\rho) \, d\rho + Q G^{-1} B(t) u(t),
\]

\[
(\hat{M}_{t_0} u)(t) = t \int_{t_0}^{t} P(t) \Phi(t, \rho) P G^{-1} B(\rho) u(\rho) \, d\rho \tag{3.4}
\]

for all \( t \geq t_0 \geq 0, u \in L_p(0, \infty; K^m) \). The first operator is called the input–output operator associated with (1.3).

It is easy to verify the following auxiliary results.

**Lemma 2.** Let the Assumptions A1–A2 hold. The following properties are true:

(a) \( \hat{L}_{t_0}, \hat{M}_{t_0}, \check{L}_{t_0} \in \mathcal{L}(L_p(t_0, \infty; K^m), L_p(t_0, \infty; K^q)), \quad M_{t_0} \in \mathcal{L}(L_p(t_0, \infty; K^m), L_p(t_0, \infty; K^n)) \),

(b) \( \| \hat{L}_t \| \leq \| \hat{L}_s \|, \quad t \geq s \geq 0 \),

(c) \( \| \check{L}_{t_0} \| = \text{ess sup}_{t \geq t_0} \| C Q G^{-1} B(t) \| \leq \| \check{L}_{t_0} \| \).

There exist constants \( M_2, M_3 \geq 0 \) such that

(d) \( \| (\check{M}_s u)(t) \| \leq M_2 \| u \|_{L_p(s, t; K^m)}, \quad t \geq s \geq 0, \quad u \in L_p(s, t; K^m) \),

(e) \( \| C(\cdot) \Phi(\cdot, s) P(t_0) x_0 \|_{L_p(t_0, \infty; K^n)} \leq M_3 \| P(t_0) x_0 \|, \quad t_0 \geq 0, \quad x_0 \in K^n \).

**Remark 4.** We note that the assumption on the boundedness of \( P G^{-1} \) and \( Q G^{-1} \) is only a sufficient condition for properties (a)–(c). So, there remains a possibility to relax this restrictive assumption.

**Definition 5.** Let Assumptions A1, A2 hold. The trivial solution of (1.3) is said to be globally \( L_p \)-stable if there exist constants \( M_4, M_5 > 0 \) such that

\[
\| P(t) x(t_0, x_0) \|_{K^n} \leq M_4 \| P(t_0) x_0 \|_{K^n},
\]

\[
\| x(\cdot; t_0, x_0) \|_{L_p(t_0, \infty; K^n)} \leq M_5 \| P(t_0) x_0 \|_{K^n}, \tag{3.5}
\]

for all \( t \geq t_0, \quad x_0 \in K^n \).

Due to the following proposition, we will see that the global \( L_p \)-stability property does not depend on the choice of projectors \( P(Q) \).

**Proposition 1.** Let Assumptions A1–A2 hold. The following two statements are equivalent:

(a) The trivial solution of (1.3) is globally \( L_p \)-stable.
The trivial solution of (1.3) is output stable, i.e., there exists a constant $M_6 > 0$ such that for all $t_0 \geq 0$, $x_0 \in \mathbb{K}^n$, we have

$$\|C(\cdot)x(\cdot; t_0, x_0)\|_{L_p(t_0, \infty; \mathbb{K}^n)} \leq M_6 \|P(t_0)x_0\|_{\mathbb{K}^n}. \quad (3.6)$$

**Proof.** (a) $\Rightarrow$ (b). Easy to see.

(b) $\Rightarrow$ (a). Due to the exponential stability, the estimate (1.2) holds. For all $t \geq t_0 \geq 0$, $x_0 \in \mathbb{K}^n$, we have

$$\|P(t_0)x_0\| + M_2 \|\Delta\| \|M_6\| P(t_0)x_0 \leq M_4 \|P(t_0)x_0\|,$$

where $M_4 := \overline{M} + M_2 \|\Delta\| M_6$. Furthermore, we have

$$\|x(\cdot; t_0, x_0)\|_{L_p(t_0, \infty; \mathbb{K}^n)} \leq \|\Phi(\cdot, t_0)P(t_0)x_0 + M_4 \|\Delta\| \|M_6\| P(t_0)x_0\|$$

$$\leq \left( \int_{t_0}^{\infty} M_5 e^{-p\omega(t-t_0)} \|P x_0\|^p \, dt \right)^{1/p} + \|M_4 \|\|\Delta\| M_6\| P(t_0)x_0\|$$

$$\leq M_5 \|P(t_0)x_0\|,$$

where $M_5 := M(p\omega)^{-1/p} + \|M_4 \|\|\Delta\| M_6$. The proof is complete. $\square$

### 4. A formula of the stability radius

First, the notion of the stability radius introduced in [7,10,14] is extended to time-varying differential–algebraic system (1.1).

**Definition 6.** Let Assumptions A1–A2 hold. The complex (real) structured stability radius of (1.1) subjected to linear, dynamic and causal perturbation in (1.3) is defined by

$$r_{\mathbb{K}}(E, A; B, C) = \inf \{ \|\Delta\|, \text{ the trivial solution of (1.3) is not globally } L_p-\text{stable or (1.3) is not index-1} \},$$

where $\mathbb{K} = \mathbb{C}, \mathbb{R}$, respectively.

**Remark 5.** It is worth to remark that if the perturbed system looses index-1 property, then the well-posedness of the initial value problem cannot be expected. Hence, it is quite natural to require the index-1 property for the perturbed system (1.3).
Proposition 2. Let Assumptions A1–A2 hold. If \( \Delta \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^q), L_p(0, \infty; \mathbb{K}^m)) \) is causal and satisfies
\[
\| \Delta \| < \min\left\{ \sup_{t_0 \geq 0} \| \bar{L}_{t_0} \|^{-1}, \| \bar{\bar{L}}_{t_0} \|^{-1} \right\},
\]
then system (1.3) is index-1 and its trivial solution is globally \( L_p \)-stable.

Proof. By assumption, we have
\[
\| \Delta \| < \| \bar{\bar{L}}_{t_0} \|^{-1} = \left( \text{ess sup}_{t \geq 0} \| C Q G^{-1} B(t) \| \right)^{-1}.
\]
Invoking Lemma 1 and using Definition 2, it is clear that system (1.3) is of index-1. Consequently, it admits a unique mild solution
\[
\begin{align*}
x(t; t_0, x_0) & \quad \text{for all } t_0 \geq 0, x_0 \in \mathbb{K}^n.
\end{align*}
\]
We will prove the output stability. Let \( T \geq t_0 \) be arbitrarily given. As a consequence of the proof of Theorem 1, there exists \( M_7 > 0 \) such that
\[
\begin{align*}
\| C P x(\cdot, t_0, x_0) \|_{L_p(t_0, T; \mathbb{K}^q)} &= \| C u(\cdot) \|_{L_p(t_0, T; \mathbb{K}^q)} \leq M_7 \| P(t_0)x_0 \|.
\end{align*}
\]
Also by the arguments used in the proof of Theorem 1, we have
\[
\begin{align*}
\| C Q x(t; t_0, x_0) = C v(t) = (\nabla u)(t). \text{ Hence}
\end{align*}
\]
\[
\begin{align*}
\| C Q x(\cdot ; t_0, x_0) \|_{L_p(t_0, T; \mathbb{K}^q)} & \leq \| \nabla \| M_7 \| P(t_0)x_0 \|.
\end{align*}
\]
Setting \( M_8 = (1 + M_7)\| \nabla \| \), we obtain
\[
\begin{align*}
\| C x(\cdot ; t_0, x_0) \|_{L_p(t_0, T; \mathbb{K}^q)} & \leq M_8 \| P(t_0)x_0 \|. \quad (4.1)
\end{align*}
\]
Now fix a number \( T > t_0 \) such that \( \| \Delta \| \| \bar{L}_T \| < 1 \). Due to the assumption on \( \| \Delta \| \), such a \( T \) exists. Then it follows from (3.3) that
\[
\begin{align*}
C(t)x(t; t_0, x_0) &= C(t)\Phi(t, T)P(T)x(T; t_0, x_0) + \left( \bar{L}_T \left( \Delta(\pi_T[\bar{C}x]_{t_0}) \right) \right)(t) \\
& \quad + \left( \bar{L}_T \left( \Delta([C x]_T) \right) \right)(t)
\end{align*}
\]
for \( t \geq T \). Hence,
\[
\begin{align*}
\| C x(\cdot ; t_0, x_0) \|_{L_p(T, \infty; \mathbb{K}^q)} & \leq \| C(\cdot)\Phi(\cdot, T)P(T)x(T; t_0, x_0) \|_{L_p(T, \infty; \mathbb{K}^q)} + \left( \bar{L}_T \left( \Delta(\pi_T[\bar{C}x]_{t_0}) \right) \right)(\cdot) \|_{L_p(T, \infty; \mathbb{K}^q)} \\
& \quad + \| \bar{L}_T \left( \Delta([C x]_T) \right) \right)(\cdot) \|_{L_p(T, \infty; \mathbb{K}^q)} \\
& \quad + \| L_T \| \| \Delta \| \left( \pi_T[\bar{C}x]_{t_0} \right) (\cdot) \|_{L_p(t_0, T; \mathbb{K}^q)} + \| L_T \| \| \Delta \| \left( [C x]_T \right) (\cdot) \|_{L_p(T, \infty; \mathbb{K}^q)} \\
& \quad + \| L_T \| \| \Delta \| \left( \pi_T[\bar{C}x]_{t_0} \right) (\cdot) \|_{L_p(t_0, T; \mathbb{K}^q)} + \| L_T \| \| \Delta \| \left( [C x]_T \right) (\cdot) \|_{L_p(T, \infty; \mathbb{K}^q)}
\end{align*}
\]
or equivalently

\[
\left( 1 - \|L_T\| \|\Delta\| \right) \left\| \begin{array}{c} Cx(\cdot; t_0, x_0) \\ L_p(T, \infty; \mathbb{K}^q) \end{array} \right\| \leq M_3 M_1 \left\| P(t_0)x_0 \right\| + \|L_T\| \|\Delta\| \left( \pi_T [Cx]_0 (\cdot) \right) \left\| L_p(0, T; \mathbb{K}^q) \right\|
\]

which implies that

\[
\left\| \begin{array}{c} Cx(\cdot; t_0, x_0) \\ L_p(T, \infty; \mathbb{K}^q) \end{array} \right\| \leq \frac{(1 - \|L_T\| \|\Delta\|)}{M_3 M_1 + \|L_T\| \|\Delta\| M_8} \left\| P(t_0)x_0 \right\|.
\] (4.2)

By (4.1), (4.2), and setting \(M_6 := M_8 + \frac{(1 - \|L_T\| \|\Delta\|)}{M_3 M_1 + \|L_T\| \|\Delta\| M_8}\) we obtain

\[
\left\| \begin{array}{c} Cx(\cdot; t_0, x_0) \\ L_p(T, \infty; \mathbb{K}^q) \end{array} \right\| \leq M_6 \left\| P(t_0)x_0 \right\|.
\]

The proof is complete. \(\square\)

**Remark 6.** If \(E(t) = I\), that is, system (1.1) is simply an explicit system of ordinary differential equations, then Proposition 2 reduces to [7, Theorem 4.3]. Here, thank to the Gronwall–Bellman inequality and the estimate given in Theorem 1, we have given a significantly shorter proof than that based on induction given in [7].

So, by Proposition 2, the inequality

\[
r_{\mathbb{K}}(E, A; B, C) \geq \min \left\{ \sup_{t_0 \geq 0} \|L_{t_0}\|^{-1}, \|\tilde{L}_{t_0}\|^{-1} \right\}
\]

holds. Next, our aim is to prove the inverse inequality. To this end, we recall some auxiliary results introduced in [7], see also [15].

**Definition 7.** We say that a causal operator \(Q \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^m), L_p(0, \infty; \mathbb{K}^q))\) has a finite memory if there exists a function \(\Psi : [0, \infty) \to [0, \infty)\) such that \(\Psi(t) \geq t\) and \((I - \pi_{\Psi(t)})Q \pi_t = 0\) for all \(t \geq 0\). The function \(\Psi\) is called the finite-memory function associated with \(Q\).

Since \(L_0 = \tilde{L}_0 + \tilde{L}_0\), the following lemma is simply a consequence of [7, Lemma 4.6].

**Lemma 3.** There exists a sequence of causal operator \(Q_n \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^m), L_p(0, \infty; \mathbb{K}^q))\) with finite memory such that

\[
\lim_{n \to \infty} \|L_0 - Q_n\| = 0.
\]

**Lemma 4.** [7, Lemma 4.7] Suppose \(f_1 \in L_p(0, \infty; \mathbb{K}^q)\), \(f_2 \in L_p(0, \infty; \mathbb{K}^m)\) with \(\text{supp} f_1 \subseteq [T_1, T_2]\) and \(\text{supp} f_2 \subseteq [T_3, T_4]\), where \(0 \leq T_1 < T_2 < T_3 < T_4\). Then there exists a causal operator \(P \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^q), L_p(0, \infty; \mathbb{K}^m))\) satisfying:

(a) \(P f_1 = f_2\),
(b) \(\text{supp} P f \subseteq [T_3, T_4]\) for all \(f \in L_p(0, \infty; \mathbb{K}^q)\).
(c) if \( f \in L_p(0, \infty; \mathbb{K}^q) \) with \( \text{supp} \, f \cap [T_1, T_2] = \emptyset \), then \( \mathbb{P} \, f = 0 \).

(d) \( \| \mathbb{P} \| = \| f_2 \| / \| f_1 \| \).

**Lemma 5.** [7, Lemma 4.8] Suppose that \( Q \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^m), L_p(0, \infty; \mathbb{K}^q)) \) is causal and has finite memory. Let \( \beta > \sup_{t \geq 0} \| Q \, S_t \|^{-1} \). Then there exist an operator \( \mathbb{P} \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^q), L_p(0, \infty; \mathbb{K}^m)) \), functions \( f, g \in L^\text{loc}_p(0, \infty; \mathbb{K}^q) \) and a natural number \( N_0 \) such that

(a) \( \| \mathbb{P} \| < \beta \), \( \mathbb{P} \) is causal and \( \mathbb{P} \) has finite memory,

(b) \( f \in L^\text{loc}_p(0, \infty; \mathbb{K}^m) \setminus L_p(0, \infty; \mathbb{K}^m) \) and \( Q \, f \in L^\text{loc}_p(0, \infty; \mathbb{K}^q) \setminus L_p(0, \infty; \mathbb{K}^q) \),

(c) \( \text{supp} \, g \subseteq [0, N_0] \) and \( \text{supp} \, Q \, g \subseteq [0, N_0] \),

(d) \( \mathbb{P}(y)(t) = 0 \) for every \( t \in [0, N_0] \) and all \( y \in L_p(0, \infty; \mathbb{K}^m) \),

(e) \( (I - \mathbb{P} \, Q) \, f = g \).

**Lemma 6.** Let Assumption A1 hold, \( \Delta \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^q), L_p(0, \infty; \mathbb{K}^m)) \) be causal, \( t > 0 \) and \( x_0 \in L_p(0, \infty; \mathbb{K}^n) \). Then the function \( u \) defined by

\[
u(\rho) := P(t) \, x(t; \rho, x_0(\rho)), \quad \rho \in [0, t],\]

satisfies \( u \in L_p(0, t; \mathbb{K}^n) \).

**Proof.** Due to the proof of Theorem 1, \( P(t) \, x(t; \rho, x_0(\rho)) \) satisfies the INHODE. Invoking [7, Lemma 4.9], it follows that \( u(\cdot) := P(t) \, x(t; \cdot, x_0(\cdot)) \in L_p(0, t; \mathbb{K}^n) \). \( \square \)

**Proposition 3.** If \( \sup_{t_0 \geq 0} \| L_{t_0} \|^{-1} < \| \widetilde{L}_0 \|^{-1} \) then for every \( \alpha \), \( \sup_{t_0 \geq 0} \| L_{t_0} \|^{-1} < \alpha < \| \widetilde{L}_0 \|^{-1} \), there exists a causal operator \( \Delta \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^q), L_p(0, \infty; \mathbb{K}^m)) \) with \( \| \Delta \| < \alpha \) such that the trivial solution of (1.3) is not globally \( L_p \) stable.

**Proof.** Proceeding in the same way as the proof of [7, Theorem 4.10], first of all, we choose a number \( \beta \) such that \( \alpha > \beta > \sup_{t \geq 0} \| L_0 \|^{-1} \). By Lemma 3, there exists a sequence \( (Q_n)_n \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^m), L_p(0, \infty; \mathbb{K}^q)) \), where every \( Q_n \) is causal and has finite memory, such that \( \lim_{n \to \infty} \| L_0 - Q_n \| = 0 \). There exits a number \( N_1 \) such that

\[ \sup_{s \geq 0} \| Q_n \, S_s \|^{-1} < \beta \quad \text{and} \quad \| Q_n - L_0 \| < 1 / \beta \]

for all \( n \geq N_1 \). By Lemma 5, there exist operators \( \Delta_n \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^q), L_p(0, \infty; \mathbb{K}^m)) \), functions \( f_n, g_n \in L^\text{loc}_p(0, \infty; \mathbb{K}^m) \) and a natural numbers \( N_{0,n}, \, n \geq N_1 \), such that the properties (a)–(e) of Lemma 5 hold. Then

\[
(I - (Q_n - L_0) \, \Delta_n)^{-1} = \sum_{k=0}^{\infty} ((Q_n - L_0) \, \Delta_n)^k \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^q), L_p(0, \infty; \mathbb{K}^m))
\]

for \( n \geq N_1 \). Furthermore, there exists \( N \geq N_1 \) such that

\[
\Delta := \Delta_N (I - (Q_N - L_0) \, \Delta_N)^{-1} \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^q), L_p(0, \infty; \mathbb{K}^m)) \quad \text{(4.3)}
\]
satisfies $\|\Delta\| < \alpha$. It is easy to see that $\Delta$ is causal and has finite memory as well. Moreover,

$$\Delta(u)(t) = 0 \quad \text{for } t \in [0, N_0, N] \text{ and } u \in L_p(0, \infty; \mathbb{K}^m).$$

Let us define

$$\tilde{y} := (I - (Q_N - L_0)\Delta N)Q_N f_N.$$

Using that $Q_N f_N \in L_p^\text{loc}(0, \infty; \mathbb{K}^q) \setminus L_p(0, \infty; \mathbb{K}^q)$ and $(I - (Q_N - L_0)\Delta N)$ is invertible in $L(L_p(0, \infty; \mathbb{K}^q))$ we get $\tilde{y} \in L_p^\text{loc}(0, \infty; \mathbb{K}^q) \setminus L_p(0, \infty; \mathbb{K}^q)$ and by property (e) of Lemma 5 we obtain

$$(I - L_0\Delta)\tilde{y} = (I - Q_N\Delta N)Q_N f_N = Q_N g_N.$$

Defining

$$f := \tilde{y}|_{[0,N_0,N]} \quad \text{and} \quad y := \tilde{y}|_{[N_0,N,\infty)},$$

for $t \geq N_0 := N_0, N_0$ we have

$$y(t) = \tilde{y}(t) = (L_0\Delta\tilde{y})(t) + (Q_N g_N)(t) = (L_0\Delta\tilde{y})(t) + (L_0\Delta y)(t).$$

Define

$$x_y(t) := (M_{N_0}\Delta f)(t) + (M_{N_0}\Delta y)(t), \quad t \geq N_0. \quad (4.4)$$

It is clear that

$$C(\cdot)x_y(\cdot) = (L_{N_0}\Delta f)(\cdot) + (L_{N_0}\Delta y)(\cdot) = y(\cdot) \in L_p^\text{loc}(0, \infty; \mathbb{K}^q) \setminus L_p(0, \infty; \mathbb{K}^q).$$

The assumption $C(\cdot) \in L_\infty(0, \infty; \mathbb{K}^{q \times n})$ implies $x_y(\cdot) \in L_p^\text{loc}(0, \infty; \mathbb{K}^n) \setminus L_p(0, \infty; \mathbb{K}^n)$. Furthermore, it is easy to see that $x_y(\cdot)$ is a (unique) mild solution to the system

$$\begin{cases}
(P_{xy})' = (P' + PG^{-1}\bar{A})Px_y + PG^{-1}B\Delta(Cx_y) + PG^{-1}B\Delta(f), \\
Q_{xy} = QG^{-1}\bar{A}Px_y + QG^{-1}B\Delta(Cx_y) + QG^{-1}B\Delta(f),
\end{cases} \quad (4.5)$$

with the initial condition $P(N_0)x_y(N_0) = 0$. Due to the assumption $\|\Delta\| < \alpha < \|\tilde{L}_0\|^{-1}$, this system of functional differential–algebraic equations is index-1 and the operator $I - CQG^{-1}B\Delta$ is invertible with the bounded inverse. Hence

$$C Q x_y = (I - CQG^{-1}B\Delta)^{-1}(C QG^{-1}\bar{A}Px_y + C QG^{-1}B\Delta(CPx_y) + C QG^{-1}B\Delta(f)).$$

Substituting into the first equation, we have

$$(P_{xy})' = (P' + PG^{-1}\bar{A})Px_y + PG^{-1}B\Delta(CPx_y) + PG^{-1}B\Delta((I - CQG^{-1}B\Delta)^{-1}$$

$$\times (C QG^{-1}\bar{A}Px_y + C QG^{-1}B\Delta(CPx_y))) + h, \quad (4.6)$$
where
\[
    h(t) = PG^{-1}B\Delta((I - CQG^{-1}B\Delta)^{-1}CQG^{-1}B\Delta(f))(t) + PG^{-1}B\Delta(f)(t), \quad t \geq N_0.
\] (4.7)

Since \( \Delta \) as well as \((I - CQG^{-1}B\Delta)^{-1} = \sum_{k=0}^{\infty}(CQG^{-1}B\Delta)^k\) are finite-memory operators and \( f \) has compact support, it is easy to see that \( h(\cdot) \) has compact support, too. By some manipulations, one can verify that

\[
    Pxy(t) = \int_{N_0}^{t} P(t)x(t; \rho, h(\rho)) d\rho.
\]

Indeed, let \( x_z \) be defined by

\[
    \begin{align*}
    P_{x_z}(t) &= \int_{N_0}^{t} P(t)x(t; \rho, h(\rho)) d\rho, \\
    Q_{x_z} &= QG^{-1}\bar{A}P_{x_z} + QG^{-1}B\Delta(C_{x_z}) + QG^{-1}B\Delta(f).
    \end{align*}
\] (4.8)

It is clear that \( x_z \) is well-defined and \( x_z \in L^1_{\rho}(N_0, \infty; K^n) \). Furthermore, for \( t \geq N_0 \) we can check by calculations that

\[
    \begin{align*}
    \int_{N_0}^{t} P(t)x(t; \rho, h(\rho)) d\rho &= \int_{N_0}^{t} \left[ P(t)\Phi(t, \rho)h(\rho) + \int_{\rho}^{t} P(t)\Phi(t, \tau)PG^{-1}B\Delta(C(\cdot) x(\cdot, \rho, h(\rho)))(\tau) d\tau \right] d\rho \\
    &= \int_{N_0}^{t} P(t)\Phi(t, \rho)h(\rho) d\rho + \int_{N_0}^{t} \int_{\rho}^{t} P(t)\Phi(t, \tau)PG^{-1}B\Delta(C(\cdot) x(\cdot, \rho, h(\rho)))(\tau) d\tau d\rho \\
    &= \int_{N_0}^{t} P(t)\Phi(t, \rho)h(\rho) d\rho + \int_{N_0}^{t} \int_{N_0}^{t} P(t)\Phi(t, \tau)PG^{-1}B\Delta(C(\cdot) x(\cdot, \rho, h(\rho)))(\tau) d\rho d\tau \\
    &= \int_{N_0}^{t} P(t)\Phi(t, \rho)h(\rho) d\rho + \int_{N_0}^{t} \int_{N_0}^{t} P(t)\Phi(t, \tau)PG^{-1}B\Delta\left( C(\cdot) \int_{N_0}^{\tau} P(\cdot) x(\cdot, \rho, h(\rho)) d\rho \right)(\tau) d\tau + \int_{N_0}^{t} P(t)\Phi(t, \tau)PG^{-1}B\Delta\left( C(\cdot) \int_{N_0}^{\tau} Q(\cdot) x(\cdot, \rho, h(\rho)) d\rho \right)(\tau) d\tau \\
    &= \int_{N_0}^{t} P(t)\Phi(t, \rho)h(\rho) d\rho + \int_{N_0}^{t} P(t)\Phi(t, \tau)PG^{-1}B\Delta(C(\cdot) x_z(\cdot))(\tau) d\tau + \int_{N_0}^{t} P(t)\Phi(t, \tau)P G^{-1}B\Delta\left( (I - CQG^{-1}B\Delta)^{-1}(CQG^{-1}Ax_z(\cdot) + CQG^{-1}B\Delta(C(\cdot)x_z(\cdot))) \right)(\tau) d\tau.
    \end{align*}
\]
It follows that $P_{xz}$ is a mild solution to (4.6), too. Due to the solution uniqueness, the equality $P_{xz} = P_{xy}$ holds. By their definition, we obtain $x_y = x_z$.

Now we assume the trivial solution of (1.3) is globally $L_p$-stable. This would imply $P_{xy} (\cdot) \in L_p (0, \infty; \mathbb{K}_q)$. To this end, we use the following estimates

\[ \| P_{xy} (\cdot) \|_{L_p(N_0, \infty; \mathbb{K}_q)} \leq \left[ \int_{N_0}^{\infty} \left( \int_{N_0}^{t} \| P(t)x(t; \rho, h(\rho)) \|_{\mathbb{K}_q} d\rho \right)^p dt \right]^{1/p} \leq M_4 \int_{N_0}^{\infty} \| h(\rho) \| d\rho < +\infty \quad \text{(because $h(\cdot)$ has compact support)}.
\]

Consequently, both $C P_{xy} (\cdot)$ and $C Q_{xy} (\cdot)$ would belong to $L_p (0, \infty; \mathbb{K}_q)$, which contradicts that $C x_y (\cdot) \in L_p^{\text{loc}} (0, \infty; \mathbb{K}_q) \setminus L_p (0, \infty; \mathbb{K}_q)$. Thus, the trivial solution of (1.3) is not globally $L_p$-stable. The proof is complete. \( \square \)

**Proposition 4.** For arbitrary small $\varepsilon > 0$ there exist $T > 0$ and a causal operator $\Delta \in L(L_p (0, T; \mathbb{K}_q), L_p (0, T; \mathbb{K}_m))$ such that $\| \Delta \| \leq \| \tilde{L}_0 \|^{-1} + \varepsilon$ and the operator $I - \Delta C Q G^{-1} B$ is not stable.

**Proof.** To verify the statement, first, we choose $T > 0$ such that

\[ \left( \text{ess sup}_{0 \leq t \leq T} \| C Q G^{-1} B \| \right)^{-1} \leq \left( \text{ess sup}_{t \geq 0} \| C Q G^{-1} B \| \right)^{-1} + \varepsilon/3 = \| \tilde{L}_0 \|^{-1} + \varepsilon/3. \quad (4.9) \]

Then, we construct a strictly monotone sequence $\{T_n\}_{n=0}^{\infty} \subset [0, T]$ such that

\[ \left( \text{ess sup}_{t \in [T_n, T_{n+1}]} \| C Q G^{-1} B \| \right)^{-1} \leq \| \tilde{L}_0 \|^{-1} + 2\varepsilon/3 \]

for $n = 0, 1, \ldots$. We proceed as follows. Due to the definition of the essential supremum, there exists a positive measured set $X \subseteq [0, T]$ such that

\[ \| C Q G^{-1} B(t) \| \geq \left( \| \tilde{L}_0 \|^{-1} + 2\varepsilon/3 \right)^{-1} \forall t \in X. \]

Let denote by $\mu(X) > 0$ the measure of $X$ and let $a = \inf\{t, t \in X\}, b = \sup\{t, t \in X\}$. It is obvious that $0 \leq a < b \leq T$. Set $T_0 = a$. For $n = 0, 1, \ldots$ choose $T_{n+1} > T_n$ such that the measure of $[T_n, T_{n+1}] \cap X$ is equal to $\mu(X)/2^{n+1}$. It is easy to see that the sequence $\{T_n\}_{n=0}^{\infty}$ fulfills the above requirements.
Next, it is clear that there exists a sequence of \( f_n \in L_p(0,T;K^m) \) with \( \supp f_n \subseteq [T_n, T_{n+1}] \), \( \| f_n \| = 1 \) and \( \| C Q G^{-1} B f_n \| \geq (\| L_0 \|^{-1} + \varepsilon)^{-1} \). By a slightly modified variant of Lemma 4, there exists causal operators \( \Delta_n \in \mathcal{L}(L_p(0,T;K^q), L_p(0,T;K^m)) \), \( n = 0, 1, \ldots \), such that

- \( \Delta_n(C Q G^{-1} f_n) = f_{n+1} \),
- \( \| \Delta_n \| \leq \| \tilde{L}_0 \|^{-1} + \varepsilon \),
- \( \supp \Delta_n h \subseteq [T_{n+1}, T_{n+2}] \), for all \( h \in L_p(0,T;K^q) \),
- if \( h \in L_p(0,T;K^q) \) with \( \supp h \cap (T_n, T_{n+1}) = \emptyset \) then \( \Delta_n h = 0 \).

We define \( f := \sum_{n=0}^\infty f_n \), \( \Delta h := \sum_{n=0}^\infty \Delta_n h \) for \( h \in L_p(0,T;K^q) \) and \( g := f_0 \). It is easy to see that \( \Delta \in \mathcal{L}(L_p(0,T;K^q), L_p(0,T;K^m)) \), \( \Delta \) is causal, \( \| \Delta \| \leq \| \tilde{L}_0 \|^{-1} + \varepsilon \) and

\[
(I - \Delta C Q G^{-1} B) f = f_0 = g.
\]

It follows from \( f \notin L_p(0,T;K^m) \), \( g \in L_p(0,T;K^m) \) that the operator \( I - \Delta C Q G^{-1} B \) has no bounded inverse in \( \mathcal{L}(L_p(0,T;K^m), L_p(0,T;K^m)) \). \( \square \)

**Remark 7.** We note that the problem of constructing a destabilizing operator \( \Delta \) is well known and could be solved in a less complicated manner, e.g., see the proof of Theorem 1.2 in [5]. Here, a very important point is the causality of the destabilizing operator \( \Delta \) which makes a difference between the above construction and others.

By Propositions 2–4, we obtain a formula for the stability radius.

**Theorem 2.** Let Assumptions A1, A2 hold. Then

\[
r_{\mathcal{K}}(E, A; B, C) = \min \left\{ \sup_{t_0 \geq 0} \| L_{t_0} \|^{-1}, \| \tilde{L}_0 \|^{-1} \right\}.
\]

**Corollary 1.** Let the data \( E, A, B, C \) be real and Assumptions A1, A2 hold. Then

\[
r_{\mathcal{C}}(E, A; B, C) = r_{\mathbb{R}}(E, A; B, C).
\]

**Remark 8.** We remark that due to the monotone property of \( \| L_t \| \) as a function of \( t \) (see Lemma 2(b)), we have

\[
\sup_{t_0 \geq 0} \| L_{t_0} \|^{-1} = \lim_{t_0 \to \infty} \| L_{t_0} \|^{-1}.
\]

Comparing to (1.4), we see that the extra term \( \| \tilde{L}_0 \|^{-1} \) is the measure for index-1 property robustness, in fact. This yields an essential difference between DAEs and ODEs.
5. Special cases

5.1. Semi-explicit systems

Let system (1.1) be given in the so-called semi-explicit form, i.e.,

\[
E = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix},
\]

(5.1)

where \( I_{n_1} \) is the identity matrix of indicated size, \( A_{ij} \) (\( 1 \leq i, j \leq 2 \)) are the submatrices of appropriate dimensions. The index-1 assumption means exactly that \( A_{22}(t) \) is invertible almost everywhere in \([0, \infty)\). In lots of applications, systems of DAEs occur in the semi-explicit form. One may set \( Q = \text{diag}(0, I_{n_2}) \) and easily obtain

\[
G = \begin{pmatrix} I_{n_1} & -A_{12} \\ 0 & -A_{22} \end{pmatrix}, \quad Q_s = \begin{pmatrix} 0 & 0 \\ A_{22}^{-1} A_{21} & I \end{pmatrix}.
\]

Furthermore, we have

\[
\Phi(t, s) = \begin{pmatrix} \Phi(t, s) \\ -A_{22}^{-1} A_{21} \Phi(t, s) \end{pmatrix},
\]

where \( \Phi(t, s) \) is the evolution operator generated by the so-called essentially underlying ordinary differential equation

\[
y' = (A_{11} - A_{12} A_{22}^{-1} A_{21}) y,
\]

(5.2)

which is supposed to be exponentially stable. Assumption A2 is equivalently to the assumptions on the essential boundedness of \( A_{22}^{-1}, A_{22}^{-1} A_{21} \) and \( A_{12} A_{22}^{-1} \). Let the structure matrices \( B, C \) be rewritten into the decomposed form as follows

\[
B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix},
\]

where the submatrices have the appropriate dimensions. By some matrix calculations, we obtain

\[
(\mathbb{L}_{L_0} u)(t) = (C_1 - C_2 A_{22}^{-1} A_{21}) \int_{t_0}^{t} \Phi(t, \rho) (B_1(\rho) - A_{12} A_{22}^{-1} B_2(\rho)) u(\rho) d\rho - C_2 A_{22}^{-1} B_2 u(t),
\]

\[
(\mathbb{L}_{\tilde{L}_0} u)(t) = C_2 A_{22}^{-1} B_2 u(t), \quad t \geq t_0.
\]

(5.3)

By Theorem 2, we have

\[
\kappa(E, A; B, C) = \min \left\{ \lim_{t_0 \to \infty} \| \mathbb{L}_{L_0} \|^{-1}, \left( \text{ess sup}_{t \geq 0} \| C_2 A_{22}^{-1} B_2(t) \| \right)^{-1} \right\}.
\]
5.2. Fully implicit regular systems and purely algebraic systems

First, we consider system (1.1) with the almost everywhere nonsingular leading term $E$. In addition, we suppose that $E^{-1}$ is essentially bounded in $[0, \infty)$. By multiplying both sides of system (1.1) with $E^{-1}$ we obtain an explicit regular system investigated in [7]. By applying Theorem 2 with the unique and trivial choice $P = I$, $Q = 0$, the formula of the stability radii obtained here coincides with that stated in [7].

Another degenerate case occurs when $E = 0$, that is system (1.1) is, in fact, a purely algebraic system

$$0 = A(t)x(t), \quad t \geq 0.$$ 

We set $Q = I$, $P = 0$. The set of assumptions equivalently means that $A(\cdot)$ is invertible almost everywhere in $[0, \infty)$ and the inverse is essentially bounded. It is obvious to see that the homogeneous system has the unique trivial solution. The algebraic system is stable in the sense that for any function $q \in L^p(0, \infty; \mathbb{K}^n)$ the inhomogeneous system

$$A(t)x(t) = q(t)$$

has the unique solution $x \in L^p(0, \infty; \mathbb{K}^n)$ and the solution depends continuously on the right-hand side (with respect to $L^p$-norm). In meaning of the results in Section 4, for any causal perturbation operator $\Delta$ with

$$
\|\Delta\| < \|L_0\|^{-1} = \left(\text{ess sup}_{t \geq 0} \|CA^{-1}B(t)\|\right)^{-1},
$$

the perturbed system remains stable and (5.4) gives the best bound, i.e., for any $\varepsilon > 0$ there exists a causal $\Delta$ with $\|\Delta\| \leq \|L_0\|^{-1} + \varepsilon$ destabilizing the algebraic system.

5.3. Time-invariant systems

Now, suppose that all the matrices $E, A, B, C$ are time-invariant. It is clear that Assumption A2 becomes unnecessary. By Fourier–Plancherel transformation technique as in [10], the following statement, which is, in fact, an extension of Theorem 2.1 in [10] to index-1 systems of DAEs, can be proven.

**Proposition 5.** Let $E, A, B, C$ be time-invariant, system (1.1) be index-1 and exponentially stable. If $p = 2$ is chosen (that is the $L^2$-stability is considered) then

$$
\|L_0\| = \|L_0\| = \sup_{\omega \in i\mathbb{R}} \|C(\omega E - A)^{-1}B\|.
$$

The function $C(\omega E - A)^{-1}B$ is called the artificial transfer functions associated with (1.1). We remark that the exponential stability of time-invariant system (1.1) means exactly that all generalized eigenvalues of matrix pencil $(E, A)$ have negative real part. Hence, the transfer function is well-defined on the imaginary axis $i\mathbb{R}$ of the complex plane. Consequently, now Theorem 2 can be reformulated as follows.
Theorem 3. Let $E, A, B, C$ be time-invariant, system (1.1) be index-1 and exponentially stable. If $p = 2$ is chosen then

$$r_C(E, A; B, C) = \|L_0\|^{-1} = \left(\sup_{\omega \in \mathbb{R}} \|C(\omega E - A)^{-1}B\|\right)^{-1}.$$ 

Proof. It remains to show $\|\tilde{L}_0\| \leq \|L_0\|$. Due to Lemma 2(c), it is obvious. Alternatively, we can verify the limit

$$\lim_{|\omega| \to +\infty} \|C(\omega E - A)^{-1}B\| = \|CQG^{-1}B\|$$

by transforming the coefficient matrix pair $E, A$ into either the Weierstrass–Kronecker normal form (see [2,6]) or the semi-explicit form (5.1) and then using direct matrix calculations. Thus

$$\sup_{\omega \in \mathbb{R}} \|C(\omega E - A)^{-1}B\| \geq \|CQG^{-1}B\| = \|\tilde{L}_0\|. \quad \Box$$

By Theorem 3, the computation of the stability radius for time-invariant systems leads to a global optimization problem and it can be solved numerically, e.g., see [1,14]. Finally, we note that the equalities in Theorem 3 means exactly that the complex stability radius with respect to dynamic perturbation investigated in this paper coincides with the complex stability radius with respect to static perturbation (i.e., $\Delta$ is a time-invariant matrix) considered in [3,4,14]. Thus, Theorem 3 generalizes a previous result for linear systems of ODEs obtained in [9].

Acknowledgment

The authors like to thank an anonymous referee for useful comments in the course of revising the paper.

References


