ADVANCES IN MATHEMATICS 69, 32–92 (1988)

Relations between Young's Natural and The Kazhdan–Lusztig Representations of S_n

A. M. GARSIA AND T. J. MCLARNAN*

Department of Mathematics, University of California, San Diego, La Jolla, California 92903

Our main result here is that, under a suitable order of standard tableaux, the classical representation of S_n introduced by Young (*in* "The Collected Papers of Alfred Young, 1873–1940," Univ. of Toronto Press, Toronto) (QSA IV), and usually referred as the *Natural* representattion, the the more recently discovered (*Invent. Math.* **53** (1979), 165–184) Kazhdan–Lusztig (K–L) representation are related by an upper triangular integral matrix with unit diagonal elements. We have been led to this discovery by a numerical exploration. We noted it in each of the irreducible representations of S_n up to $n=6$. The calculations in these cases were carried out by constructing the corresponding Kazhdan–Lusztig graphs from tables (M. Goresky, Tables of Kazhdan–Lusztig polynomials, unpublished) of K–L polynomials. To extend the calculations to $n=7$ we have used graphs obtained by means of an algorithm given by Lascoux and Schützenberger (Polynomes de Kazhdan & Lusztig pour les Grassmanniennes, preprint). Remarkably, the same property holds also for these graphs. These findings appear to confirm the assertion made by these authors that their algorithm does indeed yield K–L graphs. For the case of hook shapes we have obtained an explicit construction of the transforming matrices, a result which was also suggested by our numerical data. For general shapes, the transforming matrices are less explicit and our proof is based on certain properties of the Kazhdan–Lusztig representations given in their article (*Invent. Math.* **53** (1979), 165–184) and on a purely combinatorial construction of the natural representation.

INTRODUCTION

In a fundamental work [13] Kazhdan and Lusztig have obtained an algorithm for constructing a complete set of irreducible representations of S_n. These representations are entirely new, and what is particularly interesting is that they are relatively easy to compute once certain labeled graphs have been obtained. The result has a highly combinatorial flavor, and indeed all the K–L graphs that have so far been constructed can be given a completely combinatorial description. Unfortunately, these graphs

* Work supported by the National Science Foundation under a grant at the University of California, San Diego, and by the Office of Naval Research under a grant at the Massachusetts Institute of Technology.

0001-8708/88 $7.50

are explicitly available only for hook shapes (a result which antedates the work of Kazhdan and Lusztig; see [3]), for diagrams with no more than two rows (or columns) (see [18]) and for all Ferrer's diagrams with at most six squares.

The general construction of the K–L graphs depends on the knowledge of the so-called K–L polynomials, whose definition is quite intricate and not easily amenable to computation. This circumstance has prompted the search of a more direct, possibly combinatorial characterization of the K–L graphs. Lascoux and Schützenberger (L–S) in [17] (see also [18]), have given an algorithm for the construction of graphs of the K–L type. By this we mean labelled graphs yielding representations of S_n by the construction we describe in Section 2.

There are three basic (progressively weaker) questions arising from this work of Lascoux and Schützenberger:

(1) Are the K–L and L–S graphs the same for any given shape?

(2) Do the L–S graphs give the irreducible representation of S_n corresponding to the shape yielding the graph?

(3) Do the L–S graphs give at the very least a representation?

From personal exchanges with Lascoux (see also [17]), it appears that the answers to questions (1) and (2) are not known in full generality, while question (3) has been answered in the affirmative only for the cases that have been verified by computer. Our numerical data yield a positive answer to question (2) up to $n = 7$.

Very few authors today have much familiarity with Young's natural representation. The various presentations of Specht modules (see, for instance, [11, 12]) and the work of Garnir [5] tend to hide the simplicity and beauty of Young's construction. This circumstance has led some authors (see [17]) to be mistaken belief that the Kazhdan–Lusztig representation is easier to construct than the natural. This is not so at all, for, even if we are given the graphs, the K–L construction yields the representing matrices only for adjacent transpositions. On the other hand, as we shall see here, Young's natural can be constructed at once by a very simple combinatorial procedure which applies to all permutations. Moreover, the proof that the procedure is valid is actually quite short and elementary.

Other desirable properties, such as block structures resulting from restrictions to Young subgroups, are also shared by the natural itself. Indeed, some of the fundamental properties of the irreducible representations of S_n, such as Pieri's rule or the Murnagham–Nakayama rule, for instance, are easier to derive by working directly with the natural than from any other approach (see [8–9]).

Nevertheless, the elusiveness of the K–L graphs and their remarkable properties prompts our desire for a better understanding of their nature and a more direct way of constructing them. The original point of departure in our research was the belief that ultimately, the inner structure of the K–L graphs must somehow be encoded within Young's natural representation.

It is well known that if two representations A and B have the same character then there is a matrix which transforms one into the other. That is, we have a non-singular matrix W, such that

$$A(\sigma)\, W = WB(\sigma) \qquad \text{for all} \quad \sigma \in S_n. \tag{I.1}$$

This matrix is uniquely determined (up to a factor) by the $n-1$ systems of linear homogeneous equations obtained from (I.1) by letting σ describe in turn each of the adjacent transpositions of S_n. Actually, as we shall see, using the fact that A and B are representations, we can write down a formula for W which is relatively easy to program on the computer.

Starting from this fact we embarked on the task of carrying out a computer exploration to see what kind of transforming matrices we do obtain when we specialize A and B in (I.1) to be respectively Young's natural and the K–L representations corresponding to a given shape.

The transforming matrices should also reveal (as was pointed out to us by A. Joyal) whether or not the natural and the K–L representations are equivalent *over the integers*. That is, to be precise, since both A and B in this case always have integer entries, it is clearly of interest to know whether or not the solution to (I.1) after proper scaling, may be chosen to have integer entries and determinant one. It is to be noted that already in the case of the shape (1, 2) there are two integral representations of S_3 that are not equivalent over the integers (see [4]). In fact, we can easily verify by simple calculation that this happens between the natural and its *transpose*.

Our numerical exploration resulted in findings that went far beyond our most optimistic expectations. Indeed, the remarkable fact emerged that when we order tableaux according to either of two orders systematically used by A. Young, the transforming matrices come out upper triangular with non-negative integer entries and diagonal elements equal to 1. Moreover, even a brief look at the resulting matrices reveals a tantalizing combinatorial structure.

A closer study enabled us to formulate a simple recursive procedure for obtaining all of the transforming matrices for the representations corresponding to hook shapes. Using the fact that the K–L graphs are entirely known for hooks, we have succeeded in putting together a proof of the general validity of the procedure.

The numerical evidence gave us a very strong incentive to look for a proof of a triangularity result valid for all shapes, for it was difficult to believe that all of the above-mentioned properties of the matrices we calculated were purely accidental. Now it turns out that, except for the non-negativity of the entries, we were able to establish precisely such a result (see Theorem 5.3) by combining our combinatorial construction of the natural with some basic properties of the K–L representations. In fact, our final result (Theorem 5.4 below) gives an algorithm for the construction of the entries in the transforming matrices which is quadratic in the number of standard tableaux of the given shape.

Our numerical data further suggest that some very beautiful combinatorial mechanism is involved. It is not unlikely that an explicit construction of the transforming matrices may be obtained for all shapes. To formulate the appropriate conjectures for arbitrary shapes, it would be desirable to get numerical data for larger values of n. Unfortunately, it is difficult to go any further in this direction at this point without an explicit construction of the K–L graphs.

At any rate, we have put together a complete printout of the graphs and the resulting transforming matrices, in the hope that it may inspire further research in the connections between these two remarkable representations of the symmetric group. This printout may be obtained upon request.

The contents of this paper are divided into five sections. In the first we give our construction of Young's natural; in the second we describe the K–L representation; in the third we give the list of our transforming matrices and formulate some conjectures; in the fourth we give our result for hooks; and the fifth and final section we prove our triangularity result.

1. A Combinatorial Recipe for Young's Natural

To state the result we need some notation. As is customary, we shall identify partitions with Ferrer's diagrams. The latter will be represented by left justified rows of dots. The notation $\lambda \vdash n$ indicates that λ is a partition of n, or that the corresponding diagram has n dots. An *injective* tableau of shape $\lambda \vdash n$ is obtained by replacing the dots of the Ferrers' diagram corresponding to λ by the integers $1, 2, \ldots, n$ using each of them once and only once. We say that the tableau is *standard* if the entries are in increasing order from left to right on the rows and from bottom to top on the columns.

In the picture below we depict the Ferrers' diagram corresponding to the partition $\lambda = (1, 3, 3, 4)$, an injective and a standard tableau of shape λ

$$\begin{array}{llll} \bullet & & & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet \end{array} \qquad \begin{array}{llll} 1 & & & \\ 9 & 2 & 3 & \\ 6 & 11 & 7 & \\ 5 & 10 & 4 & 8 \end{array} \qquad T = \begin{array}{llll} 6 & & & \\ 4 & 8 & 10 & \\ 2 & 5 & 9 & \\ 1 & 3 & 7 & 11 \end{array} \quad . \tag{1.1}$$

Let T_1 and T_2 be two standard tableaux of same shape. If the integers $1, 2, \ldots, i$ occupy the same positions in T_1 and T_2 but $i+1$ does not, then we call $i+1$ the *first letter of disagreement* between T_1 and T_2.

This given, we shall say that T_1 precedes T_2, if the first letter of disagreement between T_1 and T_2 is higher in T_2 than it is in T_1. This total order of standard tableaux is introduced in QSA IV (p. 258). Young refers to it as the *first letter* order. Similarly, we can define a total order of standard tableaux based on the last letter of disagreement. More precisely, we shall say that T_1 precedes T_2 in the *last letter order* if the last letter of disagreement is higher in T_1 than in T_2. Hereafter, unless otherwise specified, we let

$$T_1^\lambda, T_2^\lambda, \ldots, T_{n_\lambda}^\lambda \tag{1.2}$$

denote the standard tableaux of shape λ arranged in the last letter order.

Given two injective tableaux T_1 and T_2 we define as the *intersection* of T_1, T_2 the rectangular figure obtained by placing in the i, j position of the rectangle the intersection of the ith row of T_1 with the jth column of T_2.

We illustrate below two examples:

$$T_2 \to \begin{array}{lll} 4 & & \\ 3 & 1 & \\ 2 & 5 & 6 \end{array} \qquad\qquad T_2 \to \begin{array}{lll} 6 & & \\ 5 & 4 & \\ 1 & 3 & 2 \end{array}$$

$$T_1 \to \begin{array}{lll} 1 & & \\ 4 & 5 & \\ 3 & 2 & 6 \end{array} \qquad \begin{array}{ccc} \varnothing & 1 & \varnothing \\ 4 & 5 & \varnothing \\ 3,2 & \varnothing & 6 \end{array} \qquad T_1 \to \begin{array}{lll} 1 & & \\ 4 & 5 & \\ 3 & 2 & 6 \end{array} \qquad \begin{array}{ccc} 1 & \varnothing & \varnothing \\ 5 & 4 & \varnothing \\ 6 & 3 & 2 \end{array}$$

In the first example there is a pair of elements which are in the same row of T_1 and in the same column of T_2. In the second, no such event occurs and the resulting intersection is actually a tableau. Note further that in the second case the intersection tableau T' can be obtained from T_1 by a permutation α_1 of the row group of T_1 and T_2 can be obtained from T' by a permutation β_2 of the column group of T_2. Here, as customary, by the *row* and *column* groups of a tableau T we mean the collections of permutations leaving invariant the rows and columns of T, respectively. In summary we have

$$T_2 = \beta_2 \alpha_1 T_1.$$

This is typical of what happens when the tableaux have the same shape.

The general result, which is fundamental for all developments concerning the representation theory of S_n, can be stated as follows.

LEMMA 1.1. *Let T_1 and T_2 be two injective tableaux of the same shape. Then the intersection of T_1 and T_2 is a tableau if and only if there are two pairs of permutations α_1, β_1, α_2, β_2, with α_i (resp. β_i) in the row (resp. column) group of T_i satisfying the following three equivalent conditions:*

$$T_2 = \beta_2 \alpha_1 T_1; \qquad T_2 = \alpha_1 \beta_1 T_1; \qquad T_2 = \alpha_2 \beta_2 T_1.$$

The proof is not difficult and may be found in [21].

This result allows us to set

$$c(T_1, T_2) = \begin{cases} \operatorname{sign}(\beta_2) & \text{if the intersection is a tableau,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for any permutation $\sigma \in S_n$ let $C^\lambda(\sigma)$ denote the matrix whose i, j entry is given by the expression

$$C^\lambda_{ij}(\sigma) = c(T^\lambda_i, \sigma T^\lambda_j).$$

We are now finally in a position to give our combinatorial construction of the natural representation. It may be stated as follows:

THEOREM 1.1. *The map*

$$\sigma \to C^\lambda(\sigma)\, C^\lambda(\varepsilon)^{-1} \qquad (\varepsilon \text{ denotes the identity of } S_n) \tag{1.3}$$

defines an irreducible integral representation of S_n whose character is the one indexed by λ in the Young (or Frobenius) indexing.

It would be quite desirable at this point to give a purely combinatorial proof of this result. For the case of hook shapes such a proof can be found in [8]. At present, the easiest way we know to derive the general case is by algebraic manipulations of Young idempotents. However, let us note first that

LEMMA 1.2. *The matrix $C^\lambda(\varepsilon)$ is lower triangular with unit diagonal elements.*

Proof. We should point out that the same result holds true if the symbols in (1.2) denote the standard tableaux of shape λ in the *first* letter order. In fact, in this case the proof is immediate. For, if $i<j$ and the first letter of disagreement between T_i and T_j is s then there is a letter a that is in the same row as s in T_i and the same column as s in T_j; this forces $c(T_i, T_j) = 0$. The statement about diagonal elements is trivial in any case. To show the result for the *last* letter order we need a few additional considerations useful in other contexts. Let T_1 and T_2 be two standard tableaux of the same shape λ, and let $s+1$ be their last letter of dis-

agreement. Let $T_1|_s$ and $T_2|_s$ denote the subtableaux of T_1 and T_2 containing the letters 1, 2, ..., s. If T_1 precedes T_2 in the last letter order then the shape of $T_1|_s$ is larger in the dominance order than that of $T_2|_s$. Our desired conclusion then follows immediately from the following useful lemma.

LEMMA 1.3. *If T_1 and T_2 are two injective tableaux of shapes λ and μ, respectively, and if λ strictly dominates μ, then there is a pair of letters that is horizontal in T_1 and vertical in T_2.*

Proof. This result is known but we shall include the proof since it is quite short. Assume that there are no such pairs and construct the intersection of T_1 and T_2. In each position of the rectangle there will then be at most one letter. Erase the empty sets and slide the letters down their columns until they are tightly packed with no gaps. We see that the number of letters in the first i rows before the slide is the same as in T_1, and that is

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i.$$

After the slide, since the resulting shape is μ, the number of letters in the first i rows is

$$\mu_1 + \mu_2 + \cdots + \mu_i.$$

Now this number must be at least as large as the former for all i; and this implies that μ dominates λ, which contradicts the hypothesis of the lemma.

This completes the proofs of both Lemmas 1.2 and 1.3.

Let us now go back to the proof of Theorem 1.1. Clearly, the matrix on the right-hand side of (1.3) has integral entries, since Lemma 1.2 implies that $C^\lambda(\varepsilon)$ has an integral inverse. Thus we need only show that it gives a representation of S_n and that it has the right character. To this end, we need to recall the construction of the Young idempotents.

For a given set A of integers we denote by $[A]$ the formal sum of all permutations of A. In other words, if S_A denotes the symmetric group of A then

$$[A] = \sum_{\sigma \in S_A} \sigma.$$

It is convenient to write these permutations as products of cycles, omitting the fixed points. For instance, we write

$$[248] = \varepsilon + (2, 4) + (2, 8) + (4, 8) + (2, 4, 8) + (2, 8, 4),$$

and this can be interpreted as an element of the group algebra of any S_n as long as $n \geqslant 8$. We also set

$$[A]' = \sum_{\sigma \in S_A} \operatorname{sign}(\sigma)\, \sigma. \tag{1.4}$$

In particular

$$[248]' = \varepsilon - (2,4) - (2,8) - (4,8) + (2,4,8) + (2,8,4).$$

This given, if the rows of a tableau T are $R_1, R_2, ..., R_h$ and its columns are $C_1, C_2, ..., C_k$ then we set

$$P(T) = [R_1][R_2] \cdots [R_h], \qquad N(T) = [C_1]'[C_2]' \cdots [C_k]'.$$

Thus for the tableau T given in (1.1) we have

$$P(T) = [1\ 3\ 7\ 11][2\ 5\ 9][4\ 8\ 10]$$

and

$$N(T) = [1\ 2\ 4\ 6]'[3\ 5\ 8]'[7\ 9\ 10]'.$$

For an injective tableau T of shape λ, we finally set

$$\gamma(T) = \frac{P(T)\, N(T)}{h_\lambda}, \tag{1.5}$$

where h_λ is the *product of the hooks* of the Ferrers' diagram of λ. The constant h_λ is the integer that makes $\gamma(T)$ into an idempotent. It can also be defined as the ratio

$$h_\lambda = \frac{n!}{n_\lambda},$$

where n_λ denotes the number of standard tableaux of shape λ.

Now let the symbols in (1.2) denote the standard tableaux of shape λ arranged in either of the two Young orders, and let σ_{ij}^λ be the permutation that sends the tableau T_i^λ into the tableau T_j^λ. For simplicity, we shall sometimes omit the superscript λ when dealing only with tableaux of the same shape.

From Lemma 1.2 we can easily derive that

$$N(T_j)\, P(T_i) = 0, \qquad \text{for all} \quad j > i. \tag{1.6}$$

Indeed, the following result is worth noting.

LEMMA 1.4. *Let T_1 and T_2 be injective tableaux of the same shape, and let $\sigma_{T_1T_2}$ be the permutation that sends T_2 into T_1. Then for any permutation σ we have*

$$P(T_1)\,\sigma N(T_2)=c(T_1,\sigma T_2)\,P(T_1)\,\sigma_{T_1T_2}N(T_2). \tag{1.7}$$

Proof. Note that the left-hand side can be written as

$$P(T_1)\,N(\sigma T_2)\,\sigma.$$

From Lemma 1.1 we get that either this product is zero or there are permutations α_1, α_2; β_1, β_2 of the row and column groups of T_1 and T_2 such that

$$\sigma T_2=\alpha_1\beta_1 T_1=\alpha_2\beta_2 T_1=\beta_2\alpha_1 T_1.$$

Thus in the latter case (since $\beta_1^{-1}\alpha_1^{-1}\sigma=\sigma_{T_1T_2}$) we have

$$\begin{aligned} P(T_1)\,N(\sigma T_2)\,\sigma &= P(T_1)\,\alpha_1\beta_1 N(T_1)\,\beta_1^{-1}\alpha_1^{-1}\sigma \\ &= \operatorname{sign}(\beta_1)\,P(T_1)\,N(T_1)\,\sigma_{T_1T_2}. \end{aligned} \tag{1.8}$$

Now we see that

$$\operatorname{sign}(\beta_1)=\operatorname{sign}(\alpha_1^{-1}\beta_2\alpha_1)=c(T_1,\sigma T_2).$$

Thus, when $c(T_1,\sigma T_2)\neq 0$, (1.7) follows from (1.8). However, since $c(T_1,\sigma T_2)$ is zero whenever the left-hand side of (1.7) is zero, we must conclude that (1.7) holds true in all cases.

This result has the following useful corollary.

LEMMA 1.5. *For any two injective tableaux T_1 and T_2 of the same shape and any permutation σ, we have*

$$P(T_1)\,\sigma_{T_1T_2}N(T_2)|_\sigma=c(T_1,\sigma T_2). \tag{1.9}$$

Proof. Equation (1.8) may be rewritten as

$$P(T_1)\,\sigma N(T_2)=c(T_1,\sigma T_2)\,P(T_1)\,N(T_1)\,\sigma_{T_1T_2}.$$

This gives

$$P(T_1)\,\sigma N(T_2)\,\sigma_{T_2T_1}=c(T_1,\sigma T_2)\,P(T_1)\,N(T_1).$$

Taking coefficients of the identity we get

$$\begin{aligned} c(T_1,cT_2)&=P(T_1)\,\sigma N(T_2)\,\sigma_{T_2T_1}|_\varepsilon \\ &=\sigma N(T_2)\,\sigma_{T_2T_1}P(T_1)|_\varepsilon=N(T_2)\,\sigma_{T_2T_1}P(T_1)|_{\sigma^{-1}}, \end{aligned}$$

and this is another way of writing the left-hand side of (1.9).

Following customary notation let us set

$$E_{ij}^{\lambda} = P(T_i^{\lambda})\, \sigma_{ij}^{\lambda} N(T_j^{\lambda}).$$

A basic fact in this theory is that these units satisfy the following multiplication rules.

THEOREM 1.2.

$$E_{ij}^{\lambda} E_{rs}^{\mu} = \begin{cases} h_\lambda c(T_r^{\lambda}, T_j^{\lambda})\, E_{is}^{\lambda} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

Proof. In a different notation, this result is given in [21]. However, we prove it here since it is an immediate consequence of Lemmas 1.3 and 1.4. Note first that if T_1 and T_2 are injective tableaux of shapes λ and μ, respectively, and σ is any permutation then Lemma 1.3 gives that the expression

$$P(T_1)\, \sigma N(T_2) = P(T_1)\, N(\sigma T_2)\, \sigma \tag{1.11}$$

vanishes unless μ dominates λ. On the other hand for the same reason the expression

$$N(T_1)\, \sigma P(T_2) \tag{1.12}$$

vanishes unless λ dominates μ. Now we see that the product

$$E_{ij}^{\lambda} E_{rs}^{\mu} = P(T_i^{\lambda})\, \sigma_{ij}^{\lambda} N(T_j^{\lambda})\, P(T_r^{\mu})\, \sigma_{rs}^{\mu} N(T_s^{\mu}) \tag{1.13}$$

may be expanded as a linear combination of expressions such as those in (1.11) with $T_1 = T_i^{\lambda}$ and $T_2 = T_s^{\mu}$ as well as expressions such as those in (1.12) with $T_1 = T_j^{\lambda}$ and $T_2 = T_r^{\mu}$. Thus our product must necessarily vanish if $\lambda \neq \mu$. For the case $\lambda = \mu$, we apply Lemma 1.4 and get

$$N(T_j^{\lambda})\, P(T_r^{\lambda}) = c(T_r^{\lambda}, T_j^{\lambda})\, N(T_j^{\lambda})\, \sigma_{jr}^{\lambda} P(T_r^{\lambda}).$$

Substituting this in (1.13) and omitting the superscripts we get

$$\begin{aligned} E_{ij} E_{rs} &= c(T_j, T_r)\, P(T_i)\, \sigma_{ij} N(T_j)\, \sigma_{jr} P(T_r)\, \sigma_{rs} N(T_s) \\ &= c(T_j, T_r)\, P(T_i)\, N(T_i)\, P(T_i)\, N(T_i)\, \sigma_{ij} \sigma_{jr} \sigma_{rs} \\ &= c(T_j, T_r)\, h_\lambda P(T_i)\, N(T_i)\, \sigma_{is}. \end{aligned}$$

The last step here uses the classical Young idempotency result; the latter may be easily derived from Lemma 1.4 (see [9] or [7]).

Theorem 1.2 has a most remarkable corollary.

THEOREM 1.3. *The group algebra elements* $\{E_{ij}^{\lambda}\}$ *for* $\lambda \vdash n$ *and* $i, j = 1, ..., n_\lambda$ *form a basis for* $A(S_n)$.

Proof. Let

$$f = \sum_{\lambda \vdash n} \sum_{i,j=1}^{n_\lambda} b_{ij}^{\lambda} E_{ij}^{\lambda};$$

then from Theorem 1.2 we get

$$fE_{rs}^{\mu} = h_\mu \sum_{i,j=1}^{n_\mu} b_{ij}^{\mu} c(T_r^\mu, T_j^\mu) E_{is}^{\mu}. \tag{1.14}$$

Taking coefficients of the identity (and using Lemma 1.5) gives

$$fE_{rs}^{\mu}|_{\varepsilon} = h_\mu \sum_{i,j=1}^{n_\mu} b_{ij}^{\mu} c(T_r^\mu, T_j^\mu)\, c(T_i^\mu, T_s^\mu).$$

Now, from Lemma 1.2 we see that, for each μ, the matrix of the coefficients $\|b_{ij}^{\mu}\|$ is related by a non-singular transformation to the matrix $\|fE_{rs}^{\mu}|_{\varepsilon}\|$. This shows that the E_{ij}^{λ} are independent. Now their number is

$$\sum_{\lambda \vdash n} n_\lambda^2 = n!, \tag{1.15}$$

which is the dimension of $A(G)$; thus they form a basis as asserted. The identity in (1.15) was proved by Young in [25] by an elementary recursive argument. This completes our proof.

We are finally in a position to complete the proof of Theorem 1.1.

To this end, note first that (1.14) implies that for each μ and any fixed $s = 1, ..., n_\mu$ the subspace E_s^μ spanned by the elements

$$E_{1,s}^{\mu}, E_{2,s}^{\mu}, ..., E_{n_\mu,s}^{\mu}$$

is invariant under left multiplication. Define then the matrix

$$A^\mu(\sigma) = \|a_{ij}^{\mu}(\sigma)\|$$

by the condition that

$$\sigma\langle E_{1,s}^{\mu}, ..., E_{n_\mu,s}^{\mu}\rangle = \langle E_{1,s}^{\mu}, ..., E_{n_\mu,s}^{\mu}\rangle A^\mu(\sigma).$$

That is, for all $j = 1, ..., n_\mu$ we have

$$\sigma E_{j,s}^{\mu} = \sum_{i=1}^{n_\mu} E_{i,s}^{\mu} a_{ij}^{\mu}(\sigma). \tag{1.16}$$

Now clearly, by its very definition the map

$$\sigma \to A^{\mu}(\sigma)$$

is a representation. On the other hand, equating the coefficients of the identity in (1.16) and using Lemma 1.5 we get

$$c(T_j^{\mu}, \sigma^{-1}T_s^{\mu}) = \sum_{i=1}^{n_{\mu}} c(T_i^{\mu}, T_s^{\mu})\, a_{ij}^{\mu}(\sigma).$$

We may write this in matrix notation as

$$C(\sigma^{-1}) = {}^{t}A^{\mu}(\sigma)\, C(\varepsilon),$$

from which we derive that

$$C(\sigma)\, C(\varepsilon)^{-1} = {}^{t}A^{\mu}(\sigma^{-1}).$$

This gives that the right-hand side of (1.3) is also a representation.

The irreducibility of this representation follows again from formula (1.14). Indeed, letting $B^{\mu} = \|b_{ij}^{\mu}\|$ we can rewrite (1.14) in matrix form as

$$f\langle E_{1,s}^{\mu}, \ldots, E_{n_{\mu},s}^{\mu}\rangle = h_{\mu}\langle E_{1,s}^{\mu}, \ldots, E_{n_{\mu},s}^{\mu}\rangle\, B^{\mu}\, {}^{t}C(\varepsilon).$$

Now the matrix B^{μ} can be completely arbitrary, and since ${}^{t}C(\varepsilon)$ is non-singular, their product can also be chosen at will. This implies that upon left multiplication by a suitable f we can send any non-zero element of E_s^{μ} into any other element. Thus E_s^{μ} has no non-trivial invariant subspaces.

To calculate the character of A^{λ} let us for convenience set

$$C(\varepsilon) = I - M.$$

Lemma 1.2 then gives that M is strictly lower triangular, and thus we get

$${}^{t}A^{\lambda}(\sigma^{-1}) = C(\sigma) + C(\sigma)\, M + C(\sigma)\, M^2 + \cdots \tag{1.17}$$

with only n^{λ} non-zero terms. Introducing then the group algebra matrices

$$A^{\lambda} = \sum_{\sigma} {}^{t}A^{\lambda}(\sigma^{-1})\, \sigma \qquad \text{and} \qquad E^{\lambda} = \|E_{ij}^{\lambda}\|$$

and using Lemma 1.5, we can write (1.17) in the form

$$A^{\lambda} = E^{\lambda} + E^{\lambda}M + E^{\lambda}M^2 + \cdots .$$

This gives

$$\text{trace } A^{\lambda} = E_{11}^{\lambda} + \cdots + E_{n_{\lambda}, n_{\lambda}}^{\lambda} + \sum_{i} \sum_{s>i} E_{is}^{\lambda}(M_{si} + M_{si}^2 + \cdots). \tag{1.18}$$

Now, it is easy to verify that Young's group algebra operator

$$\Gamma f = \frac{1}{n!} \sum_{\sigma} \sigma f \sigma^{-1}$$

preserves class functions and kills every E^{λ}_{is} when $i < s$. Applying it to (1.18) we finally derive that

$$\text{trace } A^{\lambda} = \sum_{\sigma} \sigma \frac{P(T^{\mu}_s) N(T^{\mu}_s)}{h_{\mu}} \sigma^{-1} \qquad \text{(for any } s\text{)}, \tag{1.19}$$

which is precisely Young's formula for the characters of S_n (see [21], [7], or [9]).

This completes our proof of Theorem 1.1.

It should be noted that the representation occuring in Theorem 1.1 is not exactly the natural in its original definition but rather the *inverse transpose* of the natural. Of course, these two representations have the same character.

Remark 1.1. It is worthwhile observing that the units

$$F^{\lambda}_{ij} = N(T^{\lambda}_i) \sigma^{\lambda}_{ij} P(T^{\lambda}_j) \tag{1.20}$$

have properties entirely analogous to the E^{λ}_{ij}. More precisely we have the following result.

THEOREM 1.4. *The group algebra elements* $\{F^{\lambda}_{ij}\}$ *for* $\lambda \vdash n$ *and* $i, j = 1, ..., n_{\lambda}$ *form a basis for* $A(S_n)$. *Moreover, for any* $\sigma \in S_n$ *and for any* $s = 1, ..., n_{\lambda}$ *we have*

$$\sigma \langle F^{\lambda}_{1,s}, F^{\lambda}_{2,s}, ..., F_{n_{\lambda},s} \rangle = \langle F^{\lambda}_{1,s}, F^{\lambda}_{2,s}, ..., F_{n_{\lambda}s} \rangle A(\sigma), \tag{1.21}$$

where

$$A(\sigma) = C^{-1}(\varepsilon) C(\sigma), \tag{1.22}$$

and thus it is also an irreducible representation of S_n *with character* χ^{λ} *in the Young indexing.*

This theorem is proved by imitating step by step what we did in the case of the E^{λ}_{ij}. Thus the proof will be omitted. It is interesting to note, though, that we shall use it in Section 5 to prove our triangularity result.

2. THE KAZHDAN–LUSZTIG REPRESENTATION

To conform with the notation in [13] we shall let W and S denote respectively the symmetric group S_n and the set of simple transpositions.

We shall also let s_i denote the transposition $(i, i+1)$. For a permutation w, the total number of inversions will be denoted by $l(w)$ and will be called the *length* of w. Its is customary to call an expression

$$w = a_1 a_2 \cdots a_k \qquad \text{with} \quad a_i \in S \ (i = 1, 2, ..., k)$$

reduced, if and only if $k = l(w)$. We recall that the Bruhat order on W (sometimes referred to as the *strong* order), denoted here by the familiar inequality sign $<$, is the transitive closure of the relation

$$x \to_B y \leftrightarrow \begin{cases} \text{a) } y = (i, j)\, x & \text{for some } i < j \\ \text{b) } l(y) > l(x). \end{cases}$$

In other words $x \to_B y$ means that x and y differ only by the transposition of two entries, which are in the right order in x and in the wrong order in y.

We shall make use of a number of properties of the Bruhat order that we must take for granted. The standard source is [2]; however, the reader may find the information needed here in [15]. The two basic facts we most often use are the *subword property* (SWP) and the *zigzag property* (ZZP). They may be stated as follows:

SWP. *For $x, y \in W$ we have $x < y$ if and only if for every (or for some) reduced expression*

$$y = a_1 a_2 \cdots a_m$$

we can find a reduced expression for x of the form

$$x = a_{i_1} a_{i_2} \cdots a_{i_k} \qquad \text{with} \quad 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m.$$

ZZP. *If for $x, w \in W$ and $s \in S$ we have $sw < w$ and $x < sx$ then any of the following three conditions implies the others.*

$$\begin{aligned} &\text{(a)} \quad x < w, \\ &\text{(b)} \quad x < sw, \\ &\text{(c)} \quad sx < w. \end{aligned} \tag{2.1}$$

The ZZP is an immediate consequence of the SWP, although it is not hard to prove directly from the definition of Bruhat order. Note also that from the SWP we can immediately derive that for any two elements $x, y \in W$ we have

$$x < y \leftrightarrow x^{-1} < y^{-1}. \tag{2.2}$$

This latter fact in particular implies that the ZZP holds as well for *right* multiplication by s. In fact, even when not explicitly mentioned, the derivation of a *right* result from a *left* result will be always tacitly assumed.

The construction of the Kazhdan–Lusztig representations depends on certain polynomials $P_{xw}(q)$ indexed by pairs $x, w \in W$. Certain properties of these polynomials are crucial in our developments. Unfortunately, explicit formulas for the $P_{xw}(q)$ have been given only for very special sets of pairs x, w, and even in these cases (see [18]) the properties needed here are difficult to extricate. So far most of what is known about the $P_{xw}(q)$, including tables (up to $n=6$), comes from various sets of recursions given in [13].

In order to make this paper to some degree self-contained, we will adopt as a starting point the simplest and most informative set of recursions we can extract from the contents of [13].

To begin we list three defining properties of the $P_{xw}(q)$:

(i) $P_{ww}(q) \equiv 1$,

(ii) $P_{xw}(q) \equiv 0$ when $x \not< w$,

(iii) degree $P_{xw}(q) \leqslant \frac{1}{2}(l(w) - l(x) - 1)$ (provided $x \neq w$).

Kazhdan and Lusztig set

$$\mu(x,w) = P_{xw}(q)\,|_{q^{(l(w)-l(x)-1)/2}}. \tag{2.3}$$

In other words, $\mu(x, w)$ gives the coefficient of the term of highest possible degree in $P_{xw}(q)$. Note that such a degree can only be achieved for $l(w) - l(x) = \text{odd}$. Thus we can state:

FACT 1.

$$\mu(x, w) = 0 \qquad \textit{when} \quad l(w) - l(x) \textit{ is even.} \tag{2.4}$$

This given, it is not difficult to derive from the original definition given in [13] that the polynomials $P_{xw}(q)$ can be constructed from the following algorithm. Assume we are in possession of P_{zy} whenever $l(y) < l(w)$ and whenever $l(y) - l(z) < l(w) - l(x)$. Then to construct $P_{xw}(q)$ we just find a simple reflection $s \in S$ such that $v = sw < w$ and set

$$P_{xw}(q) = \begin{cases} \text{(a) } P_{sx,w}(q) & \text{if } \quad sx > x \\ \text{(b) } P_{sx,v}(q) + qP_{x,v}(q) - \sum_{sz<z} q^{l(w)/2 - l(z)/2} P_{x,z}(q)\,\mu(z, v) & \\ & \text{if } \quad sx < x. \end{cases} \tag{2.5}$$

It is an important fact that every other choice of s which brings w down always delivers the same answer for $P_{xw}(q)$. We shall have to refer the reader to [13] for the proof of this as well as other properties of the $P_{xw}(q)$ that do not follow readily from (2.5). For the moment the recursions in (2.5) give us a simple and convenient way to derive most of what we need. In particular we can easily recover properties (i), (ii), and (iii) starting with the case $w =$ identity of (i) and proceeding inductively. The only delicate point in the proof of (iii) is to note that the leading term in the second member of (2.5b) is cancelled by the term coming from the sum for $z = x$.

Of course, as always, there is a parallel set of *right* recursions, namely, when $ws = v < w$:

$$P_{xw}(q) = \begin{cases} \text{(a) } P_{xs,w}(q) & \text{if } \ sx > x \\ \text{(b) } P_{xs,v}(q) + qP_{x,v}(q) - \sum_{zs<z} q^{l(w)/2 - l(z)/2} P_{x,z}(q)\,\mu(z,v) & \\ & \text{if } \ xs < x. \end{cases} \tag{2.6}$$

An important result can be immediately deduced from (2.5).

FACT 2.

$$\begin{aligned} sw < w \quad & \textit{and} \quad sx > x \to \mu(x,w) = 0 \quad \textit{or} \quad x = sw \\ ws < w \quad & \textit{and} \quad xs > x \to \mu(x,w) = 0 \quad \textit{or} \quad x = ws. \end{aligned} \tag{2.7}$$

Note that if $x \neq sw$ then the polynomial $P_{sx,w}$ has degree at most $(l(w) - l(sx) - 1)/2 = (l(w) - l(x) - 2)/2$. Combined with (2.5a) this implies that P_{xw} cannot have its highest degree and the first assertion follows. The other comes from (2.6a).

It is easy to see that (2.5) yields

$$P_{xw}(q)|_{q^0} = P_{xs,w}|_{q^0} \qquad \text{(for all } x < w).$$

Thus from (i) we get

$$P_{xw}(q)|_{q^0} = 1. \tag{2.8}$$

In other words we have

FACT 3. *The constant term in $P_{xw}(q)$ is always equal to* 1.

Note that because of (iii), $P_{xw}(q)$ must be of degree zero when $l(w) - l(x) = 1$; combining this with (2.8) we derive

FACT 4.

$$l(w) - l(x) = 1 \to \mu(x,w) = 1.$$

In particular for any $s \in S$ we have

$$\begin{aligned} &\text{(a)} \quad sx > x \to \mu(x, sx) = 1, \\ &\text{(b)} \quad xs > x \to \mu(x, xs) = 1. \end{aligned} \tag{2.9}$$

For convenience let use set

$$p_{xy} = P_{xy}(1), \tag{2.10}$$

and

$$c_w = \sum_{x \leqslant w} \varepsilon_x \varepsilon_w p_{xw} x. \tag{2.11}$$

The elements $\{c_w\}_{w \in W}$ constitute a most remarkable basis for $A(W)$ (the group algebra of S_n). Indeed, Eqs. (2.5) and (2.6) yield that the c_w's transform in a very beautiful and simple way upon multiplication by simple transpositions. This fact may be stated as follows:

THEOREM 2.1. *For any $w \in W$ and any $s \in S$ we have*

$$sc_w = \begin{cases} -c_w & (\textit{if } sw < w), \\ c_{sw} + c_w + \sum_{sz < z} \mu(z, w)\, c_z & (\textit{if } sw > w), \end{cases} \tag{2.12}$$

as well as

$$c_w s = \begin{cases} -c_w & (\textit{if } ws < w), \\ c_{ws} + c_w + \sum_{zs < z} \mu(z, w)\, c_z & (\textit{if } ws > w). \end{cases} \tag{2.13}$$

Before proceeding with the proof some remarks are in order. First of all we can easily see that the c_w's are a basis since the matrix $\|p_{xy}\|$ which relates them to the trivial basis $\{w\}_{w \in W}$ is upper triangular with unit diagonal elements (briefly, unitriangular). As a matter of fact it is shown in [13] that the inverse of the matrix $\|p_{xy}\|$ is given by the matrix $\|\varepsilon_x \varepsilon_y p_{w_0 y, w_0 x}\|$, where w_0 denotes the permutation

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}. \tag{2.14}$$

In this connection we should mention that the proof (see [13]) of this latter result yields also the following useful fact, which we wish to record for later use:

FACT 5. *For all $x \leqslant y$ in W we have*

$$\mu(x, y) = \mu(w_0 y, w_0 x) = \mu(y w_0, x w_0). \tag{2.15}$$

We should also mention that it was shown in [14] (by quite sophisticated machinery) that in our case ($W = S_n$) the coefficients of the $P_{xw}(q)$ are all non-negative. This implies that all our p_{xw} are greater than or equal to 1. Although this particular fact is not needed here, it would be desirable to have an elementary proof of it.

Finally, we note for later reference that the equations in (2.5) for $q=1$ reduce to

$$p_{xw}=\begin{cases}\text{(a) } p_{sx,w} & (\text{if } sx>x)\\ \text{(b) } p_{sx,v}+p_{x,v}-\sum_{sz<z} p_{x,z}\mu(z,v) & (\text{if } sx<x)\end{cases} \tag{2.16}$$

The parallel set of *right* equations can be derived from (2.6). This brings us to a further important property of the coefficients p_{xw}, namely:

FACT 6. *For all $x<w$ we have*

$$sw<w \to p_{sx,w}=p_{x,w}, \qquad ws<w \to p_{xs,w}=p_{x,w}. \tag{2.17}$$

This given, we can proceed with our treatment.

Proof of Theorem 2.1. Clearly, we need only establish (2.12). To this end, to be consistent with the notation in (2.5), again let $v=sw<w$. We can then write (since by the ZZP $x<w \leftrightarrow sx<w$)

$$c_w=\sum_{\substack{x<w\\ sx>x}} \varepsilon_x\varepsilon_w p_{x,w}(x-sx),$$

which gives

$$sc_w=\sum_{\substack{x>w\\ sx>x}} \varepsilon_x\varepsilon_w p_{x,w}(sx-x)=-c_w.$$

This establishes the first part of (2.12).

In the same vein, using (2.16b) we can write

$$c_w=\sum_{\substack{x\leqslant w\\ sx>x}} \varepsilon_x\varepsilon_w(x-sx)\left[p_{sx,v}+p_{x,v}-\sum_{sz<z} p_{x,z}\mu(z,v)\right]. \tag{2.18}$$

Note that Fact 6 yields that $p_{x,z}=p_{sx,z}$; thus we may rearrange (2.18) into

$$\begin{aligned} c_w=&-\sum_{\substack{x\leqslant v\\ sx>x}} \varepsilon_x\varepsilon_v(xp_{x,v}-sxp_{sx,v})\\ &+\sum_{\substack{x\leqslant v\\ sx>x}} \varepsilon_x\varepsilon_v(-xp_{sx,v}+sxp_{x,v})\\ &+\sum_{sz<z} \mu(z,v)\sum_{\substack{x\leqslant z\\ sx>x}} \varepsilon_x\varepsilon_v(xp_{x,z}-sxp_{sx,z}). \end{aligned}$$

Here we have again used the ZZP to replace the summation restriction $x \leqslant w$ by $x \leqslant v$ and $x \leqslant z$.

Now we note that Fact 1 implies that when $\mu(z, v) \neq 0$ we must have $\varepsilon_v = -\varepsilon_z$. We can thus simplify all this into

$$c_{sv} = -c_v + sc_v - \sum_{sz < z} \mu(z, v)\, c_z,$$

or better,

$$sc_v = c_{sv} + c_v + \sum_{sz < z} \mu(z, v)\, c_z,$$

which is the second equation in (2.12) written for v instead of w. This completes our proof.

Equations (2.13) can thought of as a recipe for the construction of the left regular representation matrices relative to the basis $\{c_w\}_{w \in W}$. It develops that these matrices have a remarkable block structure which follows quite closely the combinatorics of the Robinson–Schensted correspondence. To describe it we need to review some basic facts. Proofs and details about our assertations are unfortunately somewhat hard to reference since many of them have become part of combinatorial folklore. Good sources are [16, 22, 6], where some of the notation used here is given a more leisurely introduction.

Given a permutation σ we denote by $P(\sigma)$, $Q(\sigma)$, or simply P, Q, the left and right tableaux obtained by row inserting σ. Symbolically, we write

$$(\varnothing \leftarrow \sigma) = (P, Q). \tag{2.19}$$

Column insertion of σ will be denoted by the symbol $(\sigma \rightarrow \varnothing)$. It is a standard result that (2.19) is equivalent to either of the following two assertions,

$$(\varnothing \leftarrow \sigma^{-1}) = (Q, P), \tag{2.20}$$

$$(\sigma \rightarrow \varnothing) = (P, Q^{S}), \tag{2.21}$$

where the map $Q \rightarrow Q^{S}$ is the Schützenberger operation of *evacuation*.

It will be convenient to use the symbols $(P, |)$ and $(|, Q)$ to denote respectively the collections of permutations with left tableau equal to P and right tableau equal to Q. We shall refer to them here as *left* and *right Knuth classes*.

Given a tableau T, the word obtained by reading the entries of T row by row from left to right and from top to bottom will be referred to as *the word of T* and denoted by the symbol $w(T)$.

The standard tableau SS_I obtained by filling the rows the Ferrer's diagram of shape I with $1, 2, ..., n$ in succession from left to right and from bottom to top will be referred to as the *superstandard* tableau of that shape. The tableau $(SS_I)^S$ (obtained by evacuating SS_I) will be denoted by R_I and referred to as the *reading* tableau of shape I. We give below the superstandard and the reading tableaux of shape $I = (1\ 2\ 3\ 3)$.

$$SS_I = \begin{matrix} 9 & & \\ 7 & 8 & \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{matrix} \qquad R_I = \begin{matrix} 7 & & \\ 4 & 8 & \\ 2 & 5 & 9 \\ 1 & 3 & 6 \end{matrix} .$$

It is not difficult to see that a permutation σ is equal to the word of a tableau if and only if its right tableau under column insertion is superstandard. In view of the equivalence of (2.19) and (2.21) we get also that σ is the word of a tableau if and only if, under row insertion, its right tableau is a reading tableau.

This given, one way to obtain the evacuation T^S of a tableau T is to column insert the inverse of the word of T and take the right tableau. In symbols, for any tableau T of shape I, we have the following sequence of equivalent statements:

$$(\varnothing \leftarrow w(T)) = (T, R_I),$$

$$(\varnothing \leftarrow w^{-1}(T)) = (R_I, T),$$

$$(w^{-1}(T) \rightarrow \varnothing) = (R_I, T^S).$$

We recall that a descent of a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}$$

is an index i such that $\sigma_i > \sigma_{i+1}$. Clearly, an index i is a descent of σ^{-1} if and only if $i+1$ precedes i in σ. The descent sets of σ and σ^{-1} will be denoted by $D_R(\sigma)$ and $D_L(\sigma)$. We shall refer to them as the *right* and *left* descent sets, respectively. It is easy to see that we have

$$D_L(\sigma) = \{i\colon s_i\sigma < \sigma\} \qquad \text{and} \qquad D_R(\sigma) = \{i\colon \sigma s_i < \sigma\}.$$

This may explain the reason for this terminology.

For a standard tableau T, the index i is said to be a *descent* if $i+1$ is *north-west* (strictly N and weakly W) of i in T. The descent set of a tableau T will be denoted by $D(T)$. This may look a bit strange; however, it is easy to see that, under this definition, the descents of T are precisely the descents of $w^{-1}(T)$. Also consistent with this notation is the very important fact that if

$$(\varnothing \leftarrow \sigma) = (P, Q)$$

then

$$D_{\mathrm{L}}(\sigma) = D(P) \qquad \text{and} \qquad D_{\mathrm{R}}(\sigma) = D(Q). \tag{2.23}$$

The edges of the Hasse diagram of the Bruhat order on S_n are pairs of permutations related by the transposition of two letters i and j which are separated by entries all lying outside the interval which has i and j as its end points. These edges and the corresponding transpositions will be respectively called *Bruhat edges* and *Bruhat transpositions*. We give below a pair of Bruhat edges.

$$3415627 \leftrightarrow 3215647 \qquad (4 \text{ and } 2 \text{ separated by } 1, 5, 6),$$

$$4732615 \leftrightarrow 6732415 \qquad (4 \text{ and } 6 \text{ separated by } 7, 3, 2).$$

We are now ready to describe the block structure of the left regular representation matrices corresponding to (2.12). To write these equations in a more compact form, we set

$$\mu[x, y] = \begin{cases} \mu(x, y) & \text{if } \ x < y, \\ \mu(y, x) & \text{if } \ x > y. \end{cases} \tag{2.24}$$

In [13] Kazhdan and Lusztig introduce the graph with vertex set W and edges the pairs x, y for which $\mu[x, y] \neq 0$. We shall refer to it here as the *K–L graph* and denote its edge set by E.

Then let $L(s) = \|L_{xy}(s)\|$ denote the transformation matrix corresponding to (2.12). That is, $L_{xy}(s) = sc_y|_{c_x}$. We can describe its entries as follows:

$$L_{xx} = \begin{cases} -1 & \text{if } \ sx < x \\ \ \ 1 & \text{if } \ sx > x \end{cases}$$

and (2.25)

$$L_{xy} = \begin{cases} \mu[x, y] & \text{if } sx < x,\ sy > y, \text{ and } x, y \in E \\ 0 & \text{otherwise.} \end{cases}$$

It is part of the folklore of this topic to suspect that in our setting ($W = S_n$) the coefficients $\mu(x, y)$ take only the values 0 or 1. We shall refer to this as the 0, 1-*conjecture*. If this is true, the matrices $L(s)$ would then take an even simpler form, depending only on the K–L graph. Our numerical exploration confirms this possibility, although having checked it up to $n = 7$, we cannot consider it significant.

At any rate, we can use (2.25) and extend the definition of L to any $w \in W$ by taking a reduced expression

$$w = a_1 a_2 \cdots a_k$$

and setting

$$L(w) = L(a_1)\, L(a_2) \cdots L(a_k). \tag{2.26}$$

Note that a term c_x $(x \neq w)$ occurs in the expansion of an image of c_w upon left multiplication by a simple transposition if and only if

$$\begin{aligned} &\text{(a)} \quad D_{\mathrm{L}}(x) \not\subseteq D_{\mathrm{L}}(w) \\ &\text{(b)} \quad x, w \in E. \end{aligned} \tag{2.27}$$

This circumstance prompted Kazhdan and Lusztig to define the relation $w = L \Rightarrow x$ to mean that properties (2.27a) and (2.27b) hold for this pair. We can easily see then that an element x occurs in the expansion of an image of c_y upon left multiplication by a permutation w only if there is a sequence x_i, $i = 1, ..., k$ $(k \leqslant l(y))$ with

$$x = L \Rightarrow x_1 = L \Rightarrow x_2 = L \Rightarrow \cdots = L \Rightarrow x_k = y. \tag{2.28}$$

Let us say that there is an L-path joining x to y if (2.28) holds for some k. If L-paths did not loop we could make the matrices $L(w)$ upper triangular by ordering the elements of W in a manner that is compatible with the relation $= L \Rightarrow$. We can come close to that and make these matrices block triangular by grouping the permutations into cells and ordering the cells in a manner which is compatible with $= L \Rightarrow$. To this end Kazhdan and Lusztig define two elements x, y to be *L-equivalent* if they can be joined to each other in either direction by an L-path. The equivalence classes into which W decomposes are then called *left cells* or, briefly, *L-cells.*

The crucial circumstance that facilitates the ordering of these cells and indeed completely characterizes them is

FACT 7.

$$x = L \Rightarrow y \rightarrow D_{\mathrm{R}}(x) \supseteq D_{\mathrm{R}}(y). \tag{2.29}$$

Proof. If $x = sy > y$ and $yt < y$ then there is a reduced expression for x of the form $x = sa_1 \cdots a_k t$ which yields that $xt < x$. So the assertion is true in this case. On the other hand if $x < y$ then from Fact 2 we derive that the only way we can have $yt < y$, $xt > x$, and $\mu(x, y) \neq 0$ is that $x = yt$ but in that case we would have $D_{\mathrm{L}}(y) \supseteq D_{\mathrm{L}}(x)$, which plainly contradicts (2.27a).

We should mention that this result is proved in [13]. We included the proof here since it is so crucial for our developments and its proof is quite accessible.

Fact 7 implies in particular that permutations within the same L-cell have the same right descent set. Actually, much more than that is true. Indeed, what happens here is definitely one of the most remarkable and exciting combinatorial developments in the representation theory of S_n.

FACT 8. *The L-cells are none other than the right Knuth classes.*

It is not difficult to show that permutations with the same right tableau lie in the same L-cell. In fact, the sequence of *dual* Knuth transformations (see below, and [6]) joining one to the other gives precisely an L-path (in either direction). For, each step is a simple transposition relating two permutations having descent sets neither of which contains the other. This gives (2.27a) at each step. Fact 4 gives (2.27b). The difficult part is to prove the implication in the other direction. Since this result is of crucial importance in the proof of triangularity and the justification given in [13] is extremely brief, we shall have to give at least an indication of why it is true. In order not to unduly interrupt our flow here we shall postpone doing this till Section 5.

Accepting Fact 8 we see that we can label L-cells by standard tableaux, the label of the cell $(\,|, Q)$ being simply Q. It will also be convenient to refer to the shape of Q as *the shape of the cell.* We also see from Fact 7 that if there is an L-path from an element of cell $(\,|, Q_1)$ to an element of cell $(\,|, Q_2)$ then $D(Q_1) \supseteq D(Q_2)$. Thus the proper way of ordering L-cells is by decreasing descent sets of their labels.

To give an idea of what this ordering looks like we have depicted below the partial order of cells of S_4 by increasing descent sets (ties, whenever they occur, are to be broken by means of the relation $=L\Rightarrow$). The descent sets are given here immediately below the corresponding tableaux.

$$\begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \\ \{1,2,3\} \end{array}$$

$$\begin{array}{ll} 3 & \\ 2 & \\ 1 & 4 \\ \multicolumn{2}{c}{\{1,2\}} \end{array} \qquad \begin{array}{ll} 2 & 4 \\ 1 & 3 \end{array} \quad \begin{array}{ll} 4 & \\ 2 & \\ 1 & 3 \end{array} \;\; \{1,3\} \qquad \begin{array}{ll} 4 & \\ 3 & \\ 1 & 2 \\ \multicolumn{2}{c}{\{2,3\}} \end{array}$$

$$\begin{array}{lll} 2 & & \\ 1 & 3 & 4 \\ \multicolumn{3}{c}{\{1\}} \end{array} \qquad \begin{array}{ll} 3 & 4 \\ 1 & 2 \end{array} \quad \begin{array}{lll} 3 & & \\ 1 & 2 & 4 \end{array} \;\; \{2\} \qquad \begin{array}{lll} 4 & & \\ 1 & 2 & 3 \\ \multicolumn{3}{c}{\{3\}} \end{array}$$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \multicolumn{4}{c}{\varnothing} \end{array}$$

Accordingly we shall total order the tableaux (and their corresponding cells) as they come by, reading the rows from top to bottom and from left to right. Let $T_1, T_2, \ldots, T_{10}$ denote the tableaux in this order. That is,

$$T_1 = \begin{array}{l} 4 \\ 3 \\ 2 \\ 1 \end{array}, \quad T_2 = \begin{array}{ll} 4 & \\ 3 & \\ 1 & 2 \end{array}, \quad T_3 = \begin{array}{ll} 4 & \\ 2 & \\ 1 & 3 \end{array}, \quad T_4 = \begin{array}{ll} 2 & 4 \\ 1 & 3 \end{array}, \quad T_5 = \begin{array}{ll} 3 & \\ 2 & \\ 1 & 4 \end{array}, \ldots .$$

We see that the cardinalities of the corresponding cells are

$$1, 3, 2, 3, 3, 3, 2, 3, 3, 1.$$

The matrices $L(s) = \|L_{xy}(s)\|$ will all have this block structure. Moreover, since $L_{xy} \neq 0 \rightarrow x = L \Rightarrow y$ the subdiagonal blocks will all be zero.

However, there is a further surprise in store here. We have a natural map from cell $(|, Q_1)$, to cell $(|, Q_2)$, we simply map (P, Q_1) to (P, Q_2). Let us refer to this as the *change of label map*. It develops that

FACT 9. *Every change of label map preserves edges of the K–L graph. Moreover, these maps preserve $\mu[x, y]$ as well. That is, if x, y corresponds to x', y' under such a map then*

$$\mu[x, y] = \mu[x', y'].$$

This remarkable result is implicit in [13]. In Section 5 we shall derive it from basic identities given in [13].

The immediate consequence of this is that the subgraphs obtained by restricting the *K–L* graph to *L*-cells are isomorphic for cells of the same shape. Moreover, the matrices appearing in the corresponding diagonal blocks are identical. In particular, for the example above, the diagonal blocks corresponding to T_2, T_4, T_5 are the same. Similarly for T_6, T_8, T_9 and T_3, T_7.

This given, hereafter the diagonal block corresponding to a cell of shape λ will be denoted by B^λ and will be referred to as the Kazhdan–Lusztig representation indexed by λ. For a moment let ξ^λ denote the character of B^λ and let χ^λ denote the character of the Young natural A^λ. Fact 9 implies that

$$\text{trace } L = \sum_\lambda n_\lambda \xi^\lambda.$$

On the other hand since L is a realization of the left regular representation we must have

$$\text{trace } L = \sum_\mu n_\mu \chi^\mu.$$

Since $\{A^\mu\}_{\mu \vdash n}$ is a complete set of irreducibles we must have non-negative integers $p_{\lambda, \mu}$ such that

$$\xi^\lambda = \sum_\mu p_{\lambda, \mu} \chi^\mu. \tag{2.30}$$

Combining these last three equations gives the identities

$$n_\mu = \sum_\lambda n_\lambda p_{\lambda, \mu}.$$

Evaluating (2.30) at the identity gives

$$n_\lambda = \sum_\mu p_{\lambda, \mu} n_\mu.$$

These two equations give that the sum of the squares of the rows of the matrix $P = \|p_{\lambda, \mu}\|$ is less than or equal to 1. They also imply the same result for columns. Since trivially neither the row sums nor the column sums can be zero, the conclusion is that P must be a permutation matrix! In other words the representations $\{B^\lambda\}_{\lambda \vdash n}$ form a complete set of irreducibles and so do their set characters $\{\xi^\lambda\}_{\lambda \vdash n}$.

We might be tempted to refer to this as the *K–L labeling* of the characters of A_n; however, here again we have another surprise. Namely

FACT 10. *The K–L labeling and Young's labeling are identical. That is,*

$$\xi^\lambda = \chi^\lambda. \tag{2.31}$$

This will be a byproduct of the proof of the triangularity result.

Going back to the matrices B^λ we see from Fact 9 that they can be constructed from any of the subgraphs obtained by restricting the K–L graph to an L-cell of shape λ. The most interesting cell, as we shall see, is the one corresponding to what we call the *column superstandard* tableau of that shape. This is the tableau obtained by filling the shape with 1, 2, ..., n up the columns and from left to right. We shall denote this tableau by CS_I. For instance,

$$CS_{(1,3,3)} = \begin{matrix} 3 & & \\ 2 & 5 & 7 \\ 1 & 4 & 6 \end{matrix}$$

It is easy to see that for any P of shape I the permutation which row inserts to the pair (P, CS_I) is simply the word obtained by reading P down columns and from left to right. We shall denote this word by $w_c(P)$.

For any chosen cell of shape λ, the matrix B^λ is simply given by the formulas in (2.25) with both x and y restricted to vary in this cell.

As we see the construction of this set of irreducible representations is quite simple (at least for elementary transpositions). The only problem is: How do we get the graphs. Of course, if the 0, 1-conjecture is false then we should also want the $\mu[x, y]$'s. For the calculations up to $n=6$ we have used graphs that have been informally circulated. For these values of n the 0, 1-conjecture holds. All this can be verified also from the tables given in [10].

Our calculations for $n=7$ took a different turn since they have been based on the 0, 1-conjecture and on a conjecture of Lascoux and Schützenberger which we shall presently describe.

First of all we need to introduce the *K–L* classes R_j and L_i. These are essentially amalgamations of what used to be called *Knuth* and *dual Knuth* classes. More precisely, R_j consists of the permutations which in positions j, $j+1$, $j+2$ have the three entries $a<b<c$ in any one of the following patterns:

$$\begin{array}{c} \ldots\ldots\ b \quad a \quad c\ \ldots\ldots \\ \ldots\ldots\ b \quad c \quad a\ \ldots\ldots \\ \ldots\ldots\ a \quad c \quad b\ \ldots\ldots \\ \ldots\ldots\ c \quad a \quad b\ \ldots\ldots \end{array} \tag{2.32}$$

The class L_i has the *dual* definition. That is, L_i consists of the permutations in which the triplet i, $i+1$, $i+2$ occurs in any one of the following patterns:

$$\begin{array}{ccccccc} \ldots & i & \ldots & i+2 & \ldots & i+1 & \ldots \\ \ldots & i+1 & \ldots & i+2 & \ldots & i & \ldots \\ \ldots & i+1 & \ldots & i & \ldots & i+2 & \ldots \\ \ldots & i+2 & \ldots & i & \ldots & i+1 & \ldots \end{array} \tag{2.33}$$

K–L define involutions R_j and L_i acting on these classes which consist in interchanging a and c in the case of R_j and the left and right-most of i, $i+1$, $i+2$ in the case of L_i. These are, of course, the well-known *Knuth* [22] and *dual Knuth* transformations. So in this sense they are not new with the Kazhdan–Lusztig paper. However, formerly, in either case, the class defined by the first two patterns was considered separate from that defined by the second two. The contribution of the Kazhdan–Lusztig paper is that they should be collapsed into one in each case. In Section 5 we shall see the remarkable combinatorial significance of this.

And now we have one more surprise. It turns out that the transformations L_i and R_j send edges into edges! More precisely, we have

FACT 11. *If both x and y are in L_i then*

$$\mu[x, y] = \mu[L_i x, L_i y]. \tag{2.33}$$

Similarly if x and y $\in R_j$ then

$$\mu[x, y] = \mu[xR_j, yR_j]. \tag{2.34}$$

We let R_j act on the right for reasons that will appear clearer in Section 5. The proof of this result is based on the recurrences (2.5) and (2.6). It is elementary and somewhat tedious but it is given in full detail by Kazhdan and Lusztig in [13] (see also [15]). Thus there is no reason to repeat it here.

This given, we can produce edges from edges within an L-cell subgraph by repeated applications of the involutions L_i for various values of i whenever applicable. We may refer to this process as the *transport of edges*.

Lascoux and Schützenberger have ventured the following

Conjecture. Every edge is the transport of a (weak) Bruhat edge.

Here *weak* refers to edges produced by simple transpositions. This conjecture was comunicated verbally to us along with a description of the algorithm they have used to produce large K–L L-cell subgraphs. This algorithm consists in starting with the Hasse diagram of the (weak) Bruhat order on an L-cell and then performing edge transport until it remains stable.

All the graphs used in our calculations for $n = 7$ were obtained in this manner. After stabilization, the matrices corresponding to the elementary transpositions were constructed. The Coxeter relations were then checked and invariably they turned out to be satisfied. It also turned out that the corresponding representation was unitriangularly related to the natural in each case.

We should point out that Fact 11 implies in particular that the L–S conjecture implies the 0, 1-conjecture. So if the L–S algorithm is valid we only need the graphs to construct the matrices.

There are only three further properties we wish to mention before closing this section. First note that Fact 4 implies

FACT 12. *Each L-cell subgraph is an extension of the Hasse diagram of the Bruhat order on the L-cell.*

Note also that if $w > wt$ is a Bruhat edge and $sw < w$ for some $s \in S$ then either $swt = w$, which makes this a weak Bruhat edge, or the element $z = wt$ satisfies $sz < z$. In either case, we see that this edge is used in the construction of at least one representation matrix. That is we have $w = L \Rightarrow wt$ or $wt = L \Rightarrow w$.

Another shortcut in the construction of the L-cells subgraphs is given by

FACT 13. *Each L-cell subgraph has an involutory automorphism given by evacuation.*

Since evacuation preserves shape, the map

$$(P, Q) \to (P^S, Q)$$

defines an involution on the cell $(\,|\,, Q)$. The fact that it preserves edges is due to the fact that it can be decomposed in the sequence of maps

$$(P, Q) \to (P^S, Q^S) \to (P^S, Q),$$

each of which preserves edges. We note that the first is simply the map $x \to w_0 x w_0$. Thus the assertion in this case comes from Fact 5.

We close with the following rather pretty fact.

FACT 14. *For L-cells of hook shape the* K–L *graph reduces to the Hasse diagram of the weak Bruhat order on the cell. In particular it is isomorphic with that Hasse diagram of partitions contained in a rectangle.*

It is sufficient to show that $\mu(x, y) \neq 0 \to y = sx$ for some $s \in S$. To this end note that if P_1 and P_2 are any two standard tableaux of the same hook shape, then if they are different, there is an index i which is a descent for P_1 and not for P_2. For the descents of a hook tableau are the predecessors of the elements in the first column. Thus the fact that the only edges are Bruhat follows from Fact 2.

3. NUMERICAL DATA

We now turn our attention to the construction of the transformation matrices W^λ which relate the natural and the K–L representations of shape λ. These matrices are determined up to a constant multiplicative factor by the requirement that the relation

$$A^\lambda(\sigma)\, W^\lambda = W^\lambda B^\lambda(\sigma) \tag{3.1}$$

holds for all $\sigma \in S_n$. Equation (3.1) is a system of linear equations with rational coefficients for the entries of W^λ. As discussed in the Introduction, W^λ can be taken to be an integer matrix. If A^λ and B^λ are arbitrary equivalent representations, however, there is no reason for W^λ to have integral entries and determinant 1. Should this happen, we say that the representation A^λ and B^λ are equivalent over Z.

In order to explore the relation between the natural and the K–L representations we have computed the matrices W^{λ} for all partitions $\lambda \vdash n$ with $n \leqslant 7$ and for a few partitions of 8. As mentioned before, Kazhdan–Lusztig graphs were available to us only for $n \leqslant 6$. For $n = 7$ and 8 we constructed the representations B^{λ} from graphs produced by the Lascoux–Schützenberger algorithm.

For ease in programming, we did not obtain the matrices W^{λ} by solving the system of linear equations (3.1), which is highly redundant even if σ is restricted to vary only over the generators of S_n. Instead, we used an explicit formula for the W matrices which can be stated as follows:

THEOREM 3.1. *Let $A = \|A_{ij}\|$ and $B = \|B_{ij}\|$ be two irreducible representations of dimension m of a finite group G and let $W = \|W_{ij}\|$ be an invertible matrix which satisfies*

$$AW = WB \qquad (\textit{for all } \sigma \in G). \tag{3.2}$$

Then if the entries in the inverse of W are denoted W_{ij}^{-1}, we have

$$W_{ir}^{-1} W_{sj} = A_{ij} B_{sr} |_{\varepsilon}. \tag{3.3}$$

Proof. This equation is an immediate consequence of the following basic result from elementary representation theory:

LEMMA 3.1. *Let $B^{\lambda}(\sigma) = \|B_{ij}(\sigma)\|$ be any irreducible representation of a finite group G. Then the entries of B as elements of the group algebra satisfy the equations*

$$B_{xy} B_{sr} |_{\varepsilon} = h\chi(x = r)\, \chi(y = s),$$

where $h = |G|/m$ (this is the product of the hooks for the corresponding shape when $G = S_n$).

To get (3.3) we start with (3.2) written in the form

$$A_{ij}(\sigma) = \sum_{x,\, y} W_{ix}^{-1} B_{xy}(\sigma)\, W_{yj}.$$

Regarding this as an equation in the group algebra, we multiply both sides by B_{sr} and take the coefficient of the identity to get

$$A_{ij} B_{sr} |_{\varepsilon} = \sum_{x,\, y} W_{ix}^{-1} W_{yj} B_{xy} B_{sr} |_{\varepsilon}.$$

Lemma 3.1 let us write this as

$$A_{ij} B_{sr} |_{\varepsilon} = h W_{ir}^{-1} W_{sj},$$

which is our desired relation.

To compute the matrix $\|W_{sj}\|$, then, we select i and r such that $W_{ir}^{-1} \neq 0$. Since W^λ is determined only up to a constant factor, we may then assume that $h_\lambda W_{ir}^{-1} = 1$, so that

$$W_{sj} = \sum_{\sigma \in S_n} A_{ij}(\sigma)\, B_{sr}(\sigma^{-1}). \tag{3.4}$$

By systematically constructing the elements of S_n as products of elementary transpositions, one can generate the sum in (3.4) quite efficiently. In particular, only the matrices for the elementary transpositions need be stored. Some care in arranging calculations is, however, required to keep the labor manageable for large representations of large groups.

Of course, there is no good way to tell in advance which (i, r) will result in $W_{ir}^{-1} \neq 0$, though since W^λ has non-zero determinant, we may always choose $i = 1$. Improper selection of r will, however, result in the calculated matrix W^λ being identically zero. Should this happen, one simply selects another r and tries again. In fact, for all the examples we have considered it turned out that $W_{11}^{-1} \neq 0$. As an indication of the computational effort expended, calculating W^λ for a 35-dimensional representation of S_7 with our programs requires about 20 minutes of VAX time.

The computation for each shape λ is carried out as follows:

Step (1). Construct the natural matrices $A^\lambda(k, k+1)$ for every elementary transposition $(k, k+1)$; from a file containing the K–L graph for the shape λ, construct the K–L matrices $B^\lambda(k, k+1)$.

Step (2). Check that the Coxeter relations for S_n are satisfied by the generators from part (1). This guards against any errors in the calculation and also assures us that the graphs constructed for S_7 and S_8 by the Lascoux–Schützenberger algorithm actually give rise to representations. Since this is only a conjecture, Step (2) cannot be omitted.

Step (3). Use formula (3.4) with $i = r = 1$ to calculate the elements of W^λ. Check that the matrix so calculated in fact satisfies (3.1) for σ running over the generators of S_n. It is convenient to print both the raw matrix $\|W_{sj}\|$ obtained from (3.4) and the result of dividing the entries of this matrix by their gcd.

Both the A and B matrices depend upon the order in which we list the tableaux of shape λ which index the rows and columns of the matrices. We have done the calculations both with the first letter order and the last letter order on the tableaux (of course, the matrices with one order are transformed to those with the other by conjugating by the permutation matrix relating the two orders).

The transformation matrices W^λ for a few shapes using the last letter order are shown in Table I. A complete list is available from the authors.

The matrices printed are gotten by dividing the matrices produced from (3.4) by the gcd of their entries.

By scrutinizing all these matrices and those using the first letter order (which are in some ways not quite so nice) we are led to the following observations.

PROPERTY 1. *The transformation matrices W^{λ} are always upper triangular with all diagonal entries equal to* 1. *This holds in either the last letter order or the first letter order. Further, the integers appearing in these matrices are small.*

TABLE I

Transforming Matrix $W^{(1,2,3)}$

1	1	1	1	2	1	1	1	1	1	0	1	2	0	0	0
0	1	1	1	1	0	1	1	1	1	0	1	1	1	1	0
0	0	1	0	1	0	0	1	0	1	0	0	1	0	1	1
0	0	0	1	1	0	0	0	1	1	0	1	1	1	1	0
0	0	0	0	1	0	0	0	0	1	0	0	1	0	1	0
0	0	0	0	0	1	1	1	0	0	0	1	2	0	0	0
0	0	0	0	0	0	1	1	1	1	0	1	1	1	1	0
0	0	0	0	0	0	0	1	0	1	1	0	1	0	1	1
0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	0
0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

$$W^{(2,3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad W^{(1,1,3)} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$W^{(1,2,3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note. The top matrix is the transforming matrix $W^{(1,2,3)}$ defined by Eq. 3.1. The smaller matrices are the transforming matrices for the three shapes obtained by removing squares from the boundary of (1, 2, 3). They form the diagonal blocks of the big matrix. In computing these matrices, tableaux are listed in the last letter order.

Our initial hopes that the natural and the K–L representations might prove to be equivalent over Z are thus borne out in the most spectacular way possible. As we shall see in Section 5, this property turns out to be true in full generality. Moreover, our data led us to formulate an explicit construction of the transforming matrices in the hook case. This will be given in the next section. The evidence is strong that a construction of the same type as that given in Section 4 may actually be possible in general. This also suggests that some very close combinatorial tie must exist between the constructions of Young and of Kazhdan and Lusztig.

The matter of the size of the matrix entries provides a cautionary note to the practitioner of this sort of numerical experimentation. The entries in the matrices we have obtained range from -1 to 4. Given that these numbers are produced by summing over S_n, they are indeed quite small. Had we ended our numerical experimentation before we did, however, we would certainly have been led to make false conjectures. For all shapes $\lambda \vdash 1, 2, 3,$ or 4, W^λ is a 0–1 matrix. For the partitions of 5 and 6, the entries of W^λ are limited to 0, 1, and 2. It is only in S_7 that the first negative entry occurs. The wise numerical explorer is thus cautioned to continue calculating until a theorem is proved or until stopped by the accountants.

PROPERTY 2. *Let $\lambda_1^-, \lambda_2^-, ..., \lambda_m^-$ be the shapes obtained from λ by removing a square from the boundary, listed in decreasing order by dominance. In the last letter order, the matrix W^λ has a block decomposition in which the diagonal blocks are given by Eq.* 3.5.

$$W^\lambda = \begin{pmatrix} W^{\lambda_1^-} & & & * \\ & W^{\lambda_2^-} & & \\ & & W^{\lambda_3^-} & \\ 0 & & & \\ & & & W^{\lambda_m^-} \end{pmatrix}. \tag{3.5}$$

For example, Table I shows that

$$W^{(1,2,3)} = \begin{pmatrix} W^{(2,3)} & * & * \\ 0 & W^{(1,1,3)} & * \\ 0 & 0 & W^{(1,2,2)} \end{pmatrix}.$$

This observation holds only for the last letter order, not for the first letter order.

The reader who studies the example in Table I and our other computational results will discern numerous tantalizing bits of patterns in the off-diagonal blocks of the W^λ. For hook shapes, we discuss these patterns in Section 4, where we present a simple recursive description of the transformation matrices. For general shapes, we do not yet have simple descriptions of the off-diagonal blocks.

PROPERTY 3. *When the matrix* $W^\lambda = \|W_{sj}\|$ *is constructed using* (3.4) *with* $i = r = 1$, *the* gcd *of the entries is equal to* h_λ, *the product of the hooks of shape* λ.

This observation was made by Luc Favreau (personal communication) upon perusal of our tables. We do not know whether Property 3 represents a mere curiosity of our algorithm for W^λ or a deep fact about these matrices.

That the natural and the K–L representations appear to lie in the same equivalence class over Z is in itself a surprising fact. It reminds one of the earlier surprise that the Young and Frobenius labellings of the characters of S_n, though arising from very different analyses, proved for deep reasons to be identical. The very close relations between the natual and K–L representations suggested by the properties above seem to point to some beautiful combinatorial connection.

4. THE TRANSFORMING MATRICES FOR HOOK SHAPES

In the case that λ is a hook shape $\lambda = (1^j, n-j)$, the matrix W^λ can be constructed by a simple combinatorial procedure. This will suffice to prove all the properties stated in the previous section in the case of hook shapes. Although we shall prove in Section 5 that Property 1 holds for all shapes λ, the explicit recursions obtained for the transforming matrices of hook shapes are interesting in their own right. We therefore follow the pattern of our own discovery and discuss hooks in some detail before moving on to prove upper triangularity in general.

We begin by introducing some notation. If $\lambda = (1^j, n-j) \vdash n$ is a hook, we let $\lambda_- = (1^j, n-j-1) \vdash n-1$ and $\lambda^- = (1^{j-1}, n-j) \vdash n-1$. In other words, λ_- is the hook obtained by removing one square from the bottom row of λ, and λ^- is the hook obtained by removing the single square in the top row of λ. With this notation, we can describe the transforming matrices for hooks.

THEOREM 4.1. *If* λ *is a hook shape and if the representations* A^λ *and* B^λ *are constructed using the last letter order on the tableaux, then the transforming matrix* W^λ *is given in block form by the recurrence*

$$W^\lambda = \begin{pmatrix} W^{\lambda^-} & U^\lambda \\ 0 & W^{\lambda_-} \end{pmatrix}. \tag{4.1}$$

The off-diagonal block U^λ is in turn given by the recurrence

$$U^\lambda = \begin{pmatrix} 0 & 0 \\ W^{\lambda=} & U^{\lambda-} \end{pmatrix}. \tag{4.2}$$

If A^λ and B^λ are constructed using the first letter order, then W^λ and U^λ are given by the recurrences

$$W^\lambda = \begin{pmatrix} W^{\lambda-} & U^\lambda \\ 0 & W^{\lambda-} \end{pmatrix} \tag{4.3}$$

$$U^\lambda = \begin{pmatrix} U^{\lambda-} & 0 \\ W^{\lambda=} & 0 \end{pmatrix}. \tag{4.4}$$

Before proving this result, we state some immediate corollaries.

COROLLARY 4.1. *The matrices W^λ for hook shapes are all* 0–1 *matrices.*

This follows at once from the recurrences and the fact that the transforming matrices for row or column shapes are 1-dimensional identity matrices.

COROLLARY 4.2. *In either the first letter or the last letter order we have the equations*

$$W^{(1^{n-2},2)} = \begin{bmatrix} 1 & 1 & 0 & & & 0 \\ & 1 & 1 & 0 & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & 1 & 0 \\ & 0 & & & 1 & 1 \\ & & & & & 1 \end{bmatrix} \tag{4.5}$$

$$W^{(1,n-1)} = \begin{bmatrix} 1 & 1 & 1 & & & 1 \\ & 1 & 1 & 1 & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & 1 & 1 \\ & 0 & & & 1 & 1 \\ & & & & & 1 \end{bmatrix}. \tag{4.6}$$

Again, this is an easy consequence of the recurrences above. It is also not difficult to prove by direct calculation.

Finally, we remark that if we apply the recurrence in Eq. (4.1) to the matrix $W^{\lambda-}$ which occupies the lower right corner of W^λ, we see that the non-zero portion of U^λ consists exactly of the top two blocks of $W^{\lambda-}$. A similar construction produces U^λ in the first letter order.

The proof of Theorem 4.1 is a straightforward but tedious inductive calculation. We decompose the matrices $A^{\lambda}(k, k+1)$ and $B^{\lambda}(k, k+1)$ into blocks of the same size as those in (4.1)–(4.4), calculate the products AW and WB, and verify that they are equal. Unfortunately, several special cases need to be considered separately, complicating the argument. We give the proof only for the case of the last letter order, leaving to the interested reader the onerous task of repeating the calculations for the first letter order.

We begin by giving a block decomposition for the matrices $C^{\lambda}(k, k+1)$, recalling that for hooks we have

$$A^{\lambda}(k, k+1) = {}^{t}[C^{\lambda}(k, k+1)].$$

In the case that $k \neq 1$ or $n-1$, the intersection diagram used in producing the matrix $C^{\lambda}(k, k+1)$ is shown schematically in Fig. 1. The tableaux with n at the top precede those with n at the bottom in the last letter order. Since the permutation $(k, k+1)$ leaves both 1 and n invariant, the positions of those numbers in the schematic tableaux $(k, k+1)\, T_j$ are as shown in the figure. The upper left and lower right blocks in the matrix are obtained by realizing that the intersections in those blocks are the same as would be obtained if the largest entry, n, were removed from the tableaux. The off-diagonal blocks are zero because even the entries 1 and n suffice to show that the intersections in those blocks will not be tableaux.

An identical analysis gives the decomposition for $C^{\lambda}(n-1, n)$ shown in Fig. 2. The obvious notation used there is that I^{μ} is the identity matrix of size dim A^{μ}.

The decomposition of $C^{\lambda}(1, 2)$ is slightly more complicated. The rough picture is shown in Fig. 3. The three blocks on and above the diagonal are

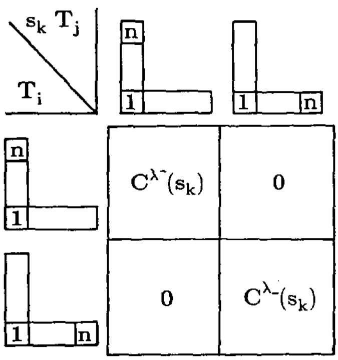


FIG. 1. Schematic intersection diagram for computing $C^{\lambda}(k, k+1)$ in the case $k \neq 1$ or $n-1$.

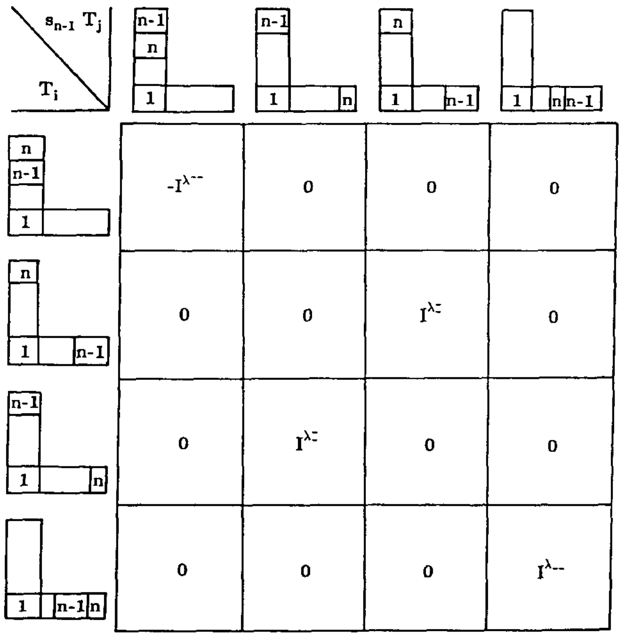


FIG. 2. Schematic intersection diagram for computing $C^{\lambda}(n-1, n)$.

obtained exactly as in Fig. 1. Now, however, that the lower left block is not forced to be identically zero. This block M^{λ}, however, can itself be decomposed into blocks as shown in Fig. 4. It is trivial that the off-diagonal blocks in Fig. 4 are zero. The form of the diagonal blocks arises by comparing the intersection tableaux with those obtained when $n-1$ is removed from both T_i and $(1, 2)\, T_j$. The minus sign in the upper left block occurs

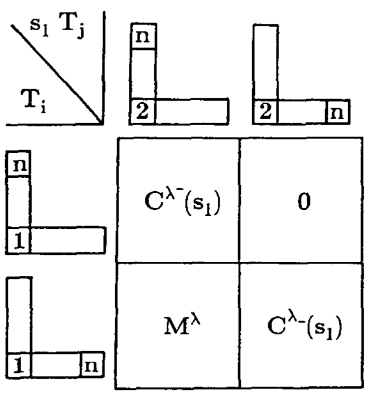


FIG. 3. Schematic intersection diagram for computing $C^{\lambda}(1, 2)$.

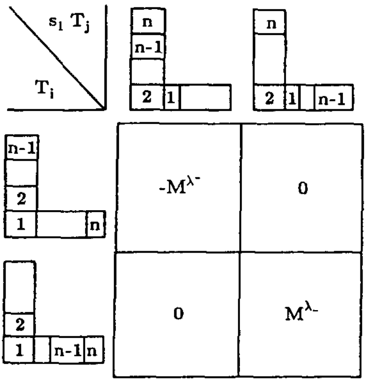


FIG. 4. Schematic intersection diagram for computing the subdiagonal block M^λ of $C^\lambda(1, 2)$. Only pairs of tableaux with 2 in the positions shown can intersect to form a tableau.

because removing $n-1$ changes the length of the single cycle comprising β in that block.

The matrices $B^\lambda(k, k+1)$ have similar block decompositions. The typical case, which arises when $k \neq n-2$ or n, is shown in Fig. 5. The reasoning producing the Figure is as follows. By Fact 14, an element $B^\lambda_{ij}(k, k+1)$ not lying on the diagonal of $B^\lambda(k, k+1)$ can be non-zero only if T_i and T_j are related by an adjacent transposition. For the pair (i, j) lying in one of the off-diagonal blocks, this transposition must be $(n-1, n)$. Further, such a $B^\lambda_{ij}(k, k+1)$ can be non-zero only if k is a descent of T_j but not of T_i. This can occur only if either k or $k+1$ is moved by $(n-1, n)$, which cannot happen if $k \neq n-2$ or $n-1$. The off-diagonal blocks are therefore identically zero. On the other hand, if (i, j) lies in one of the diagonal blocks, then the procedure for evaluating $B^\lambda_{ij}(k, k+1)$ in no way depends upon the

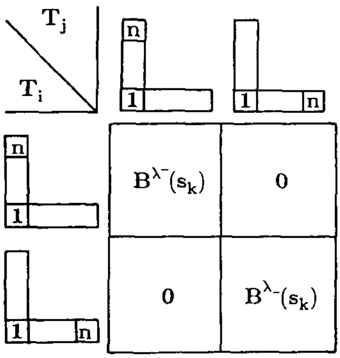


FIG. 5. Schematic diagram for computing $B^\lambda(k, k+1)$ in the case $k \neq n-2$ or $n-1$.

presence of n in the top or bottom row of both T_i and T_j, which means that the diagonal blocks have the form shown.

The forms of the matrices $B^{\lambda}(n-2, n-1)$ and $B^{\lambda}(n-1, n)$ are shown in Figs. 6 and 7, respectively. That these figures are correct is an easy exercise.

Given all these block decompositions, we can finally present the proof that for all k and λ, $A^{\lambda}(k, k+1)\, W^{\lambda} = W^{\lambda} B^{\lambda}(k, k+1)$. When only moderate ambiguity arises, we shall drop all superscripts and arguments to write this as $AW = WB$. It is easy to see that this relation holds whenever λ is a row shape or a column shape (that is, whenever $\lambda = n$ or 1^n). We assume that it holds for all shapes $\lambda \vdash m$ with $m < n$. The inductive argument now breaks up into four cases.

Case 1. $k \neq 1$, $n-2$, or $n-1$. Then by Figs. 1 and 5 we have

$$A = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

and

$$B = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$

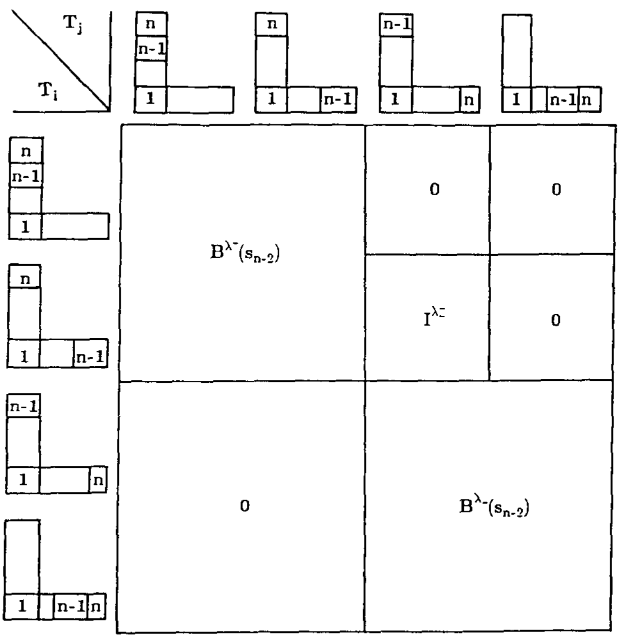


FIG. 6. Schematic diagram for computing $B^{\lambda}(n-2, n-1)$.

For convenience we have omitted the superscripts λ_- and λ^- in the equations above. Now if we let

$$W = \begin{pmatrix} W & U \\ 0 & W \end{pmatrix},$$

we can write

$$AW = \begin{pmatrix} AW & AU \\ 0 & AW \end{pmatrix}, \qquad WB = \begin{pmatrix} WB & UB \\ 0 & WB \end{pmatrix}.$$

We must show that these matrices are equal. Since the diagonal blocks in AW and WB are equal by induction, this reduces to showing that the upper right blocks are equal. The most direct way to do this is to decompose all our matrices one step further:

$$A = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix}, \qquad W = \begin{pmatrix} W & U & 0 & 0 \\ 0 & W & W & U \\ 0 & 0 & W & U \\ 0 & 0 & 0 & W \end{pmatrix},$$

$$B = \begin{pmatrix} B & J & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & J \\ 0 & 0 & 0 & B \end{pmatrix}.$$

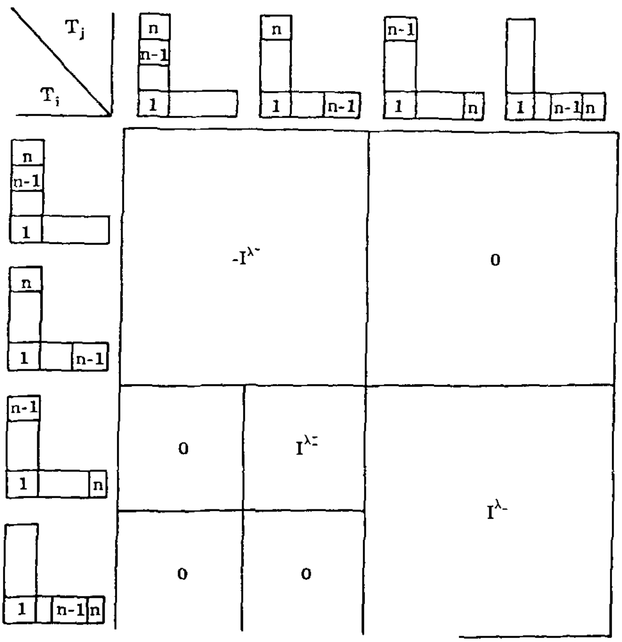


FIG. 7. Schematic diagram for computing $B^\lambda(n-1, n)$.

Here the matrix J (for "junk") is zero unless $k=n-2$, in which case it contains some non-zero elements whose nature need not concern us. In terms of these blocks we can then write

$$AW=\begin{pmatrix} AW & AU & 0 & 0 \\ 0 & AW & AW & AU \\ 0 & 0 & AW & AU \\ 0 & 0 & 0 & AW \end{pmatrix},$$

$$WB=\begin{pmatrix} WB & WJ+UB & 0 & 0 \\ 0 & WB & WB & WJ+UB \\ 0 & 0 & WB & WJ+UB \\ 0 & 0 & 0 & WB \end{pmatrix}.$$

The two non-zero blocks in the upper right corners of these matrices are equal to the blocks immediately below them. Since these blocks lie in the portions of the two matrices which have already been shown to be equal, it follows at once that the upper right corners of the two matrices are equal as well, so that $AW=WB$.

Case 2. $k=n-1$. Figures 2 and 7 then imply that

$$A=\begin{pmatrix} -I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \qquad W=\begin{pmatrix} W & U & 0 & 0 \\ 0 & W & W & U \\ 0 & 0 & W & U \\ 0 & 0 & 0 & W \end{pmatrix},$$

$$B=\begin{pmatrix} -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & I & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Direct multiplication now gives

$$AW=\begin{pmatrix} -W & -U & 0 & 0 \\ 0 & W & W & U \\ 0 & W & W & U \\ 0 & 0 & 0 & 0 \end{pmatrix}=WB.$$

Case 3. $k=n-2$. This is just like Case 2 except that the matrices must be expanded slightly farther:

$$A=\begin{bmatrix} -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$W=\begin{bmatrix} W & U & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & W & W & U & 0 & 0 & 0 & 0 \\ 0 & 0 & W & U & W & U & 0 & 0 \\ 0 & 0 & 0 & W & 0 & W & W & U \\ 0 & 0 & 0 & 0 & W & U & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & W & W & U \\ 0 & 0 & 0 & 0 & 0 & 0 & W & U \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & W \end{bmatrix},$$

$$B=\begin{bmatrix} -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & I & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

Again, direct multiplication yields

$$AW=\begin{bmatrix} -W & -U & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & W & U & W & U & 0 & 0 \\ 0 & W & W & U & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W & 0 & W & W & U \\ 0 & 0 & 0 & 0 & -W & -U & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & W & U \\ 0 & 0 & 0 & 0 & 0 & W & W & U \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & W \end{bmatrix} = WB.$$

Case 4. $k=1$. Referring to Figs. 3 and 5, and letting $N={}^tM$, we get the decompositions

$$A=\begin{pmatrix} A & N \\ 0 & A \end{pmatrix}, \quad W=\begin{pmatrix} W & U \\ 0 & W \end{pmatrix}, \quad B=\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$

As a consequence,

$$AW=\begin{pmatrix} AW & AU+NW \\ 0 & AW \end{pmatrix} \quad \text{and} \quad WB=\begin{pmatrix} WB & UB \\ 0 & WB \end{pmatrix}.$$

The diagonal blocks of AW and WB are equal by induction. All that remains is to show that $AU+NW=UB$. As usual, this is done inductively by block decomposing all the matrices involved. Figures 3, 4, and 5 tell us that

$$A=\begin{pmatrix} A & N \\ 0 & A \end{pmatrix}, \quad N=\begin{pmatrix} -N & 0 \\ 0 & N \end{pmatrix}, \quad B=\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix},$$

and

$$U=\begin{pmatrix} 0 & 0 \\ W & U \end{pmatrix}, \quad W=\begin{pmatrix} W & U \\ 0 & W \end{pmatrix}.$$

We therefore have

$$UB=\begin{pmatrix} 0 & 0 \\ WB & UB \end{pmatrix}$$

and

$$AU+NW=\begin{pmatrix} 0 & 0 \\ AW & AU+NW \end{pmatrix},$$

so that $AU+NW=UB$ by induction.

This completes the fourth case, and with it the proof of the theorem. It is easy to see that the validity of the theorem for the pure row and column shapes $\lambda=n$ and $\lambda=1^n$, combined, if you like, with the explicit calculations in Section 3 for all $n\leqslant 7$, provides a basis for all these inductions.

Although the recurrences in Theorem 4.1 describe the transforming matrices rather neatly, one cannot help but feel that the proof we have given is not exactly illuminating. To generalize it to arbitrary shapes also does not seem too promising. We therefore turn in Section 5 to describing the K–L representation and the transforming matrices in more algebraic terms. This description leads not only to a proof of the upper triangularity of the transforming matrices for arbitrary shapes, but also to a new set of recurrences describing them.

5. Triangularity

The developments in this section depend crucially on Facts 8, 9, and 10. Unfortunately, the derivation of these facts from the few terse paragraphs dedicated to them in [13] is not an effortless matter. Fact 10 (the equality of Young's and K–L character labeling), for instance, is not even stated there, although readers familiar with the subject might suspect this particular result to be true.

We have personally benefited considerably from a sequence of seminar lectures given by A. Björner, who systematically filled in the combinatorics pertaining to the $W = S_n$ case of the Kazhdan–Lusztig work. Notes from these seminars are in preparation [1]. The readers should find there a more leisurely and detailed derivation of all the material needed here.

Nevertheless, pending the appearance of these notes, the reader is left with quite a bit of work to do for a thorough understanding of the present work. This given, it will be good to give here at least brief outlines of proofs. Before we can proceed, however, we need to go over a few more properties of the Robinson–Schensted correspondence.

Let $w = w_1 w_2 \cdots w_n$ be a permutation and let for a moment P_k be the left tableau resulting from the row insertion of $w_1 w_2 \cdots w_k$. Set

$$w^{(k)} = w(P_k)\, w_{k+1} \cdots w_n$$

In other words $w^{(k)}$ is the concatenation of the word of P_k with what is left of w. It is well known that the step

$$w^{(k)} \rightarrow w^{(k+1)}$$

can be carried out by a sequence of Knuth transformations [16, 22]. Thus, assuming that

$$(\varnothing \leftarrow w) = (P, Q), \tag{5.1}$$

and that P and Q have shape I, we shall always have a sequence of indices $j_1, j_2, ..., j_s$ such that

$$w(P) = (P, R_I) = wR_{j_1} R_{j_2} \cdots R_{j_s}. \tag{5.2}$$

It will be convenient to identify a permutation with the pair of tableaux obtained by row insertion. Thus we shall write $w = (P, Q)$ instead of (5.1) or even write (P, Q) for w. We see then from (5.2) that given any triplet P, Q, Q' of standard tableaux of the same shape, we can always find a sequence of Knuth transformations connecting (P, Q) to (P, Q').

Let us recall that the *column superstandard* tableau of shape I, denoted by CS_I, is the tableau obtained by filling I with $1, 2, ..., n$ column by column from bottom to top and from left to right. A sequence $R_{j_1} R_{j_2} \cdots R_{j_m}$ giving

$$(P, CS_I) = (P, Q)\, R_{j_1} R_{j_2} \cdots R_{j_m} \tag{5.3}$$

will be referred to as *canonical for* (P, Q).

The tableau CS_I is of particular interest to us here since it has a number of peculiar properties. We shall state them in the form of propositions.

PROPOSITION 5.1. *If T is any standard tableau of shape J and*

$$D(T) \supseteq D(CS_I) \tag{5.4}$$

then we can have only one of the following two possibilities:

(i) *$T = CS_I$ and of course $J = I$, or*

(ii) *$T \neq CS_I$ but then J is strictly below I in the dominance order.*

Proof. Let $c_1, c_2, ..., c_k$ be the lengths of the successive columns of I. It develops that the tableaux T of shape J satisfying the requirement in (5.4) are in bijection with the *row strict* tableaux of shape J and content $1^{c_1} 2^{c_2} \cdots k^{c_k}$. These are the tableaux obtained by filling the shape J with c_1 ones, c_2 twos, etc., in succession so that no two equal letters fall in the same row. In fact, given any of the latter tableaux we can obtain one of our desired T's by simply replacing the 1's from bottom to top, then the 2's in the same manner, etc., by the successive letters $1, 2, ..., n$. Now it is easy to see that if $J = I$, the resulting row strict tableau is simply the one obtained by filling the ith column of I with i's whose bijective image is CS_I itself. On the other hand any other of our row strict tableaux with the desired property must have a shape J with enough room in its first i columns to contain the $c_1 + c_2 + \cdots + c_i$ letters less or equal to i. This implies that J must be dominated by I, which is our desired conclusion.

This proof establishes also another important fact for us here, namely:

PROPOSITION 5.2. *The number of standard tableaux T of shape J satisfying the condition*

$$D(T) \supseteq D(CS_I)$$

is equal to the Kotska number

$$K_{\tilde{I}, \tilde{J}}.$$

Here the symbol "$\sim$" refers to conjugation. The definition of Kotska numbers may be found in [20].

Given a tableau T the word obtained by reading the columns of T from top to bottom starting from the first column and proceeding to the right will be denoted by $cw(T)$ and referred to as the *column word* of T. In the figure below we illustrate the column superstandard tableau of shape (2, 2, 3), a standard tableau of the same shape, and its corresponding column word.

$$CS_{(2,3,3)} = \begin{matrix} 3 & 6 & \\ 2 & 5 & \\ 1 & 4 & 7 \end{matrix} \qquad T = \begin{matrix} 5 & 7 & \\ 3 & 6 & \\ 1 & 2 & 4 \end{matrix} \qquad \mathrm{cw}(T) = 5\ \ 3\ \ 1\ \ 7\ \ 6\ \ 2\ \ 4$$

The reader is invited to check that

$$(\varnothing \leftarrow 5\ \ 3\ \ 1\ \ 7\ \ 6\ \ 2\ \ 4) = \left(\begin{matrix} 5 & 7 & \\ 3 & 6 & \\ 1 & 2 & 4 \end{matrix}\ ,\ \begin{matrix} 3 & 6 & \\ 2 & 5 & \\ 1 & 4 & 7 \end{matrix} \right).$$

It should be clear that this is a general fact. That is, for any standard tableau T of shape I we have

$$(\varnothing \leftarrow \mathrm{cw}(T)) = (T, CS_I). \tag{5.5}$$

We are now a position to outline a proof of Facts 8, 9, and 10. It develops that the following Proposition simplifies considerably the derivation of Facts 8 and 9. The reader interested in the combinatorics of the Robinson–Schensted correspondence should find it a very welcome addition to the early tools introduced in the subject.

Note that the classes L_i and R_j can be defined entirely in terms of the *left* (resp. *right*) tableau of a permutation. More precisely, we see that L_i consists of the permutations whose left descent set contains one and only one of the two indices i, $i+1$. In other words we have

$$L_i = \{w = (P, Q): |D(P) \cap \{i, i+1\}| = 1\}. \tag{5.6}$$

Similarly we see that

$$R_j = \{w = (P, Q): |D(Q) \cap \{j, j+1\}| = 1\}. \tag{5.7}$$

Now, it is well known (see [16]) that the map R_j preserves the left tableau. From Schützenberger's result (Eq. (2.20)) and the identity

$$(wR_j)^{-1} = L_j w^{-1}$$

we can easily derive that likewise the map L_i preserves the right tableau. This means that both L_i and R_j map the set $L_i \cap R_j$ into itself. In fact, they are both involutions on this set. The remarkable fact is that they commute. More precisely we have

PROPOSITION 5.3. *For any i, j and any permutation $w \in L_i \cap R_j$ we have*

$$L_i(wR_j) = (L_i w)\, R_j. \tag{5.8}$$

Proof. It is convenient to give separate arguments according to the cardinality of the set of entries of w that are affected by these two transformations. To this end, using the same notation as in Section 2, let us put

$$k = |\{i, i+1, i+2\} \cup \{a, b, c\}|.$$

When $k = 6$ the result is completely trivial since L_i and R_j acts on different sets of entries and (5.1) is simply a consequence of associativity of permutation multiplication. The case $k = 5$ is again a consequence of associativity as the reader can easily discover by simple experimentation. The main reason is that, because the indices i, $i+1$, $i+2$ are consecutive, even if one of them is equal to a, b, or c, the action of L_i does not alter the action of R_j. More precisely, the simple transpositions expressing these actions by left (resp. right) multiplication remain the same.

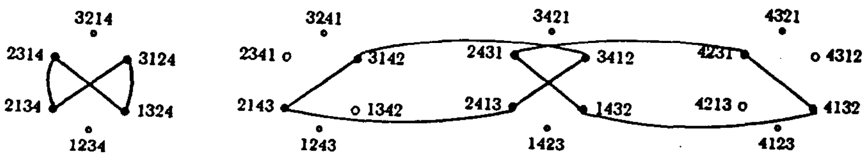
The case $k \leqslant 4$ is a little less obvious. We can shorten the argument considerably by making a few preliminary observations. First note that, since at most four entries are involved in this case, we can delete all the other entries and reduce w to a four-element permutation. We thus need only verify (5.8) in S_4. Second, because of the identities

$$\begin{cases} (1)\ w_0(wR_j) = (w_0 w)\, R_j \\ (2)\ (wR_j)\, w_0 = (ww_0)\, R_{n-j-1} \end{cases} \qquad \begin{cases} (1)\ w_0(L_i w) = L_{n-i-1}(w_0 w) \\ (2)\ (L_i w)\, w_0 = L_i(ww_0) \end{cases}$$

we can immediately see that once the identity (5.8) has been established for all i, j for a permutation w it will automatically hold for $w_0 w$, ww_0, and $w_0 ww_0$ as well (where w_0 is given by (2.14)). This observation permits us to assure that

$$\{i, i+1, i+2\} = \{j, j+1, j+2\} = \{1, 2, 3\}.$$

In the figure below we display the permutations of S_4 grouped according to the left cosets of the subgroup generated by (1, 2), (2, 3). Solid circles indicate the elements of $L_1 \cap R_1$. The straight and curved lines are to respectively represent the action of L_1 and R_1 on this set.



We see clearly from this picture that successive applications of L_1, R_1, L_1, R_1 always loop back to the point of departure. But this is precisely the result we want to establish.

This proposition can be translated into a beautiful result which entirely reveals the nature of the transformations L_i and R_j. To this end let us define a transformation acting directly on standard tableaux by setting

$$m_i T = \begin{cases} (i, i+1)\, T & \text{if } i+2 \text{ separates } i, i+1 \text{ in } \mathrm{cw}(T), \\ (i+1, i+2)\, T & \text{if } i \text{ separates } i+1, i+2 \text{ in } \mathrm{cw}(T). \end{cases} \tag{5.9}$$

We see that m_i is an involution on the class

$$T_i = \{T : |D(T) \cap \{i, i+1\}| = 1\}.$$

This given we have

THEOREM 5.1. *If* $w = (P, Q)$ *then*

$$\begin{aligned} &\text{(a)} \quad (\textit{when } w \in L_i)\; L_i w = (m_i P, Q) \\ &\text{(b)} \quad (\textit{when } w \in R_j)\; wR_j = (P, m_j Q). \end{aligned} \tag{5.10}$$

Proof. Let $w \in L_i$ and let P and Q be of shape I. Note that we have defined m_i in order to have

$$L_i \mathrm{cw}(P) = (m_i P, CS_I). \tag{5.11}$$

Let us now take a canonical sequence for (P, Q) giving

$$\mathrm{cw}(P) = wR_{j_1} R_{j_2} \cdots R_{j_m}.$$

Applying our commutativity result we then get

$$L_i \mathrm{cw}(P) = L_i(wR_{j_1} R_{j_2} \cdots R_{j_m}) = (L_i w)\, R_{j_1} R_{j_2} \cdots R_{j_m}.$$

Since Knuth transformations preserve the left tableau, we must conclude that the left tableaux of $L_i \mathrm{cw}(P)$ and $L_i w$ are the same. This proves (5.10a).

To prove (5.10b) we again use the relation

$$(wR_j)^{-1} = L_j w^{-1}$$

and get from (5.10a)

$$wR_j = [L_j w^{-1}]^{-1} = [(m_j Q, P)]^{-1} = (P, m_j Q).$$

This completes our proof.

Now let $w_1=(P_1, Q)$ and $w_2=(P_2, Q)$ be two permutations of shape I and in the same dual Knuth cell, and let Q' be any standard tableau of shape I. As observed above, we can find a sequence $R_{j1}R_{j2}\cdots R_{jk}$ giving

$$(P_1, Q')=(P_1, Q)\,R_{j1}R_{j2}\cdots R_{jk}.$$

From Theorem 5.1 we then deduce that

$$Q'=m_{jk}\cdots m_{j2}m_{j1}Q,$$

which in turn (again from Theorem 5.1) implies that

$$(P_2, Q')=(P_2, Q)\,R_{j1}R_{j2}\cdots R_{jk}.$$

This means that the change of label map corresponding to the replacement $Q\to Q'$ can be carried out by a sequence of elementary change of label maps of the form

$$(P, Q)\to(P, m_jQ).$$

In particular we see that the equality

$$\mu[x, y]=\mu[x', y'],$$

when x, y and x', y' correspond under a change of label map, is a consequence of the equality

$$\mu[x, y]=\mu[xR_j, yR_j],$$

In other words Theorem 5.1 reduces Fact 9 to Fact 11.

Remark 1.1. Fact 8 can also be reduced to Fact 11 in a similar manner. Some of the algebraic steps needed in this reduction, including a very leisurely proof of Fact 11, can be found in lecture notes by R. King [15]. The combinatorial steps were first written up by A. Björner. Note that Theorem 5.1 essentially says that the effect on the left (resp. right) tableau of the transformation L_i (resp. R_j) does not depend on the right (resp. left) tableau. The fact that the Kazhdan–Lusztig paper leads to this remarkable property of the Robinson–Schensted correspondence was first pointed out by Björner in the above-mentioned writeup. Our contribution here is a precise formulation of Proposition 5.1 and the reduction of the proof of Fact 8 to Theorem 5.1 and Proposition 5.1. We give here only our steps and refer the reader to [1, 15] for further details.

Let $w_1=(P_1, Q_1)$ and $w_2=(P_2, Q_2)$ be two permutations of shapes I and J. We are to show that if w_1 and w_2 are L-equivalent (as defined in

Section 2) then $I=J$ and $Q_1=Q_2$. To this end the first step is to show that if $x, y \in R_j$ then

$$x=L\Rightarrow y \rightarrow xR_j=L\Rightarrow yR_j.$$

The proof of this can be found in [13, 1, 15]. It is a consequence of Fact 11.

From this it is not difficult to derive that if x, y are L-equivalent and $x \in R_j$ then xR_j, yR_j are also L-equivalent. The fact that $x \in R_j \rightarrow y \in R_j$ is a consequence of Fact 7. This given, let $R_{j_1} R_{j_2} \cdots R_{j_k}$ be a canonical sequence for w_1. This gives

$$(P_1, CS_I)=(P_1, Q_1)\, R_{j_1} R_{j_2} \cdots R_{j_k}$$

and by Theorem 5.1

$$CS_I = m_{j_k} \cdots m_{j_2} m_{j_1} Q_1$$

or equivalently

$$Q_1 = m_{j_1} m_{j_2} \cdots m_{j_k} CS_I.$$

From the remarks above, and Theorem 5.1 again, it then follows that (P_1, CS_I) and $(P_2, m_1 m_2 \cdots m_k Q_2)$ are also L-equivalent. But as observed in Section 2, this implies that these two permutations have the same right descent set. This means that

$$D(CS_I)=D(m_1 m_2 \cdots m_k Q_2).$$

However, Proposition 5.1 now tells us either that $I=J$ and

$$m_{j_1} m_{j_2} \cdots m_{j_k} Q_2 = CS_I$$

and thus

$$Q_2 = m_{j_k} \cdots m_{j_2} m_{j_1} CS_I = Q_1 \tag{5.12}$$

or that I strictly dominates J. Since we have assumed nothing special about our labeling of w_1 and w_2, the denial of the first alternative, and reversal of the roles of w_1, w_2, would also give that J strictly dominates I, which is absurd. Thus (5.12) is the only valid alternative and Fact 8 must necessarily hold true as asserted.

To establish Fact 10 and the triangularity result we need some preliminary remarks about the element c_w defined by (2.11).

Let I be a set of simple transpositions and let W_I be the subgroup of W generated by the elements of I. Set

$$V_I = \{v \colon vs > v \text{ for all } s \in I\}.$$

It is well known (see [2] or [23]) that the elements of V_I are the minimal elements of their left W_I-cosets. They may be taken as coset representatives, that is, every element $w \in W$ has a unique factorization

(i) $w = vu$ with $v \in V_I$ and $u \in W_I$.

Moreover, if we denote these factors by $v_I(w)$ and $u_I(w)$, respectively, we have

(ii) $l(w) = l(v_1(w)) + l(u_I(w))$,

(iii) $w_1 \leqslant w_2 \to v_I(w_1) \leqslant v_I(w_2)$.

Now we let $w \in W$ be given and set

$$I = \{s \in S\colon ws < w\} = \{s_i\colon i \in D_{\mathrm{R}}(w)\}. \tag{5.13}$$

It develops from properties (i) and (iii) that the set of elements below w decomposes into the product of the portion of V_I below w by all of W_I. More precisely we have

PROPOSITION 5.4. *If $w \in W$ and I is as in (5.13) then an element x is below w if and only if it can be written in the form*

$$x = vu \qquad \text{with} \quad v \in V_I,\ v \leqslant v_I(w) \qquad \text{and} \qquad u \in W_I. \tag{5.14}$$

Moreover this decomposition is unique.

Proof. The *only if* part is immediate from (i) and (iii) and we know that uniqueness holds for every element of W. To prove the *if* part let

$$x = vu \qquad \text{with} \quad u \in W_I,\ v \in V_I \qquad \text{and} \qquad v \leqslant w$$

and let $u = a_1 a_2 \cdots a_k$ be a reduced decomposition of u with $a_i \in I$ for $i = 1, 2, ..., k$. Set

$$x_0 = v \qquad \text{and} \qquad x_i = va_1 \cdots a_i.$$

Now since $a_1 \cdots a_i$ is also reduced, uniqueness of factorization and property (ii) give that

$$l(x_i) = l(v) + i.$$

Thus $x_{i+1} = x_i a_{i+1} > x_i$. Now we see that, since $wa_{i+1} < w$, then $x_i < w$ and the ZZP imply that $x_{i+1} \leqslant w$. Note that we cannot have $x_{i+1} = w$ except maybe when $i + 1 = k$, for w is always brought down by any of the a_i's. Thus since $x_0 = v < w$, all the x_i's must be below w. In particular the same must be true for $x_k = x$ itself. This completes the proof.

Note that this proposition implies that $u_I(w)$ is the top element of W_I. The immediate corollary of this proposition is a factorization of every element c_w.

PROPOSITION 5.5. *Let $w \in W$ and I be as in* (5.13), *then*

$$c_w = Q_w W'_I, \tag{5.15}$$

where

$$Q_w = \varepsilon_w \sum_{\substack{v \in V_I \\ v \leqslant v_I(w)}} \varepsilon_v p_{v,w} v \qquad \textit{and} \qquad W'_I = \sum_{u \in W_I} \varepsilon_u u. \tag{5.16}$$

Proof. Proposition 5.4 gives that

$$c_w = \sum_{\substack{v \in V_I \\ v \leqslant v_I(w)}} \sum_{u \in W_I} \varepsilon_w \varepsilon_v \varepsilon_u p_{vu,w} vu$$

and (5.15), (5.16) follow immediately from the fact that

$$ws < w \rightarrow p_{ys,w} = p_{y,w} \qquad \text{(for all } y\text{)},$$

which is Fact 6.

Now let μ be a fixed partition of n and let

$$T_1, T_2, ..., T_{n_\mu}$$

here and in the rest of this section denote the standard tableaux of shape μ labelled in the order of increasing *column words.* Set

$$w_i = (T_i, T_1).$$

It is easy to see that $T_1 = CS_\mu$, and thus we must have

$$w_i = \text{cw}(T_i).$$

Specializing w to w_i in Proposition 5.5 and setting

$$I = \{s_i : i \in D(CS_\mu)\} \tag{5.17}$$

we immediately get the factorizations

$$c_{w_1} = Q_{w_1} N(T_1) \qquad \text{for} \quad i = 1, 2, ..., n_\mu. \tag{5.18}$$

The identification of W'_I with $N(T_1)$ (as defined in Section 1) is due to the fact that, when the right tableau Q of w is superstandard, the group W_I is identical with the column group of Q.

The identity in (5.18) has numerous consequences. In particular Fact 10 follows immediately from it. Indeed, note that for w_1 the factorization (5.16) reduces to

$$c_{w_1} = \varepsilon_{w_1} N(T_1) \tag{5.19}$$

This is because when I is as in (5.17) the representative $v_I(w)$ of any permutation w is simply obtained by rearranging in increasing order the elements of w which lie within unbroken strings of descents of w_1. An example will best get across what we are trying to say here. Let $\mu = (2, 3, 3)$. Then

$$w_1 = \mathrm{cw}(CS_\mu) = 3\ 2\ 1 \quad 6\ 5\ 4 \quad 8\ 7,$$

where we have separated by spaces the *unbroken strings of descents* of w_1. Now given a permutation $w \in S_8$, to construct $v_I(w)$ we simply rearrange in increasing order the elements of w occupying positions 1, 2, 3, then those occupying positions 4, 5, 6, and finally those in positions 7, 8. Thus

$$v_I(8\ 3\ 2 \quad 6\ 1\ 7 \quad 4\ 5) = 2\ 3\ 8 \quad 1\ 6\ 7 \quad 4\ 5.$$

Using this fact on w_1 immediately gives that

$$v_I(w_1) = \text{the identity permutation.}$$

Thus the sum in Q_{w_1} reduces to a single term and (5.19) follows.

Remark 5.2. We should point out that, from the remarks above, we can easily see that $v_I(w_i)$ is none other than the word obtained by reading T_i column from bottom to top rather than from top to bottom as is done to get w_i. It is also easy to derive (from property (iii)) that

$$w_i \leqslant_B w_j \leftrightarrow v_I(w_i) \leqslant_B v_I(w_j). \tag{5.20}$$

Let us now set, for any $f \in A(S_n)$

$$L_{xy}(f) = fc_y|_{c_x}.$$

Clearly, c_x is in the expansion of fc_y only if there is an L-path joining x to y. In view of Fact 7 we deduce that $L_{xy}(f) \neq 0$ implies that $D_{\mathrm{R}}(x) \supseteq D_{\mathrm{R}}(y)$. This gives that

$$A(S_n)\, c_w \subseteq L[c_x : D_{\mathrm{R}}(x) \supseteq D_{\mathrm{R}}(y)].$$

L here stands for *linear span*. Thus the character of the representation corresponding to the action on the left ideal $A(S_n)\, c_w$ is dominated by the sum of the characters of the representations B^λ corresponding to cells indexed by tableaux Q whose descent set contains the right descent set of w. That is,

$$\operatorname{char} A(S_n)\, c_w \ll \sum_\lambda \xi^\lambda \,\#\, \{Q\colon D(Q) \supseteq D_{\mathrm{R}}(w) \text{ and } \operatorname{shape}(Q) = \lambda\}. \tag{5.21}$$

Here *domination* and "$\ll$" mean that corresponding coefficients of the irreducible characters are individually related by the same inequality.

This given, we see that the dimension of $A(S_n)\, c_w$ as a vector subspace of $A(S_n)$ satisfies the inequality

$$\dim A(S_n)\, c_w \leqslant \sum_\lambda n_\lambda \,\#\, \{Q\colon D(Q) \supseteq D_{\mathrm{R}}(w) \text{ and } \operatorname{shape}(Q) = \lambda\}. \tag{5.22}$$

If we now choose $w = w_1$, formula (5.19) gives us another way of calculating this dimension. Indeed, it is well known (see [7] or [9] for a simple proof), that the character of the representation corresponding to left action on the left ideal of an idempotent $N(T)$, for any tableau T of shape μ, has the expansion

$$\operatorname{char} A(S_n)\, N(T) = \sum_\lambda K_{\bar{\lambda}, \bar{\mu}} \chi^\lambda, \tag{5.23}$$

where here the χ^λ is the irreducible character indexed by λ in the Young labeling (the character given by formula (1.19)).

Evaluating (5.23) at the identity gives

$$\dim A(S_n)\, N(T) = \sum_\lambda K_{\bar{\lambda}, \bar{\mu}} n_\lambda. \tag{5.24}$$

On the other hand, with this choice of w we have

$$D_{\mathrm{R}}(w) = D(CS_\mu).$$

Thus we can use Proposition 5.2 and get that

$$\#\, \{Q\colon D(Q) \supseteq D_{\mathrm{R}}(w) \text{ and } \operatorname{shape}(Q) = \lambda\} = K_{\bar{\lambda}, \bar{\mu}}.$$

Substituting this in (5.22) yields

$$\dim A(S_n)\, c_w \leqslant \sum_\lambda K_{\bar{\lambda}, \bar{\mu}} n_\lambda.$$

Comparing with (5.24) we see that in this case (5.22), and thus (5.21) as well, must be equalities. Equating the right-hand sides of (5.21) and (5.23) yields the equation

$$\sum_\lambda K_{\bar{\lambda}, \bar{\mu}} \xi^\lambda = \sum_\lambda K_{\bar{\lambda}, \bar{\mu}} \chi^\lambda.$$

Now it is well know that the matrix $\|K_{\tilde{\lambda},\tilde{\mu}}\|$ is invertible. This also follows from our proof of Proposition 5.2 since it gives that

$$K_{\tilde{\lambda},\tilde{\mu}} \neq 0 \rightarrow \lambda \leqslant \mu. \tag{5.25}$$

Moreover, it is easily seen that all the diagonal elements are equal to one.

We must then conclude that

$$\xi^{\mu} = \chi^{\mu} \qquad \text{(for all } \mu\text{)}.$$

This establishes Fact 10.

To proceed with the proof of triangularity we need two auxiliary facts. The first is purely algebraic:

PROPOSITION 5.6. *Let $\{A^{\lambda}\}$ be a complete set of irreducible representations of a finite group G and let $\{\chi^{\lambda}\}$ be the corresponding characters. Let θ be an arbitrary element of the group algebra $A(G)$. Then the character of the representation induced by the action of G on the left ideal $A(G)\,\theta$ is given by the formula*

$$\text{char } A(G)\,\theta = \sum_{\lambda} \text{rank } A^{\lambda}(\theta)\,\chi^{\lambda}. \tag{5.26}$$

The proof is quite straightforward and will be omitted.

The second fact is purely combinatorial; to state it we need some notation. Given a tableau T, let $v(T)$ denote the word obtained by reading T column by column from *bottom to top* (not from top to bottom as we do for $\text{cw}(T)$) and let $c(T_1, T_2)$ be as defined in Section 1.

PROPOSITION 5.7. *If T_1 and T_2 have the same shape and T_1 is standard then*

$$c(T_1, T_2) \neq 0 \rightarrow v(T_1) \leqslant_B v(T_2). \tag{5.27}$$

A proof is given in [8] and will not be repeated here. We should point out that originally Proposition 5.7 came out of efforts (see [8]) to develop a purely combinatorial treatment of skew representations. Curiously, it turns out that it now gives a crucial step in the proof of the triangularity result.

Going back to our tableaux $T_1, T_2, \ldots, T_{n_\mu}$ we note that when I is as defined in (5.17) we have that

$$v_I(w_i) = v(T_i). \tag{5.28}$$

Thus (5.27) combined with our convention on labeling the T_i's and Remark 5.2 in particular implies that the matrix

$$C(\varepsilon) = \|c(T_i, T_j)\|$$

is upper triangular. Again let σ_{ij} denote the permutation that sends T_j into T_i and set as before

$$F_{ij} = N(T_i)\,\sigma_{ij} P(T_j).$$

As observed in Section 1 (Remark 1.1) we have

$$\sigma\langle F_{1,s}, F_{2,s}, ..., F_{n_\mu,s}\rangle = \langle F_{1,s}, F_{2,s}, ..., F_{n_\mu,s}\rangle\, A^\mu(\sigma) \qquad (5.29)$$

with

$$A^\mu(\sigma) = C^{-1}(\varepsilon)\, C(\sigma) \qquad \text{and} \qquad C(\sigma) = \|c(T_i, \sigma T_j)\|.$$

Let us now set

$$f_j = c_{w_j} P(T_1) = Q_{w_j} F_{1,1}. \qquad (5.30)$$

It develops that these units afford the Kazhdan–Lusztig representation B^μ. More precisely, setting

$$B^\mu_{ij} = L_{w_i, w_j}, \qquad B^\mu = \|B^\mu_{ij}\|$$

we have

THEOREM 5.2.

$$\sigma\langle f_1, f_2, ..., f_{n_\mu}\rangle = \langle f_1, f_2, ..., f_{n_\mu}\rangle\, B^\mu(\sigma). \qquad (5.31)$$

Proof. Note that we can write

$$\sigma c_{w_j} = \sum_{i=1}^{n_\mu} c_{w_i} B^\mu_{ij}(\sigma) + \sum\nolimits^{(*)} c_x L_{x, w_j}(\sigma),$$

where the ($*$) is to indicate that this summation is to be carried out only over permutations x which lie outside the L-cell of w_1. Multiplying both sides of this equation by $P(T_1)$ gives

$$\sigma f_j = \sum_{i=1}^{n_\mu} f_i L^\mu_{ij}(\sigma) + \sum\nolimits^{(*)} c_x P(T_1)\, L_{x, w_j}(\sigma).$$

It is thus clear that to show (5.31) we need only establish that the second summation vanishes identically. For convenience, let us extend the meaning of the symbol "$x = L \Rightarrow y$" to mean that there is an L-path from x to y. This given, the following result, interesting in its own merits, is all that is needed to complete the proof of the theorem.

PROPOSITION 5.8. *If x is not in the L-cell of w_1 and $x = L \Rightarrow w_1$ then*

$$c_x P(T_1) = 0.$$

We break up the proof into three lemmas.

LEMMA 5.1. *Under the hypotheses of Proposition* 5.8, *a term involving* c_y *can occur in the expansion of an element* fc_x $(f \in A(S_n))$ *only if the shape of* y *is strictly dominated by* μ.

Proof. If such a term does occur then $y = L \Rightarrow x$. This combined with the hypothesis $x = L \Rightarrow w_1$ gives (Fact 7)

$$D_{\mathrm{R}}(y) \supseteq D(CS_\mu). \tag{5.32}$$

Proposition 5.1 can thus be applied to the right tableau of y. However, we cannot have the first alternative here. For if the right tableau of y is CS_μ then y and w_1 are in the same L-cell and we would then have all three relations

$$x = L \Rightarrow w_1, \qquad w_1 = L \Rightarrow y, \qquad y = L \Rightarrow x,$$

which would imply that x and w_1 are in the same L-cell, contrary to our hypotheses. The second alternative gives that the shape of y is strictly dominated by μ as desired.

Let A^λ still denote Young's natural representation corresponding to shape λ.

LEMMA 5.2. *Under the hypotheses of Proposition* 5.8, *we have*

$$\operatorname{rank} A^\lambda(c_x) \leqslant \begin{cases} K_{\tilde{\lambda}, \tilde{\mu}} & \text{if } \lambda < \mu \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From Lemma 5.1 and (5.32) we derive that

$$A(S_n)\, c_x \subseteq L(c_y\colon \text{shape } y < \mu \text{ and } D_{\mathrm{R}}(y) \supseteq D(CS_\mu)).$$

This implies that the character of the representation corresponding to the action of S_n on the left ideal $A(S_n)\, c_x$ is dominated by the sum of the characters of the representations B^λ coming from cells indexed by standard tableaux Q satisfying the two conditions

$$\text{shape } Q < \mu \qquad \text{and} \qquad D(Q) \supseteq D(CS_\mu).$$

In other words, using Proposition 5.2 and Fact 10, we must have

$$\operatorname{char} A(S_n)\, c_x \ll \sum_{\lambda < \mu} K_{\tilde{\lambda}, \tilde{\mu}} \chi^\lambda,$$

where again, χ^λ denotes the character of A^λ. Comparing with (5.26) we see that the desired conclusion follows from Proposition 5.6.

The final step is given by

LEMMA 5.3. *For any tableau T of shape μ and any λ we have*

$$\operatorname{rank} A^{\lambda}(P(T)) = \begin{cases} K_{\lambda,\mu} & \text{if } \lambda \geqslant \mu \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows immediately from Proposition 5.6 and the identity

$$\operatorname{char} A(S_n)\, P(T) = \sum_{\lambda \geqslant \mu} K_{\lambda,\mu} \chi^{\lambda}$$

which is well known and is usually referred to as *Young's Rule* (see [20]). It is also essentially another version of formula (5.23).

We can now easily complete the proof of Proposition 5.8. Indeed, the combination of Lemmas 5.2 and 5.3 yields that for all λ

$$A^{\lambda}(c_x P(T_1)) = 0. \tag{5.33}$$

This is because in order for this matrix not to vanish both matrices

$$A^{\lambda}(c_x) \qquad \text{and} \qquad A^{\lambda}(P(T_1))$$

must have ranks different from zero. However, for the first (by Lemma 5.2) this happens only when $\lambda < \mu$ and for the second (by Lemma 5.3) only when $\lambda \geqslant \mu$. Thus (5.33) must hold true as asserted. But, since the $\{A^{\lambda}\}$ form a complete system of irreducibles for S_n, the vanishing of all the matrices in (5.33) implies the vanishing of $c_x P(T_1)$ itself. This completes the proof of Proposition 5.8 and Theorem 5.2.

Note now that (5.29) (for $s = 1$) gives

$$\sigma F_{11} = \sum_{i=1}^{n_\mu} F_{i,1} A^{\mu}_{i,1}(\sigma).$$

Thus, if we extend by linearity the definition of A^{μ} to all of $A(S_n)$, formula (5.30) in terms of the natural representation becomes

$$f_j = \sum_{i=1}^{n_\mu} F_{i,1} A^{\mu}_{i,1}(Q_{w_j})$$

or in matrix form

$$\langle f_1, f_2, ..., f_{n_\mu} \rangle = \langle F_{1,1}, F_{2,1}, ..., F_{n_\mu,1} \rangle \| A^{\mu}_{i,1}(Q_{w_j}) \|. \tag{5.34}$$

Equations (5.29), (5.31), and (5.34) give us a matrix transforming the Kazhdan–Lusztig representation into Young's natural, and a key to the triangularity result, namely:

THEOREM 5.3. *The matrix* $W = \|w_{ij}\|$ *with coefficients*

$$w_{ij} = A^{\mu}_{i1}(Q_{w_j}) = \varepsilon_{w_j} \sum_{\substack{v \in V_I \\ v \leqslant v_I(w_j)}} \varepsilon_v p_{v, w_j} A^{\mu}_{i, 1}(v). \tag{5.35}$$

is the solution of the equation

$$A^{\mu}(\sigma) = WB^{\mu}(\sigma) W^{-1}. \tag{5.36}$$

Moreover, W is upper triangular with unit diagonal elements.

Proof. Eliminating the f_i's and the F_{i1}'s from (5.29), (5.31), and (5.34) we get (5.35) with

$$w_{ij} = A^{\mu}_{i1}(Q_{w_j}).$$

The second expression in (5.35) follows from (5.16) by linearity. To prove the final assertion, suppose $w_{ij} \neq 0$. Then from (5.35) it follows that there must be a $v \leqslant v_I(w_j)$ such that $A^{\mu}_{i1}(v) \neq 0$. But since

$$A^{\mu}_{i1}(v) = \sum_{s=1}^{n_\mu} C^{-1}_{is}(\varepsilon)\, C_{s1}(v) \tag{5.37}$$

there must be an s for which we have both

(a) $C^{-1}_{is}(\varepsilon) \neq 0$ and
(b) $C_{s1}(v) \neq 0$.

But (a) combined with the upper triangularity of $C(\varepsilon)$, implies $i \leqslant s$ while (b) means that

$$c(T_s, vT_1) \neq 0,$$

and this in turn (by Proposition 5.7) implies that

$$v(T_s) \leqslant_B v(vT_1).$$

Now we easily see that $v(vT_1) = v$, and since we have $v \leqslant_B v_I(w_j) = v(T_j)$ we finally get that

$$v(T_s) \leqslant_B v(T_j).$$

However, (5.20) and (5.28) combined with our convention on the labeling of the tableaux T_k give us then that $s \leqslant j$. Since we already had $i \leqslant s$, we deduce that $i \leqslant j$ as desired.

To complete the proof we need to find out what happens when $i = j$. Going through the above string of inequalities, we see that we must have $i = s = j$ as well, and thus also

$$v(T_i) \leqslant v \leqslant v(T_i),$$

which forces $v = v_I(w_i)$ to be the only contributor to the sum in (5.35). But for this term we have

$$p_{v,\, w_i} = p_{w_i,\, w_i} = 1.$$

Moreover, since the term with $s = i$ is the only contributor to the sum in (5.37) we derive that

$$A^{\mu}_{i1}(v) = A^{\mu}_{i1}(v(T_i)) = C^{-1}_{ii}(\varepsilon)\, c(T_i, T_i) = 1.$$

The only thing unaccounted for is the sign, which is

$$\varepsilon_{w_i} \varepsilon_{v_I(w_i)}.$$

However, since

$$w_i = v_I(w_i)\, u_0,$$

where u_0 is the top element of W_I, we see that this sign is the same for all i and is equal to the sign of u_0.

We end by showing that the elements w_{ij} satisfy recursions analogous to the equations (2.16) satisfied by the p_{xw}'s themselves. To see this let us change notation slightly and set, when $x = w_i$, $y = w_j$,

$$f_x = f_i, \qquad F_x = F_{i,1}, \qquad \text{and} \qquad q_{xy} = w_{ij}.$$

Note that with this notation

$$f_y = \sum_{x \leqslant y} F_x q_{xy}. \tag{5.38}$$

Similarly, (5.29) becomes

$$\sigma F_x = \sum_z F_z A^{\mu}_{zx}(\sigma). \tag{5.39}$$

This given we have

THEOREM 5.4. *If w is the column word of a standard tableau of shape μ and $y = sw < w$ is also such a word then*

$$q_{xw} = \sum_{z \leqslant y} A_{xz}(s)\, q_{zy} - q_{xy} - \sum_{sz < z} \mu(z, y)\, q_{xz}, \tag{5.40}$$

where all these sums are carried out only over column words of standard tableaux of shape μ.

Proof. Using the second equation in (2.12) with w replaced by y and sw replaced by w and multiplying both sides on the right by $P(T_1)$, Proposition 5.8 gives

$$sf_y = f_w + f_y + \sum_{sz<z} f_z \mu(z, y).$$

Substituting into it Eqs. (5.38) and (5.39) gives

$$\sum_{z \leqslant y} \left(\sum_x F_x A^{\mu}_{xz}(s) \right) q_{zy} = \sum_{x \leqslant w} F_x q_{xw} + \sum_{x \leqslant y} F_x q_{xy} + \sum_{sz<z} \sum_{x \leqslant z} F_x q_{xz} \mu(z, y).$$

Equating coefficients of F_x yields our desired recursion (5.40).

We should point out that, since Eq. (5.40) involves only elements of a single cell, it gives a considerably less laborious algorithm for the calculations of the transforming matrix W than the one described in Section 3.

Acknowledgments

We wish to express our gratitude here to two people who have contributed greatly to our work: Jeff Remmel, whose active participation in endless conversations and seminars on each of the various phases of our research gave us a unique mechanism for bringing our initially vague ideas to solid fruition, and Anders Björner, who, in several seminars in the years following the appearance of the Kazhdan–Lusztig paper, has painstakingly endeavoured to translate its contents into a language more accessible to a combinatorial audience. In fact, the portions of Section 2 and the very beginning of Section 5 which review the purely combinatorial properties of the Kazhdan–Lusztig representations are mostly based on information obtained from him.

References

1. A. Björner, The combinatorics of the Kazhdan–Lusztig representations, Lecture notes, in preparation.
2. N. Bourbaki, "Groupes et algèbres de Lie," Chap. 4–6, Hermann, Paris, 1968.
3. C. W. Curtis, N. Iwahori, and R. Kilmoyer, Hecke algebras and characters of parabolic type of finite groups with (B, N) pair, *Inst. Hautes Etudes Sci. Publ. Math.* **40** (1972), 81–116.
4. C. W. Curtis and Reiner, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York, 1962.
5. H. Garnir, Théorie de la représentation lineaire des groupes symétriques, *Mém. Soc. Roy. Belg. Cl. Sci. Mé. Coll.* in 8 (2) **26** (1951), No. 1615.
6. A. M. Garsia and J. B. Remmel, Shuffles of permutations and the Kronecker product, *Graphs Combin.* **1** (1985), 217–263.
7. A. M. Garsia and J. Remmel, Symmetric functions and raising operators, *Linear and Multilinear Algebra* **10** (1981), 15–43.

8. A. M. Garsia and M. Wachs, Combinatorial aspects of skew representations of the symmetric groups, to appear.
9. A. M. Garsia, Raising operators and Young's rule, *in* "Proceedings, Montreal Colloquium in Enumerative Combinatorics, June 1985."
10. M. Goresky, Tables of Kazhdan–Lusztig polynomials, unpublished.
11. G. D. James, "The representation Theory of the Symmetric Groups," Lecture Notes in Mathematics, Vol. 682, Springer-Verlag, Berlin/New York/Heidelberg, 1978.
12. G. D. James and A. Kerber, The representation theory of the symmetric group, *in* "Encyclopedia of Mathematics," No. 16, Addison–Wesley Reading, MA, 1981.
13. D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.
14. D. Kazhdan and G. Lusztig, Schubert varieties and Poincaré duality, *in* Geometry of the Laplace Operator (R. Osserman and A. Weinstein, Eds.), Proc. Symp. Pure Math., Vol. 36, Amer. Math. Soc., Providence, RI, 1980.
15. R. King, Lecture notes, La Jolla, CA, 1983.
16. D. E. Knuth, "The Art of Computer Programming," Addison–Wesley, Reading, MA, 1975.
17. A. Lascoux and M.-P. Schützenberger, La représentation de Kazhdan & Lusztig du groupe symétrique, preprint.
18. A. Lascoux and M.-P. Schützenberger, Polynomes de Kazhdan & Lusztig pour les Grassmanniennes, preprint.
19. G. Lusztig, On a theorem of Benson and Curtis, *J. Algebra* **71** (1981), 490–498.
20. I. G. MacDonald, "Symmetric Functions and Hall Polynomials," Oxford Univ. Press (Clarendon), Oxford, 1979.
21. D. E. Rutherford, "Substitutional Analysis," Edinburgh Univ. Press, Edinburgh, 1948.
22. M.-P. Schützenberger, La correspondance de Robinson, *in* "Combinatoire et représentation du groupe symétrique," Actes Table Ronde C.N.R.S. Strasbourg, 1976.
23. R. Steinberg, "Lectures on Chevalley Groups," Lecture Notes, Department of Mathematics, Yale University, 1968.
24. D. Vogan, A generalized τ-invariant for the primitive spectrum of a semisimple Lie algebra, preprint, Princeton University, 1978.
25. A. Young, "The Collected Papers of Alfred Young, 1873-1940," Math. Expositions No. 21, Univ. of Toronto. Toronto Press.