Mathematical model of wave transformation over radial sand ridge field on continental shelf of South Yellow Sea

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Abstract: According to a deformed mild-slope equation derived by Guang-wen Hong and an enhanced numerical method, a wave refraction-diffraction nonlinear mathematical model that takes tidal level change and the high-order bathymetry factor into account has been developed. The deformed mild-slope equation is used to eliminate the restriction of wave length on calculation steps. Using the hard disk to record data during the calculation process, the enhanced numerical method can save computer memory space to a certain extent, so that a large-scale sea area can be calculated with high-resolution grids. This model was applied to wave field integral calculation over a radial sand ridge field in the South Yellow Sea. The results demonstrate some features of the wave field: (1) the wave-height contour lines are arc-shaped near the shore; (2) waves break many times when they propagate toward the shore; (3) wave field characteristics on the northern and southern sides of Huangshayang are different; and (4) the characteristics of wave distribution match the terrain features. The application of this model in the region of the radial sand ridge field suggests that it is a feasible way to analyze wave refraction-diffraction effects under natural sea conditions.

Key words: wave transformation; mathematical model; radial sand ridge field; South Yellow Sea

1 Introduction

A sand ridge field is a large-scale group of radial sandbanks above the waterline and underwater sandy ridges, including tidal channels. The South Yellow Sea sand ridge field is located off the northern Jiangsu coast on the southern continental shelf of the Yellow Sea, from Sheyang Estuary to Haozhigang, which is south of the Yangtze River. It has a radial fan shape with Jianggang as the center. The relief map was shown in Fig. 1. The length of the sand ridge field is 200 km from north to south, and the width is 140 km from east to west. The ridge field fanning out to the sea consists of more than 70 sand ridges and tidal channels. The water depth of the area ranges from 0 to 25 m. The main tidal channels that cut sand ridges are Xiyang, Xiaobeicao, Chenjiawucao, Caomishuyang, Kushuiyang, Huangshayang, Lanshayang, and Xiaomiaohong. The average water depth of these large tidal channels is more than 10 m and increases with distance from the shore. With the development of Jiangsu Province, many ports
will be built in this region. Calculations of wave fields are essential in port construction. However, the area under consideration is large, and the terrain changes rapidly. There has been little research of wave fields’ characteristics in this region. Therefore, a new wave model has been established in this study.

In a shallow sea area, refraction, diffraction, and breaking are the main wave phenomena. In order to study wave transformation, researchers have developed a large number of wave models. Generally speaking, the numerical models can be divided by their computational domain into three types: small-scale, middle-scale, and large-scale models. The three-dimensional Navier-Stokes (N-S) equation is one of the typical small-scale models (Wu and Yuan 2007). It can be used to calculate linear and nonlinear wave transformation over complicated terrain. At the same time, it can provide the vertical distribution of velocity. However, due to the limitations of computer capacity and speed, it is mainly used in small-scale areas. The middle-scale model can be represented by the Boussinesq equation and the Berkhoff mild-slope equation. In the Boussinesq equation, the wave transformation is described by water level and velocity, so the refraction and diffraction of wave transformation can be calculated accurately. As the Boussinesq equation depicts the wave surface elevation with time, the time step is usually 1/30 to 1/24 the length of the wave period, and the space step is 1/12 to 1/8 the wave length. Thus, this sort of equation is only suitable for middle-scale areas (Li et al. 2005). For example, it can be used to compute wave fields in a harbor basin. As the required solution item is a velocity potential function in the Berkhoff mild-slope equation, with eight to ten calculation points for one wave length, the computational domain is constrained by finite computer memory (Zheng et al. 2009). Large-scale models can be represented by the action balance equation (Zheng et al. 2008). This kind of equation can be used to calculate large-scale wave fields. However, the precondition is an increased space step and reduced grid resolution. The radial sand ridge field of the South Yellow Sea is large. The topography of the radial sand ridge field fluctuates so dramatically that a method with very large space steps cannot simulate the real topography. Furthermore, with this kind of model, it is hard to calculate the wave diffraction. Considering these factors, a large-scale fine-mesh mathematical model for wave refraction and diffraction was proposed, and used in this study to analyze the overall distribution characteristics of the wave field.
2 Extended mild-slope equations

2.1 Governing equations

The governing equations of this model are based on the mild-slope equations further developed by Hong (1996). He assumed that the wave number vector was irrotational, and transformed the mild-slope equation into a wave-activity equation and eikonal equation. In this way, the solution of the velocity potential function, which varies rapidly in space, can be changed into the solution of a wave action and wave number that vary slowly, so that the space step is not constrained by wave length but by the level of the acute topography change underwater, which enlarges the calculation domain of the mathematical model. Equations are as follows:

The nonlinear dispersion equation is

\[ \omega^2 = gk \tanh kh + F_1 \frac{H}{2h} + F_2 \left( \frac{H}{2h} \right)^2 + F_3 \left( \frac{H}{2h} \right)^3 \]  

where \( \omega \) is the angular frequency, \( g \) is the acceleration of gravity, \( k \) is the wave number, \( h \) is the water depth, \( H \) is the wave height, and \( F_1, F_2, \) and \( F_3 \) are the nonlinear dispersion coefficients derived by Li and Lee (2000).

The wave number vector irrotational equation is

\[ \frac{\partial K \sin \alpha}{\partial x} - \frac{\partial K \cos \alpha}{\partial y} = 0 \]  

where \( K \) is the wave number vector, and \( \alpha \) is the angle between wave direction and the normal vector of the boundary.

The wave action conservation equation is

\[ \frac{\partial}{\partial x} \left( A \frac{\tilde{C}_g}{\tilde{\sigma}} K \cos \alpha \right) + \frac{\partial}{\partial y} \left( A \frac{\tilde{C}_g}{\tilde{\sigma}} K \sin \alpha \right) = -W^* A \]  

where \( A = \frac{H^2}{8\sigma} \); \( \tilde{\sigma} = gk \tanh kh \); \( \tilde{C} = \frac{\tilde{\sigma}}{k} \); \( \tilde{C}_g = \frac{\tilde{C}}{2} \left( 1 + \frac{2kh}{\sinh 2kh} \right) \); \( W^* \) represents dissipation due to the bottom effect, \( W^* = \frac{4f_w H}{\pi g} \left( \frac{k\omega}{k \sinh kh} \right)^3 \); and \( f_w \) is the friction coefficient. In this mathematical model the wave breaking index is 0.78.

The eikonal equation is

\[ K^2 = k^2 + \frac{1}{\tilde{C} \tilde{C}_g R} \left[ \frac{\partial}{\partial x} \left( \tilde{C} \tilde{C}_g \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial y} \left( \tilde{C} \tilde{C}_g \frac{\partial R}{\partial y} \right) - \frac{R}{4} W^* \right] + \frac{gG}{\tilde{C} \tilde{C}_g} \]  

where \( R = \frac{H}{2\omega} \), and \( G \) is the high-order bathymetry variation derived by Pan et al. (2000).

2.2 Initial conditions and boundary conditions

The initial wave height \( H_0 \), wave period \( T \), and wave direction \( \alpha_0 \) of the open sea are
given directly. The wave height boundary condition derived by Pan et al. (2008) is
\[
\frac{\partial H}{\partial n} = B_i H
\]  
(5)
where \( n \) is the external interface normal direction, \( B_i = K \cos \alpha_i \frac{2R_i \sin \varepsilon_i}{1 + 2R_i \cos \varepsilon_i + R_i^2} \), \( \alpha_i \) is the angle between the incident wave and interface normal direction, \( R_i \) is the reflection coefficient, and \( \varepsilon_i \) is the phase difference.

The wave direction boundary condition derived by Feng and Hong (2000) is
\[
\alpha^* = \theta^* + \arcsin \left( \frac{-B_k}{K} \right)
\]  
(6)
where \( \alpha^* \) is the angle of reflection, \( \theta^* \) is the interface tangent direction angle, and \( B_k = K \cos \alpha \frac{R^2 - 1}{1 + 2R \cos \varepsilon + R^2} \). In the case of total reflection, \( R = 1 \) and \( \varepsilon = 0 \). In the case of an open boundary, \( R = 0 \).

3 Calculation method for large-scale sea area
3.1 Newly developed bookkeeping procedure

The wave number irrotational vector equation and wave action conservation equation are nonlinear hyperbolic partial differential equations, so a finite difference scheme should be used. In this study, the backward difference semi-implicit scheme, which can save the computer memory by using a hard disk to record intermediate data, was adopted to write source codes. This method can not only enlarge the calculation domain but also enhance the grid resolution at the same time. This is a newly developed bookkeeping procedure. It is suitable for calculating the sea area of complicated large-scale underwater topography.

Eq. (2) is used as an example here to indicate the implementation process of the developed bookkeeping procedure, and the solution of other equations are the same. After the backward difference semi-implicit scheme is carried out, Eq. (2) can be discretized as follows:
\[
\begin{align*}
&-\max \left( \frac{K_{i+1,j} \sin \alpha_{i+1,j}}{\Delta y}, 0 \right) \alpha_{i+1,j} + \left( \frac{K_{i,j} \cos \alpha_{i,j}}{\Delta x} + \frac{K_{i,j} \sin \alpha_{i,j}}{\Delta y} \right) \alpha_{i,j} - \max \left( -\frac{K_{i,j} \sin \alpha_{i,j}}{\Delta y}, 0 \right) \alpha_{i,j+1} \\
&= \frac{K_{i,j} \cos \alpha_{i,j} - K_{i,j} \sin \alpha_{i,j}}{\Delta x} \alpha_{i-1,j} + \frac{K_{i,j} - K_{i,j-1} \cos \alpha_{i,j}}{\Delta y} \alpha_{i,j} - \frac{K_{i,j} - K_{i,j-1} \sin \alpha_{i,j}}{\Delta x} \sin \alpha_{i,j}
\end{align*}
\]  
(7)
where \( \Delta x \) and \( \Delta y \) are the space steps, and the subscripts \( i \) and \( j \) indicate variables for the \( i \)th row and \( j \)th column in the grid computing domain. The iterative method is used to solve the nonlinear hyperbolic equations. From the discretization scheme, we know the values of \( \alpha_{i+1,j} \) can be calculated with \( \alpha_{i,j} \), while \( \alpha_{i-1,j} \) is not involved in the following computations. Thus, the values of \( \alpha_{i-1,j} \) can be recorded on the hard disk, and the computation memory space can be saved. Then, the newly developed bookkeeping procedure is realized. Using this method we can calculate the large-range wave field with a high resolution grid.
3.2 Water depth data variation with tidal level

It is necessary to consider the impact of the water level on the wave field, because the area of the radial sand ridge field is large. If we take 8 s as the wave period to calculate the velocity in a deep water area, then it will require at least 3 h for waves to reach the shore from the deep sea. If the incident waves are at a low tidal level in a deep water area, the sea surface may reach the mean sea level when waves propagate to the shore. Thus, the variation in tidal level cannot be ignored.

The model in this study uses water depth data varying with the water level to reflect its impact. First of all, the whole area is divided into three small regions: northern, central, and southern regions. In the center, a point is selected, the water level process of which is used to approximately represent the situation throughout the area. Secondly, the time of wave propagation from the $i$th row to the $(i+1)$th row in the calculation domain is calculated as waves move from deep to shallow waters. Finally, interpolation is performed based on the existing tidal process lines. We can determine the changing tide value at corresponding moments, and add it to the original water depth data. Water depth data change with the tide, incorporating the effects of changing water level on the wave field.

4 Verification of mathematical model

The Berkhoff physical model was used to evaluate the calculation accuracy of the mathematical wave model. In the physical model, the slope coefficient was 50, and the angle between the gradient direction of the slope and wave-crest line of the incident wave was 70°. There was an elliptic shoal on the slope, whose center point was $(x_0, y_0) = (10 \text{ m}, 10 \text{ m})$. The major half axis and minor half axis were, respectively, 4 m and 3 m. Here, $H_0 = 0.0464 \text{ m}$, $T = 1.0 \text{ s}$, the direction of the incident wave moved along the $+x$-axis, and the space steps in the $x$ and $y$ directions were, respectively, 0.05 m and 0.1 m. Fig. 2 is a sketch of a submerged shoal and locations of the measurement sections from 1# through 8#. The water depth in the physical model was 0-0.4 m.

Fig. 3 shows the comparison of mathematical model results with Berkhoff physical model test results. Overall, they match well. It should be noted that behind the shallow areas where the wave energy was concentrated, the measured wave height on both sides was close
to 0 m (sections 4 and 5), but the calculation results were not 0 m. This was due to the superposition of waves behind the shallow areas. In this area, the wave number vector was rotational, similar to that at non-tidal points (Berkhoff et al. 1982). The basic assumption of a large-scale wave model is that the wave number vector is irrotational; therefore, errors will appear. This is the limitation of all large-scale wave models.

5 Application of model in radial sand ridge field

5.1 Applicability of model in radial sand ridge field

The underwater topography of the radial sand ridge field of the South Yellow Sea fluctuates so dramatically that the slope coefficient reaches 30-40. Moreover, it is a vast sea, ranging from 32°00' N to 33°48' N, and from 120°40' E to 122°10' E, with an area of 200 km × 140 km.

In order to calculate a wider range of water area, the extended mild-slope equation is used in this model. The wave number vector irrotational equation is introduced based on the traditional mild-slope equation (Hong 1996). The restrictions of wave length on calculation steps can thereby be eliminated. In order to reflect the impact of tides on the wave field, this model uses time-varying water depth data. To reflect the impact of topography on the wave field, this model uses high-order bathymetry variation factors derived by Pan et al. (2000) and nonlinear dispersion relations derived by Li and Lee (2000). In order to accurately reflect the
topography, high-resolution grids are needed. If the space step is too large, a complicated landform will be smoothed. Through tentative calculation, this paper argues that a 30-m grid has been able to better reflect the topography of the area. The area of the radial sand ridge is so large that a huge amount of memory is required if we use fine grids. A 32-bit Fortran compiler cannot process it. Therefore, we invented a bookkeeping procedure.

5.2 Analysis of wave field characteristics in radial sand ridge field

According to statistical observation of the waves in this area, the probability of waves coming from the northeast is high. We calculated the wave field distribution under the condition of yearly maximum average wind velocity and the average tidal level. The average annual maximum wind speed over the sand ridge field is 17.74 m/s, and the corresponding significant incident wave height at offshore boundaries is 2.64 m; these can be considered the normal conditions (Zhang et al. 1999). During the calculation, we considered an incident wave direction of NE, a wave period of 8 s, and a wave height of 2.64 m to be the initial conditions.

5.2.1 Arc-shaped wave-height contour lines

Table 1 shows the calculated results of Zhang et al. (1999) and the model in this paper. The calculation points are plotted in Fig. 4. From the table we can see that there are only three points (10, 16, and 19) that show a certain difference; others match well. The comparison shows that the calculated wave heights are reliable.

<table>
<thead>
<tr>
<th>Point No.</th>
<th>Zhang’s wave height (m)</th>
<th>Calculated wave height (m)</th>
<th>Point No.</th>
<th>Zhang’s wave height (m)</th>
<th>Calculated wave height (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.48</td>
<td>2.15</td>
<td>13</td>
<td>2.46</td>
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<td>2.42</td>
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</tr>
<tr>
<td>3</td>
<td>1.22</td>
<td>0.90</td>
<td>15</td>
<td>2.45</td>
<td>2.62</td>
</tr>
<tr>
<td>4</td>
<td>0.44</td>
<td>0.40</td>
<td>16</td>
<td>2.25</td>
<td>0.98</td>
</tr>
<tr>
<td>25</td>
<td>0.00</td>
<td>0.00</td>
<td>17</td>
<td>2.56</td>
<td>1.99</td>
</tr>
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<td>5</td>
<td>2.59</td>
<td>2.45</td>
<td>18</td>
<td>2.55</td>
<td>1.80</td>
</tr>
<tr>
<td>6</td>
<td>2.36</td>
<td>2.63</td>
<td>19</td>
<td>1.10</td>
<td>0.52</td>
</tr>
<tr>
<td>7</td>
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<td>2.99</td>
<td>20</td>
<td>2.51</td>
<td>2.34</td>
</tr>
<tr>
<td>8</td>
<td>2.57</td>
<td>2.69</td>
<td>21</td>
<td>1.88</td>
<td>1.79</td>
</tr>
<tr>
<td>9</td>
<td>2.61</td>
<td>2.59</td>
<td>22</td>
<td>2.57</td>
<td>2.46</td>
</tr>
<tr>
<td>10</td>
<td>2.28</td>
<td>1.06</td>
<td>23</td>
<td>2.80</td>
<td>2.46</td>
</tr>
<tr>
<td>11</td>
<td>0.61</td>
<td>0.67</td>
<td>24</td>
<td>1.94</td>
<td>2.09</td>
</tr>
<tr>
<td>12</td>
<td>2.54</td>
<td>2.69</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 5 shows the wave-height contour line distribution of the radial sand ridge field, where the solid line represents the 1-m wave-height contour line, and the dotted line is the 2-m wave-height contour line. As can be seen from the figure, the 1-m wave-height contour line surrounds the radial sand ridge vertex, Jianggang, with an arc-shaped distribution. At the same time, a 2-m wave-height contour line surrounds Jianggang, with a sawtooth-shaped distribution.
The reason for the arc-shaped distribution of the 1-m wave height contour line is the same as that of the arc distribution of depth contour lines. Close to the top of the radial sand ridge field where Jianggang is located, the water is relatively shallow. Waves that propagate from the open sea break many times before reaching the shallow water area. Therefore, the wave energy dissipates and the wave height decreases accordingly. However, the water is deep in both the southern and northern parts of Jianggang; waves do not break and wave height does not decrease until it propagates to the near-shore area. That is how the special characteristics of wave-height contour lines form.

5.2.2 Wave-breaking zone

When waves break, the turbulence and vortex of the water area are very strong. The wave-breaking zone is not only the main area of wave energy dissipation, but also the area where sediment moves most violently, so study of the wave-breaking zone is of great significance to understanding the evolution of the seabed. Fig. 6 shows the distribution of the wave-breaking zone in this area. The black area near the shore represents the breaking points of waves and the grey outline is the edge of the breaking zone. In the figure, the edge of the breaking zone in eastern Jianggang is farther away from the coastline than other parts of the edge. This is because the existence of the large radial sand ridges lessens the water depth. Meanwhile,
this sea area is a converging area, where the wave height increases. When wave height increases somewhat, waves will break, so the edge of the breaking zone here is farther away from the coastline than other parts of the edge. With shoreward propagation, waves break many times and wave energy decreases quickly. Thus, wave height decreases, a large-scale area with small wave height forms near the shore, and wave breaking is the main way that the sand ridge field responds to the wave field.

5.2.3 Two wave fields with different characteristics

Fig. 7 shows the wave direction distribution. We can see the wave direction of Xiyang (points 1-4), the Chengjiawucao (points 8-11), the Caomishuyang (points 12 and 13), the Kushuiyang (points 14-16), and the Huangshayang (17-19 points) turn gradually to Jianggang when waves propagate toward the coastline. This is mainly because the distribution of depth contour lines in this area is arc-shaped, with Jianggang at the center. Fig. 8 is the schematic diagram of wave direction and the depth contour lines. The black curve is the generalized depth contour line. Because of the refraction effect caused by topography, wave energy is concentrated in this area. Wave direction lines propagate along the direction perpendicular to the isobaths, so the wave direction lines have a radial distribution around Jianggang. However, the wave directions in the area of the Lanshayang (points 20-22) and Xiaomiaohong (points 23 and 24) do not turn toward Jianggang significantly, because of the refraction of the seabed. Waves propagate along the direction perpendicular to the depth contour lines, and the isobaths in this area are generally consistent with the coastline, so wave direction lines are perpendicular to the coastline. In summary, due to the influence of the depth contour line, the wave-crest lines in the north are different from those in the south.
6 Conclusions

In view of the wide area of the radial sand ridge field in the South Yellow Sea and the complicated underwater terrain characteristics, a new mathematical wave model has been established. In order to calculate a large-scale sea area, the extended mild-slope equation is used in this model. The restrictions of wave length on calculation steps can thereby be eliminated. In order to consider the impact of tides on the wave field, the model uses tidal-level-varying water depth data. For the purpose of accurately reflecting the influence of topography on wave fields, this model also uses high-order bathymetry factors and nonlinear dispersion relations. In order to accurately reflect the topography, high-resolution grids are needed. The bookkeeping procedure takes the large range and fine-mesh requirements into account at the same time. Compared with the N-S equation, Boussinesq equation, and mild-slope equation, the model in this study is effective in calculating the large-scale wave fields. It can also be used with a higher resolution grid rather than the dynamic spectral balance equation.

The model was applied to the calculation of a radial sand ridge field in the South Yellow Sea, and the results showed the following:

(1) Under average tidal level conditions, the deep-water waves propagate into the radial sand ridge field. Wave height contour lines surrounding Jianggang have an arc-shaped distribution due to the underwater topography.

(2) Waves break many times, wave energy is consumed quickly, and wave height decreases gradually when waves propagate into shallow water areas near the vertex of sand ridges. Thus, a large-scale area with a small wave height is formed near the shore.

(3) Wave field characteristics on the northern and southern sides of Huangshayang are different. In the northern area near the tidal inlet, wave direction changes significantly, generally toward Jianggang, while the wave-crest line in the southern waters is basically parallel to the coastline.

References


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