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# Projection bodies and valuations

Monika Ludwig

*Abteilung für Analysis, Technische Universität Wien, Wiedner Hauptstraße 8-10/1142, 1040 Wien, Austria*

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**Abstract**

Let $\Pi$ be the projection operator, which maps every polytope to its projection body. It is well known that $\Pi$ maps the set of polytopes, $\mathcal{P}^n$, in $\mathbb{R}^n$ into $\mathcal{P}^n$, that it is a valuation, and that for every $P \in \mathcal{P}^n$, $\Pi P$ is affinely associated to $P$. It is shown that these properties characterize the projection operator $\Pi$. This proves a conjecture by Lutwak.


## 0. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (i.e., of compact, convex sets) in Euclidean $n$-space $\mathbb{R}^n$ and let $\mathcal{P}^n$ denote the set of convex polytopes in $\mathbb{R}^n$. A convex body $K \in \mathcal{K}^n$ is determined by its support function, $h(K, \cdot)$, on the unit sphere $S^{n-1}$, where $h(K, u) = \max\{u \cdot x : x \in K\}$ and where $u \cdot x$ denotes the standard inner product of $u$ and $x$. The projection body, $\Pi K$, of $K$ is the convex body whose support function is given for $u \in S^{n-1}$ by

$$h(\Pi K, u) = \mathrm{vol}(K|u^\perp),$$

where vol denotes $(n-1)$-dimensional volume and $K|u^\perp$ denotes the image of the orthogonal projection of $K$ onto the subspace orthogonal to $u$.

Projection bodies were introduced by Minkowski at the turn of the last century in connection with Cauchy's surface area formula. They are an important tool for studying projections and have also proved to be useful in other ways and in other subjects.

One important aspect here is the range of the operator $\Pi$. Projection bodies of convex polytopes are special polytopes called zonotopes. These are important due to

*E-mail address:* monika.ludwig@tuwien.ac.at.

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the connection to oriented matroids, hyperplane arrangements, aspects of optimization, computational geometry, and other areas (cf. [5,35]). Projection bodies of convex bodies are highly symmetric centered convex bodies called zonoids. These arise in a number of guises; for example, the zonoids in $\mathbb{R}^n$ are precisely the ranges of non-atomic $\mathbb{R}^n$-valued measures, and they are precisely the polars of the unit balls of $n$-dimensional subspaces of $L_1([0,1])$ (cf. the surveys [6,11,31]).

Here, we focus on the operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$ itself. Starting with Aleksandrov's classical projection theorem, there have been many important results on $\Pi$ including those by Petty [25] and Schneider [27] (see [9]). These have applications in the local theory of Banach spaces (see [7]) and in Minkowski geometry (see [32]).

One reason that the operator $\Pi$ is so useful in these areas is that projection bodies of affinely equivalent convex bodies are affinely equivalent. Specifically,

$$\Pi(\phi K) = |\det \phi| \phi^{-t} \Pi K \quad \text{and} \quad \Pi(K + x) = \Pi K \tag{1}$$

for every $K \in \mathcal{K}^n$, $\phi \in \mathrm{GL}(n)$, and $x \in \mathbb{R}^n$. Here $\mathrm{GL}(n)$ denotes the group of general linear transformations in $\mathbb{R}^n$, $\det \phi$ denotes the determinant of $\phi$, and $\phi^{-t}$ denotes the inverse of the transpose of $\phi$. This was proved by Petty [25]. It follows from (1) that the volume of $\Pi K$ and of the polar of $\Pi K$ are affine invariants, and there are important affine isoperimetric inequalities for these quantities (see [10,20,22,26,33] and Lutwak's survey [21]). Recently, Zhang [34] derived from these results an affine invariant Sobolev inequality that is stronger than the classical Sobolev inequality.

A basic property of the operator $\Pi$ is that it is a valuation. In general, a function $Z$ defined on $\mathcal{K}^n$ and taking values in an Abelian semi-group is called a valuation if

$$ZK_1 + ZK_2 = Z(K_1 \cup K_2) + Z(K_1 \cap K_2),$$

whenever $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}^n$. A classical result by Hadwiger [12] states that the continuous, rigid motion invariant, real valued valuations on $\mathcal{K}^n$ are precisely the linear combinations of intrinsic volumes. In recent years, many new results on real and tensor valued valuations have been obtained (see, for example, [1–3,13–16,19,30], and Klain and Rota's book [17]), including Alesker's proof [4] of McMullen's 20-year-old conjecture on the classification of translation invariant valuations.

For operators taking values in $\mathcal{P}^n$ and $\mathcal{K}^n$, it is natural to consider valuations with respect to Minkowski addition. With this operation $\mathcal{P}^n$ and $\mathcal{K}^n$ are Abelian semi-groups, and

$$\Pi K_1 + \Pi K_2 = \Pi(K_1 \cup K_2) + \Pi(K_1 \cap K_2), \tag{2}$$

whenever $K_1, K_2, K_1 \cup K_2 \in \mathcal{K}^n$, i.e., $\Pi$ is a valuation. Lutwak asked whether (1) and (2) characterize the projection operator $\Pi$. We obtain the following results.

**Theorem.** *An operator $Z : \mathcal{P}^n \to \mathcal{P}^n$ is a valuation such that*

$$Z(\phi P) = |\det \phi| \phi^{-t} ZP \quad \textit{and} \quad Z(P + x) = ZP \tag{3}$$

*for every* $\phi \in \mathrm{GL}(n)$ *and* $x \in \mathbb{R}^n$, *if and only if there is a constant* $c \geqslant 0$ *such that*

$$ZP = c\Pi P$$

*for every* $P \in \mathscr{P}^n$.

The projection operator is continuous and it is monotone increasing, i.e., if $K \subset L$ then $\Pi K \subset \Pi L$. This immediately implies the following for operators on $\mathscr{K}^n$.

**Corollary 1.** *An operator* $Z : \mathscr{K}^n \to \mathscr{K}^n$ *is a monotone increasing valuation such that*

$$Z(\phi K) = |\det \phi| \phi^{-t} ZK \quad \textit{and} \quad Z(K + x) = ZK$$

*for every* $\phi \in \mathrm{GL}(n)$ *and* $x \in \mathbb{R}^n$, *if and only if there is a constant* $c \geqslant 0$ *such that*

$$ZK = c\Pi K$$

*for every* $K \in \mathscr{K}^n$.

For the extension to continuous operators on $\mathscr{K}^n$, we use an argument by Schneider [28] and obtain the following.

**Corollary 2.** *An operator* $Z : \mathscr{K}^n \to \mathscr{K}^n$ *is a continuous valuation such that*

$$Z(\phi K) = |\det \phi| \phi^{-t} ZK \quad \textit{and} \quad Z(K + x) = ZK$$

*for every* $\phi \in \mathrm{GL}(n)$ *and* $x \in \mathbb{R}^n$, *if and only if there is a constant* $c \geqslant 0$ *such that*

$$ZK = c\Pi K$$

*for every* $K \in \mathscr{K}^n$.

For additional information regarding projection bodies, see the books by Gardner [9], Leichtweiß [18], Schneider [29], and Thompson [32].

## 1. Proof of the Theorem

We assume that $Z : \mathscr{P}^n \to \mathscr{P}^n$ is a valuation for which (3) holds and will show that there is a constant $c \geqslant 0$ such that $ZP = c\Pi P$ for every $P \in \mathscr{P}^n$.

We work in $n$-dimensional Euclidean space $\mathbb{R}^n$ with origin $o$, basis $e_1, \ldots, e_n$, and use coordinates $x = (x_1, \ldots, x_n)^t$ for $x \in \mathbb{R}^n$. Let $\mathrm{SL}(n)$ denote the group of special linear transformations in $\mathbb{R}^n$, i.e., of linear transformations with determinant 1, and let $\mathrm{O}(n)$ denote the group of orthogonal transformations in $\mathbb{R}^n$.

The affine hull of a polytope $P$ is the smallest affine subspace containing $P$, and the dimension of $P$, $\dim P$, is defined as the dimension of the affine hull of $P$.

**Lemma 1.** *If $P\in\mathcal{P}^n$ and $\dim P<(n-1)$, then $ZP=\{o\}$. If $P\in\mathcal{P}^n$ and $\dim P=(n-1)$, then $ZP$ is a segment in the one-dimensional subspace orthogonal to the affine hull of $P$.*

**Proof.** Let $P\in\mathcal{P}^n$ with $P\subset H$ where $H$ is the $k$-dimensional subspace with equation $x_{k+1}=\cdots=x_n=0$. Since every $P'\in\mathcal{P}^n$ with $\dim P'=k$ is an affine image of such a polytope $P$, (3) implies that it suffices to prove the lemma in this case. Let

$$\phi=\begin{pmatrix} I & B \\ 0 & A \end{pmatrix},$$

where $I$ is the $k\times k$ identity matrix, 0 is the $(n-k)\times k$ null matrix, $B$ is a $k\times(n-k)$ matrix, and $A$ is an $(n-k)\times(n-k)$ matrix with determinant 1. Then $\phi\in\mathrm{SL}(n)$ and

$$\phi^{-t}=\begin{pmatrix} I & 0 \\ C & A^{-t} \end{pmatrix}$$

with $C=-A^{-t}B^t$. Since $P\subset H$,

$$\phi P=P. \tag{4}$$

Write $x=\binom{x'}{x''}$ with $x'=(x_1,\dots,x_k)^t$ and $x''=(x_{k+1},\dots,x_n)^t$ for $x\in\mathbb{R}^n$. Let $x\in ZP$. It follows from (4) and (3) that $y=\phi^{-t}x\in ZP$. Therefore,

$$\begin{pmatrix} y' \\ y'' \end{pmatrix}=\begin{pmatrix} x' \\ Cx'+A^{-t}x'' \end{pmatrix}\in ZP. \tag{5}$$

This is true for every $k\times(n-k)$ matrix $B$ and every $(n-k)\times(n-k)$ $A$ matrix with determinant 1. If $x'\neq o'$, this implies that $y''$ can be an arbitrary vector. Since $ZP$ is bounded, this implies that $x'=o'$. Thus $ZP$ lies in the orthogonal complement of $H$. If $k=(n-1)$, this proves the lemma. So let $k<(n-1)$. Then $x'=o'$ and (5) holds for every $(n-k)\times(n-k)$ matrix $A$ with determinant 1. Since $ZP$ is bounded and $(n-k)\geqslant 2$, this implies that $x''=o''$. $\square$

For a polytope $P$, an outer normal vector ($\neq o$) to a facet (i.e., an $(n-1)$-dimensional face) is called a *facet normal*. Denote by $\mathrm{vol}(P,v)$ the $(n-1)$-dimensional volume of the facet with facet normal $v$, and call a facet normal $v$ *scaled*, if it has length $\mathrm{vol}(P,v)$. We recall some simple facts about projection bodies of polytopes (see, for example, [9] or [8]). If $P\in\mathcal{P}^n$ with $\dim P<(n-1)$, then $\Pi P=\{o\}$. If $P\in\mathcal{P}^n$ with $\dim P\geqslant(n-1)$ and scaled facet normals $v_1,\dots,v_m$, then

$$\mathrm{vol}(P|u^\perp)=\tfrac{1}{2}\sum_{i=1}^m |v_i\cdot u|.$$

For $x \in \mathbb{R}^n$, the support function of the segment $[-x, x]$ with endpoints $-x$ and $x$ is given by $h([-x, x], u) = |u \cdot x|$. Thus if $P$ is a polytope with scaled facet normals $v_1, \dots, v_m$ then

$$\Pi P = \tfrac{1}{2} \sum_{i=1}^{m} [-v_i, v_i]. \tag{6}$$

In the next lemma, we use the following well-known characterization of volume (cf. [24]). If $v : \mathscr{P}^{n-1} \to \mathbb{R}$ is a simple, translation invariant, non-negative valuation then there is a constant $c \geqslant 0$ such that

$$v(P) = c\,\mathrm{vol}(P) \tag{7}$$

for every $P \in \mathscr{P}^{n-1}$. Here a valuation $v : \mathscr{P}^{n-1} \to \mathbb{R}$ is called *simple*, if it vanishes on polytopes $P$ with $\dim P < (n-1)$.

**Lemma 2.** *There is a constant $c \geqslant 0$ such that $ZP = c\Pi P$ for every $P \in \mathscr{P}^n$ with* $\dim P = (n-1)$.

**Proof.** Let $P \in \mathscr{P}^n$ with $P \subset H$ where $H$ is the subspace orthogonal to $e_n$. By Lemma 1, there are $v_1(P), v_2(P) \in \mathbb{R}$ such that

$$ZP = [v_1(P)e_n, v_2(P)e_n].$$

Let $\phi \in \mathrm{GL}(n)$ be such that $\phi e_i = e_i$ for $i = 1, \dots, (n-1)$, and $\phi e_n = -e_n$. Then $P = \phi P$, and it follows from (3) that $v_1(P) = -v_2(P)$. Thus

$$Z(P) = [-v(P)e_n, v(P)e_n]$$

with $v(P) \geqslant 0$. The functional $v$ is defined for every $P \in \mathscr{P}^n$ with $P \subset H$. We identify $H$ and $\mathbb{R}^{n-1}$ and have $v : \mathscr{P}^{n-1} \to [0, \infty)$. Since $Z$ is a valuation, so is $v$. By Lemma 1, $ZP = \{o\}$ if $\dim P < (n-1)$. This implies that $v$ is simple. If $x \in \mathbb{R}^{n-1}$, then it follows from (3) that $Z(P + x) = ZP$. Therefore $v$ is translation invariant. Thus, we obtain by (7) that there is a constant $c \geqslant 0$ such that $v(P) = 1/2c\,\mathrm{vol}(P)$. By (6),

$$\Pi P = \tfrac{1}{2}[-\mathrm{vol}(P)e_n, \mathrm{vol}(P)e_n].$$

Thus $ZP = c\Pi P$ for every $P \subset H$. Since every $P' \in \mathscr{P}^n$ with $\dim P' = (n-1)$ is an affine image of a polytope $P \in \mathscr{P}^{n-1}$, this combined with (3) completes the proof of the lemma. $\square$

For a polytope $P$, denote by $\mathscr{N}_{\mathrm{F}}(P)$ the set of facet normals of $P$. We recall some simple facts about Minkowski sums of polytopes that will be used in the next lemma (cf. [29] for more details). Let $P, P_1, P_2 \in \mathscr{P}^n$. Since $P_1 + P_2 = \{x + y : x \in P_1, y \in P_2\}$, it is easy to see that

$$v \in \mathcal{N}_{\mathrm{F}}(P_1) \text{ implies that } v \in \mathcal{N}_{\mathrm{F}}(P_1 + P_2). \tag{8}$$

If $v \in \mathcal{N}_{\mathrm{F}}(P + [-x, x])$, $x \in \mathbb{R}^n$, and if $v \notin \mathcal{N}_{\mathrm{F}}(P)$, then $v$ is a normal vector to a facet with an edge parallel to $x$, i.e.,

$$v \in \mathcal{N}_{\mathrm{F}}(P + [-x, x]) \backslash \mathcal{N}_{\mathrm{F}}(P) \text{ implies that } v \cdot x = 0. \tag{9}$$

We also need the following fact about the projection body of a simplex (cf. [23] or [8]). Let $T$ be an $n$-dimensional simplex given as the convex hull of the points $x_0, \ldots, x_n$. Let $v_0, \ldots, v_n$ be the scaled facets normals of $T$ labeled such that the facet with normal $v_k$ does not contain $x_k$. Then $v_k \cdot (x_i - x_j) = 0$ for $k \neq i,j$. Combined with (6) this shows that $\mathcal{N}_{\mathrm{F}}(\Pi T)$ consists of the vectors

$$x_i - x_j, \quad i \neq j, \ i,j = 1, \ldots, n \tag{10}$$

and their multiples ($\neq 0$).

**Lemma 3.** *For every simplex $T$, $\mathcal{N}_{\mathrm{F}}(ZT) \subset \mathcal{N}_{\mathrm{F}}(\Pi T)$.*

**Proof.** Let $S$ be the simplex that is the convex hull of $o, e_1, \ldots, e_n$. By (10), $\mathcal{N}_{\mathrm{F}}(\Pi S)$ consists of all multiples ($\neq 0$) of $e_i$ for $i = 1, \ldots, n$, and of $e_i - e_j$ for $i,j = 1, \ldots, n$, $i \neq j$. We show that only these vectors (and their multiples) can be elements of $\mathcal{N}_{\mathrm{F}}(ZS)$.

Let $H$ be the $(n-1)$-dimensional subspace orthogonal to $e_1 - e_2$. Then $H$ contains $e_1 + e_2$ and $e_k$ for $k = 3, \ldots, n$, and dissects $S$ into two simplices $S_1$ and $S_2$, i.e.,

$$S = S_1 \cup S_2 \quad \text{and} \quad S_1 \cap S_2 \subset H.$$

Since $Z$ is a valuation, this implies

$$Z(S_1) + Z(S_2) = Z(S) + Z(S_1 \cap S_2). \tag{11}$$

Lemma 2 shows that

$$Z(S_1 \cap S_2) = c\Pi(S_1 \cap S_2) = [-x, x], \tag{12}$$

where the segment $[-x, x]$ is orthogonal to $H$ and therefore parallel to $e_1 - e_2$. Define $\phi, \psi \in \mathrm{GL}(n)$ by $\phi e_2 = 1/2(e_1 + e_2)$ and $\phi e_k = e_k$ for $k = 1, \ldots, n$, $k \neq 2$, and $\psi e_1 = 1/2(e_1 + e_2)$ and $\psi e_k = e_k$ for $k = 2, \ldots, n$. Then

$$\phi S_1 = S \quad \text{and} \quad \psi S_2 = S. \tag{13}$$

We set $P = ZS$ and obtain from (13), (11), (3), and (12) that

$$\tfrac{1}{2}\phi^t P + \tfrac{1}{2}\psi^t P = P + [-x, x]. \tag{14}$$

Let $v \in \mathcal{N}_{\mathrm{F}}(P)$. Then $\phi v \in \mathcal{N}_{\mathrm{F}}(\phi^t P)$. By (14) and (8), this implies that $\phi v \in \mathcal{N}_{\mathrm{F}}(P + [-x, x])$. By (9) we obtain that if $\phi v \in \mathcal{N}_{\mathrm{F}}(P + [-x, x]) \backslash \mathcal{N}_{\mathrm{F}}(P)$ then

$$\phi v \cdot x = v \cdot \phi^t x = 0.$$

So if $\phi v \in \mathcal{N}_{\mathrm{F}}(P + [-x, x])$ and $v \cdot \phi^t x \neq 0$, then $\phi v \in \mathcal{N}_{\mathrm{F}}(P)$. Using this argument repeatedly, we obtain that if $v \in \mathcal{N}_{\mathrm{F}}(P)$ and if $v \cdot (\phi^t)^k x \neq 0$ for $k = 1, \dots, m$, then $\phi^m v \in \mathcal{N}_{\mathrm{F}}(P)$. Since $P$ is a polytope and has only finitely many facets and since $\phi^m$ has the same eigenvectors as $\phi$, this implies the following. If $v \in \mathcal{N}_{\mathrm{F}}(P)$ and $v \cdot (\phi^t)^k x \neq 0$ for every positive integer $k$, then $v$ has to be an eigenvector of $\phi$. The eigenvectors of $\phi$ are the vectors $v$ where the coordinate $v_2$ vanishes and the multiples of $e_1 - e_2$. The equation $v \cdot (\phi^t)^k x = 0$ can also be written in the following way. We represent the map $\phi^t$ for the relevant first and second coordinates by the matrix

$$\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

and use that

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^k \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2^k \\ 2(2^{k-1} - 1) \end{pmatrix}.$$

This shows that $v \cdot (\phi^t)^k x = 0$ is equivalent to $2^{k-1} v_1 + (2^{k-1} - 1) v_2 = 0$. Thus for $v \in \mathcal{N}_{\mathrm{F}}(P)$ we obtain that either

$$2^{k-1} v_1 + (2^{k-1} - 1) v_2 = 0 \tag{15}$$

for a positive integer $k$, or

$$v_2 = 0 \tag{16}$$

or

$$v = t(e_1 - e_2)$$

with $t \neq 0$. Similarly, we use $\psi$ and obtain that if $v \in \mathcal{N}_{\mathrm{F}}(P)$ then either $v \cdot (\psi^t)^m x = 0$, i.e.,

$$(2^{m-1} - 1) v_1 + 2^{m-1} v_2 = 0 \tag{17}$$

for a positive integer $m$, or

$$v_1 = 0 \tag{18}$$

or

$$v = t(e_1 - e_2)$$

with $t \neq 0$. If (15) and (17) hold, then $v_1 = v_2 = 0$. If (15) and (18) hold, then $v_1 = 0$. Note that for $k = 1$ this is the only condition we get. If (16) and (17) hold, then

$v_2 = 0$. Note that for $m = 1$ this is the only condition we get. Therefore if $v \in \mathcal{N}_{\mathrm{F}}(P)$, then

$$v_1 = 0 \quad \text{or} \quad v_2 = 0 \quad \text{or} \quad v = t(e_1 - e_2)$$

with $t \neq 0$.

For every pair of basis vectors $e_i, e_j$, $i,j = 1, \ldots, n$, $i \neq j$, the $(n-1)$-dimensional subspace orthogonal to $e_i - e_j$ dissects $S$ into two simplices. Using the same argument as for $e_1, e_2$, we obtain the following. If $v \in \mathcal{N}_{\mathrm{F}}(P)$, then

$$v_i = 0 \quad \text{or} \quad v_j = 0 \quad \text{or} \quad v = t(e_i - e_j)$$

with $t \neq 0$. Multiples of the vectors $e_i$, $i = 1, \ldots, n$, and of $e_i - e_j$, $i,j = 1, \ldots, n$, $i \neq j$, are the only vectors for which these conditions hold simultaneously. This completes the proof of the lemma for the simplex $S$. Since every simplex is an affine image of $S$, (3) implies that the lemma holds for every simplex. $\square$

In the next lemma, we use Minkowski's uniqueness theorem that states that a polytope is determined up to translation by its outer normal vectors and the $(n-1)$-dimensional volume of its facets (cf. [29, p. 397]).

**Lemma 4.** *There is a constant $c' \geqslant 0$ such that $ZT = c' \Pi T$ for every $n$-dimensional simplex $T$.*

**Proof.** Let $S$ be a fixed regular simplex with centroid at the origin and vertices $x_0, \ldots, x_n$. By (10), $\mathcal{N}_{\mathrm{F}}(\Pi S)$ consists of all multiples ($\neq 0$) of $x_i - x_j$ for $i \neq j$, $i,j = 1, \ldots, n$, and by Lemma 3, these vectors are the only possible facet normals of $ZS$. Let $v, v' \in \mathcal{N}_{\mathrm{F}}(\Pi S)$ be such that $v = x_i - x_j$, $i \neq j$, $1 \leqslant i,j, \leqslant n$, and $v' = x_{k_i} - x_{k_j}$, $k_i \neq k_j$, $1 \leqslant k_i, k_j \leqslant n$. Since $S$ is a regular simplex, there is a $\phi \in \mathrm{O}(n)$ such that $\phi x_i = x_{k_i}$, $\phi x_j = x_{k_j}$, and $\phi S = S$. Therefore $\phi v = v'$. By (3), this implies that $ZS = \phi^t ZS$ and

$$\mathrm{vol}(ZS, v) = \mathrm{vol}(\phi^t ZS, v) = \mathrm{vol}(ZS, \phi v) = \mathrm{vol}(ZS, v').$$

Thus, all facets of $ZS$ as well as of $\Pi S$ have the same $(n-1)$-dimensional volume. We apply Minkowski's uniqueness theorem and obtain that there is a constant $c' \geqslant 0$ and a vector $x \in \mathbb{R}^n$ such that

$$ZS = c' \Pi S + x. \tag{19}$$

For every $\phi \in \mathrm{O}(n)$ with $\phi S = S$, this implies by (3) and (1) that

$$\phi^{-t} ZS = c' \phi^{-t} \Pi S + x.$$

Thus $x = \phi^t x$ and $x = o$ in (19). Combined with (3) this completes the proof of the lemma. $\square$

Lemmas 1 and 2 show that

$$ZP = c\Pi P \tag{20}$$

for every $P \in \mathscr{P}^n$ with $\dim P \leqslant (n-1)$. For an $n$-dimensional simplex $T$, Lemma 4 shows that $ZT = c'\Pi T$. We dissect $T$ into two simplices $T_1$, $T_2$, use that $Z$ is a valuation, and obtain

$$ZT + Z(T_1 \cap T_2) = ZT_1 + ZT_2.$$

Since $\dim(T_1 \cap T_2) = (n-1)$ and since $\Pi$ is a valuation, it follows from (20) and Lemma 4 that

$$c'\,\Pi T + c\Pi(T_1 \cap T_2) = c'\Pi T_1 + c'\Pi T_2 = c'\Pi T + c'\Pi(T_1 \cap T_2).$$

Thus $c = c'$.

Now let $P$ be an $n$-dimensional polytope. We have to show that

$$ZP = c\Pi P. \tag{21}$$

Suppose we can dissect $P$ into $P_1, P_2 \in \mathscr{P}^n$ for which (21) hold. Then using that $Z$ and $\Pi$ are valuations, we obtain

$$ZP + Z(P_1 \cap P_2) = ZP_1 + ZP_2 = c\Pi P_1 + c\Pi P_2 = c\Pi P + c\Pi(P_1 \cap P_2).$$

Since $\dim(P_1 \cap P_2) = (n-1)$, using (20) implies

$$ZP + c\Pi(P_1 \cap P_2) = c\Pi P + c\Pi(P_1 \cap P_2).$$

Thus (21) holds also for $P$. Since (21) holds for simplices, using this argument repeatedly completes the proof of the Theorem. $\square$

## 2. Proof of Corollary 2

Since the operator $Z$ is continuous and since

$$Z(\phi P) = |\det \phi| \phi^{-t} ZP$$

holds for every $\phi \in \mathrm{GL}(n)$, we have

$$Z(\psi P) = \hat{\psi} ZP \tag{22}$$

for every singular linear transformations $\psi$, where $\hat{\psi}$ is the matrix of the algebraic complements of the entries of $\psi$. Let $\psi_u$ be the matrix corresponding to the orthogonal projection to $u^\perp$. Then $\hat{\psi}_u$ corresponds to the projection to the line with direction $u$ and the definition of the support function implies that

$$h(ZK,u) = h(\hat{\psi}_u ZK, u).$$

By (22) we have

$$Z(\psi_u K) = \hat{\psi}_u ZK.$$

Therefore, $Z$ is already determined by its values for $(n-1)$-dimensional convex sets. Since these values are known by Lemma 2, this completes the proof of the corollary. □

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