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Generic fibers of the generalized Springer resolution of type A

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Abstract

It is well known that when the Lie algebra is of type A, D, E the Springer fiber above a subregular nilpotent element is described by the Dynkin diagram and is called the Dynkin curve of the Lie algebra. On the other hand, the closure of the minimal nilpotent orbit is obtained by collapsing the zero section of a cotangent bundle of a projective space $\mathbb{P}^k$. In this article, we are interested in the study of the generalized Springer resolution of type A, we give a complete description of the generalized Springer fiber above a generic singularities showing that it is isomorphic to a Dynkin curve or to a projective space.

Keywords: Springer fibers; Special linear group; Simple singularity

1. Introduction and notations

In 1970, E. Brieskorn has discovered a connection between the rational double points singularities with the complex Lie algebra theory, (cf. [5]). His result is the following: let G be a simple algebraic group of type A, D, E with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$. Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$. The variety $\mathcal{N}$ is exactly the closure of an unique nilpotent orbit $\mathcal{O}_{\mathrm{reg}}$ called the *regular* nilpotent orbit. There is an unique nilpotent orbit $\mathcal{O}_{\mathrm{s-reg}}$ of codimension 2 in $\mathcal{N}$ such that $\overline{\mathcal{O}_{\mathrm{s-reg}}} = \mathcal{N} - \mathcal{O}_{\mathrm{reg}}$ ($\mathcal{O}_{\mathrm{s-reg}}$ is called the *subregular* nilpotent orbit). Let T_x denote a transverse slice in $\mathfrak{g}$ to the orbit $\mathcal{O}_{\mathrm{s-reg}}$ at the point

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$x \in \mathcal{O}_{\mathrm{s-reg}}$. Then $(T_x \cap \mathcal{N}, x)$ is a normal surface with an isolated rational double point of type corresponding to $\mathfrak{g}$. Few years later Esnault [10] has obtained the same result with a geometric point of view which consists to the study of the Springer resolution, $f_{\mathfrak{b}} : T^*(G/B) \to \mathcal{N}$, where B is a Borel subgroup of G [23]: the Springer fiber above $x \in \mathcal{O}_{\mathrm{s-reg}}$ is well known as a finite union of projective lines which corresponds to the *Dynkin curve* of $\mathfrak{g}$, and it was originally obtained by Tits (see [26]); H. Esnault shows that each projective line of the Springer fiber above a subregular nilpotent element has a self-intersection -2, this proves that the Springer resolution restricts to the minimal resolution of the generic singularities of $\mathcal{N}$ and shows again that these singularities are rational double points of same type as $\mathfrak{g}$.

On the other hand, there is an other interesting singularity arising from the closure of the *minimal* nilpotent orbit $\mathcal{O}_{\min}$ in $\mathfrak{g}$ corresponding to the unique (non-zero) nilpotent orbit which is contained in the closure of all non-zero nilpotent orbit, and $\overline{\mathcal{O}_{\min}} = \mathcal{O}_{\min} \cup \{0\}$ is normal and has an isolated singularity. In case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ such singularity is exactly obtained by collapsing the cotangent bundle of $\mathbb{P}^{n-1}$, so the fiber above such singularity is exactly the zero section of this cotangent bundle.

In the present work, we are interested in the study of the fibers of the *generalized Springer resolution*, $f_{\mathfrak{p}} : T^*(G/P) \to \overline{\mathcal{O}_{\mathfrak{p}}}$, where P is a parabolic subgroup of a semisimple complex algebraic group G and $\mathcal{O}_{\mathfrak{p}}$ denotes the Richardson orbit associated to $\mathrm{Lie}(P) = \mathfrak{p}$. Firstly, we obtain a result on the dimension of the fibers of $f_{\mathfrak{p}}$, (cf. Theorem 2.1) which is a generalization of a Steinberg's work [26,27], the last result will allow us to describe some irreducible components of the fibers of $f_{\mathfrak{p}}$ (cf. Proposition 2.4). Next, we restrict our study to the case $G = \mathbf{SL}(n, \mathbb{C})$; we will give a description of the intersection $\mathcal{O}_{\mathfrak{p}}$ with the nilpotent radical of $\mathfrak{p}$ (cf. Theorem 3.3), this will help us to describe the closure of the intersection of the nilpotent radical of $\mathfrak{p}$ with every *adjacent* nilpotent orbit to $\mathcal{O}_{\mathfrak{p}}$ (cf. Theorem 3.7), and we will give a complete description of the generalized Springer fibers above the elements of such orbit (cf. Theorem 3.9) showing that those fibers are isomorphic to a Dynkin curve or to a projective space. Finally, by adopting Esnault's work we will find in some cases that the generalized Springer resolution restricts to the minimal resolution of some rational double points of type A (cf. Theorem 4.6).

Let G be a semisimple (connected) complex algebraic group with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ on which G acts by the adjoint action. Fix a Cartan subalgebra $\mathfrak{h}$. Let $\mathcal{W}$ denote the associated Weyl group. We have the Chevalley–Cartan decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha,$$

where $\mathcal{R}$ is the root system of $\mathfrak{g}$ relatively to $\mathfrak{h}$. Let S be a set of simple roots of $\mathcal{R}$. Denote $\mathcal{R}^+$ (resp., $\mathcal{R}^-$) the positive roots (resp., negative roots) (w.r.t. S). Let $\mathfrak{b} := \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha$ be the standard Borel subalgebra (w.r.t. S). Let B be the Borel subgroup of G with $\mathrm{Lie}(B) = \mathfrak{b}$. Let P be a standard parabolic subgroup with $\mathrm{Lie(P)} = \mathfrak{p}$. The parabolic subalgebra $\mathfrak{p}$ is determined by a subset $S_{\mathfrak{p}} \subset S$. Denote $\mathcal{R}_{\mathfrak{p}}$ the root subsystem generated by $S_{\mathfrak{p}}$ and $\mathcal{W}_{\mathfrak{p}}$ the subgroup of $\mathcal{W}$ generated by the simple reflexions s_α

with $\alpha \in \mathcal{R}_{\mathfrak{p}}$. We also have

$$\mathfrak{p} = \mathfrak{l}_{\mathfrak{p}} \oplus \mathfrak{n}_{\mathfrak{p}} \text{ with } \mathfrak{l}_{\mathfrak{p}} := \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}_{\mathfrak{p}}} \mathfrak{g}_{\alpha} \text{ and } \mathfrak{n}_{\mathfrak{p}} := \sum_{\alpha \in \mathcal{R}^{+} - \mathcal{R}_{\mathfrak{p}}} \mathfrak{g}_{\alpha},$$

where $\mathfrak{l}_{\mathfrak{p}}$ is the Levi component and $\mathfrak{n}_{\mathfrak{p}}$ is the nilpotent radical of $\mathfrak{p}$. Denote $L_{\mathfrak{p}}$ and $U_{\mathfrak{p}}$, the connected algebraic subgroups of G with $\mathrm{Lie}(L_{\mathfrak{p}}) = \mathfrak{l}_{\mathfrak{p}}$ and $\mathrm{Lie}(U_{\mathfrak{p}}) = \mathfrak{n}_{\mathfrak{p}}$. Denote $\mathcal{W}^{S_{\mathfrak{p}}}$ (resp., ${}^{S_{\mathfrak{p}}}\mathcal{W}^{S_{\mathfrak{p}}}$) the set of the representatives of minimal length of the classes of $\mathcal{W}/\mathcal{W}_{\mathfrak{p}}$ (resp., of $\mathcal{W}_{\mathfrak{p}} \setminus \mathcal{W}/\mathcal{W}_{\mathfrak{p}}$). We have

$$\mathcal{W}^{S_{\mathfrak{p}}} = \{w \in \mathcal{W};\ w(S_{\mathfrak{p}}) \subset \mathcal{R}^{+}\}, \tag{1.1}$$

$${}^{S_{\mathfrak{p}}}\mathcal{W}^{S_{\mathfrak{p}}} = \{w \in \mathcal{W};\ w(S_{\mathfrak{p}}) \subset \mathcal{R}^{+} \text{ and } w^{-1}(S_{\mathfrak{p}}) \subset \mathcal{R}^{+}\}. \tag{1.2}$$

For every $\alpha \in \mathcal{R}$ denote U_{α} the unique unipotent subgroup of G such that $\mathrm{Lie}(U_{\alpha}) = \mathfrak{g}_{\alpha}$. For every $w \in \mathcal{W}^{S_{\mathfrak{p}}}$, let n_w denote a representative of w in $\mathrm{Norm}_G(\mathfrak{h})$; define

$$N_{\mathfrak{p}}(w) := \{\alpha \in \mathcal{R}^{+} \mid w^{-1}(\alpha) \in \mathcal{R}^{-} - \mathcal{R}_{\mathfrak{p}}\} \tag{1.3}$$

and $U_{\mathfrak{p},w}$ the unipotent subgroup of G generated by the subgroups U_{α} with $\alpha \in N_{\mathfrak{p}}(w)$. We have the well-known Bruhat–Tits decomposition (see [2, p.100]). Every element g in G can be uniquely written as the product $g = un_w p$, with $w \in \mathcal{W}^{S_{\mathfrak{p}}}$, $u \in U_{\mathfrak{p},w}$, and $p \in P$, and we also have

$$G = \coprod_{w \in {}^{S_{\mathfrak{p}}}\mathcal{W}^{S_{\mathfrak{p}}}} Pn_w P. \tag{1.4}$$

From general nilpotent orbit theory, recall that there is a unique nilpotent G-orbit $\mathcal{O}_{\mathfrak{p}}$ such that the set $\mathcal{O}_{\mathfrak{p}} \cap \mathfrak{n}_{\mathfrak{p}}$ is open and dense in $\mathfrak{n}_{\mathfrak{p}}$. Moreover, $\mathcal{O}_{\mathfrak{p}} \cap \mathfrak{n}_{\mathfrak{p}}$ is exactly a P-orbit and we have $\dim(\mathcal{O}_{\mathfrak{p}}) = 2\dim(\mathfrak{n}_{\mathfrak{p}})$ (cf. [19,26]). $\mathcal{O}_{\mathfrak{p}}$ is called the *Richardson orbit* associated to $\mathfrak{p}$.

Let $G \times^P \mathfrak{n}_{\mathfrak{p}}$ be the space obtained as the quotient of $G \times \mathfrak{n}_{\mathfrak{p}}$ by the right action of P given by $(g, x).p := (gp, p^{-1}.x)$ with $g \in G$, $x \in \mathfrak{n}_{\mathfrak{p}}$ and $p \in P$. By the Killing form we get the following identification $G \times^P \mathfrak{n}_{\mathfrak{p}} \simeq T^*(G/P)$. Let $g * x$ denote the class of (g, x) and $\mathscr{P} := G/P$. The map $G \times^P \mathfrak{n}_{\mathfrak{p}} \to \mathscr{P} \times \mathfrak{g}, g * x \mapsto (gP, g.x)$ is an embedding which identify $G \times^P \mathfrak{n}_{\mathfrak{p}}$ with the following closed subvariety of $\mathscr{P} \times \mathfrak{g}$ (see [21, p. 19]):

$$\mathcal{Y} := \{(gP, x) \mid x \in g.\mathfrak{n}_{\mathfrak{p}}\}. \tag{1.5}$$

The map $f_{\mathfrak{p}} : G\times^P \mathfrak{n}_{\mathfrak{p}} \to G, g * x \mapsto g.x$ is called the *generalized Springer resolution* and we have the following commutative diagram:

$$\begin{array}{ccc} G\times^P \mathfrak{n}_{\mathfrak{p}} & \xrightarrow{\simeq} & \mathcal{Y} \\ & {\scriptstyle f_{\mathfrak{p}}} \searrow \quad \swarrow {\scriptstyle pr_2} & \\ & \mathfrak{g} & \end{array} \tag{1.6}$$

where pr_2 is the second projection of $\mathscr{P}\times\mathfrak{g}$ on $\mathfrak{g}$. The map $f_{\mathfrak{p}}$ is proper (because G/P is complete) and its image is exactly $G.\mathfrak{n}_{\mathfrak{p}} = \overline{\mathcal{O}_{\mathfrak{p}}}$. Moreover, the fiber of $f_{\mathfrak{p}}$ above points of $\mathcal{O}_{\mathfrak{p}}$ is finite; it is a birational map when $G^x \subset P$, where G^x is the stabilizer of x in G and $x \in \mathcal{O}_{\mathfrak{p}}$, this happens in the particular case $P = B$ is a Borel subgroup of G [26, Theorem 1, p. 129], and in this case the map $f_{\mathfrak{b}}$ is a desingularization of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ and is called the *Springer resolution* [23].

In case $G = \mathbf{SL}(n,\mathbb{C})$, the generalized Springer resolution is birational [3], moreover every nilpotent orbit is a Richardson orbit for an appropriate parabolic subgroup of $\mathbf{SL}(n,\mathbb{C})$ [6, p. 112]. So the generalized Springer resolutions are the desingularizations of the closures of the nilpotent orbits.

Let x be a nilpotent element in $\mathfrak{n}_{\mathfrak{p}}$. By (1.6) we have

$$\begin{aligned} \mathscr{P}_x &:= f_{\mathfrak{p}}^{-1}(x) = \{g.P \in \mathscr{P} \mid x \in g.\mathfrak{n}_{\mathfrak{p}}\}, \\ \mathscr{P}_x &= \{g.P \in \mathscr{P} \mid g^{-1}.x \in \mathfrak{n}_{\mathfrak{p}}\}. \end{aligned} \tag{1.7}$$

Following [22], let G^x be the stabilizer of x in G, G^{xo} denote its neutral component and $A(x) := G^x/G^{xo}$ the component group. Let $\{C_\sigma\}_{\sigma\in H}$ (resp., $\{D_i\}_{1\leqslant i\leqslant m}$) denote the set of the irreducible components of $\mathscr{P}_x$ (resp., of $G.x \cap \mathfrak{n}_{\mathfrak{p}}$). We have a surjective map $\pi : H \to \{1,\dots,m\}$ such that for every $1\leqslant i\leqslant m$ the set $H_i := \pi^{-1}(i)$ is exactly an orbit under the action of $A(x)$.

In case $G = \mathbf{SL}(n,\mathbb{C})$, the subgroup $G^x \subset \mathbf{GL}(n,\mathbb{C})^x$ is always connected (it is an open set of the space $\{g \in \mathfrak{gl}(n,\mathbb{C}) \mid gx = xg\}$), then π is a bijection between the irreducible components of $\mathscr{P}_x$ and the irreducible components of $G.x \cap \mathfrak{n}_{\mathfrak{p}}$.

Let us give a brief outline of the contents of the paper.

- In Section 2, we will give a fundamental result about the dimension of the generalized Springer resolution; this will help us to give a description of some irreducible components of the generalized Springer fibers.
- In Section 3, we are interested in case $G = \mathbf{SL}(n,\mathbb{C})$. Our main result gives a complete description of the generalized Springer fibers for elements in an *adjacent* nilpotent orbit to $\mathcal{O}_{\mathfrak{p}}$.

- In Section 4, we adopt Esnault's work to show in some cases that the generalized Springer resolution restricts to the minimal resolution of some rational double points of type *A*.

2. Generalities

The Springer resolution has been intensively studied by many mathematicians as N. Spaltenstein, G. Kempf, R. Steinberg, P. Slodowy,…. R. Steinberg has established the following formula which related the dimension of the Springer fiber above a nilpotent element with the dimension of the stabilizer of x in G: $\dim(f_{\mathfrak{b}}^{-1}(x)) = \frac{1}{2}(\dim(G^x) - r))$, where r is the rank of G (cf. [27, p. 133; 27, p. 217]). By studying his proof we have obtained the following generalization:

Theorem 2.1. *For every element* $x \in \overline{\mathcal{O}}_{\mathfrak{p}} = \operatorname{Im} f_{\mathfrak{p}}$ *we have*

$$\dim(f_{\mathfrak{p}}^{-1}(x)) \leqslant \tfrac{1}{2}(\dim(G^x) - \dim(\mathfrak{l}_{\mathfrak{p}})).$$

Proof. For every $x \in \overline{\mathcal{O}}_{\mathfrak{p}}$, denote $\mathcal{O}_x$ the G-orbit of x. Consider the subvariety V of $\mathfrak{g} \times \mathscr{P} \times \mathscr{P}$ defined by

$$V := \{(y, g.P, g'.P) \in \mathcal{O}_x \times \mathscr{P} \times \mathscr{P};\ \ y \in g.\mathfrak{n}_{\mathfrak{p}} \cap g'.\mathfrak{n}_{\mathfrak{p}}\}.$$

Then V is a closed G-variety and is a fibration above $\mathcal{O}_x$ whose fibers are isomorphic to $\mathscr{P}_x \times \mathscr{P}_x$. We deduce that

$$\dim(V) = 2\dim(\mathscr{P}_x) + \dim(\mathcal{O}_x). \tag{2.1}$$

By the Bruhat–Tits decomposition we have a disjoint union $V = \coprod_{w \in {}^{S_{\mathfrak{p}}}\mathcal{W}^{S_{\mathfrak{p}}}} V_w$. Let $w \in {}^{S_{\mathfrak{p}}}\mathcal{W}^{S_{\mathfrak{p}}}$ and let n_w be a representative of w in $\operatorname{Norm}_G(\mathfrak{h})$, then $V_w := \{(y, g.P, g.n_w.P) \in V\}$. In particular, we have

$$\dim(V) = \max_{w \in {}^{S_{\mathfrak{p}}}\mathcal{W}^{S_{\mathfrak{p}}}} \dim(V_w).$$

We can identify V_w with a subvariety of $\mathcal{O}_x \times [G/(P \cap n_w.P)]$ by the following morphism:

$$\phi : V_w \to \mathcal{O}_x \times [G/(P \cap n_w.P)],$$

$$(y, g.P, g.n_w.P) \mapsto (y, g(P \cap n_w.P)).$$

Moreover, the projection $p : V_w \to G/(P \cap n_w.P)$ allows us to see that the fiber above $g(P \cap n_w.P)$ is exactly $\mathcal{O}_x \cap g\mathfrak{n}_{\mathfrak{p}} \cap gn_w.\mathfrak{n}_{\mathfrak{p}} \simeq \mathcal{O}_x \cap \mathfrak{n}_{\mathfrak{p}} \cap n_w.\mathfrak{n}_{\mathfrak{p}}$. Then V_w is a G-bundle

above the space $G/(P \cap n_w.P)$ with the fibers isomorphic to $\mathcal{O}_x \cap \mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}$. We deduce that

$$\dim(V_w) = \dim(G) - \dim(P \cap n_w.P) + \dim(\mathcal{O}_x \cap \mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}). \tag{2.2}$$

But

$$\dim(P \cap n_w.P) = \dim(\mathfrak{p} \cap n_w.\mathfrak{p}),$$

$$\dim(\mathfrak{p} \cap n_w.\mathfrak{p}) = \dim(\mathfrak{p} \cap n_w.\mathfrak{l}_\mathfrak{p}) + \dim(\mathfrak{l}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}) + \dim(\mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}).$$

The element $w \in \mathcal{W}$ permutes the roots, so $n_w.\mathfrak{l}_\mathfrak{p} = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}_\mathfrak{p}} \mathfrak{g}_{w(\alpha)}$ and the lines $\mathfrak{g}_{w(\alpha)}$ which are not contained in $\mathfrak{p}$ are exactly those for which $w(\alpha) \in \mathcal{R}^- - \mathcal{R}_\mathfrak{p}^-$. We deduce that

$$\dim(\mathfrak{p} \cap n_w.\mathfrak{l}_\mathfrak{p}) = \dim(\mathfrak{l}_\mathfrak{p}) - \operatorname{card}\{\alpha \in \mathcal{R}_\mathfrak{p};\ \ w(\alpha) \in \mathcal{R}^- - \mathcal{R}_\mathfrak{p}^-\ \}.$$

As $w \in {}^{S_\mathfrak{p}}\mathcal{W}^{S_\mathfrak{p}}$, for every $\alpha \in \mathcal{R}_\mathfrak{p}^+$ by (1.2) we have $w(\alpha) \in \mathcal{R}^+$, then

$$\{\alpha \in \mathcal{R}_\mathfrak{p};\ \ w(\alpha) \in \mathcal{R}^- - \mathcal{R}_\mathfrak{p}^-\} = \{\alpha \in \mathcal{R}_\mathfrak{p}^-;\ \ w(\alpha) \in \mathcal{R}^- - \mathcal{R}_\mathfrak{p}^-\}.$$

This remark gives us

$$\dim(\mathfrak{p} \cap n_w.\mathfrak{l}_\mathfrak{p}) = \dim(\mathfrak{l}_\mathfrak{p}) - \operatorname{card}\{\alpha \in \mathcal{R}_\mathfrak{p}^-;\ \ w(\alpha) \in \mathcal{R}^- - \mathcal{R}_\mathfrak{p}^-\ \}.$$

By symmetry we get

$$\dim(\mathfrak{p} \cap n_w.\mathfrak{l}_\mathfrak{p}) = \dim(\mathfrak{l}_\mathfrak{p}) - \operatorname{card}\{\alpha \in \mathcal{R}_\mathfrak{p}^+;\ \ w(\alpha) \in \mathcal{R}^+ - \mathcal{R}_\mathfrak{p}^+\ \}. \tag{2.3}$$

With the same argument we get

$$\begin{aligned}\dim(\mathfrak{l}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}) &= \operatorname{card}\{\beta \in \mathcal{R}_\mathfrak{p};\ \ \beta = w(\alpha)\ \alpha \in \mathcal{R}^+ - \mathcal{R}_\mathfrak{p}^+\ \} \\ &= \operatorname{card}\{\beta \in \mathcal{R}_\mathfrak{p};\ \ w^{-1}(\beta) \in \mathcal{R}^+ - \mathcal{R}_\mathfrak{p}^+\ \} \\ &= \operatorname{card}\{\beta \in \mathcal{R}_\mathfrak{p}^+;\ \ w^{-1}(\beta) \in \mathcal{R}^+ - \mathcal{R}_\mathfrak{p}^+\ \}. \end{aligned} \tag{2.4}$$

With the same argument with $w \in {}^{S_\mathfrak{p}}\mathcal{W}^{S_\mathfrak{p}}$ (cf. 1.2) we get

$$\dim(\mathfrak{p} \cap n_{w^{-1}}.\mathfrak{l}_\mathfrak{p}) = \dim(\mathfrak{l}_\mathfrak{p}) - \operatorname{card}\{\alpha \in \mathcal{R}_\mathfrak{p}^+;\ \ w^{-1}(\alpha) \in \mathcal{R}^+ - \mathcal{R}_\mathfrak{p}^+\ \} \tag{2.5}$$

and

$$\dim(\mathfrak{l}_\mathfrak{p} \cap n_{w^{-1}}.\mathfrak{n}_\mathfrak{p}) = \mathrm{card}\{\beta \in \mathcal{R}_\mathfrak{p}^+;\ \ w(\beta) \in \mathcal{R}^+ - \mathcal{R}_\mathfrak{p}^+\ \}. \tag{2.6}$$

On the other hand,

$$2\dim(\mathfrak{p} \cap n_w.\mathfrak{p}) = \dim(\mathfrak{p} \cap n_w.\mathfrak{p}) + \dim(\mathfrak{p} \cap n_{w^{-1}}.\mathfrak{p}).$$

With (2.3), (2.4), (2.5) and (2.6) we get

$$2\dim(\mathfrak{p} \cap n_w.\mathfrak{p}) = 2\dim(\mathfrak{l}_\mathfrak{p}) + 2\dim(\mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}). \tag{2.7}$$

With the relations (2.2) and (2.3) we get

$$\dim(V_w) = \dim(G) - \dim(\mathfrak{l}_\mathfrak{p}) - \dim(\mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}) + \dim(\mathcal{O}_x \cap \mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}).$$

As $\dim(\mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}) - \dim(\mathcal{O}_x \cap \mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}) \geqslant 0$, with (2.1) and $\dim(\mathcal{O}_x) = \dim(G) - \dim(G^x)$ we deduce that

$$\dim(\mathscr{P}_x) \leqslant \tfrac{1}{2}(\dim(G^x) - \dim(\mathfrak{l}_\mathfrak{p})). \qquad \square$$

Remark 2.2. The author thanks the anonymous referee for indicating that the last theorem was a special case of a result obtained by Springer (see [24, Lemma 4.2; 18, Proposition 1.2]).

Moreover, the above relation is an equality if $\dim(\mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}) - \dim(\mathcal{O}_x \cap \mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}) = 0$, so we have

Corollary 2.3. *For every element* $x \in \overline{\mathcal{O}_\mathfrak{p}}$ *we have*: $\dim(f_\mathfrak{p}^{-1}(x)) = \frac{1}{2}(\dim(G^x) - \dim(\mathfrak{l}_\mathfrak{p}))$ *if and only if* $\mathcal{O}_x \cap \mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}$ *is dense in* $\mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}$ *for an element* $w \in {}^{S_\mathfrak{p}}\mathcal{W}^{S_\mathfrak{p}}$.

An immediate application of this theorem is the possibility to describe certain irreducible components of the fibers of the generalized Springer resolution $f_\mathfrak{p}$.

Proposition 2.4. *Let* P *be a standard parabolic subgroup. Let* Q *be a parabolic subgroup which contains* P *and let* Q' *be a parabolic subgroup in the conjugacy class of* Q. *Denote* $\mathrm{Lie}(P) = \mathfrak{p}$, $\mathrm{Lie}(Q) = \mathfrak{q}$ *and* $\mathrm{Lie}(Q') = \mathfrak{q}'$. *Let* $\mathfrak{n}_{\mathfrak{q}'}$ *be the nilpotent radical of* $\mathfrak{q}'$. *Let* $\mathcal{O}_\mathfrak{q}$ *be the Richardson orbit associated to* $\mathfrak{q}$. *Let* x *be a nilpotent element. Then we have the following equivalences*:

(i) $x \in \mathfrak{n}_{\mathfrak{q}'} \cap \mathcal{O}_\mathfrak{q}$.
(ii) $x \in \mathfrak{n}_{\mathfrak{q}'}$ and $\dim(Q/P) = \dim(f_\mathfrak{p}^{-1}(x))$.

(iii) *$g.Q/P$ is an irreducible component of $f_{\mathfrak{p}}^{-1}(x)$ where g is an element in G such that $gQg^{-1} = Q'$.*

Proof. (i) $\Rightarrow$ (ii) Let S be the set of simple roots of $\mathcal{R}$. Denote $S_{\mathfrak{q}} := \{\alpha \in S \mid \alpha \in \mathcal{R}_{\mathfrak{q}}\}$. Then $S_{\mathfrak{q}}$ is a basis of the root subsystem $\mathcal{R}_{\mathfrak{q}}$. Relatively to S (resp., $S_{\mathfrak{q}}$), denote l (resp., $l_{\mathfrak{q}}$) the length function on $\mathcal{W}$ (resp., $\mathcal{W}_{\mathfrak{q}}$). Denote $\mathfrak{n}^+ := \sum_{\alpha\in\mathcal{R}^+} \mathfrak{g}_\alpha$. Let $\mathrm{w}_{\mathfrak{q}}$ the unique element in $\mathcal{W}_{\mathfrak{q}}$ such that $\mathrm{w}_{\mathfrak{q}}(\mathcal{R}_{\mathfrak{q}}^+) = \mathcal{R}_{\mathfrak{q}}^-$. As $\mathrm{w}_{\mathfrak{q}} \in \mathcal{W}_{\mathfrak{q}}$, then we have [15, p. 114]

$$l_{\mathfrak{q}}(\mathrm{w}_{\mathfrak{q}}) = \mathrm{card}(\{\alpha \in \mathcal{R}_{\mathfrak{q}}^+ \mid \mathrm{w}_{\mathfrak{q}}^{-1}(\alpha) \in \mathcal{R}_{\mathfrak{q}}^-\}) = \mathrm{card}(\mathcal{R}_{\mathfrak{q}}^+).$$

But $l_{\mathfrak{q}}$ is only the restriction of l on $\mathcal{W}_{\mathfrak{q}}$, [15, p. 19], then we deduce that

$$l_{\mathfrak{q}}(\mathrm{w}_{\mathfrak{q}}) = l(\mathrm{w}_{\mathfrak{q}}) = \mathrm{card}(\{\alpha \in \mathcal{R}^+ \mid \mathrm{w}_{\mathfrak{q}}^{-1}(\alpha) \in \mathcal{R}^-\}).$$

As consequence we have $\mathcal{R}^+ \cap \mathrm{w}_{\mathfrak{q}}(\mathcal{R}^+) = \mathcal{R}^+ - \mathcal{R}_{\mathfrak{q}}^+$. Denote $w_{\mathfrak{q}}$ the unique element in $\mathcal{W}$ of minimal length in the double class of $\mathrm{w}_{\mathfrak{q}}$ in $\mathcal{W}_{\mathfrak{p}}\backslash\mathcal{W}/\mathcal{W}_{\mathfrak{p}}$. Then we have $\mathrm{w}_{\mathfrak{q}} = w_1 w_{\mathfrak{q}} w_2$ with $w_1, w_2 \in \mathcal{W}_{\mathfrak{p}}$. We deduce that for every $\alpha \in \mathcal{R}_{\mathfrak{q}}^+$ we have $w_{\mathfrak{q}} w_2(\alpha) \in \mathcal{R}_{\mathfrak{p}}$. By the same argument we have $w_2(\mathcal{R}_{\mathfrak{q}}^+ - \mathcal{R}_{\mathfrak{p}}^+) = \mathcal{R}_{\mathfrak{q}}^+ - \mathcal{R}_{\mathfrak{p}}^+$, we deduce that $w_{\mathfrak{q}}(\mathcal{R}_{\mathfrak{q}}^+ - \mathcal{R}_{\mathfrak{p}}^+) \subset \mathcal{R}_{\mathfrak{p}}$. As consequence we have

$$\mathfrak{n}_{\mathfrak{p}} \cap w_{\mathfrak{q}}(\mathfrak{n}_{\mathfrak{p}}) = \mathfrak{n}_{\mathfrak{q}}. \tag{2.8}$$

By Corollary 2.3 we have

$$\dim(f_{\mathfrak{p}}^{-1}(x)) = \tfrac{1}{2}(\dim(G^x) - \dim(\mathrm{l}_{\mathfrak{p}})).$$

With properties on Richardson orbit we can verify that $\dim(G^x) = \dim(\mathrm{l}_{\mathfrak{q}})$.

$$\dim(f_{\mathfrak{p}}^{-1}(x)) = \tfrac{1}{2}(\dim(\mathrm{l}_{\mathfrak{q}}) - \dim(\mathrm{l}_{\mathfrak{p}})) = \dim(Q/P).$$

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i) Say that $g.Q/P$ is an irreducible component of $f_{\mathfrak{p}}^{-1}(x)$ is equivalent to say that Q/P is an irreducible component of $f_{\mathfrak{p}}(g^{-1}.x)$, by (1.7) we have $g^{-1}.x \in \mathfrak{n}_{\mathfrak{p}}$. By (2.8) we can conjugate $g^{-1}.x$ with an element of Q to assume that we have $g^{-1}.x \in \mathfrak{n}_{\mathfrak{q}}$ (if not by (1.7) we would have $f_{\mathfrak{p}}^{-1}(x) = \emptyset$), and we have

$$\dim(Q/P) = \dim(f_{\mathfrak{p}}^{-1}(x)).$$

Then we have

$$\dim(\mathrm{l}_{\mathfrak{p}}) + 2\dim(f_{\mathfrak{p}}^{-1}(x)) = \dim(G^x) \geqslant \dim(P^{g^{-1}.x}).$$

As consequence

$$\begin{aligned}\dim(Q.(g^{-1}.x)) &= \dim(Q) - \dim(Q^{g^{-1}.x}) \\ &\geqslant \dim(Q) - (\dim(\mathfrak{l}_{\mathfrak{p}}) + 2\dim(f_{\mathfrak{p}}^{-1}(x)) = \dim(\mathfrak{n}_{\mathfrak{q}}).\end{aligned}$$

Now by properties on Richardson orbit we get the result. □

Remark 2.5. Let x be an element of $\mathfrak{g}$. A *polarization* of x, is a Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ such that $\kappa(x, [\mathfrak{q}, \mathfrak{q}]) = 0$ and $2\dim(\mathfrak{q}) = \dim(\mathfrak{g}^x) + \dim(\mathfrak{g})$ where $\kappa(\ ,\ )$ is the Killing form. Then every polarization is necessary a parabolic subalgebra and the nilpotent elements which admit polarizations are exactly the nilpotent elements of Richardson orbits, [7, p. 46]. Then the above proposition says that the different polarizations of x which contain $\mathfrak{p}$, give certain irreducible components of $f_{\mathfrak{p}}^{-1}(x)$.

3. Study in $\mathfrak{sl}(n, \mathbb{C})$

Now consider the case $G = \mathbf{SL}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. The subalgebra $\mathfrak{h}$ (resp., $\mathfrak{b}$) can be identified with the subvariety which consists of the diagonal matrices (resp., upper triangular matrices) of $\mathfrak{sl}(n, \mathbb{C})$. Denote $E_{i,j}$ the elementary matrices. The one-dimensional vector subspaces $\mathfrak{g}_\alpha$ are generated by the elementary matrices $E_{i,j}$ with $i \neq j$. For every $1 \leqslant i, j \leqslant n$, denote $p_{i,j}$ the coordinate projection corresponding to the line $\mathfrak{g}_\alpha$ generated by the elementary matrix $E_{i,j}$. The roots are given by the following linear forms $\{p_{i,i} - p_{j,j}\}$, with $i \neq j$. The simple roots $\{\alpha_i\}_{i=1,\ldots,n-1}$ are the linear form $\{p_{i,i} - p_{i+1,i+1}\}_{i=1,\ldots,n-1}$, and the Weyl group is identified with the symmetric group $\mathfrak{S}_n$, [4, p. 250/251]. Let s_k be the elementary transposition of $\mathfrak{S}_n$ which interchanges k and $k+1$.

The reasons to consider $\mathfrak{sl}(n, \mathbb{C})$ are on the one hand the generalized Springer resolution is a desingularization, and on the other hand every nilpotent orbit is a Richardson orbit for a suitable parabolic subalgebra. [6, p. 112].

Definition 3.1. A partition of n is a sequence of integers $\mathbf{p} = (p_1, p_2, \ldots, p_l)$ such that $p_i \geqslant 1$ and $\sum_{i=1}^{l} p_i = n$.

The standard parabolic subalgebras of $\mathfrak{sl}(n, \mathbb{C})$ are in bijective correspondence with the partitions of n. If $\mathbf{p} = (p_1, p_2, \ldots, p_l)$ is a partition of n, then the corresponding standard parabolic subalgebra has the following shape:

$$\begin{pmatrix} L_1 & * & \ldots & * \\ 0 & L_2 & * & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & L_l \end{pmatrix},$$

where $L_i \in \mathbf{GL}_{p_i \times p_i}(\mathbb{C})$. Two partitions $\mathbf{p} = (p_1, p_2, \ldots, p_l)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_l)$ of n are called *associated* if there is permutation $\sigma \in \mathfrak{S}_l$ such that $q_i = p_{\sigma(i)}$. A partition $\mathbf{p} = (p_1, p_2, \ldots, p_l)$ of n is said *ordered* if $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_l$. To the partition $\mathbf{p}$ corresponds the *Young diagram* whose rows are composed respectively, of $p_1, p_2, \ldots, p_l$ squares. If $\mathbf{p} = (p_1, p_2, \ldots, p_l)$ is an ordered partition of n we define its *dual partition* as the partition $\hat{\mathbf{p}} = (\hat{p_1}, \hat{p_2}, \ldots, \hat{p_t})$ with $\hat{p_i} := \mathrm{card}\{j;\ p_j \geqslant i\}$. We can notice that the dual partition is also ordered.

The nilpotent orbits in $\mathfrak{sl}(n, \mathbb{C})$ are parameterized by the ordered partitions of n [6, p. 32] corresponding to the lengths of the Jordan blocs arranged in decreasing order; if $\mathbf{p}$ is an ordered partition of n, we denote $\mathcal{O}_{\mathbf{p}}$ the corresponding nilpotent orbit. We have the following identities (see [6, p. 94]):

$$\dim(\ker(x^j)) = \sum_{i=1}^{j} \hat{p_i}$$

and

$$\mathrm{rank}(x^j) = \sum_{i>j} \hat{p_i}. \tag{3.1}$$

If $\mathbf{p} = (p_1, p_2, \ldots, p_l)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_k)$ are two-ordered partition of n, we denote $\mathbf{p} \geqslant \mathbf{q}$ if

$$\sum_{i=1}^{j} p_i \geqslant \sum_{i=1}^{j} q_i \quad \text{for every } j. \tag{3.2}$$

If $\mathbf{p} \geqslant \mathbf{q}$, we will say that the partition $\mathbf{p}$ *dominates* the partition $\mathbf{q}$.
The geometric interpretation of this order is given by

Proposition 3.2 (*Gerstenhaber [6, p. 95; 11]*).

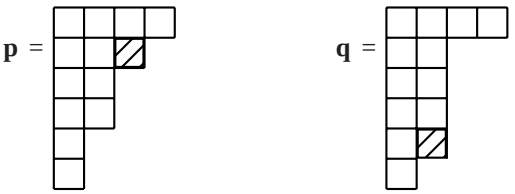
(i) $\mathbf{p} \geqslant \mathbf{q}$ *if and only if* $\overline{\mathcal{O}_{\mathbf{p}}} \supset \mathcal{O}_{\mathbf{q}}$.
(ii) *If* $\mathbf{p} \geqslant \mathbf{q}$ *such that for every* $\mathcal{O}_{\mathbf{q}} \subset \overline{\mathcal{O}} \subset \overline{\mathcal{O}_{\mathbf{p}}}$ *we have* $\mathcal{O} = \mathcal{O}_{\mathbf{q}}$ *or* $\mathcal{O} = \mathcal{O}_{\mathbf{p}}$. *Then we have*:

Case: 1 *there is an integer* i *such that* $p_k = q_k$ *for* $k \neq i, i+1$ *and* $q_i = p_i - 1 \geqslant q_{i+1} = p_{i+1} + 1$. *Then we have* $\mathrm{codim}_{\overline{\mathcal{O}_{\mathbf{p}}}}(\mathcal{O}_{\mathbf{q}}) = 2$.
Case: 2 *there is two integers* $i < j$ *such that* $p_k = q_k$ *for* $k \neq i, j$ *and* $q_i = p_i - 1 = q_j = p_j + 1$. *Then we have* $\mathrm{codim}_{\overline{\mathcal{O}_{\mathbf{p}}}}(\mathcal{O}_{\mathbf{q}}) = 2(j - i)$.

Such partitions are called "adjacent".

We can see the two cases by theirs Young diagrams in the following manner:
Case: 1

Case: 2



The first case consists to move a box in a corner to the next row, and the second case consists to move a box in a corner to the previous column.

If $\mathbf{p}$ is an ordered partition of n, then the nilpotent orbit $\mathcal{O}_{\mathbf{p}}$ is the Richardson orbit for every standard parabolic subalgebra whose corresponding partition is associated to $\hat{\mathbf{p}}$ (cf. [6, p. 112]), in particular we have $\mathcal{O}_{\mathbf{p}} = \mathcal{O}_{\mathfrak{p}}$.

Let $\mathbf{p}$ be an ordered partition of n and let $\mathfrak{p}$ be the standard parabolic subalgebra corresponding to the partition $\hat{\mathbf{p}}$. We have the decomposition $\mathfrak{p} = \mathfrak{l}_{\mathfrak{p}} \oplus \mathfrak{n}_{\mathfrak{p}}$. As the subalgebra $[\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}]$ is stable under $L_{\mathfrak{p}}$ which is a reductive group, there is a vector subspace $V_{\mathfrak{p}}$ such that $\mathfrak{n}_{\mathfrak{p}} = V_{\mathfrak{p}} \oplus [\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}]$ and $V_{\mathfrak{p}}$ is stable under $L_{\mathfrak{p}}$. In fact $V_{\mathfrak{p}}$ is unique, it is the direct sum of the subspaces $\mathfrak{g}_{\alpha}$, where α is the sum of simple roots in $S_{\mathfrak{p}}$ and of a unique simple root in $S - S_{\mathfrak{p}}$, [6, p. 123]. Here is the first important result:

Theorem 3.3. *Let* $\mathbf{p} = (p_1, p_2, \ldots, p_l)$ *an ordered partition of n. Let* $\mathcal{O}_{\mathbf{p}}$ *be the nilpotent orbit corresponding to* $\mathbf{p}$. *Let* $\mathfrak{p}$ *be the parabolic subalgebra corresponding to* $\hat{\mathbf{p}} = (\hat{p_1}, \hat{p_2}, \ldots, \hat{p_t})$. *Then we have*:

(i) *The subvariety* $\mathcal{O}_{\mathbf{p}} \cap V_{\mathfrak{p}}$ *is reduced to a unique* $L_{\mathfrak{p}}$*-orbit which is open and dense in* $V_{\mathfrak{p}}$.
(ii) $\mathcal{O}_{\mathbf{p}} \cap \mathfrak{n}_{\mathfrak{p}} = (\mathcal{O}_{\mathbf{p}} \cap V_{\mathfrak{p}}) \oplus [\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}] := \{x + y;\ \ x \in \mathcal{O}_{\mathbf{p}} \cap V_{\mathfrak{p}},\ \ y \in [\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}]\}$.

Proof. (i) Let $x = x_1 + x_2 \in \mathfrak{n}_{\mathfrak{p}} = V_{\mathfrak{p}} \oplus [\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}]$ be an element in $\mathfrak{n}_{\mathfrak{p}}$ with $x_1 \in V_{\mathfrak{p}}$ and $x_2 \in [\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}]$, then $x \in \mathcal{O}_{\mathbf{p}}$ *if and only if* $[\mathfrak{p}, x] = \mathfrak{n}_{\mathfrak{p}}$. Now $V_{\mathfrak{p}}$ is stable under

$\mathfrak{l}_\mathfrak{p}$ we have $[\mathfrak{l}_\mathfrak{p}, x_1] \subset V_\mathfrak{p}$, the condition

$$[\mathfrak{p}, x] = [\mathfrak{l}_\mathfrak{p}, x_1] + [\mathfrak{l}_\mathfrak{p}, x_2] + [\mathfrak{n}_\mathfrak{p}, x_1] + [\mathfrak{n}_\mathfrak{p}, x_2] = \mathfrak{n}_\mathfrak{p}$$

with

$$[\mathfrak{l}_\mathfrak{p}, x_1] \subset V_\mathfrak{p},$$

$$[\mathfrak{l}_\mathfrak{p}, x_2] + [\mathfrak{n}_\mathfrak{p}, x_1] + [\mathfrak{n}_\mathfrak{p}, x_2] \subset [\mathfrak{n}_\mathfrak{p}, \mathfrak{n}_\mathfrak{p}]$$

implies $[\mathfrak{l}_\mathfrak{p}, x_1] = V_\mathfrak{p}$, but this last equality is equivalent to the fact that the orbit $\mathcal{O}_2$ of x_1 under $L_\mathfrak{p}$ is open and dense in $V_\mathfrak{p}$. So if we denote $p : \mathfrak{n}_\mathfrak{p} = V_\mathfrak{p} \oplus [\mathfrak{n}_\mathfrak{p}, \mathfrak{n}_\mathfrak{p}] \to V_\mathfrak{p}, x_1 + x_2 \mapsto x_1$ the first projection, we get $p(\mathcal{O}_\mathbf{p} \cap \mathfrak{n}_\mathfrak{p}) \subset \mathcal{O}_2$, in particular we deduce that $L_\mathfrak{p}.p(\mathcal{O}_\mathbf{p} \cap \mathfrak{n}_\mathfrak{p}) = \mathcal{O}_2$. We can easily verify that the elements $x_1 \in V_\mathfrak{p}$ have the following shape:

$$\begin{pmatrix} 0 & M_1^1 & 0 & \dots & 0 \\ 0 & 0 & M_2^1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 & M_{t-1}^1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{3.3}$$

with $M_i^1 \in \mathrm{Mat}_{\hat{p}_i \times \hat{p}_{i+1}}(\mathbb{C})$ for $1 \leqslant j \leqslant t-1$, we can identify $V_\mathfrak{p} \simeq \mathrm{Mat}_{\hat{p}_1 \times \hat{p}_2}(\mathbb{C}) \times \cdots \times \mathrm{Mat}_{\hat{p}_{t-1} \times \hat{p}_t}(\mathbb{C})$ and write

$$x_1 = (M_1^1, \dots, M_{t-1}^1) \in \mathrm{Mat}_{\hat{p}_1 \times \hat{p}_2}(\mathbb{C}) \times \cdots \times \mathrm{Mat}_{\hat{p}_{t-1} \times \hat{p}_t}(\mathbb{C}). \tag{3.4}$$

Now it is easy to see that if we consider $x_1 \in V_\mathfrak{p}$ with the configuration (3.3), then we have

$${x_1}^2 = \begin{pmatrix} 0 & 0 & M_1^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & M_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & M_{t-2}^2 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 \end{pmatrix} \tag{3.5}$$

with $M_i^2 := M_i^1 M_{i+1}^1 \in \mathrm{Mat}_{\hat{p}_i \times \hat{p}_{i+2}}(\mathbb{C})$, so we can write $x_1^2 = (M_1^2, \ldots, M_{t-2}^2) \in \mathrm{Mat}_{\hat{p}_1 \times \hat{p}_2}(\mathbb{C}) \times \cdots \times \mathrm{Mat}_{\hat{p}_{t-2} \times \hat{p}_t}(\mathbb{C})$. Then by induction we can verify that

$$x_1{}^k = (M_1^k, \ldots, M_{t-k}^k) \in \mathrm{Mat}_{\hat{p}_1 \times \hat{p}_{k+1}}(\mathbb{C}) \times \cdots \times \mathrm{Mat}_{\hat{p}_{t-k} \times \hat{p}_t}(\mathbb{C}) \tag{3.6}$$

with $M_i^k := M_i^1 M_{i+1}^1 \ldots M_{i+k-1}^1$. Suppose that x_1 is of maximal rank. As $\mathrm{rank}(x_1) = \sum_{1 \leqslant i \leqslant t} \mathrm{rank}(M_i^1)$ and as $M_i^1 \in \mathrm{Mat}_{\hat{p}_i \times \hat{p}_{i+1}}(\mathbb{C})$ with $\hat{p}_i \geqslant \hat{p}_{i+1}$, we deduce that $\mathrm{rank}(x_1) = \sum_{i \geqslant 2} \hat{p}_i$. Likewise for every integer k, we have $\mathrm{rank}(x_1^k) = \sum_{1 \leqslant i \leqslant t-k+1} \mathrm{rank}(M_i^k)$, and as $M_i^k \in \mathrm{Mat}_{\hat{p}_i \times \hat{p}_{i+k}}(\mathbb{C})$ with $\hat{p}_i \geqslant \hat{p}_{i+k}$ we get $\mathrm{rank}(M_i^k) = \hat{p}_{i+k}$, so we get $\mathrm{rank}(x_1{}^k) = \sum_{i \geqslant k+1} \hat{p}_i$. By (3.1) and Proposition 3.2, we deduce that $x_1 \in \mathcal{O}_2$ if and only if $x_1 \in \mathcal{O}_{\mathbf{p}}$ if and only if $p(\mathcal{O}_{\mathbf{p}} \cap \mathfrak{n}_{\mathfrak{p}}) = \mathcal{O}_2 = \mathcal{O}_{\mathbf{p}} \cap V_{\mathfrak{p}}$ and this shows (i).

(ii) If we write $x = x_1 + x_2$ with $x_1 \in V_{\mathfrak{p}}$ and $x_2 \in [\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}]$, by the proof of (i) we have $x \in \mathcal{O}_{\mathbf{p}}$ if and only if $x_1 \in \mathcal{O}_{\mathbf{p}}$, and the result follows. □

Remark 3.4. (i) The above theorem give a characterization of the elements of the Richardson orbit $\mathcal{O}_{\mathfrak{p}}$; this will allow us to give a characterization of $\mathfrak{n}_{\mathfrak{p}} - (\mathcal{O}_{\mathfrak{p}} \cap \mathfrak{n}_{\mathfrak{p}})$, in particular this will help us to find out the irreducible components of $G.x \cap \mathfrak{n}_{\mathfrak{p}}$ when x is in an adjacent orbit to $\mathcal{O}_{\mathfrak{p}}$ (cf. Theorem 3.7).

(ii) We can also notice that this result is not always true for an other parabolic subalgebra corresponding to a partition associated to $\hat{\mathbf{p}}$. In the proof we use the fact that $\hat{\mathbf{p}}$ is ordered, this permits us to show that $\mathcal{O}_{\mathbf{p}} \cap V_{\mathfrak{p}} \neq \emptyset$.

Let M be an irreducible subvariety contained in the nilpotent cone $\mathcal{N}$ of $\mathfrak{sl}(n, \mathbb{C})$. As $\mathcal{N}$ is a finite union of nilpotent orbits, there is a unique nilpotent orbit $\mathcal{O}_M$ such that $\mathcal{O}_M \cap M$ is dense in M.

Definition 3.5. We will call $\mathcal{O}_M$ the orbit induced by M.

Now let $\mathbf{p} \geqslant \mathbf{q}$ be two adjacent ordered partitions of n. Let $\mathfrak{p}$ be the standard parabolic subalgebra corresponding to the partition $\hat{\mathbf{p}}$. As the image of the generalized Springer resolution $f_{\mathfrak{p}}$ is exactly $\overline{\mathcal{O}_{\mathfrak{p}}} = G.\mathfrak{n}_{\mathfrak{p}}$ $(= \overline{\mathcal{O}_{\mathbf{p}}})$, we deduce that $\mathcal{O}_{\mathbf{q}}$ has a non-empty intersection with $\mathfrak{n}_{\mathfrak{p}}$. On the other hand as $\mathbf{q}$ is adjacent to $\mathbf{p}$, the orbit $\mathcal{O}_{\mathbf{q}}$ is necessarily induced by every irreducible component of $\mathfrak{n}_{\mathfrak{p}} - (\mathcal{O}_{\mathbf{p}} \cap \mathfrak{n}_{\mathfrak{p}})$ for which the intersection with $\mathcal{O}_{\mathbf{q}}$ is non-empty.

Denote $M(m, n) := \mathrm{Mat}_{m \times n}(\mathbb{C})$. For every $0 \leqslant l \leqslant \min(m, n)$, denote $M_l(m, n) := \{x \in M(m, n) \mid \mathrm{rank}(x) \leqslant l\}$; the subvariety $M_l(m, n)$ is called a *determinantal variety*, it is an irreducible normal subvariety of codimension $(m-l)(n-l)$ in $M(m, n)$, moreover $M_l(m, n)$ coincides with the closure of the subvariety of $M(m, n)$ which consists of matrices of rank equal to l [1, Chapter II].

For every $1 \leqslant k \leqslant t-1$ let Y_k denote the subvariety of $\mathfrak{n}_{\mathfrak{p}}$ defined by

$$Y_k := \{x_1 + x_2 \in V_{\mathfrak{p}} + [\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}];\ \mathrm{rank}(M_k^1) < \hat{p}_{k+1}\}. \tag{3.7}$$

We can notice that Y_k is exactly the direct sum of a determinantal variety and a vector space, therefore it is an irreducible normal subvariety of codimension $\hat{p}_k - \hat{p}_{k+1} + 1$ in $\mathfrak{n}_\mathfrak{p}$.

Remark 3.6. (i) By Theorem 3.3, the subvarieties Y_k are exactly the irreducible components of $\mathfrak{n}_\mathfrak{p} - (\mathcal{O}_\mathbf{p} \cap \mathfrak{n}_\mathfrak{p})$. In case $\hat{p}_k = 1$, Y_k is a hyperplane in $\mathfrak{n}_\mathfrak{p}$ and coincides with the nilpotent radical of a parabolic subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ containing $\mathfrak{p}$.

(ii) Let $\mathbf{p} \geqslant \mathbf{q}$ be two-adjacent ordered partitions of n. If $\{Y_k\}_{k \in I}$ denotes the set of the irreducible components of $\mathfrak{n}_\mathfrak{p} - (\mathcal{O}_\mathbf{p} \cap \mathfrak{n}_\mathfrak{p})$ which induce $\mathcal{O}_\mathbf{q}$, then we have $(\mathcal{O}_\mathbf{q} \cap \mathfrak{n}_\mathfrak{p}) \subset \bigcup_{k \in I} Y_k$, in particular every irreducible component of $\mathcal{O}_\mathbf{q} \cap \mathfrak{n}_\mathfrak{p}$ is (at least) contained in a subvariety Y_k for a certain $k \in I$, and for every $k \in I$ the subvariety Y_k contains a unique irreducible component of $\mathcal{O}_\mathbf{q} \cap \mathfrak{n}_\mathfrak{p}$ which is dense in Y_k. As consequence, we have an injection from the set I to the set of the irreducible components of $\mathcal{O}_\mathbf{q} \cap \mathfrak{n}_\mathfrak{p}$. In particular, there is an injection from the set I to the set of the irreducible components of $f_\mathfrak{p}^{-1}(x)$ for $x \in \mathcal{O}_\mathbf{q}$. In fact, we will see at the end of the proof of Theorem 3.9 that we have in fact a bijection between these two sets.

Theorem 3.7. *With the notations above. Denote* $i_0 := \min\{j;\ \hat{p}_j \neq \hat{q}_j\}$ *and* $m_0 := \min\{j > i_0;\ \hat{p}_j \neq \hat{q}_j\}$. *Then the irreducible components of* $\mathfrak{n}_\mathfrak{p} - (\mathcal{O}_\mathbf{p} \cap \mathfrak{n}_\mathfrak{p})$ *which induce* $\mathcal{O}_\mathbf{q}$ *are all isomorphic and are the subvarieties* $\{Y_k\}_{i_0 \leqslant k \leqslant m_0 - 1}$. *Moreover we have* $\dim(\mathcal{O}_\mathbf{q} \cap \mathfrak{n}_\mathfrak{p}) = \frac{1}{2}\dim(\mathcal{O}_\mathbf{q})$.

Proof. We will consider two cases:

(i) *Case* $p_1 > q_1$: let $x = x_1 + x_2$ a nilpotent element in $\mathfrak{n}_\mathfrak{p}$, where $x_1 \in V_\mathfrak{p}$ with the formula

$$x_1 = (M_1^1, \ldots, M_{t-1}^1) \in \mathrm{Mat}_{\hat{p}_1 \times \hat{p}_2}(\mathbb{C}) \times \cdots \times \mathrm{Mat}_{\hat{p}_{t-1} \times \hat{p}_t}(\mathbb{C}).$$

(cf. (3.4)) and $x_2 \in [\mathfrak{n}_\mathfrak{p}, \mathfrak{n}_\mathfrak{p}]$. By Remark 3.6 (i), $x \in \mathfrak{n}_\mathfrak{p} - (\mathcal{O}_\mathbf{p} \cap \mathfrak{n}_\mathfrak{p})$ if and only if there is an integer $j \in \{1, \ldots, t-1\}$ such that $\mathrm{rank}(M_j^1) < \hat{p}_{j+1}$. By hypothesis $p_1 > q_1$, and by Proposition 3.2 we get

$$q_k = \begin{cases} p_1 - 1 & \text{if } k = 1, \\ p_2 + 1 & \text{if } k = 2, \\ p_k & \text{otherwise.} \end{cases} \tag{3.8}$$

With the dual partition we get $\hat{q}_{i_0} = 2$, $\hat{p}_{i_0} = 1$, $\hat{q}_k = \hat{p}_k$ for every $i_0 < k < t$ and $\hat{q}_t = 0$, $\hat{p}_t = 1$, in particular we deduce that for every $i_0 \leqslant k \leqslant t-1$ we have $M_k^1 \in \mathbb{C}$. So for every $i_0 \leqslant k \leqslant t-1$, $Y_k = \{p_{l_k, l_{k+1}} = 0\} \cap \mathfrak{n}_\mathfrak{p}$ is a hyperplane in $\mathfrak{n}_\mathfrak{p}$ with $l_k := \sum_{j \leqslant k} \hat{p}_j$ (cf. Remark 3.6 (i)), so Y_k is exactly the nilpotent radical of the standard parabolic subalgebra corresponding to the set of simple roots $S_\mathfrak{p} \cup \{\alpha_{l_k}\}$; now we can verify that such standard parabolic subalgebras correspond to partitions associated to $\hat{\mathbf{q}}$. So these subvarieties $\{Y_k\}_{i_0 \leqslant k \leqslant t-1}$ induce the nilpotent orbit $\mathcal{O}_\mathbf{q}$. Now, we have to show that the other irreducible components of $\mathfrak{n}_\mathfrak{p} - (\mathcal{O}_\mathbf{p} \cap \mathfrak{n}_\mathfrak{p})$ do not

induce $\mathcal{O}_{\mathbf{q}}$. Consider $k < i_0$ such that $\operatorname{rank}(M_k^1) < \hat{p}_{k+1}$, by (3.6) we necessary have $\operatorname{rank}(M_1^k) = \operatorname{rank}(M_1^1 M_2^1 \dots M_k^1) < \hat{p}_{j+1}$. In particular we have $\operatorname{rank}(x^j) < \sum_{i>j} \hat{p}_i$. By (3.1), $x \in \mathcal{O}_{\mathbf{q}}$ if and only if $\operatorname{rank}(x^j) = \sum_{i>j} \hat{q}_i$. But $j < i_0$, then $\hat{p}_i = \hat{q}_i$ for every $i \leqslant j < i_0$, as consequence $\sum_{i>j} \hat{q}_i = \sum_{i>j} \hat{p}_i$ and the result follows.

(ii) *Case* $p_1 = q_1$: we have $\hat{q}_{i_0} - 1 = \hat{p}_{i_0} = \hat{p}_{i_0+1} = \cdots = \hat{p}_{m_0-1} \geqslant 2$ and $\hat{q}_{m_0} = \hat{p}_{m_0} - 1$. Denote $X_k := \{x \in \mathfrak{n}_{\mathfrak{p}} \mid \operatorname{rank}(M_k^1) = \hat{p}_{k+1} - 1\}$. Then by [1, p. 71], we have

$$Y_k = \overline{X_k}. \tag{3.9}$$

But for every $i_0 \leqslant k \leqslant m_0 - 1$, we can verify that $X_k \cap \mathfrak{n}_{\tilde{\mathfrak{p}}} \neq \emptyset$, where $\tilde{\mathfrak{p}}$ is the standard parabolic subalgebra corresponding to the partition $(\hat{p}_1, \dots, \hat{p}_{k-1}, \hat{p}_k + 1, \hat{p}_{k+1} - 1, \hat{p}_{k+2}, \dots, \hat{p}_l)$, and the last partition is associated to the partition $\hat{\mathbf{q}}$. By this remark and by (3.9) we deduce that Y_k induces a nilpotent orbit $\mathcal{O} \subset \overline{\mathcal{O}_{\mathbf{q}}}$. Let $\mathfrak{n}_{\mathfrak{b}}$ be the nilpotent radical of the standard Borel subalgebra $\mathfrak{b}$. Then we have $\dim(\mathfrak{n}_{\mathfrak{b}} \cap \mathcal{O}) = \frac{1}{2}\dim(\mathcal{O})$ [22]. As $Y_k \subset \mathfrak{n}_{\mathfrak{p}} \subset \mathfrak{n}_{\mathfrak{b}}$, we deduce that

$$\dim(Y_k) \leqslant \tfrac{1}{2}\dim(\mathcal{O}) \leqslant \tfrac{1}{2}\dim(\mathcal{O}_{\mathbf{q}}) = \dim(\mathfrak{n}_{\tilde{\mathfrak{p}}})$$

the last equality comes from the properties of Richardson orbits. But we have noticed that the partition $(\hat{p}_1, \dots, \hat{p}_{k-1}, \hat{p}_k + 1, \hat{p}_{k+1} - 1, \hat{p}_{k+2}, \dots, \hat{p}_l)$ is associated to $\hat{\mathbf{q}}$, we deduce that

$$\dim(\mathfrak{n}_{\tilde{\mathfrak{p}}}) = \dim(\mathfrak{n}_{\mathfrak{p}}) - \hat{p}_k + \hat{p}_{k+1} - 1.$$

As the variety Y_k is of codimension $\hat{p}_k - \hat{p}_{k+1} + 1$, (cf. p. 450), then we have

$$\dim(Y_k) = \tfrac{1}{2}\dim(\mathcal{O}) = \tfrac{1}{2}\dim(\mathcal{O}_{\mathbf{q}})$$

we deduce that $\mathcal{O} = \mathcal{O}_{\mathbf{q}}$.

To finish the proof we have to verify that the other irreducible components Y_j do not induce $\mathcal{O}_{\mathbf{q}}$ for $j \leqslant i_0 - 1$ or $m_0 \leqslant j$. But it is exactly the same reasoning as for the case (i) which consists to verify that $\dim(x^j) < \sum_{i>j} \hat{q}_i$ for every element $x \in Y_j$.

Finally, if $m_0 = i_0 + 1$ there is a unique irreducible component of $\mathfrak{n}_{\mathfrak{p}} - (\mathcal{O}_{\mathbf{p}} \cap \mathfrak{n}_{\mathfrak{p}})$ which induces $\mathcal{O}_{\mathbf{q}}$, and if $m_0 \geqslant i_0 + 2$ for every $i_0 \leqslant i, j \leqslant m_0$ we have $\hat{p}_i = \hat{p}_j$ and as consequence the matrices $\{M_k^1\}_{i_0 \leqslant k \leqslant m_0 - 1}$ are square matrices of same length; we deduce that the subvarieties $\{Y_k\}_{i_0 \leqslant k \leqslant m_0 - 1}$ are all isomorphic. □

Let $\alpha \in S - S_{\mathfrak{p}}$ be a simple root. Denote P_α the minimal standard parabolic subgroup associated to the simple root α.

Definition 3.8. A *projective line* of type α is a subset of G/P of the form $g.P_\alpha P/P$, where $g \in G$.

We can remark that $P_\alpha P/P \simeq P_\alpha/P_\alpha \cap P$. But $P_\alpha \cap P = B$, because α is not in $S_\mathfrak{p}$. So we get $P_\alpha P/P \simeq P_\alpha/B \simeq \mathbb{P}^1$. Two projective lines of the same type are disjoint or are equal and two projective lines of different types have at most a common point [26, p. 146].

Theorem 3.9 (*Main theorem*). *With the notations of the last theorem. Let* $\mathbf{p} \geqslant \mathbf{q}$ *be two adjacent ordered partitions. Let* $x \in \mathcal{O}_\mathbf{q} \cap \mathfrak{n}_\mathfrak{p}$.

(i) *If* $\operatorname{codim}_{\overline{\mathcal{O}_\mathbf{p}}}(\mathcal{O}_\mathbf{q}) = 2$, *then* $f_\mathfrak{p}^{-1}(x)$ *is a finite union of projective lines: for every* $i \in \{\sum_{u \leqslant k} \hat{p}_u\}_{i_0 \leqslant k \leqslant m_0 - 1}$ *there is a unique projective line of type* α_i *in* $f_\mathfrak{p}^{-1}(x)$. *Moreover,* $f_\mathfrak{p}^{-1}(x)$ *is the union of these projective lines which intersect themselves transversely. Finally, the projective lines of type* α_i *and* α_j *have a non-empty intersection if and only if* $i = \sum_{u \leqslant k} \hat{p}_u$ *and* $j = \sum_{u \leqslant l} \hat{p}_u$ *with* $l = k \pm 1$. *In particular,* $f_\mathfrak{p}^{-1}(x)$ *is isomorphic to the Dynkin curve in* $A_{m_0 - i_0}$.

(ii) *If* $\operatorname{codim}_{\overline{\mathcal{O}_\mathbf{p}}}(\mathcal{O}_\mathbf{q}) > 2$, *then* $f_\mathfrak{p}^{-1}(x)$ *is reduced to a unique irreducible component isomorphic to the projective space* $\mathbb{P}^{\hat{p}_{i_0} - \hat{p}_{m_0} + 1}$.

Proof. Like for the proof of the last theorem we will consider two cases.

(i) *Case* $p_1 > q_1$: for every $\sum_{u \leqslant i_0} \hat{p}_u \leqslant k \leqslant n-1$, denote $\mathfrak{q}_k$ the standard parabolic subalgebra whose associated parabolic subgroup Q_k is given by the subset of simples roots $S_\mathfrak{p} \cup \{\alpha_k\}$. We notice that $Q_k/P \simeq P_{\alpha_k} P/P$. Moreover, these standard parabolic subalgebras are associated to the dual partition $\hat{\mathbf{q}}$ (cf. Proof of the last theorem), by Proposition 2.4 we deduce that $f_\mathfrak{p}^{-1}(x)$ is a union of projective lines of type α_k for $\sum_{u \leqslant i_0} \hat{p}_u \leqslant k \leqslant n-1$, and for every type α_k we find a unique projective line of the same type.

Finally, for every $\sum_{u \leqslant i_0} \hat{p}_u \leqslant k, l \leqslant n-1$ such that $|k-l| \geqslant 2$ we can remark that the intersection of the two hyperplanes $\{p_{k,k+1} = 0\} \cap \mathfrak{n}_\mathfrak{p}$ and $\{p_{l,l+1} = 0\} \cap \mathfrak{n}_\mathfrak{p}$ in $\mathfrak{n}_\mathfrak{p}$ is exactly the nilpotent radical of a standard parabolic corresponding to a partition associated to $\hat{\mathbf{t}} = (\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_{i_0}, 2, 1, 1, \ldots, 1)$. Then we have $\hat{\mathbf{t}} \geqslant \hat{\mathbf{q}}$ and $\hat{\mathbf{t}} \neq \hat{\mathbf{q}}$. As consequence, we deduce that $\mathcal{O}_\mathbf{q} \cap \mathfrak{n}_{\mathfrak{q}_k} \cap \mathfrak{n}_{\mathfrak{q}_l} = \emptyset$, By Proposition 2.4 the corresponding projective lines $g_k.P_{\alpha_k} P/P$ and $g_l.P_{\alpha_l} P/P$ in $f_\mathfrak{p}^{-1}(x)$ have an empty intersection.

(ii) *Case* $p_1 = q_1$, by Proposition 3.2 if $\operatorname{codim}_{\overline{\mathcal{O}_\mathbf{p}}}(\mathcal{O}_\mathbf{q}) = 2$ we get $\hat{p}_{i_0} = \cdots = \hat{p}_{m_0 - 1} = \hat{p}_{m_0} \geqslant 2$, and if $\operatorname{codim}_{\overline{\mathcal{O}_\mathbf{p}}}(\mathcal{O}_\mathbf{q}) > 2$ then $m_0 = i_0 + 1$ and we get $\hat{p}_{i_0} > \hat{p}_{i_0+1} \geqslant 2$.

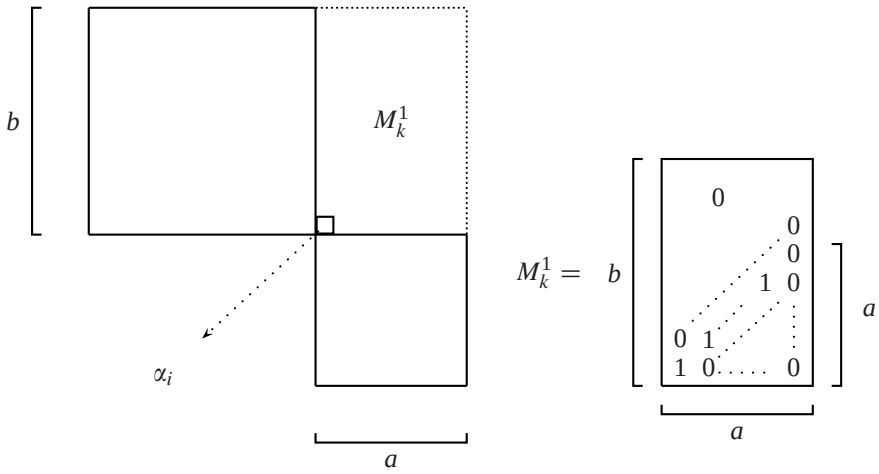
By Theorem 3.7, the irreducible components $\{Y_k\}_{i_0 \leqslant k \leqslant m_0 - 1}$ of $\mathfrak{n}_\mathfrak{p} - (\mathcal{O}_\mathbf{p} \cap \mathfrak{n}_\mathfrak{p})$ induce the nilpotent orbit $\mathcal{O}_\mathbf{q}$. Fix an integer $i_0 \leqslant k \leqslant m_0 - 1$, we now compute the irreducible component of $f_\mathfrak{p}^{-1}(x)$ corresponding to the subvariety Y_k (see Remark 3.6 (ii)).

Denote $b := \hat{p}_k$ and $a := \hat{p}_{k+1}$, we have $b \geqslant a$ and we get $b = a$ in case $m_0 \geqslant i_0 + 2$. Denote $i := \sum_{u \leqslant k} \hat{p}_u$ and let α_i the corresponding simple root. Consider $x = x_1 + x_2 \in Y_k \cap \mathcal{O}_\mathbf{q}$, with $x_1 = (M_1^1, \ldots, M_{t-1}^1) \in V_\mathfrak{p} \simeq \operatorname{Mat}_{\hat{p}_1 \times \hat{p}_2}(\mathbb{C}) \times \cdots \times \operatorname{Mat}_{\hat{p}_{t-1} \times \hat{p}_t}(\mathbb{C})$ (cf.

(3.3) and (3.4)) and $x_2 \in [\mathfrak{n}_\mathfrak{p}, \mathfrak{n}_\mathfrak{p}]$. By (3.9) we have to choose M_k^1 of rank $a-1$; for our computation we will choose x_1 with

$$M_k^1 = E_{i,i+1} + E_{i-1,i+2} + \cdots + E_{i-a+2,i+a-1}, \tag{3.10}$$

which is of rank $a-1$, i.e., M_k^1 has the following shape:



(3.11)

Let $w := w_1 w_2 w_3 \in \mathcal{W}$ be the element of the symmetric group defined by

$$w_1 := (s_{i-a+1}s_{i-a}\dots s_{i-b+1})(s_{i-a+2}s_{i-a+1}\dots s_{i-b+2})\dots(s_{i-1}s_{i-2}\dots s_{i-(b-a)-1}),$$

$$w_2 := s_{i+a-1}s_{i+a-2}\dots s_{i+1},$$

$$w_3 := s_{i-(b-a)}s_{i-(b-a)+1}\dots s_i.$$

Remark that w is written with the simple transpositions

$$s_{i-b+1}, s_{i-b+2}, \dots, s_{i+a-2}, s_{i+a-1},$$

then we deduce that

$$N_\mathfrak{b}(w) \subset \{\beta \mid \beta = \alpha_u + \alpha_{u+1} + \cdots + \alpha_v, \quad i-b+1 \leqslant u \leqslant v \leqslant i+a-1\}. \tag{3.12}$$

For every $i-b+1 \leqslant m \leqslant i-a+1$, consider the root $\beta_m := \alpha_m + \alpha_{m+1} + \cdots + \alpha_{i+a-1}$; then we get:

$$w^{-1}(\beta_m) = w_3^{-1} w_2^{-1} w_1^{-1}(\beta_m)$$

$$= w_3^{-1} w_2^{-1}(\alpha_{m+a-1} + \alpha_{m+a} + \cdots + \alpha_{i+a-1})$$
$$= w_3^{-1}(\alpha_{m+a-1} + \alpha_{m+a} + \cdots + \alpha_i).$$

Moreover by construction w_3 is written in a reduced form, then by Springer [25, p. 142] we have

$$N_{\mathfrak{b}}(w_3) = \{\alpha_{m+a-1} + \alpha_{m+a} + \cdots + \alpha_i \mid \ i - b + 1 \leqslant m \leqslant i - a + 1\}. \tag{3.13}$$

We deduce that

$$w^{-1}(\beta_m) < 0. \tag{3.14}$$

Let w be the representative of minimal length of the class of w in $\mathcal{W}/\mathcal{W}_{\mathfrak{p}}$, then we can verify that

$$N_{\mathfrak{p}}(w) = \{\beta \in \mathcal{R}^+ - \mathcal{R}_{\mathfrak{p}} \mid w^{-1}(\beta) < 0\} \tag{3.15}$$

by (3.14) we deduce that

$$\{\beta_m \mid i - b + 1 \leqslant m \leqslant i - a + 1\} \subset N_{\mathfrak{p}}(w)\}. \tag{3.16}$$

By (3.12) and (3.15) we get

$$N_{\mathfrak{p}}(w) \subset \{\beta \mid \beta = \alpha_u + \alpha_{u+1} + \cdots + \alpha_v, \quad i - b + 1 \leqslant u \leqslant i \leqslant v \leqslant i + a - 1\}. \tag{3.17}$$

On the other hand, by (3.10) we have

$$M_k^1 = \sum_{k=0}^{a-2} E_{i-k,i+k+1} \in \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{\alpha_{i-1}+\alpha_i+\alpha_{i+1}} \oplus \cdots \oplus \mathfrak{g}_{\alpha_{i-a+2}+\cdots+\alpha_{i+a-1}}. \tag{3.18}$$

For every $0 \leqslant k \leqslant a-2$ we have

$$w^{-1}(E_{i-k,i+k+1}) = w_3^{-1} w_2^{-1} w_1^{-1}(E_{i-k,i+k+1})$$
$$= w_3^{-1} w_2^{-1}(E_{i,i+k+1})$$
$$= w_3^{-1}(E_{i,i+k+2}) \in w_3^{-1}(\mathfrak{g}_{\alpha_i+\ldots\alpha_{i+k+1}}) = \mathfrak{g}_{w_3^{-1}(\alpha_i+\ldots\alpha_{i+k+1})}.$$

By (3.13) we have $w_3^{-1}(\alpha_i + \dots \alpha_{i+k+1}) > 0$. In particular we deduce that $x \in \mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}$, where n_w is a representative of w in $\mathrm{Norm}_G(\mathfrak{h})$. Let

$$u := (Id_n + \sum_{m=i-b+1}^{i-a+1} \lambda_m E_{m,i+a-1}\ ) \in \prod_{m=i-b+1}^{i-a+1} U_{\alpha_m+\dots+\alpha_{i+a}}, \tag{3.19}$$

with $\lambda_m \in \mathbb{C}$. By (3.16) we have $u \in U_{\mathfrak{p},w}$. Recall that if $v \in U_\delta$ and $Y \in \mathfrak{g}_\gamma$, where γ, δ are two roots, then $v.Y \in \sum_{t \geqslant 0} \mathfrak{g}_{\gamma+t.\delta}$, [26, p. 80], with (3.18) and (3.19) we have $uM_k^1u^{-1} = M_k^1$, and by (3.17) we deduce that

$$uxu^{-1} \in \mathfrak{n}_\mathfrak{p} \cap n_w.\mathfrak{n}_\mathfrak{p}. \tag{3.20}$$

By Theorems 2.1 and 3.7 we have

$$\begin{aligned}
\dim(f_\mathfrak{p}^{-1}(x)) &\leqslant \tfrac{1}{2}(\dim(G^x) - \dim(\mathfrak{l}_\mathfrak{p})) = \tfrac{1}{2}(\dim(G) - \dim(\mathcal{O}_\mathbf{q}) - \dim(\mathfrak{l}_\mathfrak{p})) \\
&\leqslant \tfrac{1}{2}(\dim(\mathcal{O}_\mathbf{p}) - \dim(\mathcal{O}_\mathbf{q})) = \tfrac{1}{2}(\dim(\mathcal{O}_\mathbf{p}) - 2\dim(\mathcal{O}_\mathbf{q} \cap \mathfrak{n}_\mathfrak{p})) \\
&\leqslant \dim(\mathfrak{n}_\mathfrak{p}) - \dim(\mathcal{O}_\mathbf{q} \cap \mathfrak{n}_\mathfrak{p}) = \mathrm{codim}_{\mathfrak{n}_\mathfrak{p}}(Y_k) = b - a + 1.
\end{aligned}$$

As $\mathrm{card}(\{\beta_m \mid \beta_m = \alpha_m + \alpha_{m+1} + \dots + \alpha_{i+a-1},\ i-b+1 \leqslant m \leqslant i+a-1\}) = b-a+1$, and with (3.20) we deduce that $\dim(f_\mathfrak{p}^{-1}(x)) = b-a+1$ and the irreducible component of $f_\mathfrak{p}^{-1}(x)$ corresponding to Y_k is given by the closure in G/P of the subvariety

$$\begin{aligned}
&(Id_n + \sum_{m=i-b+1}^{i-a+1} \lambda_m E_{m,i+a-1}) n_w P/P \\
&\quad = n_{w_1} n_{w_2} (Id_n + \sum_{m=i-b+1}^{i-a+1} \lambda_m E_{m+a-1,i+1}) n_{w_3} P/P.
\end{aligned}$$

Let r_d denote the simple reflexion which interchanges d and $d+1$ of the symmetric group $\mathfrak{S}_z$, with $z = b-a+2$ and let $F_{i,j}$ denote the elementary matrix in $\mathfrak{sl}(z, \mathbb{C})$. Consider P_z the maximal parabolic subgroup in $\mathbf{SL}(z, \mathbb{C})$ whose corresponding Weyl subgroup of $\mathfrak{S}_z$ is generated by $r_1, \dots, r_{z-2}$. Then we have the following isomorphism:

$$\begin{aligned}
&(Id_n + \sum_{m=i-b+1}^{i-a+1} \lambda_m E_{m+a-1,i+1}) n_{w_3} P/P \\
&\quad \simeq (Id_z + \sum_{m=1}^{z-1} \lambda_m F_{m,z}) n_{r_1 r_2 \dots r_{t-1}} P_z/P_z.
\end{aligned}$$

But the right member is exactly the big cell in $\mathbf{SL}(z,\mathbb{C})/P_z$, then we deduce that the irreducible component of $f_{\mathfrak{p}}^{-1}(x)$ corresponding to Y_k is isomorphic to $\mathbf{SL}(z,\mathbb{C})/P_z$ which is exactly the projective space of the hyperplanes in $\mathbb{C}^z$, therefore this irreducible component is isomorphic to $\mathbb{P}^{z-1} = \mathbb{P}^{b-a+1}$.

If $b = a$ we have $w_1 = s_{i-a+1}s_{i-a+2}\dots s_{i-1}$ and $w_3 = s_i$ and we get

$$\begin{aligned}(Id_n + \lambda E_{i-a+1,i+a-1})n_w P/P &= n_{w_1}(Id_n + \lambda E_{i,i+a-1})n_{w_2}n_{s_i}P/P\\ &= n_{w_1}n_{w_2}(Id_n + \lambda E_{i,i+1})n_{s_i}P/P.\end{aligned}$$

Therefore the corresponding irreducible component is a projective line of type α_i.

Let us show now that if $|k-l|\geqslant 2$, then $Y_k \cap Y_l \cap \mathcal{O}_{\mathbf{q}} = \emptyset$. As $|k-l|\geqslant 2$ we have $\hat{p}_{i_0} = \hat{p}_{i_0+1} = \dots = \hat{p}_{m_0-1} = a$. Let $x = x_1 + x_2 \in Y_k \cap Y_l$, with $x_1 = (M_1^1, \dots, M_{t-1}^1) \in V_{\mathfrak{p}}$ (cf. (3.3) and (3.4)) and $x_2 \in [\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}]$. In particular, we have $\operatorname{rank}(M_k^1) < a$ and $\operatorname{rank}(M_l^1) < a$. Let g be an element in P. We can write:

$$g = \begin{pmatrix} L_1 & * & \dots & * \\ 0 & L_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & L_t \end{pmatrix},$$

$$g\alpha g^{-1} = \begin{pmatrix} 0 & L_1M_1^1L_2 & * & \dots & * \\ 0 & 0 & L_2M_2^1L_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ \vdots & \vdots & \ddots & 0 & L_{t-1}M_{t-1}^1L_t \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{3.21}$$

with $L_i \in \mathbf{GL}(\hat{p}_i, \mathbb{C})$. Because of $|k-l|\geqslant 2$ and $\operatorname{rank}(M_k^1) < a$, $\operatorname{rank}(M_l^1) < a$, we can choose L_k, L_{k-1}, L_l and L_{l-1} such that the first column of $L_kM_k^1L_{k-1}$ and $L_lM_l^1L_{l-1}$ in $g\alpha g^{-1}$ is zero, in particular we get $g\alpha g^{-1} \in \mathfrak{n}_{\tilde{\mathfrak{p}}}$ where $\tilde{\mathfrak{p}}$ is the standard parabolic subalgebra corresponding to the partition

$$(\hat{p}_1, \dots, \hat{p}_{k-1}, \hat{p}_k + 1, \hat{p}_{k-1} - 1, \hat{p}_{k-2}, \dots, \hat{p}_{l-1}, \hat{p}_l + 1, \hat{p}_{l-1} - 1, \hat{p}_{l-2}, \dots, \hat{p}_t),$$

which is associated to the ordered partition

$$(\hat{p}_1, \dots, \hat{p}_{i_0-1}, a+1, a+1, a, \dots, a, a-1, a-1, \hat{p}_{m_0+1}, \dots, \hat{p}_t)$$

and the last partition strictly dominates the partition

$$\hat{\mathbf{q}} = (\hat{p}_1, \ldots, \hat{p}_{i_0-1}, a+1, a, a, \ldots, a, \hat{p}_{m_0+1}, \ldots, \hat{p}_t).$$

As consequence we find $Y_k \cap Y_l \cap \mathcal{O}_{\mathbf{q}} = \emptyset$, this means that the irreducible components in $f_{\mathfrak{p}}^{-1}(x)$ corresponding to Y_k and Y_l are disjoint.

Let us show now that we have a bijection between the set of the irreducible components Y_k which induce $\mathcal{O}_{\mathbf{q}}$ and the set of irreducible components of $\mathcal{O}_{\mathbf{q}} \cap \mathfrak{n}_{\mathfrak{p}}$ (cf. Remark 3.6 (ii)). By Theorem 3.7 we have $\mathcal{O}_{\mathbf{q}} \cap \mathfrak{n}_{\mathfrak{p}} \subset \bigcup_{i_0 \leqslant k \leqslant m_0-1} Y_k$, as consequence every irreducible component of $\mathcal{O}_{\mathbf{q}} \cap \mathfrak{n}_{\mathfrak{p}}$ is contained in a subvariety Y_k for a certain integer $i_0 \leqslant k \leqslant m_0 - 1$, then it suffices to show in each subvariety Y_k we only have a unique irreducible component of $\mathcal{O}_{\mathbf{q}} \cap \mathfrak{n}_{\mathfrak{p}}$. Case (i) is trivial because properties concerning Richardson orbit. Case (ii): let D_i $D_j$be two irreducible components of $\mathcal{O}_{\mathbf{q}} \cap \mathfrak{n}_{\mathfrak{p}}$ contained in Y_k. Let $x = x_1 + x_2 \in D_i$ (resp., $y = y_1 + y_2 \in D_j$) with $x_1 \in V_{\mathfrak{p}}$ and $x_2 \in [\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}]$ (resp., $y_1 \in V_{\mathfrak{p}}$ and $y_2 \in [\mathfrak{n}_{\mathfrak{p}}, \mathfrak{n}_{\mathfrak{p}}]$). By conjugating x (resp., y) by an appropriate element $g \in P$ (resp., $g' \in P$) with a good choice of the Levi component in g (resp., in g') (cf. (3.21)), we can suppose that the writing of M_k^1 in x_1 (resp., y_1) has the configuration (3.11) p. 453. The calculus which followed shows that the irreducible components C_i and C_j of $f_{\mathfrak{p}}^{-1}(x)$ corresponding to D_i and D_j are isomorphic and as consequence D_i and D_j are isomorphic. As Y_k induces the orbit $\mathcal{O}_{\mathbf{q}}$, one of the irreducible component of $\mathcal{O}_{\mathbf{q}} \cap \mathfrak{n}_{\mathfrak{p}}$ contained in Y_k is necessary dense in Y_k, if D_i is dense in Y_k we have the same property for D_j, so we necessary have $D_i = D_j$.

In cases (i) and (ii), if $|k - l| \geqslant 2$ then the irreducible components of $f_{\mathfrak{p}}^{-1}(x)$ associated corresponding to Y_k and Y_l have an empty intersection. On the other hand the generalized Springer resolution $f_{\mathfrak{p}}$ is birational and its image $\operatorname{Im} f_{\mathfrak{p}} = G.\mathfrak{n}_{\mathfrak{p}}$ is a normal variety [3, p. 448], by main Zariski Theorem [12, p. 280], the fibers of $f_{\mathfrak{p}}$ are connected, in particular we deduce that the projective lines in $f_{\mathfrak{p}}^{-1}(x)$ corresponding to Y_k and Y_l have a non-empty intersection if and only if $k = l \pm 1$, and the proof is complete. □

4. Application to the study of a germ's surface singularity

By keeping the notations of the last Section let $\mathbf{p} \geqslant \mathbf{q}$ be two-adjacent ordered partitions with $\operatorname{codim}_{\overline{\mathcal{O}_{\mathbf{p}}}}(\mathcal{O}_{\mathbf{q}}) = 2$. Let x be an element in $\mathcal{O}_{\mathbf{q}}$. This Section consists to rely the description of the singularity $(\overline{\mathcal{O}_{\mathbf{p}}} \cap T_x, x)$, where T_x is a transverse slice in $\mathfrak{g}$ to the orbit $\mathcal{O}_{\mathbf{q}}$ at the point x to the study of the fiber $f_{\mathfrak{p}}^{-1}(x)$.

Definition 4.1. Let M be a G-variety. A transverse slice in M to the orbit of x at the point x, is a locally closed subvariety T_x of M such that:

(a) $x \in T_x$;
(b) the morphism $\varphi : G \times T_x \to M, (g, Y) \mapsto g.Y$ is smooth;
(c) T_x has the minimal dimension for properties (a) and (b).

As we work with $\mathbb{C}$, then $\dim(T_x) = \operatorname{codim}_M(G.x)$, moreover if M is smooth then T_x is necessary smooth [20, p. 61].

To give such a transverse slice it is enough to take a vector subspace T_x which is supplementary to the tangent space of the orbit of x at the point x.

Let $x \in \mathcal{O}_{\mathbf{q}}$. By Jacobson–Morozov Theorem , there is a semisimple element h and a nilpotent element y in $\mathfrak{g}$ such that

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

Then by Representation theory of $\mathfrak{sl}(2, \mathbb{C})$ the subvariety $T_x := x + \mathfrak{g}^y$ is supplementary in $\mathfrak{g}$ to the orbit of $\mathcal{O}_{\mathbf{q}}$ at the point x, where $\mathfrak{g}^y$ is the centralizer of y in $\mathfrak{g}$. Denote:

$$M := \overline{\mathcal{O}_{\mathbf{p}}} \cap T_x \quad \text{and} \quad \hat{M} := f_{\mathfrak{p}}^{-1}(S).$$

Definition 4.2.

(a) Let M be a complex algebraic variety. A desingularization of M is a morphism $\pi : \hat{M} \to M$ such that π is a proper birational morphism and that $\hat{M}$ is a smooth variety.

(b) The normal variety M has rational singularities if for every desingularization $\pi : \hat{M} \to M$ we have $R^1\pi_*(\mathcal{O}_{\hat{M}}) = \{0\}$.

We have the following result:

Lemma 4.3.

(i) $M \subset \mathcal{O}_{\mathbf{p}} \cup \mathcal{O}_{\mathbf{q}}$;

(ii) *The morphism* $f_{\mathfrak{p}|\hat{M}} : \hat{M} \to M$ *is a desingularization of* M.

Proof. (i) Let us show that the elements in M come from $\mathcal{O}_{\mathbf{p}}$ and $\mathcal{O}_{\mathbf{q}}$. Let $\mathfrak{g} = \bigoplus V_i$ the decomposition of $\mathfrak{g}$ as sum of irreducible representations for $< x, h, y > \simeq \mathfrak{sl}(2, \mathbb{C})$. Every V_i contains a unique vector line $\mathfrak{g}_i$ such $[y, \mathfrak{g}_i] = 0$ (cf. [14, p. 33]), denote $n_i \in \mathbb{Z}$ the eigenvalue of ad_h for the subvariety $\mathfrak{g}_i$. By [14, p. 33] we have $\mathfrak{g}^y = \bigoplus \mathfrak{g}_i$. Denote $\lambda : \mathbb{C}^* \to G$ the unique parameter subgroup associated to the semisimple element h and let $z = x + v \in x + \mathfrak{g}^y$. We can suppose that $\mathfrak{g}^y$ is the direct sum of certain vector spaces $\mathfrak{g}_\alpha$ with $\alpha \in \mathcal{R}^-$, [6, p. 45/46], we have $y \in \mathfrak{g}_{-2} = \sum_{\alpha(h)=-2} \mathfrak{g}_\alpha$, then we have $n_i \leqslant 0$. If we write $v = \sum z_i$ with $z_i \in \mathfrak{g}_i$, then we have $\lambda(t).z = t^2 x + \sum t^{n_i} z_i$, where $z_i \in \mathfrak{g}_i$, and because of nilpotent orbits are stable under $\mathbb{C}^*$ we deduce that $z \in M$, $t \in \mathbb{C}^*$, $t^2\lambda(t^{-1}).z = x + \sum t^{2-n_i} z_i \in S \cap \mathcal{O}_z$. This shows that x is in the closure of the orbit of every element of M. But $x \in \mathcal{O}_{\mathbf{q}}$ which is adjacent to $\mathcal{O}_{\mathbf{p}}$, we deduce that $M \subset \mathcal{O}_{\mathbf{p}} \cup \mathcal{O}_{\mathbf{q}}$.

(ii) By construction we have locally $\mathfrak{g} \simeq T_x \times \mathcal{O}_{\mathbf{q}}$, and $\overline{\mathcal{O}_{\mathbf{p}}}$ is locally isomorphic to $M \times \mathcal{O}_{\mathbf{q}}$. As $\hat{M} = f_{\mathfrak{p}}^{-1}(M)$, we have

$$\hat{M} = \{(Y, g.P); \ Y \in S \cap g.\mathfrak{n}_{\mathfrak{p}}\}$$

and locally $T^*(G/P)$ is isomorphic to the space $\hat{M} \times \mathcal{O}_{\mathbf{q}}$. In particular $\hat{M}$ is smooth. And the map $f_{\mathfrak{p}|\hat{M}}$ is proper because G/P is complete; so $f_{\mathfrak{p}|\hat{M}} : \hat{M} \to M$ is a desingularization of M. □

Lemma 4.4 (*Hinich [13, p. 302]*). *$(M; x)$ is a normal surface with a rational singularity.*

Proof. We can remark that $\varphi^{-1}(\overline{\mathcal{O}_{\mathbf{p}}}) = G \times M$ and we have two smooth morphisms $\varphi_| : G \times M \to \overline{\mathcal{O}_{\mathbf{p}}}$ and $pr_2 : G \times M \to M$ at the point $(1_G, x)$. We deduce that $(\overline{\mathcal{O}_{\mathbf{p}}}, x)$ is normal if and only if the surface (M, x) is normal [8]. By [3,16] it was shown that every closure of nilpotent orbit in $\mathfrak{sl}(n, \mathbb{C})$ is normal. As consequence (M, x) is a normal surface with an isolated singularity.
By Theorem 5 in [9] we deduce that $G \times M$ has rational singularities.
The following diagram

$$\begin{array}{ccc} G \times \hat{M} & \xrightarrow{id \times f_{\mathfrak{p}|\hat{M}}} & G \times M \\ & {\scriptstyle p_1'} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & G & \end{array}$$

is a simultaneous desingularization of the fibers of p_1. By Theorem 3 in [9] we can deduce that M has rational singularities. □

Definition 4.5. A $f : \hat{M} \to (M, x)$ desingularization of a normal surface with a rational singularity x is called minimal if every irreducible component of the exceptional fiber $f^{-1}(x)$ has a self-intersection number different of -1.

The minimal desingularization exists up to isomorphism and every desingularization of (M, x) factorizes through the minimal desingularization. Moreover, the normal surfaces with a rational singularity for which every irreducible component of the exceptional fiber has a self-intersection -2 are well known and are obtained as quotients of $\mathbb{C}^2$ by finite subgroups of $\mathbf{SL}(2, \mathbb{C})$, [20, p. 72]. Such singularities are called *simple* or *rational double* points and are classified by the families A_r, D_r, E_6, E_7, E_8.

Here is the main theorem of this last Section.

Theorem 4.6. *Let $\mathbf{p} = (p_1, p_2, \ldots, p_l) \geqslant \mathbf{q} = (q_1, q_2, \ldots, q_k)$ two adjacent ordered partitions such that $\operatorname{codim}_{\overline{\mathcal{O}_{\mathbf{p}}}}(\mathcal{O}_{\mathbf{q}}) = 2$ and $p_1 > q_1$. Let $\mathfrak{p}$ be the standard subalgebra corresponding to the partition $\hat{\mathbf{p}}$. Denote $i_0 := \min\{j;\ \hat{p}_j \neq \hat{q}_j\}$ and $m_0 := \min\{j > i_0;\ \hat{p}_j \neq \hat{q}_j\}$. Let $x \in \mathcal{O}_{\mathbf{q}}$, and T_x be a transverse slice in $\mathfrak{sl}(n, \mathbb{C})$ to the orbit $\mathcal{O}_{\mathbf{q}}$ at the point x. Let $M := \overline{\mathcal{O}_{\mathbf{p}}} \cap T_x$ and $\hat{M} := f_{\mathfrak{p}}^{-1}(M)$. Then*

(i) *The morphism $f_{\mathfrak{p}|\hat{M}} : \hat{M} \to M$ is the minimal desingularization of the surface M.*
(ii) *The surface (M, x) is a normal surface with a simple singularity of type $A_{m_0 - i_0}$.*

The following calculus is exactly the same for which H. Esnault has done in the particular subregular case (see [10; 20, p. 88]). The reasoning is done in a more general context but we can apply only for the case $p_1 > q_1$:

By Theorem 3.9, $f_{\mathfrak{p}}^{-1}(x)$ is a finite union of projective lines of type α_i with $i \in \{\sum_{u \leqslant k} \hat{p}_u\}_{i_0 \leqslant k \leqslant m_0 - 1}$. To prove the theorem it remains to compute the self-intersection numbers of these projective lines in $\hat{M}$, i.e., to compute $c_1(P_{\alpha_i} P/P, \hat{M})$ the first Chern class of the normal bundle of each of these projectives lines in $\hat{M}$. Denote $N_{(P_{\alpha_i} P/P)/\hat{M}}$ the normal bundle of the projective line $P_{\alpha_i} P/P$ in $\hat{M}$. If $A \subset B \subset C$ are three smooth varieties then we have the short exact sequence of normal bundles:

$$0 \to N_{A/B} \to N_{A/C} \to N_{B/C_{|A}} \to 0.$$

We apply the last short exact sequence of normal bundles to the three smooth varieties $P_{\alpha_i} P/P \subset \hat{M} \subset T^*(G/P)$; but we have seen that $T^*(G/P)$ is locally trivial (it is locally isomorphic to $\hat{M} \times \mathcal{O}_{\mathbf{q}}$), as consequence the restrict normal bundle of $\hat{M}$ in $T^*(G/P)$ is isomorphic to the tangent bundle of $\mathcal{O}_{\mathbf{q}}$, the last one is trivial if we consider a small neighborhood of x in T_x. As consequence we have to compute $c_1(P_i/P, T^*(G/P))$.

Lemma 4.7. *Let P be standard parabolic subgroup of a semisimple complex algebraic group G. Let $\alpha_i \in S - S_{\mathfrak{p}}$ and denote P_i the parabolic subgroup corresponding to the subset of simple roots $S_{\mathfrak{p}} \cup \{\alpha_i\}$. Then the natural map $T^*(G/P) \simeq G \times^P \mathfrak{n}_{\mathfrak{p}} \to G/P_i, (g.P, Y) \mapsto g.P_i$ is a locally trivial G-fibration and we can identify $G \times^P \mathfrak{n}_{\mathfrak{p}}$ with the fiber bundle $G \times^{P_i} F_i$, where $F_i := P_i \times^P \mathfrak{n}_{\mathfrak{p}}$, $\mathfrak{n}_{\mathfrak{p}}$ is the nilpotent radical of* $\mathrm{Lie}(P)$.

Proof. The natural morphism

$$\phi : T^*(G/P) \simeq G \times^P \mathfrak{n}_{\mathfrak{p}} \to G/P_i, (g.P, Y) \mapsto g.P_i$$

is G-invariant so we have $G \times^P \mathfrak{n}_{\mathfrak{p}} \simeq G \times^{P_i} F_i$, where $F_i = \phi^{-1}(\bar{e})$ [21, p. 26]. It is easy to verify that

$$\begin{aligned} F_i &= \{(g.P, Y) \in G \times^P \mathfrak{n}_{\mathfrak{p}};\ Y \in g.\mathfrak{n}_{\mathfrak{p}} \text{ and } g.P \subset P_i\} \\ &\simeq P_i \times^P \mathfrak{n}_{\mathfrak{p}}. \qquad \square \end{aligned}$$

We can remark that the projective line $P_i P/P \simeq P_i/P$ is contained in F_i. As consequence we have $P_i/P \subset F_i \subset T^*(G/P)$ three smooth varieties. We deduce that the normal bundle of P_i/P in $T^*(G/P)$ is an extension of the normal bundle of P_i/P in F_i and of the restrict normal bundle of F_i in $T^*(G/P)$. But

$$T^*(G/P) \simeq G \times^P \mathfrak{n}_{\mathfrak{p}} \to G/P_i$$

is a locally trivial fibration over a smooth basis, then the restrict normal bundle of F_i in $T^*(G/P)$ is trivial because it is isomorphic to the trivial bundle with fiber the tangent space to G/P_i at the point $\overline{e}$, so its Chern classes are trivial.

We have to compute $c_1(P_i/P, F_i)$.

Lemma 4.8. *With the above notations. Let $x \in \mathfrak{n}_\mathfrak{p}$. Let $P_{\alpha_i}P/P$ a projective line in $f_\mathfrak{p}^{-1}(x)$. Then $c_1(P_{\alpha_i}P/P, T^*(G/P)) = -2$.*

Proof. It remains to compute $c_1(P_{\alpha_i}P/P, P_i \times^P \mathfrak{n}_\mathfrak{p})$. Let $\mathfrak{n}_{\mathfrak{p}_i}$ denote the nilpotent radical of $\mathrm{Lie}(P_i)$. Then we have the following P-invariant short exact sequence:

$$0 \to \mathfrak{n}_{\mathfrak{p}_i} \to \mathfrak{n}_\mathfrak{p} \to \mathfrak{n}_\mathfrak{p}/\mathfrak{n}_{\mathfrak{p}_i} \to 0.$$

As consequence we deduce that the short exact sequence of bundles:

$$0 \to P_i \times^P \mathfrak{n}_{\mathfrak{p}_i} \to P_i \times^P \mathfrak{n}_\mathfrak{p} \to P_i \times^P (\mathfrak{n}_\mathfrak{p}/\mathfrak{n}_{\mathfrak{p}_i}) \to 0.$$

But the P-module $\mathfrak{n}_{\mathfrak{p}_i}$ is obtained as the restriction of the P_i-module $\mathfrak{n}_{\mathfrak{p}_i}$, then the bundle $P_i \times^P \mathfrak{n}_{\mathfrak{p}_i}$ is trivial. Moreover by hypothesis $\mathfrak{n}_\mathfrak{p}/\mathfrak{n}_{\mathfrak{p}_i}$ is vector space of dimension 1 on which P acts via the simple root α_i: if we write $P = L_P.U_P$ where $L_P = C.(L_P, L_P)$ is the Levi component of P with C (resp., (L_P, L_P)) the center (resp., the derived subgroup) of L_P and U_P is its nilpotent radical. The subgroup U_P and (L_P, L_P) have no characters, the action of P on $\mathfrak{n}_\mathfrak{p}/\mathfrak{n}_{\mathfrak{p}_i}$ (induced by the adjoint action) comes from the action of C, therefore this action is reduced to the action of the maximal torus whose Lie algebra is $\mathfrak{h}$, and the differential of the action of the maximal torus is exactly given by the simple root α_i. Then we have

$$P_i \times^P (\mathfrak{n}_\mathfrak{p}/\mathfrak{n}_{\mathfrak{p}_i}) \simeq \mathbf{SL}(2, \mathbb{C}) \times^{B_2} \mathfrak{n}_2 \simeq T^*(\mathbf{SL}(2, \mathbb{C})/B_2)$$

where B_2 is the Borel subgroup of $\mathbf{SL}(2, \mathbb{C})$ and $\mathfrak{n}_2$ is the nilpotent radical of $\mathrm{Lie}(B_2)$. But $\mathbf{SL}(2, \mathbb{C})/B_2 \simeq \mathbb{P}^1$. As consequence we get

$$c_1(P_i/P, P_i \times^P \mathfrak{n}_\mathfrak{p}) = c_1(T^*(\mathbb{P}^1)) = -2. \qquad \square$$

Remark 4.9. Our approach is another way to get Theorem 4.6 (ii) which was already obtained in [17].

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