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Fat points of $\mathbb{P}^n$ whose support is contained in a linear proper subspace

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Abstract

In this paper we study fat points of $\mathbb{P}^n$ whose support is contained in a linear subspace of dimension r, with $r < n$, and we determine in some cases the Hilbert function, in other cases the regularity index (or an upper bound for it) in $\mathbb{P}^n$, in terms of what is known in $\mathbb{P}^r$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\{P_1,\dots,P_s\}$ be a set of s distinct points in the projective space $\mathbb{P}^n$, over an algebraically closed field K of characteristic 0.

If we assign to each P_i a 'multiplicity' that is, a positive integer m_i, we obtain a set of 'fat points' $X = \{(P_1,m_1),\dots,(P_s,m_s)\}$ of $\mathbb{P}^n$, the study of which consists in examining the hypersurfaces passing through each P_i with multiplicity at least m_i.

The Hilbert function of $X = \{(P_1,m_1),\dots,(P_s,m_s)\} \subset \mathbb{P}^n$ is defined

$$H(X,t) = dim_K(K[x_0,\dots,x_n])_t - dim_K I_t, \quad \forall t \geq 0,$$

where $I = \mathbf{p}_1^{m_1} \cap \cdots \cap \mathbf{p}_s^{m_s}$ and $\mathbf{p}_i$ is the ideal of P_i in $K[x_0,\dots,x_n]$.

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In this way we study the linear systems we are interested in, by using the homogeneous pieces of the graded ideal I. This approach allows us to use some of the technical advantages that are inherent in studying the Hilbert function of a graded ring.

In this paper, which started off from the thesis of one of the authors ([6]), we study s (distinct or fat) points of $\mathbb{P}^n$, whose support actually lies on a linear subspace isomorphic to a smaller $\mathbb{P}^r$, tacitly assuming the minimality of r, equivalently that the points generate $\mathbb{P}^r$ (in particular $s \geq r+1$).

We write out the Hilbert function of such points in $\mathbb{P}^n$ in terms of the one in $\mathbb{P}^r$ (Theorem 3.1). From this we deduce a comparison between the respective regularity indices and we give a necessary and sufficient condition for their equality.

In several cases in which the Hilbert function or the regularity index (or an upper bound for it) are known in $\mathbb{P}^r$, this allows us to explicitly compute the one in $\mathbb{P}^n$.

In particular, by using results by Catalisano [1], Catalisano and Gimigliano [2], Davis and Geramita [4] and Giuffrida [8], we are able to explicitly compute (or to give an algorithm for the computation of) the Hilbert function of special configurations of fat points of $\mathbb{P}^n$.

Similarly, by using results by Catalisano [1], Catalisano et al. ([3]) and Fatabbi [5], we give an upper bound for the regularity index of fat points of $\mathbb{P}^n$ with support lying on a plane or in general position in $\mathbb{P}^r (r < n)$, and we precisely compute it for fat points of $\mathbb{P}^n$ with support lying on a non-singular plane conic or on the rational normal curve of $\mathbb{P}^r (r < n)$.

2. Preliminaries and notation

Let K be an algebraically closed field of characteristic 0 and let $R = K[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n(K)$.

Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of points in $\mathbb{P}^n$ each with an assigned multiplicity m_i.

We can associate to X the ideal $I = \mathbf{p}_1^{m_1} \cap \cdots \cap \mathbf{p}_s^{m_s} \subset R$, where $\mathbf{p}_i$ is the homogeneous prime ideal of height n (generated by n linear forms) in $K[x_0, \ldots, x_n]$ of the point P_i.

We denote the homogeneous coordinate ring of X, R/I, by $A = \bigoplus_{i \geq 0} A_i$.

It is well known that A is a graded Cohen-Macaulay ring of Krull dimension 1, and its multiplicity is

$$\delta(A) = \sum_{i=1}^{s} \binom{m_i + n - 1}{n},$$

which is by definition the degree of X, denoted by $\delta(X)$.

We may then assume that x_0 is not a 0-divisor *modulo* I (i.e., its image in A is not a 0-divisor).

It is also known that the Hilbert function of X (or of A), which is defined by

$$H(X, t) = dim_K A_t = dim_K R_t - dim_K I_t,$$

strictly increases until it reaches the degree of X, and keeps constant thereafter (see, for instance, [9]).

We denote by

$$\tau(X) = \min\{t \mid H(X,t) = \delta(X)\}$$

the *regularity index* of X, and by $\sigma(X) = \tau(X) + 1$.

Sometimes we also denote $\sigma(X)$ (or $\tau(X)$) by $\sigma(I)$ (resp. $\tau(I)$), where I is the (unique saturated) ideal associated to X.

If $X = \{(P_1, m_1), \dots, (P_s, m_s)\}$ is a set of points of $\mathbb{P}^n$ whose support is contained in a linear subspace of dimension r, for simplicity of notation, we shall sometimes abusively say "X is contained in $\mathbb{P}^r$" (and abusively denote $X \subset \mathbb{P}^r$).

Up to a linear change of coordinates we may think of such a linear subspace isomorphic to $\mathbb{P}^r$ as the one with equations $x_{r+1} = \cdots = x_n = 0$, i.e., we may assume $P_i = [a_{i_0} : \cdots : a_{i_r} : 0 : \cdots : 0]$, for all $i = 1, \dots, s$. Then, when we talk of "X as a subset of $\mathbb{P}^r$", we really mean that we are considering the corresponding set of fat points $\tilde{X} = \{(\tilde{P}_1, m_1), \dots, (\tilde{P}_s, m_s)\}$, where $\tilde{P}_i = [a_{i_0} : \cdots : a_{i_r}] \in \mathbb{P}^r$.

Now we introduce some further notation and facts that we shall use.

Let R and S be two polynomial rings:

$$R = K[x_0, \dots, x_n], \qquad S = K[y_0, \dots, y_n].$$

Following [7], which is based on [10–12], we shall think of the polynomials of S as representing differential operators and the polynomials of R as the polynomials on which the differential operators act.

In order to describe such an action, we first write a monomial of the ring R as x^B and a monomial of the ring S as y^A, where $B = (b_0, \dots, b_n)$, with $b_i \in Z$ and $b_i \geq 0$, and $A = (a_0, \dots, a_n)$, with $a_i \in Z$ and $a_i \geq 0$.

Then we say

$$A \preceq B \text{ (or } y^A \preceq x^B) \Leftrightarrow a_i \leq b_i, \quad \forall i \Leftrightarrow x^A | x^B \text{ in } R.$$

With the notation above we obtain

$$y^A \circ x^B = \begin{cases} 0 & \text{if } A \not\preceq B, \\ \prod_{i=1}^{n} (b_i!/(b_i - a_i)!) x^{B-A} & \text{if } A \preceq B. \end{cases}$$

The action above extends, by linearity, to an action

$$S_i \times R_j \to R_{j-i},$$

$$(s_i, r_j) \mapsto s_i \circ r_j.$$

In particular, for $i = j$, we obtain a K-bilinear pairing

$$S_i \times R_i \to K.$$

Example. Let $F_3 = y_1^2 y_2 \in K[y_0, \dots, y_3]$ and $G_5 = x_1 + x_1^2 x_2 x_3 \in K[x_0, \dots, x_3]$, then

$$F_3 \circ G_5 = (\partial^3/\partial x_1 \partial x_1 \partial x_2)(G_5) = (\partial^3/\partial x_1 \partial x_1 \partial x_2)(x_1 + x_1^2 x_2 x_3) = 0 + 2x_3.$$

For simplicity of notation, we set

$$k_{n,r,l} = \binom{n-r-1+l}{n-r-1}.$$

Remark 2.1. If F is a homogeneous polynomial of degree t, it is clear that, for each fixed r $(0 < r < n)$, we can write

$$F = \sum_{l=0}^{t} \left[\sum_{i=1}^{k_{n,r,l}} M_{i,l} \cdot F_{i,l} \right],$$

where, $\forall l \geq 0$, $\{M_{i,l}\}_{i=1,\dots,k_{n,r,l}}$ are all the monomials in $(K[x_{r+1},\dots,x_n])_l$ and $F_{i,l} \in (K[x_0,\dots,x_r])_{t-l}$, $\forall i = 1,\dots,k_{n,r,l}$.

Example. Let $F \in K[x_0,x_1,x_2,x_3]$,

$$F = x_1^3 + x_2^3 + 2x_3^3 + x_0x_2^2 + x_0x_1x_2 + x_0x_1x_3 + x_1^2x_3 + x_1^2x_2 + x_0^2x_3$$

and choose $r = 1$. Then we can write

$$F = 1 \cdot (x_1^3) + x_2 \cdot (x_0x_1 + x_1^2) + x_3 \cdot (x_0x_1 + x_1^2 + x_0^2) + x_2^2 \cdot x_0 + x_2^3 \cdot 1 + x_3^3 \cdot 2.$$

Remark 2.2. Observe that, if $A = (0,\dots,0,a_{r+1},\dots,a_n)$ with $\sum_{j=0}^{n} a_j = d$ and $y^A \in K[y_0,\dots,y_n]$, then there exists an index $\underline{i}$ such that $M_{\underline{i},d} = x^A$.

Furthermore, $\forall l \leq d$, $\forall i = 1,\dots,k_{n,r,l}$ we have

$$y^A \circ M_{i,l} = \begin{cases} u_{i,l} & \text{for } i = \underline{i} \text{ and } l = d \\ 0 & \text{otherwise,} \end{cases}$$

where $u_{i,l} \in K$.

Therefore, by Remark 2.1, we obtain

$$y^A \circ F = \sum_{l=0}^{t} \left[\sum_{i=1}^{k_{n,r,l}} F_{i,l} \cdot (y^A \circ M_{i,l}) \right] = u_{\underline{i},d} F_{\underline{i},d} + \sum_{l=d+1}^{t} \left[\sum_{i=1}^{k_{n,r,l}} F_{i,l} \cdot (y^A \circ M_{i,l}) \right].$$

3. Hilbert function

Let $X = \{(P_1,m_1),\dots,(P_s,m_s)\}$ be a set of fat points of $\mathbb{P}^n$ and assume that $X \subset \mathbb{P}^r$. In order to get information in $\mathbb{P}^n$, we need to know not only the corresponding set of fat points in $\mathbb{P}^r$, $\tilde{X}$, but also a family of sets of fat points in $\mathbb{P}^r$, with the same support and all multiplicities lowered by one at each step. More precisely: for all $l \geq 0$, we define

$$I^{l,\mathbb{P}^r} = (\mathbf{p}_1^{(m_1-l)^+} \cap \cdots \cap \mathbf{p}_s^{(m_s-l)^+}) \cap K[x_0,x_1,\dots,x_r],$$

where

$$(m_i - l)^+ = \begin{cases} m_i - l & \text{if } m_i > l \\ 0 & \text{if } m_i \leq l. \end{cases}$$

Then, clearly, $I^{0,\mathbb{P}^r}$ is the ideal of the corresponding fat points in $\mathbb{P}^r$, and $I^{l,\mathbb{P}^r} = \bigoplus_{t=0}^{\infty} I_t^{l,\mathbb{P}^r}$, $\forall l \geq 0$.

With this notation, we have the following theorem:

Theorem 3.1. *Let $X=\{(P_1,m_1),\cdots,(P_s,m_s)\}$ be a set of fat points of $\mathbb{P}^n$ and assume $X \subset \mathbb{P}^r$, with $0<r<n$. Then, $\forall t \geq 1$,*

$$H(X,t) = \binom{n+t}{n} - \sum_{l=0}^{t} k_{n,r,l}\, dim_K\, I_{t-l}^{l,\mathbb{P}^r}.$$

Proof. Let $I = \mathbf{p}_1^{m_1} \cap \cdots \cap \mathbf{p}_s^{m_s} \subset K[x_0,\ldots,x_n]$ be the ideal associated to X. We want to prove that

$$dim_K\, I_t = \sum_{l=0}^{t} k_{n,r,l}\, dim_K\, I_{t-l}^{l,\mathbb{P}^r},$$

and clearly it is enough to show that, $\forall t \geq 1$,

$$I_t = \bigoplus_{l=0}^{t} \mathscr{M}_l \cdot I_{t-l}^{l,\mathbb{P}^r}, \tag{1}$$

where $\mathscr{M}_l$ is the vector space generated by all the monomials of degree l in $x_{r+1},\ldots,x_n$, (which is of dimension $k_{n,r,l}$).

Fix t and set $V_l = \mathscr{M}_l \cdot I_{t-l}^{l,\mathbb{P}^r}$ for each $l = 0,\ldots,t$. Then it is not hard to see that the sum of vector spaces $\sum_{l=0}^{t} V_l$ is, in fact, direct.

Now we want to prove the equality claimed above.

To see that $\bigoplus_{l=0}^{t} V_l \subset I_t$, observe that an element $v_l \in V_l$ is the sum of terms of type $v \cdot w$, with $v \in \mathscr{M}_l$ and $w \in I_{t-l}^{l,\mathbb{P}^r}$; thus it trivially follows (after recalling that $P_1,\ldots,P_s \in V(x_{r+1},\ldots,x_n)$) that $v \in \mathbf{p}_1^l \cap \cdots \cap \mathbf{p}_s^l$. Therefore

$$v_l \in (\mathbf{p}_1^l \cap \cdots \cap \mathbf{p}_s^l) \cdot I_{t-l}^{l,\mathbb{P}^r} \subset (\mathbf{p}_1^l \cap \cdots \cap \mathbf{p}_s^l)(\mathbf{p}_1^{(m_1-l)^+} \cap \cdots \cap \mathbf{p}_s^{(m_s-l)^+}) \subset I_t.$$

Now we have to prove the other inclusion. Let $F \in I_t$. By Remark 2.1 we have

$$F = \sum_{l=0}^{t} \left[\sum_{i=1}^{k_{n,r,l}} M_{i,l} \cdot F_{i,l} \right].$$

We have to prove that

$$F_{i,l} \in I_{t-l}^{l,\mathbb{P}^r},$$

$\forall l=1,\ldots,t$, $\forall i=1,\ldots,k_{n,r,l}$; i.e., we still have to prove that, for any given $l \in \{1,\ldots,t\}$ and $h \in \{1,\ldots,s\}$,

$$(y^B \circ F_{i,l})(P_h) = 0,$$

for each $i = 1,\ldots,k_{n,r,l}$ and each monomial $y^B \in K[y_0,\ldots,y_n]$ with $B = (b_0,\ldots, b_r,0,\ldots,0)$ and $\sum_{j=0}^{n} b_j \leq m_h - l - 1$.

Now let $y^A \in K[y_0,\dots,y_n]$, with $A=(0,\dots,0,a_{r+1},\dots,a_n)$ and $\sum_{j=0}^n a_j = d$. Then Remark 2.2 yields

$$[y^B \circ (y^A \circ F)](P_h) = u \cdot (y^B \circ F_{i,l})(P_h) + \sum_{l=d+1}^{t} \left[\sum_{i=1}^{k_{n,r,l}} (y^B \circ F_{i,l})(P_h) \cdot (y^A \circ M_{i,l})(P_h) \right]$$

which equals 0, since $F \in I$. We are done if we show that

$$\sum_{l=d+1}^{t} \left[\sum_{i=1}^{k_{n,r,l}} (y^B \circ F_{i,l})(P_h) \cdot (y^A \circ M_{i,l})(P_h) \right] = 0.$$

Note that, for each $M_{i,l}$ such that $y^A \not\preceq M_{i,l}$, we have that $y^A \circ M_{i,l} = 0$. On the other hand, if $y^A \preceq M_{i,l}$, since $l > d = \sum a_j$, there exists $j \in \{r+1,\dots,n\}$ such that $x_j | y^A \circ M_{i,l}$, hence $(y^A \circ M_{i,l})(P_h) = 0$, as $P_1,\dots,P_s \in V(x_{r+1},\dots,x_n)$. $\square$

By direct computation, after choosing $X = \emptyset$ in Theorem 3.1, we obtain the following combinatorial result:

Corollary 3.2. *Let $n \geq r > 1$. Then, $\forall t \geq 1$*

$$\binom{t+n}{n} = \sum_{i=0}^{t} \binom{n-r+i}{n-r} \binom{r-1+t-i}{r-1}.$$

As a first application of Theorem 3.1 to a geometric case, we obtain the following known result:

Corollary 3.3. *Let X be a set of distinct points of $\mathbb{P}^n$ and assume $X \subset \mathbb{P}^r$. Then the Hilbert function of X is the same either as a subset of $\mathbb{P}^n$ or as a subset of $\mathbb{P}^r$.*

Proof. With the notation of Theorem 3.1, $\forall l > 0$, $X^l = \emptyset$, hence we have that $I^{l,\mathbb{P}^r} = K[x_0,\dots,x_r]$. Thus, $\forall t \geq 1$,

$$\begin{aligned} H(X,t) &= \binom{n+t}{n} - \sum_{l=0}^{t} k_{n,r,l}\, dim_K\, I^{l,\mathbb{P}^r}_{t-l} \\ &= \binom{n+t}{n} - \left[dim_K\, I^{0,\mathbb{P}^r}_t + \sum_{l=1}^{t} k_{n,r,l}\, dim_K\, I^{l,\mathbb{P}^r}_{t-l} \right]. \end{aligned}$$

By Corollary 3.2,

$$H(X,t) = \binom{r+t}{r} - dim_K\, I^{0,\mathbb{P}^r}_t = H\left(\frac{K[x_0,\dots,x_r]}{I^{0,\mathbb{P}^r}}, t \right). \qquad \square$$

Obviously, a set of s distinct points in generic position (i.e. with maximal Hilbert function) in $\mathbb{P}^r$, is not in generic position in $\mathbb{P}^n$ with $n > r$, as

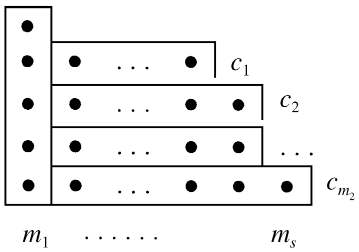
$$H(X,t) = \min \left\{ s, \binom{t+r}{r} \right\} \neq \min \left\{ s, \binom{t+n}{n} \right\}.$$

Now let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points of $\mathbb{P}^n$ with $m_1 \geq m_2 \geq \cdots \geq m_s$.

Following [4], we define $\forall i$, $1 \leq i \leq m_2$

$$c_i := card\{2 \leq j \leq s \mid m_j \geq m_2 - (i-1)\}.$$

We can think of placing side by side s columns, where the ith column has heigth m_i. Then the numbers c_i have the following graphic meaning:



Clearly, $\sum_{i=1}^{s} m_i = m_1 + c_1 + \cdots + c_{m_2}$.

Remark 3.4. Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points in $\mathbb{P}^1$, with $m_1 \geq m_2 \geq \cdots \geq m_s$ and $I = \mathbf{p}_1^{m_1} \cap \cdots \cap \mathbf{p}_s^{m_s}$ the associated ideal.

Let $I^l = \mathbf{p}_1^{(m_1-l)^+} \cap \cdots \cap \mathbf{p}_s^{(m_s-l)^+}$, $\forall l \geq 0$, the ideal which corresponds to $X^l = \{(P_1, (m_1-l)^+), \ldots, (P_s, (m_s-l)^+)\}$.

A direct computation shows that, for each $l \geq 0$

$$\delta(X^l) = \begin{cases} m_1 - l + c_1 + \cdots + c_{m_2-l}, & 0 \leq l < m_2 \\ m_1 - l, & m_2 \leq l < m_1, \\ 0, & l \geq m_1; \end{cases}$$

hence, $\forall l \geq 0$ and $\forall t \geq l$:

$$dim_K I^l_{t-l} = \begin{cases} (t+1-(m_1+c_1+\cdots+c_{m_2-l}))^+, & 0 \leq l < m_2, \\ (t+1-m_1)^+, & m_2 \leq l < m_1, \\ t+1-l, & m_1 \leq l, \end{cases}$$

The next result extends to $\mathbb{P}^n$ (Proposition 3.3 of [4]), which computes the Hilbert function of collinear fat points of $\mathbb{P}^2$.

Proposition 3.5 (The Hilbert function of collinear fat points of $\mathbb{P}^n$). *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points of $\mathbb{P}^n$, with collinear support and $m_1 \geq$*

$m_2 \geq \cdots \geq m_s$. Let c_i be defined as above. Then

$$\Delta H(X,t) = \begin{cases} \binom{n-1+t}{n-1}, & 0 \leq t < m_1, \\ \sum_{l=0}^{m_2-i} \binom{n-2+l}{n-2}, & m_1+c_1+\cdots+c_{i-1} \leq t < m_1+c_1+\cdots+c_i, \\ 0, & t \geq \sum_{j=1}^{s} m_j. \end{cases}$$

Proof. Observe that, in this case, $I_{t-l}^{l,\mathbb{P}^1}$ coincides with the space I_{t-l}^{l} introduced in Remark 3.4. Then it follows from Theorem 3.1 and Remark 3.4, that

$$\begin{aligned} H(X,t) &= \binom{n+t}{n} - \sum_{l=0}^{t} k_{n,r,l}\, dim_K\, I_{t-l}^{l} \\ &= \binom{n+t}{n} - \left[\sum_{l=0}^{m_2-1} k_{n,1,l} \cdot (t+1-(m_1+c_1+\cdots+c_{m_2-l}))^+ \right. \\ &\qquad \left. + \sum_{l=m_2}^{m_1-1} k_{n,1,l}(t+1-m_1)^+ + \sum_{l=m_1}^{t} k_{n,1,l}(t+1-l) \right]. \end{aligned}$$

By means of Corollary 3.2, we write

$$\binom{n+t}{n} = \sum_{l=0}^{t} k_{n,1,l}(t-l+1),$$

which yields

$$\begin{aligned} H(X,t) = &\sum_{l=0}^{m_2-1} k_{n,1,l}(t+1-l-(t+1-(m_1+c_1+\cdots+c_{m_2-l}))^+)^+ \\ &+ \sum_{l=m_2}^{m_1-1} k_{n,1,l}(t+1-l-(t+1-m_1)^+)^+. \end{aligned}$$

Therefore,

$$H(X,t)=\begin{cases}\dbinom{t+n}{n}, & t<m_1\\ \displaystyle\sum_{l=0}^{m_2-i}k_{n,1,l}(t+1-l) & (t\geq m_1+c_1+\cdots+c_{i-1}\\ \qquad+\displaystyle\sum_{l=m_2-i+1}^{m_2-1}k_{n,1,l}(m_1+c_1+\cdots+c_{m_2-l}-l) & \text{and}\\ \qquad+\displaystyle\sum_{l=m_2}^{m_1-1}k_{n,1,l}(m_1-l), & t<m_1+c_1+\cdots+c_i\\ \displaystyle\sum_{l=0}^{m_2-1}k_{n,1,l}(m_1+c_1+\cdots+c_{m_2-l}-l) & \\ \qquad+\displaystyle\sum_{l=m_2}^{m_1-1}k_{n,1,l}(m_1-l), & t\geq\displaystyle\sum_{j=1}^{s}m_j.\end{cases}$$

Now the result follows from direct computation. □

If $X=\{(P_1,m_1),\dots,(P_s,m_s)\}$ is a set of points on a non-singular conic of $\mathbb{P}^2$, Catalisano (in [1, Theorem 3.1]) exhibits an algorithm for computing the Hilbert function of X from the Hilbert Function of a suitable subset $X'\subset X$. By means of Theorem 3.1 it is then possible to recover the Hilbert function of fat points of $\mathbb{P}^n$ with support lying on a non-singular conic of $\mathbb{P}^2$.

As an application, we compute the Hilbert function of 4 fat points of $\mathbb{P}^n$, all with the same multiplicities and support lying on a non singular conic of $\mathbb{P}^2$. In the computation, which is omitted, we use the explicit formula provided by [4, Proposition 5.1].

Proposition 3.6. *Let $X=\{(P_1,k),\dots,(P_4,k)\}$ be a set of fat points of $\mathbb{P}^n$ with support on an irreducible plane conic. Then*

$$H(X,t)=\begin{cases}\dbinom{n+t}{n}, & t<k,\\ \displaystyle\sum_{l=0}^{k-1}\binom{n-3+l}{n-3}\binom{k+2-l}{2}, & t=k,\\ \displaystyle\sum_{l=0}^{2k-t-1}\binom{n-3+l}{n-3}\binom{t+2-l}{2} & \\ \qquad+4\displaystyle\sum_{l=2k-t}^{k-1}\binom{n-3+l}{n-3}\binom{k-l+1}{2}, & k+1\leq t<2k,\\ 4\displaystyle\sum_{l=0}^{k-1}\binom{n-3+l}{n-3}\binom{k-l+1}{2}, & t\geq 2k.\end{cases}$$

More generally, the Hilbert function of fat points of $\mathbb{P}^n$, which actually lie in a smaller $\mathbb{P}^r$, can be recovered whenever the one in $\mathbb{P}^r$ is known. This is the case, for instance, for 6 points in $\mathbb{P}^2$, no three collinear and not all six on a conic, for which there is an algorithm by Giuffrida (see [8]), or of 3 points in $\mathbb{P}^2$, computed by Davis and Geramita (see [4]), or of points on the twisted cubic of $\mathbb{P}^3$, for which an algorithm is given by Catalisano and Gimigliano (see [2]).

4. Regularity index

Equality (1) in the proof of Theorem 3.1 allows a sort of decomposition of I, after setting $m = m_1 \geq \cdots \geq m_s$

$$I = \left(\bigoplus_{l=0}^{m-1} I^{l,\mathbb{P}^r} \otimes_K \mathscr{M}_l \right) \oplus (S \otimes_K \mathscr{M}_{\geq m}),$$

where $S = K[x_0, \dots, x_r]$ and $\mathscr{M}_{\geq m}$ denotes the vector space generated by all monomials in $x_{r+1}, \dots, x_n$ of degree at least m.

This suggests that also the regularity index of I is influenced not only by that of the corresponding points in $\mathbb{P}^r$ (i.e., of $I^{0,\mathbb{P}^r}$), but also by the regularity indices of the ideals $I^{l,\mathbb{P}^r}$, as the following shows:

Theorem 4.1. *Let $X = \{(P_1), \dots, (P_s, m_s)\}$ be a set of fat points of $\mathbb{P}^n$, with $m_1 \geq m_2 \geq \cdots \geq m_s$ and let $X \subset \mathbb{P}^r \subset \mathbb{P}^n$, with $r < n$. Then*

$$\sigma(I) \geq \sigma(I^{0,\mathbb{P}^r}).$$

Moreover,

$$\sigma(I) = \sigma(I^{0,\mathbb{P}^r}) \Leftrightarrow \sigma(I^{l,\mathbb{P}^r}) \leq \sigma(I^{0,\mathbb{P}^r}) - l, \quad \forall l \geq 0.$$

Proof. Let $I = \mathbf{p}_1^{m_1} \cap \cdots \cap \mathbf{p}_s^{m_s}$ be the ideal in $K[x_0, \dots, x_n]$ which corresponds to X.

From Theorem 3.1 we have that

$$\begin{aligned} \Delta H(X,t) &= \binom{n+t}{n} - \sum_{l=0}^{t} k_{n,r,l}\, dim_K\, I_{t-l}^{l,\mathbb{P}^r} - \binom{n+t-1}{n} + \sum_{l=0}^{t-1} k_{n,r,l}\, dim_K\, I_{t-1-l}^{l,\mathbb{P}^r} \\ &= \binom{n-1+t}{n-1} - \sum_{l=0}^{t-1} k_{n,r,l}(dim_K\, I_{t-l}^{l,\mathbb{P}^r} - dim_K\, I_{t-1-l}^{l,\mathbb{P}^r}) - k_{n,r,t} dim_K\, I_0^{t,\mathbb{P}^r}. \end{aligned}$$

Then Corollary 3.2 yields

$$\begin{aligned} \Delta H(X,t) &= \sum_{l=0}^{t} k_{n,r,l} \binom{r-1+t-l}{r-1} \\ &\quad - \sum_{l=0}^{t-1} k_{n,r,l}(dim_K\, I_{t-l}^{l,\mathbb{P}^r} - dim_K\, I_{t-1-l}^{l,\mathbb{P}^r}) - k_{n,r,t}\, dim_K\, I_0^{t,\mathbb{P}^r} \end{aligned}$$

$$= k_{n,r,t} - k_{n,r,t}\, dim_K\, I_0^{t,\mathbb{P}^r} + \sum_{l=0}^{t-1} k_{n,r,l} \left[\binom{r-1+t-l}{r-1} - (dim_K\, I_{t-l}^{l,\mathbb{P}^r} - dim_K\, I_{t-1-l}^{l,\mathbb{P}^r}) \right]$$

$$= k_{n,r,t} - k_{n,r,t}\, dim_K\, I_0^{t,\mathbb{P}^r} + \sum_{l=0}^{t-1} k_{n,r,l} \cdot \Delta H(K[x_0,\dots,x_r]/I^{l,\mathbb{P}^r}, t-l).$$

In particular, $\forall t \geq m_1$, we have $dim_K\, I_0^{t,\mathbb{P}^r} = 1$, hence

$$\Delta H(X,t) = \sum_{l=0}^{t-1} k_{n,r,l} \cdot \Delta H(K[x_0,\dots,x_r]/I^{l,\mathbb{P}^r}, t-l).$$

Therefore, for $t \geq \sigma(X) \geq \alpha(I) \geq m_1$, in order to have $\Delta H(X,t) = 0$, we must necessarily have $\Delta H(K[x_0,\dots,x_r]/I^{l,\mathbb{P}^r}, t-l) = 0$, $\forall l = 0,\dots,t-1$. In other words, $\sigma(I) \geq \sigma(I^{l,\mathbb{P}^r})$, $\forall l = 0,\dots,t-1$. In particular, $\sigma(I) \geq \sigma(I^{0,\mathbb{P}^r})$.

Now we prove the second part of the statement.

In order to show that the condition is necessary, we have to prove that, if $\exists l \geq 0$ such that $\sigma(I^{l,\mathbb{P}^r}) > \sigma(I^{0,\mathbb{P}^r}) - l$, then $\sigma(I) > \sigma(I^{0,\mathbb{P}^r})$; equivalently, that $\exists t \geq \sigma(I^{0,\mathbb{P}^r}) \geq m_1$ such that $\Delta H(X,t) > 0$. We have already seen that, for such t,

$$\Delta H(X,t) = \sum_{l=0}^{t-1} k_{n,r,l} \cdot \Delta H(K[x_0,\dots,x_r]/I^{l,\mathbb{P}^r}, t-l).$$

Now, set $t = \sigma(I^{0,\mathbb{P}^r})$. Then $t - l < \sigma(I^{l,\mathbb{P}^r})$, hence

$$\Delta H(K[x_0,\dots,x_r]/I^{l,\mathbb{P}^r}, t-l) > 0,$$

and then

$$\sum_{l=0}^{t-1} k_{n,r,l} \cdot \Delta H(K[x_0,\dots,x_r]/I^{l,\mathbb{P}^r}, t-l) > 0.$$

Conversely, if $\sigma(I^{l,\mathbb{P}^r}) \leq \sigma(I^{0,\mathbb{P}^r}) - l$, $\forall l \geq 0$, then, $\forall t \geq \sigma(I^{0,\mathbb{P}^r})$, we have that $t - l \geq \sigma(I^{l,\mathbb{P}^r})$, thus

$$\binom{r-1+t-l}{r-1} = dim_K\, I_{t-l}^{l,\mathbb{P}^r} - dim_K\, I_{t-1-l}^{l,\mathbb{P}^r}.$$

We obtain $\Delta H(X,t) = 0$, $\forall t \geq \sigma(I^{0,\mathbb{P}^r}) \geq m_1$, since

$$\Delta H(X,t) = \sum_{l=0}^{t-1} k_{n,r,l} \left[\binom{r-1+t-l}{r-1} - (dim_K\, I_{t-l}^{l,\mathbb{P}^r} - dim_K\, I_{t-1-l}^{l,\mathbb{P}^r}) \right], \quad \forall t \geq m_1.$$

Therefore $\sigma(I) \leq \sigma(I^{0,\mathbb{P}^r})$, hence the desired equality. □

Remark. It should be pointed out that in many cases we will consider in this section we will find that the regularity indices in $\mathbb{P}^r$ and in $\mathbb{P}^n$ coincide, and it is rather natural to suspect that we always have the equality in Theorem 4.1. This is still under investigation (the condition in Theorem 4.1 seems rather strong, though reasonable), but so far we have found neither a proof nor a counterexample.

Proposition 4.2. *Let $X = \{(P_1,a),(P_2,b),(P_3,c)\}$ be a set of 3 fat points of $\mathbb{P}^n$ with support not on a line and $a \geq b \geq c$. Then $\sigma(X) = a+b$.*

Proof. By Corollary 4.3 in [4], $\forall l = 0,\dots,a$, we have that

$$\sigma(I^{l,\mathbb{P}^2}) = (a-l)^+ + (b-l)^+.$$

Then, $\forall l = 0,\dots,b$, we have

$$\sigma(I^{l,\mathbb{P}^2}) = a+b-2l \leq a+b-l = \sigma(I^{0,\mathbb{P}^2}) - l,$$

and, $\forall l = b,\dots,a$, that

$$\sigma(I^{l,\mathbb{P}^2}) = a-l \leq a+b-l = \sigma(I^{0,\mathbb{P}^2}) - l.$$

By Theorem 4.1, it follows that $\sigma(I) = \sigma(I^{0,\mathbb{P}^2}) = a+b$. □

Proposition 4.3. *Let $X = \{(P_1,m_1),\dots,(P_s,m_s)\}$, be a set of fat points of $\mathbb{P}^n$, with $m_1 \geq m_2 \geq \cdots \geq m_s$, and $s > 3$. If $P_1,\dots,P_s$ lie on a non-singular plane conic, then*

$$\sigma(X) = \max\left\{m_1+m_2, \left[\frac{\sum_{i=1}^s m_i}{2}\right]+1\right\}.$$

Proof. By Proposition 2.2 in [1], $\forall l = 0,\dots,m_1$, we have that

$$\sigma(I^{l,\mathbb{P}^2}) = \max\left\{(m_1-l)^+ + (m_2-l)^+, \left[\frac{\sum_{i=1}^s (m_i-l)^+}{2}\right]+1\right\}.$$

Then, $\forall l = 0,\dots,m_2-1$,

$$\begin{aligned}\sigma(I^{l,\mathbb{P}^2}) &\leq \max\left\{m_1+m_2-2l, \left[\frac{(\sum_{i=1}^s m_i)-2l}{2}\right]+1\right\}\\ &\leq \max\left\{m_1+m_2, \left[\frac{\sum_{i=1}^s m_i}{2}\right]+1\right\} - l = \sigma(I^{0,\mathbb{P}^2}) - l\end{aligned}$$

and, $\forall l = m_2, m_2+1,\dots,m_1$,

$$\begin{aligned}\sigma(I^{l,\mathbb{P}^2}) &= \max\left\{m_1-l, \left[\frac{m_1-l}{2}\right]+1\right\}\\ &\leq \max\left\{m_1+m_2, \left[\frac{\sum_{i=1}^s m_i}{2}\right]+1\right\} - l = \sigma(I^{0,\mathbb{P}^2}) - l.\end{aligned}$$

Now the result follows from Theorem 4.1 and [1, Proposition 2.2]. □

Remark 4.4. If X and X' are two sets of fat points such that $X' \subset X$, then repeated applications of [4, Proposition 2.1] yield $\sigma(X') \leq \sigma(X)$.

Proposition 4.5. *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points of $\mathbb{P}^n$ with $m_1 \geq m_2 \geq \cdots \geq m_s$ and support in general position in a $\mathbb{P}^r$, with $r < n$. Then*

$$\sigma(X) \leq \max\left\{m_1 + m_2, \left[\frac{\sum_{i=1}^s m_i + r - 2}{r}\right] + 1\right\}.$$

Proof. Denote

$$\rho = \max\left\{m_1 + m_2, \left[\frac{\sum_{i=1}^s m_i + r - 2}{r}\right] + 1\right\},$$

then, by [3, Theorem 6], we have that $\sigma(I^{0,\mathbb{P}^r}) \leq \rho$.

Now, if necessary, construct a set

$$Y = \{(P_1, m_1), \ldots, (P_s, m_s), (P_{s+1}, 1), \ldots, (P_d, 1)\}$$

by adding points of $\mathbb{P}^n$ lying in the same $\mathbb{P}^r$ containing X, in such a way that $\sigma(J^{0,\mathbb{P}^r}) = \rho$, where J is the ideal which corresponds to Y. This is possible because of [4, Proposition 2.1].

By [3, Theorem 6], $\forall l \geq 1$, we have that

$$\sigma(J^{l,\mathbb{P}^r}) \leq \max\left\{(m_1 - l)^+ + (m_2 - l)^+, \left[\frac{\sum_{i=1}^s (m_i - l)^+ + r - 2}{r}\right] + 1\right\}.$$

Then, $\forall l > 0$, $\sigma(J^{l,\mathbb{P}^r}) \leq \rho - l = \sigma(J^{0,\mathbb{P}^r}) - l$. Therefore, by Theorem 4.1, we have that $\sigma(Y) = \sigma(J^{0,\mathbb{P}^r}) = \rho$, and since $X \subset Y$, we have that $\sigma(X) \leq \rho$. □

With the same technique, and by using [5, Theorem 3.3], it is possible to prove

Proposition 4.6. *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points of $\mathbb{P}^n$ with support lying on a plane. Then*

$$\sigma(X) \leq \max\left\{h, \left[\frac{\sum_{i=1}^s m_i}{2}\right] + 1\right\},$$

where $h = \max\{\sum_{j=1}^k m_{i_j} \mid P_{i_1}, \ldots, P_{i_k}$ are collinear$\}$.

Proposition 4.7. *Let $X = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ be a set of fat points in $\mathbb{P}^n$ with $m_1 \geq m_2 \geq \cdots \geq m_s$ and support on a rational normal curve of $\mathbb{P}^r$. Then*

$$\sigma(X) = \max\left\{m_1 + m_2, \left[\frac{\sum m_i + r - 2}{r}\right] + 1\right\}.$$

Proof. Since X is on a rational normal curve in $\mathbb{P}^r$, by in [3, Proposition 7] we have that $\forall l = 0, \ldots, m_1$,

$$\sigma(I^{l,\mathbb{P}^r}) = \max\left\{(m_1 - l)^+ + (m_2 - l)^+, \left[\frac{\sum_{i=1}^s (m_i - l)^+ + r - 2}{r}\right] + 1\right\}.$$

If we show that $\sigma(I^{1,\mathbb{P}^r}) \leq \sigma(I^{0,\mathbb{P}^r}) - 1$, $\forall s > 0$ and $\forall m_1 \geq m_2 \geq \cdots \geq m_s > 0$, then we have that $\sigma(I^{l,\mathbb{P}^r}) \leq \sigma(I^{0,\mathbb{P}^r}) - l, \forall l \geq 0$ and we can use Theorem 4.1.

If $s = 1$, we have that $\sigma(I^{1,\mathbb{P}^r}) = m_1 - 1 \le \sigma(I^{0,\mathbb{P}^r}) = m_1$.

If $s \ge 2$, we have four cases:
(a) $\sigma(I^{0,\mathbb{P}^r}) = m_1 + m_2$, and $\sigma(I^{1,\mathbb{P}^r}) = m_1 + m_2 - 2$;
(b) $\sigma(I^{0,\mathbb{P}^r}) = m_1 + m_2$, and $\sigma(I^{1,\mathbb{P}^r}) = [(\sum_{i=1}^{s}(m_i - 1)^+ + r - 2)/r] + 1$;
(c) $\sigma(I^{0,\mathbb{P}^r}) = [(\sum_{i=1}^{s} m_i + r - 2)/r] + 1$, and $\sigma(I^{1,\mathbb{P}^r}) = [(\sum_{i=1}^{s}(m_i - 1)^+ + r - 2)/r] + 1$;
(d) $\sigma(I^{0,\mathbb{P}^r}) = [(\sum_{i=1}^{s} m_i + r - 2)/r] + 1$, and $\sigma(I^{1,\mathbb{P}^r}) = m_1 + m_2 - 2$.

In cases (a) and (d) it is obvious that $\sigma(I^{1,\mathbb{P}^r}) \le \sigma(I^{0,\mathbb{P}^r}) - 1$. On the other hand, since $s \ge r + 1$, in case (b) we have

$$\sigma(I^{0,\mathbb{P}^r}) - 1 \ge \left[\frac{\sum_{i=1}^{s} m_i + r - 2}{r}\right] \ge \left[\frac{\sum_{i=1}^{s} m_i + r - 2 - s}{r}\right] + 1 = \sigma(I^{1,\mathbb{P}^r}),$$

while in case (c) we have

$$\sigma(I^{0,\mathbb{P}^r}) - 1 = \left[\frac{\sum_{i=1}^{s} m_i + r - 2}{r}\right] \ge \left[\frac{\sum_{i=1}^{s} m_i + r - 2 - s}{r}\right] + 1 = \sigma(I^{1,\mathbb{P}^r}).$$

□

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