A Functional Analytic Approach
to Uniqueness of Solutions
to Some Nonlinear Cauchy Problems

RICHARD A. GRAFF

The MITRE Corporation, Bedford, Massachusetts 01730

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INTRODUCTION

We prove a powerful abstract theorem on the uniqueness of strict solutions to nonlinear differential equations in Banach spaces. In addition, we examine how to apply it to parabolic and hyperbolic systems of nonlinear partial differential equations which contain arbitrary nonlinearities in all spatial derivatives of the unknowns.

This uniqueness theorem may be viewed as a generalization of the standard uniqueness result for solutions to quasi-dissipative equations from nonlinear contractive semigroup theory. However, unlike that result, the present theorem may be applied to equations which are not quasi-dissipative in any weaker norm (such as the applications considered here). Our theorem may also be viewed as a nonlinear generalization of the abstract linear result [1, Proposition 3.5] that existence of strong solutions implies uniqueness of weak solutions.

The uniqueness theorem assumes two local conditions: that the equation possesses a special kind of differentiability, and that the time-dependent linear equations obtained by differentiating the nonlinear equation along solution curves generate (not necessarily unique) weak evolution systems. The relevant concepts of weak linear evolution systems and $\alpha$-differentiability were introduced in [1, Sections 1, 5], where they were used to study the differentiability of flows for nonlinear equations.

The theorem applies with nearly equal facility to Cauchy problems and mixed problems with constant boundary value data. To illustrate the techniques involved with both types of applications, we carry out our application to parabolic systems in detail for a mixed problem, and to hyperbolic systems in detail for the Cauchy problem. Following each application, the modifications needed to treat the other type of problem are discussed.
Existence as well as uniqueness of solutions to general nonlinear parabolic and hyperbolic systems was previously established in [3], but only for the Cauchy problem case, and under more restrictive differentiability assumptions on the initial data than required in the present article. In the mixed case, the techniques used in [3] lead to unsolvable problems; and as far as this author is aware, no previous uniqueness results have been obtained for mixed problems of this generality in the nonlinearities involved.

We have elected to discuss mixed problems with Dirichlet boundary data in this article, but no significant modification is required to deal with more general elliptic boundary value data.

Generalization to the time-dependent case is straightforward. Indeed, the original announcement [2] of the uniqueness theorem (without proof, and without the present applications) was for the time-dependent case. However, for most time-dependent applications this generality is not necessary. Differential equations in a Banach space $X$ to which the time-dependent uniqueness theorem is applicable can often be regarded as time-independent equations in $X \times R$, and treated with the present version of the theorem.

1. Preliminaries and the General Theorem

Throughout this article, we assume that $X$, $Y$, $Z$ are Banach spaces, and that $Y$ is a linear subspace of $X$ such that the inclusion of $Y$ into $X$ is continuous and dense. We will always assume that $W$ is an open subset of $Y$. $L(X, Z)$ (resp. $L(Y, Z)$) will denote the space of continuous linear maps from $X$ to $Z$ (resp. from $Y$ to $Z$) with the strong operator topology.

**Definition 1.** Let $G: W \to X$ be a continuous map (i.e., an unbounded vector field), and $y_0 \in W$. By a solution to the initial-value problem $\dot{u}(t) = G(u(t))$, $u(0) = y_0$, we will mean a curve $u(\cdot) \in C^0([0, T], W) \cap C^1([0, T], X)$ for some $T > 0$ such that $u(0) = y_0$ and $\dot{u}(t) = G(u(t))$ for $0 \leq t \leq T$ (where the derivative of the curve is taken in the space $X$). We will also refer to $u(\cdot)$ as an integral curve for $G(\cdot)$ with initial value $y_0$.

**Definition 2.** Let $A(\cdot): [a, b] \to L(Y, X)$ be strongly measurable and bounded. A weak evolution system for $A(\cdot)$ is a bounded family $\{U(t, s): a \leq s \leq t \leq b\}$ of continuous linear operators in $L(X, X^{**})$ such that:
(1) \( U(t, t) = \text{Id}_X \) for each \( t \in [a, b] \),

(2) \( U(t, \cdot) : [a, t] \to L(X, X^{**}) \) is continuous for each \( t \in (a, b] \),

(3) For each \( y \in Y \), \( U(t, \cdot)(y) \) is absolutely continuous, and
\[
(\partial/\partial s) U(t, s)(y) = -U(t, s)A(s)(y) \text{ for almost every } s.
\]

Remark 1. (i) Weak evolution systems are not unique, not even in the time-independent case. For example, if \( A \) is the generator of a strongly continuous linear semigroup \( \{e^{tA} : t \geq 0 \} \) on \( X \) such that \( Y \subseteq D(A) \), then \( (\partial/\partial s) e^{(t-s)A}y = -e^{(t-s)A}Ay \) for each \( y \in Y \). Thus, if \( X = H_0([-1, 1], R) \), \( Y = H_{2,0}([-1, 1], R) \), and \( A = \partial^2/\partial x^2 \), then \( A \) generates at least two weak evolution systems \[1, Example 1.9\]: \( U_1(t, s) = e^{(t-s)A_1} \) and \( U_2(t, s) = e^{(t-s)A_2} \), where \( A_1 \) is the one-dimensional Laplacian on \([-1, 1]\) with Neumann boundary data and \( A_2 \) is the one-dimensional Laplacian on \([-1, 1]\) with Dirichlet boundary data.

(ii) If there exists a proper evolution system \( U(\cdot, \cdot) \) for \( A(\cdot) \) (i.e., an evolution system in the conventional sense, \[1, Definition 1.5\]) which restricts to a strongly continuous evolution system on \( Y \), then \( U(\cdot, \cdot) \) is the only weak evolution system for \( A(\cdot) \) \[1, Theorem 1.8\].

DEFINITION 3. Let \( f : W \to Z \), \( p \in W \). We say that \( f \) has a Gateaux derivative at \( p \) if there exists \( l \in L(Y, Z) \) such that, for each \( y \in Y \),
\[
\lim_{t \to 0} \frac{1}{t} \| f(p + ty) - f(p) - tl(y) \|_Z = 0.
\]
We call \( l \) the Gateaux derivative of \( f \) at \( p \), and denote it by \( Df(p) \).

DEFINITION 4. Let \( f : W \to Z \). We say that \( f \) is uniformly \( \alpha \)-differentiable on \( W \) if \( f \) has a Gateaux derivative at each point of \( W \), \( Df(\cdot) : W \to L(Y, Z) \) is strongly continuous, and there exists \( k \geq 0 \) such that
\[
\| f(v) - f(u) - Df(u)(v-u) \|_Z \leq k \| v - u \|_X \quad \text{for each } u, v \in W.
\]
We say that \( f(\cdot) \) is locally uniformly \( \alpha \)-differentiable on \( W \) if, for each \( p \in W \), there exists a neighborhood \( W_p \), with \( p \in W_p \subseteq W \), such that \( f |_{W_p} \) is uniformly \( \alpha \)-differentiable on \( W_p \).

LEMMA 1. Assume that \( W \) is convex, \( f \) is \( C^2 \), and that there exists \( c \geq ( \) such that \( \| D^2f(y)(v, w) \|_Z \leq c \| v \|_Y \| w \|_X \) for each \( y \in W \), \( v, w \in Y \). Then \( f \) is locally uniformly \( \alpha \)-differentiable, and is uniformly \( \alpha \)-differentiable on bounded sets.

Proof. Immediate from Taylor's Integral Theorem.

THEOREM 1 (Uniqueness Theorem). Let \( G(\cdot) : W \to X \) be a locally uniformly \( \alpha \)-differentiable vector field, \( y_0 \in W \), \( \gamma(\cdot) : [0, T] \to W \) an integra
curve for $G(\cdot)$ with initial value $y_0$. Assume that \{DG(y(t)) : t \in [0, T]\} generates a weak evolution system \{U(t, s) : 0 \leq s \leq t \leq T\} on $X$. Then $\gamma$ is the only integral curve for $G(\cdot)$ on $[0, T]$ with initial value $y_0$.

Proof. Let $\sigma(\cdot) : [0, T] \to W$ be any integral curve for $G(\cdot)$ with initial value $y_0$. Then obviously $\sigma(0) = \gamma(0)$. To see that $\sigma = \gamma$, it suffices to show the following: if $r \in [0, T)$ and $\sigma(r) = \gamma(r)$, then there exists $\delta > 0$ such that $\sigma(t) = \gamma(t)$ for $t \in [r, r + \delta]$.

So assume that $\sigma(r) = \gamma(r)$. Since $G(\cdot)$ is locally uniformly $\alpha$-differentiable, there exists $\varepsilon > 0$, $k \geq 0$, such that $\|G(z) - G(y) - DG(y)(z - y)\|_X \leq k \|z - y\|_X$ for all $z, y \in W$ and $\|y - \gamma(r)\|_Y \leq \varepsilon$. Since $\sigma$ and $\gamma$ are both continuous, there exists $\delta > 0$ such that $\|\sigma(t) - \gamma(r)\|_Y \leq \varepsilon$ and $\|\gamma(t) - \gamma(r)\|_Y \leq \varepsilon$ for $t \in [r, r + \delta]$.

Thus, for $r \leq s \leq t \leq r + \delta$,

\[
\sigma(t) - \gamma(t) = \sigma(t) - \gamma(t) - U(t, r)(\sigma(r) - \gamma(r))
= \int_r^t \frac{\partial}{\partial s} (U(t, s)(\sigma(s) - \gamma(s))) ds
= \int_r^t U(t, s)(\sigma'(s) - \gamma'(s) - DG(\gamma(s))(\sigma(s) - \gamma(s))) ds
= \int_r^t U(t, s)(G(\sigma(s)) - G(\gamma(s)) - DG(\gamma(s))(\sigma(s) - \gamma(s))) ds,
\]

which implies that $\|\sigma(t) - \gamma(t)\|_X \leq K \int_r^t \|\sigma(s) - \gamma(s)\|_X ds$ for $t \in [r, r + \delta]$, where $K = k \|U\|_{\infty, X, X^*}$. By Gronwall's inequality, $\sigma(t) - \gamma(t) = 0$ for $t \in [r, r + \delta]$.

The remainder of this section is devoted to additional preliminaries needed for the applications in the next section. We first discuss some conditions which imply that time-dependent families of unbounded linear operators generate weak evolution systems, and then examine conditions under which smooth nonlinear partial differential operators induce smooth $\alpha$-differentiable maps between Sobolev function spaces.

DEFINITION 5. Let $A : [a, b] \to L(Y, X)$. We will call $A(\cdot)$ an $X$-stable map if each $A(t)$ has an extension to the generator $A^*(t)$ of a strongly continuous linear semigroup on $X$ such that there exist $M \geq 1$ and $\beta \in R$ with

\[
\|e^{s_n A^*_{(t_n)}} e^{s_{n-1} A^*_{(t_{n-1})}} \cdots e^{s_1 A^*_{(t_1)}}\|_{X^*, X} \leq M e^{\beta (s_1 + \cdots + s_n)}
\]

for each finite sequence $a \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq b$ and each finite sequence $s_1, \ldots, s_n$ of nonnegative numbers. The pair $(M, \beta)$ will be called stability constants for $A$. 
PROPOSITION 1. Assume that $A(\cdot): [a, b] \to (Y, X)$ is strongly measurable, bounded, and $X$-stable. Then $A(\cdot)$ generates at least one weak evolution system on $[a, b]$.

Proof: Special case of [1, Theorem 3.31].

DEFINITION 6. Let $C$ be a metric space, $p(\cdot, \cdot)$ the metric on $C$, and $h: C \to L(Y, X)$. We will call $h$ a metrically $X$-stable map if each $h(v)$ has an extension to the generator $h^*(v)$ of a strongly continuous linear semigroup on $X$ such that there exist constants $M \geq 1$, $\beta \in R$, and $c \geq 0$, with

$$\|e^{\beta h^*(v_n)}e^{\beta h^*(v_{n-1})} \cdots e^{\beta h^*(v_1)}\|_{X, X} \leq Me^{\beta(s_1 + \cdots + s_n) + c(p(v_1, v_2) + \cdots + p(v_{n-1}, v_n))}$$

for each pair of finite sequences $v_1, \ldots, v_n \in C$ and $s_1, \ldots, s_n \geq 0$. The triple $(M, \beta, c)$ will be called stability constants for $h$.

LEMMA 2. Let $C$ be a metric space, $h: C \to L(Y, X)$ a metrically $X$-stable map with stability constants $(M, \beta, c)$, and $\sigma: [a, b] \to C$ a curve of bounded variation. Then $h \circ \sigma: [a, b] \to L(Y, X)$ is $X$-stable with stability constants $(Me^{\psi(\sigma)}(s), \beta, c)$, where $\psi(\sigma)$ is the total variation of $\sigma$.

Proof: Let $a \leq t_1 \leq \cdots \leq t_n \leq b$, and $s_1, \ldots, s_n \geq 0$. Then

$$\|e^{\beta h^*(\sigma(t_n))} \cdots e^{\beta h^*(\sigma(t_1))}\|_{X, X} \leq Me^{\beta(s_1 + \cdots + s_n) + c(p(\sigma(t_1), \sigma(t_2)) + \cdots + p(\sigma(t_{n-1}), \sigma(t_n)))} \leq Me^{\psi(\sigma)\beta(s_1 + \cdots + s_n)}.$$

LEMMA 3. Let $X_i$ and $Y_i$ be Banach spaces with $Y_j \subset X_i$, $i = 1, 2$, $C$ a metric space, $h: C \to L(Y_1, X_1)$ a metrically $X_1$-stable map. Let $g: C \to L(X_2, X_1)$ be uniformly Lipschitz from $C$ to $L(X_2, X_1)$ with the uniform operator topology, such that $g(v)$ is an isomorphism for each $v \in C$, and $g(\cdot)$ and $g^{-1}(\cdot)$ are both bounded on $C$. Assume in addition that $g(v)(Y_2) = Y_1$ for each $v \in C$. Then $g^{-1}(v) h(v) g(v)$ has an extension to the generator of a strongly continuous linear semigroup on $X_2$ for each $v \in C$, and the map $f: C \to L(Y_2, X_2)$ defined by $f(\cdot) = g^{-1}(\cdot) h(\cdot) g(\cdot)$ is a metrically $X_2$-stable map.

Proof: That each $f(v)$ has an extension to a semigroup generator in $X_2$ follows from [5, Proposition 2.4]. The rest of the proof is a straightforward modification of the proof of [5, Proposition 4.4], following an application of [5, Proposition 3.3] to reduce the proof to a calculation involving semigroup generator resolvents.
Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $P$ be the subset of $L(H, H)$ consisting of those operators $B$ such that $\langle Bx, x \rangle \geq \epsilon \langle x, x \rangle$ for some $\epsilon > 0$, with the metric which $P$ inherits from $L(H, H)$ with the uniform operator topology. Note that $P$ is open in $L(H, H)$ with this operator topology. For each $B \in P$, we define the norm $\| \cdot \|_{N(B)}$ by $\| v \|_{N(B)}^2 = \langle Bv, v \rangle$. If $B$ is symmetric, note that $\langle B \cdot, \cdot \rangle$ is an inner product. We will denote $H$, reequipped with the norm $N(B)$, by $H_{N(B)}$.

**Lemma 4.** Let $X$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, $C$ a metric space, $A: C \rightarrow L(Y, X)$. Let $B: C \rightarrow P$ be locally Lipschitz, and denote the associated map into the norms on $X$ by $N(\cdot)$. Assume that, for each $v \in C$, $A(v)$ has an extension to the generator of a strongly continuous linear semigroup of type $(1, \beta)$ on $X_{N(v)}$. Then $A: C \rightarrow L(Y, X)$ is locally metrically $X$-stable.

**Proof.** Let $v \in C$. Since this is a local result, it suffices to replace $C$ with a neighborhood $V$ of $v$ such that there exists $k > 0$ with $k^{-1} \| x \|_{N(v)} \leq \| x \|_{N(u)}$ for all $x \in X, u \in V$, and to assume that there exists $c \geq 0$ such that $\| B(w) - B(u) \| \leq c \rho(v, w)$ for all $v, w \in V$, where $\rho(\cdot, \cdot)$ is the metric on $C$.

If $v, w \in V, 0 \neq x \in X,$

\[
\frac{\| x \|_{N(v)}}{\| x \|_{N(u)}} = 1 + \frac{\| x \|_{N(v)} - \| x \|_{N(u)}}{\| x \|_{N(v)}} \leq 1 + \frac{\| x \|_{N(v)} - \| x \|_{N(u)}}{\| x \|_{N(v)}} (\| x \|_{N(v)} + \| x \|_{N(u)}) \leq 1 + k^2 \| B(w) - B(v) \| \leq 1 + k^2 c(\rho(v, w)) \leq e^{k^2 c(\rho(v, w))}.
\]

Now, assume that $v_1, \ldots, v_n \in V$, and that $A^*(v_i)$ is an extension of $A(v_i)$ to the generator of a semigroup of type $(1, \beta)$ on $X_{N(v_i)}$ for each $i$. Then, it follows immediately from [5, Proposition 3.4] that, for $s_i \geq 0$,

\[
\| e^{s_i A^*(v_i)} e^{s_{i-1} A^*(v_{i-1})} \cdots e^{s_1 A^*(v_1)} \|_{N(v_n), N(v_0)} \leq e^{\rho(s_1 + \cdots + s_n) + 2k^2 c(\rho(v_1, v_2) + \cdots + \rho(v_{n-1}, v_n))}.
\]
This implies that
\[ \|e^{s_n 4\star(v_n)} e^{s_{n-1} 4\star(v_{n-1})} \ldots e^{s_1 4\star(v_1)}\|_{X,X} \leq k^2 e^{\beta(s_1 + \cdots + s_n)} + 2k^2 c (\rho(v_1,v_2) + \cdots + \rho(v_{n-1},v_n)) ,\]
which implies that \( A: V \to L(Y,X) \) is metrically \( X \)-stable with stability constants \((k^2, \beta, 2k^2c)\).

**Lemma 5.** Let \( A: C \to L(Y,X) \) be metrically \( X \)-stable, \( B: C \to L(X,X) \) bounded. If \((M, \beta, c)\) are stability constants for \( A(\cdot) \), \( K \) a bound for \( \|B(\cdot)\|_{X,X} \), then \( A(\cdot) + B(\cdot) \) is metrically \( X \)-stable with stability constants \((M, \beta + MK, c)\).

**Proof.** This is simply a rephrased version of [5, Proposition 3.53]. The proofs of the two versions are identical.

We will also make use of the criterion established by Tanabe [9] and Sobolevskii [7] for a time-dependent family of generators of analytic linear semigroups to generate an evolution system, which we state for the sake of completeness. If \( l \in L(Y,X) \), then we use the same symbol to denote the natural extension of \( l \) to a linear map between the complexifications of \( Y \) and \( X \).

**Proposition 2.** Let \( A: [a, b] \to L(Y,X) \) be uniformly Holder continuous with exponent \( \delta \in (0, 1] \). Assume that there exist real constants \( M, \omega \in \mathbb{R}, \varepsilon > 0 \), such that, for each complex \( \lambda \) with \( |\text{Arg}(\lambda - \omega)| < \pi/2 + \varepsilon \) and each \( t \in [a, b] \), \( \lambda \) is in the resolvent set of \( A(t) \) and \( \| (\lambda - A(t))^{-1} \|_{X,X} \leq M |\lambda - \omega|^{-1} \). Then \( A(\cdot) \) generates a unique evolution system on \([a, b]\).

We next turn our attention to Sobolev spaces, and to establishing criteria for the \( \alpha \)-differentiability of nonlinear partial differential operators. Let \( p \in (1, \infty) \), and let \( k \) be a nonnegative integer. Recall that \( L_p^\alpha(R^n, R) \) denotes the Banach space of \( L^p \) functions on \( R^n \) whose \( \alpha \)-order distributional derivatives are in \( L^p \) for \( i \leq k \). \( L_p^\alpha(R^n, R) \) is defined for general \( t \in \mathbb{R} \) via Bessel potentials. For \( t > n/p \), \( t - n/p \notin \mathbb{N} \), \( L_p^\alpha(R^n, R) \subset C^{(-n/p)}(R^n, R) \), the inclusion is continuous, and \( L_p^\alpha(R^n, R) \) is a Banach algebra. \( \| u \| \) will denote the norm of \( u \) in \( L_p^\alpha(R^n, R) \). Recall the basic interpolation inequality: for each \( r \leq s \leq t \), there is a constant \( C = C(r, s, t) \) such that, for each \( u \in L_p^\alpha(R^n, R) \), \( \| u \| \leq C \| u \|_r \| u \|_t^{1 - \varepsilon} \), where \( \varepsilon = (s - r)/(t - r) \).

Let \( M \) be a compact region in \( R^n \) with smooth boundary. Then \( L_p^\alpha(M, R) \) is defined for nonnegative integers \( k \) in the same way as \( L_p^\alpha(R^n, R) \). The space \( L_p^\alpha(M, R) \) is defined for general nonnegative \( t \) by complex interpolation. The embedding and multiplication theorems and interpolation
inequalities that hold for Sobolev spaces of functions on \( \mathbb{R}^n \) are also valid for the corresponding Sobolev spaces on \( M \).

If \( f: M \times \mathbb{R}^m \rightarrow \mathbb{R}^q \) is a \( C^\infty \) map, and \( t > n/p \), then there is an induced \( C^\infty \) map \( F \) from \( L^p_r(M, \mathbb{R}^m) \) to \( L^p_r(M, \mathbb{R}^q) \) defined by \( F(u) = f \circ u \) for each \( u \in L^p_r(M, \mathbb{R}^m) \). Furthermore, if we define the \( C^\infty \) map \( \delta f: M \times \mathbb{R}^m \rightarrow L^r_r(\mathbb{R}^m, \mathbb{R}^q) \) for each \( r \in \mathbb{N} \) by letting \( \delta f \) be the \( r \)th partial derivative of \( f \) with respect to the second variable, then \( \partial^r F(u)(v) = ((\delta f) \circ u)(v) \) for each \( u \in L^p_r(M, \mathbb{R}^m) \), \( v \in (L^p_r(M, \mathbb{R}^m))^r \). Finally, \( F \) and each of its derivatives are bounded maps when restricted to bounded subsets of \( L^p_r(M, \mathbb{R}^m) \).

The results in the preceding paragraph about induced mappings between function spaces hold if we replace \( M \) with \( \mathbb{R}^n \) (i.e., \( f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^q \)). However, because of the noncompactness of \( \mathbb{R}^n \), it is necessary to restrict attention to those \( C^\infty \) maps \( f \) which behave sufficiently well with respect to the \( \mathbb{R}^n \)-valued coordinate. A class of smooth maps which possess appropriate asymptotic behavior with respect to the \( \mathbb{R}^n \)-valued coordinate is defined as follows:

**Definition 7.** Let \( f(\cdot, \cdot): \mathbb{R}^n \times \mathbb{R}^m \rightarrow C^\infty \mathbb{R}^q \). We will say that \( f(\cdot, \cdot) \) is a type \( \mathcal{S}^p \) map if \( f(\cdot, 0) \in L^p(\mathbb{R}^n, \mathbb{R}^q) \) and if, for each \( i > 0 \),

1. \( D^i f(\cdot, 0): \mathbb{R}^n \rightarrow L^p(\mathbb{R}^n, L^r(\mathbb{R}^m, \mathbb{R}^q)) \), where \( D^i f(\cdot, \cdot) \) denotes the \( i \)th partial derivative of \( f(\cdot, \cdot) \) with respect to the first variable.
2. \( D^i f(\mathbb{R}^n \times B) \) is a bounded subset of \( L^p(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^q) \) for each bounded subset \( B \) of \( \mathbb{R}^m \).

If \( p = 2 \), we will omit the superscript and simply refer to \( f(\cdot, \cdot) \) as a type \( \mathcal{S} \) map.

It is often the case that nonlinear differential operators induce \( \alpha \)-differentiable maps between appropriate Sobolev spaces. We will need two results of this type. Let \( \tilde{m} = m(\sum_{j=0}^k (n+j-1)) \). For \( M \) open in \( \mathbb{R}^n \) and \( u: M \rightarrow C^k \mathbb{R}^m \), let \( j_k u: M \rightarrow C^\infty \mathbb{R}^m \) denote the \( k \)-jet extension of \( u \).

**Proposition 3.** Let \( M \) be either a compact region in \( \mathbb{R}^n \) with smooth boundary, or all of \( \mathbb{R}^n \). Assume that \( f: M \times \mathbb{R}^m \rightarrow \mathbb{R}^q \) is \( C^\infty \), and if \( M = \mathbb{R}^n \) assume in addition that \( f \) is \( \mathcal{S}^p \). For \( t > n/p + k \), define \( F: L^p_r(M, \mathbb{R}^m) \rightarrow L^p_r(\mathbb{R}^n, \mathbb{R}^q) \) by \( F(u)(x) = f(x, j_k u(x)) \) for each \( x \in \mathbb{R}^n \). Then:

(i) for \( t > n/p + 2k \), \( F \) is \( C^\infty \), and locally uniformly \( \alpha \)-differentiable from the Banach space pair \( (L^p_r(M, \mathbb{R}^m), L^p_r(\mathbb{R}^n, \mathbb{R}^q)) \) to \( L^p_r(M, \mathbb{R}^q) \);

(ii) for \( t > 2(n/p + k) \), \( F \) is \( C^\infty \), and locally uniformly \( \alpha \)-differentiable from the Banach space pair \( (L^p_r(M, \mathbb{R}^m), L^p_r(M, \mathbb{R}^m)) \) to \( L^p_r(M, \mathbb{R}^q) \).
Proof. (i) is a slightly rephrased version of [1, Theorem 5.26], and the proof of (i) is identical to the proof of the earlier version.

To establish (ii), let \( \| \cdot \|_s \) denote the \( L^p_s \)-norm function for \( s \in \mathbb{R} \), and let \( r \in (n/p, t/2 - k) \). Assume that \( B \) is a bounded set in \( L^p_0(M, \mathbb{R}^m) \), so \( j_k(B) \) is bounded in \( L^p_0(M, \mathbb{R}^m) \). This implies that there exists a constant \( c > 0 \) such that \( \| \delta^2f(\cdot, j_k u(\cdot)) \|_r \leq c \) for \( u \in B \). Then, for each \( u \in B, v \in L^p_0(M, \mathbb{R}^m) \), it follows from the interpolation inequality that

\[
\| D^2F(u)(v, v) \|_0 \leq c \| D^2F(u)(v, v) \|_r \\
= \| \delta^2f(\cdot, j_k u(\cdot))(j_k v(\cdot), j_k v(\cdot)) \|_r \\
\leq c \| j_k v \|_r^2 \leq c \| v \|_{t+k}^2 \\
\leq c \| v \|_{2(r+k)} \| v \|_0 \leq c \| v \|_r \| v \|_0.
\]

Thus Lemma 1 implies that \( F \) is locally uniformly \( \alpha \)-differentiable from the pair \( \left( L^p_t(M, \mathbb{R}^m), L^p(M, \mathbb{R}^m) \right) \) to \( L^p_t(M, \mathbb{R}^q) \).

Remark 2. Let \( f(\cdot, \cdot) \) be as in the above proposition, let \( t > n/p + k \), and let \( F(\cdot) \) be the induced map from \( L^p_t(M, \mathbb{R}^m) \) to \( L^p_{t-k}(M, \mathbb{R}^q) \). Then, as implicitly observed in the proof of the proposition, each \( DF(u) = \delta f \circ (Id, j_k u) \in L^p_{t-k}(M, L(R^m, R^q)) \) is a linear differential operator of order \( k \). Furthermore, if \( \varepsilon \in (0, t - k - n/p) \), \( DF(u) \) has coefficients which are uniformly Holder continuous with Holder exponent \( \varepsilon \). Thus the first derivative can be regarded as a smoothly parametrized family of linear \( k \)-th-order differential operators in \( C^\infty(M, L(R^m, R^q)) \) such that \( DF \) maps bounded subsets of \( L^p_t(M, \mathbb{R}^m) \) to bounded subsets of \( C^\infty(M, L(R^m, R^q)) \).

We conclude this section with some additional notation. Let \( \lambda_{0,i,j_1} : 1 \leq i \leq k, 1 \leq j_1 \leq \cdots \leq j_i \leq k \) be the natural \( R^m \)-valued projections on \( R^m \), defined so that \( \lambda_{0,i,j_1} \circ j_k u = u \) and \( \lambda_{0,i,j_1} \circ j_k u = \partial u / (\partial x_{j_1} \cdots \partial x_{j_i}) \). For \( p \in \mathbb{R}^m \), let \( p^0 = \lambda_0(p) \), and let \( p_{i_1,\ldots,i_n} = \lambda_{i_1,\ldots,i_n}(p) \).

If \( M \) is an arbitrary region in \( \mathbb{R}^n \) and \( t > 0 \), we let \( L^p_{t,0}(M, \mathbb{R}^m) \) denote the Banach subspace of \( L^p_t(M, \mathbb{R}^m) \) which consists of the closure of those \( C^\infty \) functions from \( M \) to \( \mathbb{R}^m \) whose support is compact and contained in the interior of \( M \). Recall the well-known result that \( L^p_{t,0}(R^n, \mathbb{R}^m) - L^p_t(R^n, \mathbb{R}^m) \). Similarly, for \( k \in \mathbb{N} \cup \{0\} \), we let \( C^{k,0}(M, \mathbb{R}^m) \) denote the closed subspace of \( C^k(M, \mathbb{R}^m) \) consisting of those elements \( f \) such that \( D^f \mid _{\partial M} \equiv 0 \) for \( 0 \leq i \leq k \).

Finally, for the Hilbert space \( p = 2 \), we employ the standard alternative notation \( H^t(M, \mathbb{R}^m) \) and \( H^0_t(M, \mathbb{R}^m) \) to denote the spaces \( L^2_t(M, \mathbb{R}^m) \) and \( L^2_{t,0}(M, \mathbb{R}^m) \), respectively.
2. Applications to Parabolic and Hyperbolic Systems

Parabolic Systems

Let \( k \in 2\mathbb{N} \), let \( M \) be either a compact region in \( \mathbb{R}^n \) with smooth boundary or all of \( \mathbb{R}^n \), let \( f: M \times \mathbb{R}^n \to C^\infty \mathbb{R}^m \), and let \( p \in (1, \infty) \). If \( M = \mathbb{R}^n \), assume in addition that \( f \) is an \( \mathcal{S}^p \) map.

Definition 8. We will say that \( f(\cdot, \cdot) \) is locally uniformly strongly elliptic if, for each bounded set \( B \subset \mathbb{R}^n \), there exists a constant \( c = c(B) > 0 \) such that

\[
\left( \sum_{1 \leq j_1 \leq \ldots \leq j_k \leq n} (-1)^{k/2 + 1} \frac{\partial f(x, w)}{\partial p^{j_1 \ldots j_k}} \xi_{j_1} \xi_{j_2} \ldots \xi_{j_k} v, v \right) > c \| \xi \|^k \| v \|^2
\]

for each \( (x, w) \in \mathbb{R}^n \times B, \xi \in \mathbb{R}^n, v \in \mathbb{R}^m \) (note that each partial derivative in this expression in an \( m \times m \) matrix.) We will call any \( c \) which satisfies this inequality a modulus of strong ellipticity for \( f \) on \( B \). If \( c \) satisfies the inequality for all \( w \in \mathbb{R}^n \), then we will simply say that \( c \) is a modulus of strong ellipticity for \( f \).

Remark 3. Moduli of strong ellipticity always exist for strongly elliptic linear operators, a result which does not hold in the nonlinear case. However, if \( t > n/p + k, B \) is a bounded subset of \( \mathbb{R}^n \), and \( f(\cdot, \cdot) \) is as described in Definition 8, then there exists a constant \( c > 0 \) such that \( c \) is a common modulus of strong ellipticity for the family of strongly elliptic linear operators \( \{ \delta f \circ (\text{Id}, j_k u): j_k u \in B \} \).

Theorem 2. Let \( M \) be a compact region in \( \mathbb{R}^n \) with smooth boundary, \( f: M \times \mathbb{R}^n \to C^\infty \mathbb{R}^m \) a strongly elliptic operator of order \( k \), \( t > 2(n/p + k) \), \( Y = L^p(M, \mathbb{R}^n) \cap L^{\infty}_{k/2,0}(M, \mathbb{R}^n), X = L^p(M, \mathbb{R}^m), F(\cdot): Y \to X \) the vector field induced by \( f(\cdot, \cdot) \). Then there exists at most one maximally defined integral curve for \( F(\cdot) \) for each initial value in \( Y \).

Proof. By Proposition 3(ii), \( F \) is a locally uniformly \( \alpha \)-differentiable vector field from \( Y \) to \( X \). Let \( \gamma: [0, T] \to Y \) be an integral curve for \( F \). Then, by Theorem 1, it suffices to show that \( DF(\cdot) \) generates a weak evolution system on \( [0, T] \).

Let \( s \in (n/p + k, t) \), let \( Z = L^s(M, \mathbb{R}^n) \), and let \( \tilde{Y} = L^s(M, \mathbb{R}^m) \cap L^{\infty}_{k/2,0}(M, \mathbb{R}^m) \). Since \( \gamma(\cdot) \) is continuous in \( Y \) and Lipschitz in \( X \), the interpolation inequality for norms between Sobolev spaces implies that \( \gamma \) is H"older continuous in \( Z \) with H"older exponent \( \varepsilon \), where \( \varepsilon = (t - s)/t \).

The map \( \delta f \circ (\cdot, j_k(\cdot)) \) induces a smooth map \( A(\cdot) \) from \( Z \) to \( L(\tilde{Y}, X) \). It follows from Remarks 2 and 3 and the theory of parabolic linear operators that \( A(w) \) is the generator of an analytic semigroup of linear operators on
$X$ for each $w \in Z$, and that for each bounded set $B \subset Z$ there exist real constants $M, \omega$ such that, for each $v \in B$ and complex number $\lambda$ with $\text{Re}(\lambda) > \omega$, $\lambda$ is in the resolvent set of $A(v)$ and $\| (\lambda - A(v))^{-1} \|_{X,X} \leq M | \lambda - \omega |^{-1}$.

Since $A \circ \gamma(\cdot)$ is Holder continuous with Holder exponent $\varepsilon$, Proposition 2 implies that $A(\gamma(\cdot))$ generates an evolution system $U(\cdot, \cdot)$ on $[0, T]$. Since each $A(\gamma(r))$ is an extension of $DF(\gamma(r))$, it follows that $U(\cdot, \cdot)$ is also an evolution system for $DF(\cdot)$.

**Corollary.** Let $\varepsilon > 0$, $Y = C^{2k+\varepsilon}(M, R^n) \cap C^{k/2-1,0}(M, R^n)$, $X = C^0(M, R^n)$, and let $F(\cdot)$ denote the vector field induced by $f(\cdot, \cdot)$ from $Y$ to $X$. Then there exists at most one maximally defined integral curve for $F(\cdot)$ for each initial value in $Y$.

**Proof.** Choose $p$ large enough to imply that $\varepsilon > 2n/p$, and let $t = 2k + \varepsilon$. Let $\bar{Y} = L^p_0(M, R^n) \cap L^p_{k/2,0}(M, R^n)$, and $\bar{X} = L^p(M, R^n)$. Then $Y \subset \bar{Y}$, $X \subset \bar{X}$, and it follows immediately that any integral curve in $C^0([0, T], Y) \cap C^1([0, T], X)$ is also an integral curve in $C^0([0, T], \bar{Y}) \cap C^1([0, T], \bar{X})$. The corollary thus follows from the theorem.

It is natural to ask whether $\varepsilon$ can be allowed to equal zero in the above corollary. By working in the space of continuous functions and using the Kato-Tanabe theory of evolution systems for time-dependent families of linear analytic semigroup generators with nonconstant domain [6], it is possible to prove the corollary with $\varepsilon = 0$. We sketch the proof.

**Theorem 3.** Let $M$ be a compact region in $R^n$ with smooth boundary, $f: M \times R^m \to C^0 R^m$ a strongly elliptic operator of order $k$, $Y = C^{2k}(M, R^n) \cap C^{k/2-1,0}(M, R^n)$, $X = C^0(M, R^n)$, and $F(\cdot): Y \to X$ the vector field induced by $f(\cdot, \cdot)$. Then there exists at most one maximally defined integral curve for $F(\cdot)$ for each initial value in $Y$.

**Proof.** The proof of Proposition 3(ii) adapts straightforwardly to the case of $C^1$ function spaces to show that $F(\cdot): Y \to X$ is locally uniformly $\alpha$-differentiable. So let $\gamma(\cdot): [0, T] \to Y$ be an integral curve for $F$. It suffices to show that $DF(\gamma(\cdot))$ generates a weak evolution system on $[0, T]$.

To see this, first note that, since the vector field $F(\cdot)$ is induced by a $k$th-order differential operator and $\gamma$ is a continuous curve in $C^{2k}(M, R^n)$, $\gamma$ is a $C^1$ curve in $C^k(M, R^n)$. Now, for each $t \in [0, T]$, define the $k$th-order linear differential operator $\psi(t) \in C^0(M, L(R^m, R^n))$ on $M$ by $\psi(t) = \delta f^\alpha (\text{Id}, j_k \gamma(t))$. Clearly $\psi(\cdot): [0, T] \to C^0(M, L(R^m, R^n))$ is a $C^1$ curve. It follows from [8, Theorems 2, 3] that each $\psi(t)$ induces the
generator \( A(t) \) of an analytic semigroup in \( X \) whose domain contains \( Y \), and furthermore that there exists a real constant \( \omega \) such that \( -(A(\cdot) - \omega) \) satisfies conditions E1, E2, E3 of [6]. Since \( -(A(\cdot) - \omega) \) satisfies conditions E1–E3 of [6], it follows from [6, Sect. 3] that \( \{A(\cdot)\} \) generates a weak evolution system. But since each \( A(t) \) is clearly an extension of \( DF(y(t)) \), this weak evolution system is also a weak evolution system for \( \{DF(\gamma(\cdot))\}. \)

**Remark 4.** The proof of the above theorem depends critically upon the results of H. B. Stewart [8] on the generation of analytic semigroups in the space of continuous functions by strongly elliptic operators. Stewart stated his results for the case of complex-valued strongly elliptic operators, but his proofs carry through for the case of strongly elliptic systems.

The proof of Theorem 2 and its corollary can be carried directly over to the case of strongly elliptic operators on unbounded domains (and ditto for the proof of Theorem 3). In particular, in the case of an analogue of Theorem 2 for the case \( M = \mathbb{R}^n \), it is possible to let \( X = L^p_* (\mathbb{R}^n, \mathbb{R}^m) \) for appropriately chosen \( s > 0 \), use Proposition 3(i) in place of Proposition 3(ii), and conclude the uniqueness of integral curves for slightly lower values of \( t \) than for the case of compact \( M \) (in the no-boundary case, it is easy to demonstrate the existence of the linear evolution system required by the uniqueness theorem in \( L^p_* (\mathbb{R}^n, \mathbb{R}^m) \) for appropriate \( s > 0 \)). However, that this improvement is minor is shown by the fact that this result does not allow us to obtain an improvement in the corollary. Thus we will not bother to develop this case for parabolic systems, although we will use this approach in our application to hyperbolic systems.

**Second-Order Wave Equations**

Let \( f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \) be a map of type \( \mathcal{S} \), where \( \tilde{m} = m(2 + 2n + n(n + 1)/2) \), and consider the equation

\[
\frac{\partial^2 u}{\partial t^2} (t, x) = f\left( x, u(t, x), \frac{\partial u}{\partial t} (t, x), \frac{\partial u}{\partial x_1} (t, x), \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_n} (t, x) \right) = 0
\]

in \( \mathbb{R}^{n+m} \), where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \; u(t, x) \in \mathbb{R}^m \). We transform this equation into an equivalent first-order system in the usual way by writing it as follows:
\[
\frac{\partial u}{\partial t}(t, x) = v(t, x),
\]
\[
\frac{\partial v}{\partial t}(t, x) = f\left(x, u(t, x), v(t, x), \frac{\partial u}{\partial x_1}(t, x), \ldots, \frac{\partial u}{\partial x_n}(t, x), \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_1}(t, x), \ldots, \frac{\partial^2 u}{\partial x_n \partial x_n}(t, x)\right).
\]

Let \(\lambda_0, \nu_0, \{\lambda_i, \nu_i, \lambda_{ij}, 1 \leq i \leq j \leq n\}\) be the \(R^m\)-valued projections on \(R^n\), defined so that \(\lambda_0^0(j_2u, j_1v) = u, \nu_0^0(j_2u, j_1v) = v, \lambda_i^o(j_2u, j_1v) = \partial u/\partial x_i, \nu_i^o(j_2u, j_1v) = \partial v/\partial x_i, \lambda_{ij}^o(j_2u, j_1v) = \partial^2 u/\partial x_i \partial x_j\). If \(y \in R^n\), let \(p^i = \lambda_i(y), q^j = \nu_j(y), p^{ij} = \lambda_{ij}(y)\). We assume that \(f(\cdot, \cdot)\) is locally uniformly strongly elliptic, i.e., for each bounded set \(B \subset R^n\), there exists a constant \(c = c(B) > 0\) such that

\[
\left(\sum_{1 \leq i < j \leq n} \frac{\partial f(x, y)}{\partial p^{ij}} x_i x_j + v \right) > c \|\xi\|^2 \|v\|^2
\]

for each \((x, y) \in R^n \times B, \xi \in R^n, v \in R^m\). In addition, we assume that the \(m \times m\) matrices \(\partial f(x, y)/\partial p^{ij}\) and \(\partial f(x, y)/\partial q^j\) are symmetric for each \((x, y) \in R^n \times R^n\).

**Theorem 4.** Let \(t > n/2 + 4\), \(Y = H'(R^n, R^m) \times H'^{-1}(R^n, R^m), X = H'^{-1}(R^n, R^m) \times H'^{-2}(R^n, R^m)\). Let \(G(\cdot) : Y \to H'^{-2}(R^n, R^m)\) be the map induced by \(f(\cdot, \cdot)\), and let \(F(\cdot) : Y \to X\) be the vector field defined by \(F(u, v) = (v, G(u, v))\). Then there exists at most one maximally defined integral curve for \(F(\cdot)\) for each initial value in \(Y\).

**Proof.** The proof of \(\alpha\)-differentiability for \(F(\cdot)\) is essentially the same as the proof of Proposition 3(i). Specifically, [1, Lemma 5.25] implies that, for each bounded subset \(B \subset Y\), there exists a constant \(k = k(B)\) such that \(\|D^2G(u, v)(w, z)\|_{r, -2} \leq k(\|w\|_r + \|z\|_r - 1)(\|w\|_{r, -1} + \|z\|_{r, -2})\) (for this part of the proof, we need only assume that \(t - 3 < n/2\)). It follows from Lemma 1 that \(F(\cdot)\) is locally uniformly \(\alpha\)-differentiable.

So let \(\gamma(\cdot) : [0, T] \to Y\) be an integral curve for \(F(\cdot)\). It suffices to show that \(\{DF(\gamma(\cdot))\}\) generates a weak evolution system on \([0, T]\). By Proposition 1, this reduces to showing that \(\{DF(\gamma(\cdot))\}\) is \(X\)-stable on \([0, T]\).

The map \(\delta f\) induces a \(C^\infty\) map from \(H'^{-3}(R^n, R^m)\) to \(H'^{-3}(L(R^n, R^m)) = C^1(L(R^n, R^m))\), which implies that \(\delta f^o(j_2, j_1)\) induces
For each \((u, v) \in X\), define

\[
\delta f^o (j_2 u, j_1 v) (j_2 w, j_1 z) = a^0 (u, v) w + b^0 (u, v) z + a^i (u, v) \frac{\partial w}{\partial x_i} + b^i (u, v) \frac{\partial^2 w}{\partial x_i \partial x_j},
\]

\(i \leq j\). It follows that each \(a^i\) induces a \(C^\infty\) map from \(X\) to \(C^1(L(R^n, R^n))\).

Let \(\mathcal{Z} = H^2(R^n, R^n) \times H^1(R^n, R^n), Z = H^1(R^n, R^n) \times H^0(R^n, R^n)\). For each \((u, v) \in X\), let \(A(u, v) : L(\mathcal{Z}, Z)\) be defined by \(A(u, v)(w, z) = (z, (\delta f^o (j_2 u, j_1 v))(j_2 w, j_1 z))\). Note that, for each \((u, v) \in Y\), \(A(u, v)|_Y = DF(u, v)\). Let \((\cdot, \cdot)_0\) denote the standard inner product on \(H^0(R^n, R^n)\). For each \((u, v) \in X\), define the norm \(N(u, v)\) on \(Z\) by

\[
\|(w, z)\|^2_{N(u, v)} = \sum_{i < j} \left( a^{ij}(u, v) \frac{\partial w}{\partial x_i}, \frac{\partial w}{\partial x_j} \right) + d_0 (w, w)_0 + (z, z)_0,
\]

where \(d_0 \geq 0\) is chosen sufficiently large to guarantee that \(N(u, v)\) is a norm [4, Lemma 3.2ff]. From [4, Lemma 3.5], it follows that there exists \(\beta \geq 0\) such that \(A(u, v)\) generates a semigroup of type \((1, \beta)\) on \(Z_{N(u, v)}\). Furthermore, it follows from [4, Lemmas 3.2, 3.6] that \(d_0\) and \(\beta\) can be chosen locally uniformly on neighborhoods in \(X\).

Define an isomorphism \(S: X \simeq Z\) by

\[
S = \begin{bmatrix}
(1 - A)^{(t - 2)/2} & 0 \\
0 & (1 - A)^{(t - 2)/2}
\end{bmatrix}
\]

where \(A\) is the Laplacian. Note that \(S\) induces an isomorphism between \(Y\) and \(\mathcal{Z}\). It follows from [4, Lemma 3.7] that, for each \((u, v) \in Y\), \(S(DF(u, v)) S^{-1} = A(u, v) + B(u, v)\), where \(B: Y \to L(Z, Z)\) is continuous. By Lemma 5, \(A(\cdot) + B(\cdot): Y_X \to L(\mathcal{Z}, Z)\) is metrically \(Z\)-stable on sufficiently small neighborhoods in \(Y\), where \(Y_X\) denotes the linear space \(Y\) with the metric which \(Y\) inherits as a subset of \(X\). By Lemma 3, \(DF(\cdot) = S^{-1}(A(\cdot) + B(\cdot)) S: Y_X \to L(Y, X)\) is metrically \(X\)-stable on sufficiently small neighborhoods in \(Y\). Thus Lemma 2 implies that \(\{DF(y(\cdot))\}\) is \(X\)-stable on \([0, T]\).

Existence as well as uniqueness was established in [3] for solutions to the wave equation in Theorem 4, in what is the only treatment of existence for a nonlinear wave equation of this generality I am aware of at this time. However, it was necessary to assume in [3] that \(t > n/2 + 6\), rather than
$t > n/2 + 4$ as assumed in Theorem 4. Also, as mentioned in the introduction, the techniques in [3] do not appear applicable to mixed problems. In the case of a wave equation on a compact region $M$ in $\mathbb{R}^n$ with smooth boundary, it is necessary to impose boundary conditions on the functions in $Y$ and $X$. In addition, it is necessary to modify the proof of Theorem 4 to accommodate the assumption that $X = Z$ (this actually simplifies the proof). Specifically, for each $s \geq 1$, let $H^s_0(M, \mathbb{R}^m)$ denote $H^s(M, \mathbb{R}^m) \cap H^0_0(M, \mathbb{R}^m)$. Define $Y = H^s_0(M, \mathbb{R}^m) \times H^s_0(M, \mathbb{R}^m)$, where $t$ is to be determined, $X = H^0_0(M, \mathbb{R}^m) \times H^0_0(M, \mathbb{R}^m)$, and let $F(\cdot)$ be as in Theorem 4. Then, for $t > n + 3$, a slight modification of Proposition 3(ii) shows that $F(\cdot)$ is locally uniformly $\alpha$-differentiable. To show that $\{DF(\gamma(\cdot))\}$ is $X$-stable for $\gamma(\cdot)$ an integral curve, it is sufficient to assume $t > n/2 + 4$ as in Theorem 4 (that part of the proof carries over to this case). Letting $t > \max(n + 3, n/2 + 4)$, we conclude that there exists at most one integral curve for $F(\cdot)$ for each initial value in $Y$. Note that, for $n \geq 2$, $\max(n + 3, n/2 + 4) = n + 3$, so the hypothesis on $t$ could be improved by improving the criterion for $\alpha$-differentiability in Proposition 3(ii).

REFERENCES