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Negations and contrapositions of complete lattices

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Abstract

We introduce the negation $\mathcal{C}\mathsf{L}$ of a complete lattice L as the concept lattice of the complementary context $(\mathcal{J}\mathsf{L}, \mathcal{M}\mathsf{L}, \not\leq)$, formed by the join-irreducible elements as objects and the meet-irreducible elements as attributes. We show that the double negation $\mathcal{C}\mathcal{C}\mathsf{L}$ is always order-embeddable in L, and that for finite lattices, the sequence $(\mathcal{C}^n\mathsf{L})_{n\in\omega}$ runs into a 'flip-flop' (i.e., $\mathcal{C}^n\mathsf{L} \simeq \mathcal{C}^{n+2}\mathsf{L}$ for some n). Using vertical sums, we provide constructions of lattices which are isomorphic or dually isomorphic to their own negation. The only finite distributive examples among such 'self-negative' or 'self-contrapositive' lattices are vertical sums of four-element Boolean lattices. Explicitly, we determine all self-negative and all self-contrapositive lattices with less than 11 points.

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0. Introduction

The basic notions of *Formal Concept Analysis* are *contexts* $\mathbb{K} = (J,M,I)$ and their *concept lattices* $\mathcal{B}\mathbb{K}$ (see Section 1). Since every context $\mathbb{K}$ has a natural 'negation' $\mathcal{C}\mathbb{K}$ obtained by passing to the complementary incidence relation, it is natural to ask for a corresponding notion of 'negation' for complete lattices. Thus, the 'negative small contexts' $(\mathcal{J}\mathsf{L}, \mathcal{M}\mathsf{L}, \not\leq)$ of complete lattices L, where $\mathcal{J}\mathsf{L}$ denotes the set of (completely) join-irreducible elements and $\mathcal{M}\mathsf{L}$ the set of (completely) meet-irreducible elements of L, and their concept lattices

$$\mathcal{C}\mathsf{L} = \mathcal{B}(\mathcal{J}\mathsf{L}, \mathcal{M}\mathsf{L}, \not\leq)$$

will be the main subject of our studies in the present note. The lattice $\mathcal{C}\mathsf{L}$ will be interpreted as the lattice-theoretical *negation* of L, while its dual will be referred to as

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the *contraposition* of L. (*Notational remark*: In [6], $\mathcal{C}\mathsf{L}$ denotes the context $(L, L, \leqslant)$ and not the negation of L.)

The investigation of the negation operator $\mathcal{C}$ was inspired by the following earlier observations (see [5]). The negation of the Aleksandrov completion (by lower ends) of a quasiordered set is dually isomorphic to the Dedekind–MacNeille completion (by cuts). For example, the Cantor Discontinuum D is the Aleksandrov completion of the rational line Q, while the unit interval I is the Dedekind–MacNeille completion of Q and, consequently, the negation of D. More generally, a complete lattice is a so-called Cantor lattice iff its negation is completely distributive.

Heuristic considerations of [5] suggested that for any finite lattice L, the *negation* sequence $(\mathcal{C}^n\mathsf{L})_{n\in\omega}$ should run into a 'flip-flop', i.e., a complete lattice isomorphic to its double negation. The general validity of that phenomenon was first established in [2], and a simplified proof will be given in Section 3. However, there exist infinite lattices L such that the lattices L, $\mathcal{C}\mathsf{L}$, $\mathcal{C}^2\mathsf{L}$, $\mathcal{C}^3\mathsf{L}, \ldots$ and their duals are mutually non-isomorphic (see Example 3.14).

Using the fact that under mild restrictions the contraposition operator commutes with *vertical sums*, we shall construct in Section 4 certain *self-negative* and *self-contrapositive* lattices, i.e., lattices L that are isomorphic to their own negation or contraposition, respectively. Finite self-negative lattices do not occur frequently, and self-contrapositive lattices are still much rarer: among the 14 570 non-isomorphic lattices that are MacNeille completions of posets with less than nine elements, there are 61 self-negative and only 7 self-contrapositive lattices (see Tables 1 and 2).

While we have found convenient criteria for a finite lattice to be a flip-flop (see 3.11), an open question is how to characterize (finite) self-negative or self-contrapositive lattices by simple conditions.

1. Contexts and concept lattices

For the readers convenience, we first recall a few basic notions and facts from Formal Concept Analysis (see [4, 5, 7, 8, 10] for more background).

A *context* is a triple $\mathbb{K} = (J, M, I)$, where J and M are sets and I is a relation between J and M, i.e., $I \subseteq J \times M$. The elements of $J_{\mathbb{K}} = J$ are interpreted as the *objects*, those of $M_{\mathbb{K}} = M$ as the *marks* or *attributes*, and $I_{\mathbb{K}} = I$ as the *incidence relation* of $\mathbb{K}$. With each context $\mathbb{K}$, there is associated a *Galois connection* or *polarity* (cf. [1])

$$\partial^{\mathbb{K}}: \mathcal{P}J \longrightarrow \mathcal{P}M, \quad A \longmapsto A^{\uparrow} = \{b \in M \mid \forall a \in A: aIb\},$$
$$\partial_{\mathbb{K}}: \mathcal{P}M \longrightarrow \mathcal{P}J, \quad B \longmapsto B^{\downarrow} = \{a \in J \mid \forall b \in B: aIb\}.$$

A *concept* of $\mathbb{K}$ is a pair (A, B) with $A = B^{\downarrow}$ and $B = A^{\uparrow}$. Ordered by

$$(A, B) \leqslant (C, D) \Longleftrightarrow A \subseteq C,$$

the concepts form a complete lattice, the *concept lattice* $\mathcal{B}\mathbb{K}$. The *basic functions*

$$\gamma = \gamma_{\mathbb{K}}\colon\ J \longrightarrow \mathcal{B}\mathbb{K}, \quad j \longmapsto (\{j\}^{\uparrow\downarrow}, \{j\}^{\uparrow})$$

and

$$\mu = \mu_{\mathbb{K}}\colon M \longrightarrow \mathcal{B}\mathbb{K}, \quad m \longmapsto (\{m\}^{\downarrow}, \{m\}^{\downarrow\uparrow})$$

satisfy the crucial equivalence

$$\gamma(j) \leqslant \mu(m) \Longleftrightarrow jIm.$$

The following notations are convenient:

$$AI = \{m \in M \mid jIm \text{ for some } j \in A\},$$

$$IB = \{j \in J \mid jIm \text{ for some } m \in B\}.$$

If x and y are elements of an ordered set $\mathsf{P} = (P, \leqslant)$, the set $\uparrow x = \{x\}\leqslant$ is called a *principal filter*, the set $\downarrow y = \leqslant\{y\}$ a *principal ideal*, and their intersection $[x, y] = \uparrow x \cap \downarrow y$ an *interval*. The ordered set $\mathcal{O}\mathsf{P} = (P, \geqslant)$ with $x \geqslant y$ iff $y \leqslant x$ is the *opposite* or *dual* of P. We call $\mathcal{K}\mathsf{P} = (P, P, \leqslant)$ the *large context* of P. Passing from P to $\mathcal{K}\mathsf{P}$, one may regard every ordered set as a context. Under this identification, for any subset $A \subseteq P$, the set of upper bounds is $A^{\uparrow}$, and the set of lower bounds is $A^{\downarrow}$.

Now let $\mathsf{L} = (L, \leqslant)$ be an arbitrary complete lattice. The least and the greatest element of L will be denoted by 0_{L} and 1_{L}, respectively. By a *join base* of L, we mean a *join-dense subset* of L, that is, a set $J \subseteq L$ such that each element of L is a join of members of J; an equivalent condition is that for all $x \nleqslant y$ in L, there exists a $j \in J$ with $j \leqslant x$ but $j \nleqslant y$. We call an element $j \in L$ *completely join-irreducible* (written $\vee$-*irreducible*) if it belongs to each subset X of L whose join it is. *Meet bases* (*meet-dense subsets*) and $\wedge$-*irreducible* elements are defined dually. We denote by $\mathcal{J}\mathsf{L}$ and $\mathcal{M}\mathsf{L}$ the set of all $\vee$- and $\wedge$-irreducible elements of L, respectively. Notice that our definition of irreducibility excludes 0_{L} from $\mathcal{J}\mathsf{L}$ and 1_{L} from $\mathcal{M}\mathsf{L}$; furthermore, $\mathcal{J}\mathsf{L}$ is contained in every join base, and $\mathcal{M}\mathsf{L}$ in every meet base. If $\mathcal{J}\mathsf{L}$ is a join base and $\mathcal{M}\mathsf{L}$ is a meet base of L then we speak of a *small-based* lattice (in [5], small-based lattices are called *doubly based*). Clearly, every finite lattice is small-based. The context $\mathcal{S}\mathsf{L} = (\mathcal{J}\mathsf{L}, \mathcal{M}\mathsf{L}, \leqslant)$ will be referred to as the *small context* or *standard context* of L.

Let us return to the basic functions γ and μ of an arbitrary context $\mathbb{K} = (J, M, I)$. For the concept lattice $\mathcal{B}\mathbb{K}$, a join base and a meet base are given by the images $\gamma[J]$ and $\mu[M]$, respectively. The context $\mathbb{K}$ is said to be *purified* if its basic functions γ and μ are injective. If $\mathbb{K}$ is purified with $\gamma[J] = \mathcal{J}\mathcal{B}\mathbb{K}$ and $\mu[M] = \mathcal{M}\mathcal{B}\mathbb{K}$ then $\mathbb{K}$ is said to be *reduced*. The small context of any small-based lattice is reduced, and on the other hand, the concept lattice of any reduced context is small-based. Moreover, a complete lattice L is small-based iff it is isomorphic to $\mathcal{B}\mathcal{S}\mathsf{L}$, and a context $\mathbb{K}$ is reduced iff it is isomorphic to $\mathcal{S}\mathcal{B}\mathbb{K}$. In this sense, the operators $\mathcal{B}$ and $\mathcal{S}$ induce mutually inverse bijections between (the isomorphism classes of) reduced contexts and small-based lattices.

Natural candidates for morphisms between contexts are pairs of maps rather than single maps. We will not enter this theory more deeply, but some basic definitions and facts will be crucial for later results on negations of contexts (cf. [4, 6]).

Given two contexts $\mathbb{K}$ and $\mathbb{L}$ and two maps $\alpha\colon J_{\mathbb{K}} \longrightarrow J_{\mathbb{L}}$ and $\beta\colon M_{\mathbb{K}} \longrightarrow M_{\mathbb{L}}$, we call the pair $\varphi = (\alpha, \beta)$ a *context morphism* (*from* $\mathbb{K}$ *into* $\mathbb{L}$). We say φ is *injective*, *surjective* or *bijective*, respectively, if the components α and β have the corresponding property, and φ is called a *quasi-embedding* (*of* $\mathbb{K}$ *in* $\mathbb{L}$) if for all $j \in J_{\mathbb{K}}$ and $m \in M_{\mathbb{K}}$,

$$j\, I_{\mathbb{K}}\, m \iff \alpha(j)\, I_{\mathbb{L}}\, \beta(m).$$

An injective quasi-embedding is called an *embedding*, a surjective one a *quasi-isomorphism*, and a bijective one an *isomorphism*. By definition, a map α between ordered sets P and Q is an (*order*) *embedding* iff (α, α) is a context embedding of $\mathcal{K}\mathsf{P}$ in $\mathcal{K}\mathsf{Q}$.

Given any context morphism $\varphi = (\alpha, \beta)$ from $\mathbb{K}$ into $\mathbb{L}$, define two maps

$$\alpha^{\rightarrow}\colon \mathcal{B}\mathbb{K} \longrightarrow \mathcal{B}\mathbb{L}, \quad (A,B) \longmapsto (\alpha[A]^{\uparrow\downarrow}, \alpha[A]^{\uparrow}),$$
$$\beta^{\rightarrow}\colon \mathcal{B}\mathbb{K} \longrightarrow \mathcal{B}\mathbb{L}, \quad (A,B) \longmapsto (\beta[B]^{\downarrow}, \beta[B]^{\downarrow\uparrow}).$$

It is evident that in this way we obtain a pair of isotone, i.e., order-preserving maps from $\mathcal{B}\mathbb{K}$ into $\mathcal{B}\mathbb{L}$ (which need not coincide, nor preserve joins or meets). For the subsequent theory of iterated negations, we shall use the following rule, lifting context embeddings to the concept lattices (cf. [2, 4]):

Proposition 1.1. *For any quasi-embedding* (α, β) *from a context* $\mathbb{K}$ *into a context* $\mathbb{L}$, *the maps* $\alpha^{\rightarrow}$ *and* $\beta^{\rightarrow}$ *are order embeddings from* $\mathcal{B}\mathbb{K}$ *into* $\mathcal{B}\mathbb{L}$. *If, moreover,* (α, β) *is a quasi-isomorphism then* $\alpha^{\rightarrow} = \beta^{\rightarrow}$ *is an isomorphism between* $\mathcal{B}\mathbb{K}$ *and* $\mathcal{B}\mathbb{L}$; *the inverse isomorphism sends a concept* (C, D) *of* $\mathbb{L}$ *to the concept* $(\alpha^{-1}[C], \beta^{-1}[D])$ *of* $\mathbb{K}$.

As an application of Proposition 1.1, we note:

Corollary 1.2. *For any subcontext* $\mathbb{K} = (J, M, I \cap (J \times M))$ *of a context* $\mathbb{L} = (K, N, I)$, *the inclusion maps* $\alpha\colon J \hookrightarrow K$ *and* $\beta\colon M \hookrightarrow N$ *induce embeddings* $\alpha^{\rightarrow}$ *and* $\beta^{\rightarrow}$ *of* $\mathcal{B}\mathbb{K}$ *in* $\mathcal{B}\mathbb{L}$. *In particular, for any complete lattice* $\mathsf{L} = (L, \leqslant)$ *and any context* $\mathbb{K} = (J, M, \leqslant)$ *with* $J, M \subseteq L$, *the concept lattice* $\mathcal{B}\mathbb{K}$ *is embedded in* L *by either of the maps*

$$\bigvee_{\mathbb{K},\mathsf{L}} : \mathcal{B}\mathbb{K} \longrightarrow \mathsf{L}, (A,B) \longmapsto \bigvee A \quad \textit{and} \quad \bigwedge_{\mathbb{K},\mathsf{L}} : \mathcal{B}\mathbb{K} \longrightarrow \mathsf{L}, (A,B) \longmapsto \bigwedge B.$$

Observe that if a complete lattice K is (order) embedded in a complete lattice L by a map α then K is a *retract* of L because either of the maps

$$\alpha^{\vee}\colon \mathsf{L} \longrightarrow \mathsf{K}, \quad x \longmapsto \bigvee\{y \in K \mid \alpha(y) \leqslant x\},$$
$$\alpha^{\wedge}\colon \mathsf{L} \longrightarrow \mathsf{K}, \quad x \longmapsto \bigwedge\{y \in K \mid x \leqslant \alpha(y)\}$$

is a *retraction*, i.e., isotone and left inverse to α.

2. Negations and contrapositions

With every context $\mathbb{K} = (J, M, I)$ there are associated three natural companions:
the *opposite* or *dual* context $\mathcal{O}\mathbb{K} = (M, J, \{(m, j) \mid jIm\})$,
the *complementary* or *negative* context $\mathcal{C}\mathbb{K} = (J, M, (J \times M) \setminus I)$,
and the *contrapositive context* $\mathcal{CO}\mathbb{K} = (M, J, (M \times J) \setminus \{(m, j) \mid jIm\})$.
See Fig. 1.

If one of the contexts $\mathbb{K}$, $\mathcal{C}\mathbb{K}$, $\mathcal{O}\mathbb{K}$, $\mathcal{CO}\mathbb{K}$ is purified then so are the others. However, complementation and contraposition may destroy reducedness.

Via the operators $\mathcal{S}$ and $\mathcal{B}$, negations and contrapositions may be lifted to the level of complete lattices. The *negative* (*small*) *context* of a complete lattice $\mathsf{L} = (L, \leqslant)$ is given by

$$\mathcal{CS}\mathsf{L} = (\mathcal{J}\mathsf{L}, \mathcal{M}\mathsf{L}, \nleqslant)$$

and the *negation* or *complement* of L is the concept lattice

$$\mathcal{C}\mathsf{L} = \mathcal{BCS}\mathsf{L} = \mathcal{B}(\mathcal{J}\mathsf{L}, \mathcal{M}\mathsf{L}, \nleqslant),$$

while the *contraposition* $\mathcal{CO}\mathsf{L}$ is the negation of the opposite lattice $\mathcal{O}\mathsf{L} = (L, \geqslant)$. Negation and contraposition is now applicable to lattices as well as to contexts. It will cause no confusion to use the same symbols $\mathcal{C}$ and $\mathcal{O}$ in both situations.

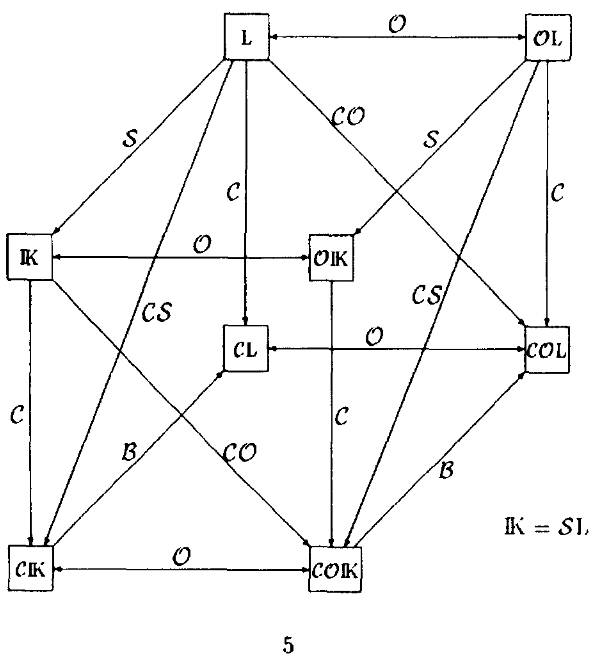


Fig. 1. A commutative cube of dualizations, negations and contrapositions.

The contraposition of L is isomorphic to the opposite of the negation $\mathcal{C}\mathsf{L}$ via the isomorphism

$$\mathcal{C}\mathcal{O}\mathsf{L} \longrightarrow \mathcal{O}\mathcal{C}\mathsf{L}, \quad (A,B) \longmapsto (B,A).$$

Moreover, the *double negation* $\mathcal{C}^2\mathsf{L}$ of any complete lattice L agrees, up to isomorphism, with the *double contraposition*:

$$(\mathcal{C}\mathcal{O})^2\mathsf{L} = \mathcal{C}\mathcal{O}\mathcal{C}\mathcal{O}\mathsf{L} \simeq \mathcal{C}\mathcal{O}\mathcal{O}\mathcal{C}\mathsf{L} = \mathcal{C}\mathcal{C}\mathsf{L} = \mathcal{C}^2\mathsf{L}.$$

Certain classical completions of (quasi-)ordered sets may be regarded as specific concept lattices, as explained, for example, in [5]. A *join completion* of an ordered set P is a complete lattice containing an isomorphic copy of P as a join base. The smallest join completion of P is the *Dedekind–MacNeille completion* or *normal completion* (*by cuts*), $\mathcal{N}\mathsf{P} = \{A^{\downarrow} \mid A \subseteq P\}$, and the largest one is the *Aleksandrov completion* (*by lower ends*), $\mathcal{A}\mathsf{P} = \{\leqslant A \mid A \subseteq P\}$. Up to isomorphisms, they are the concept lattices of the large context $\mathcal{K}\mathsf{P}$ and of its contraposition, respectively:

$$\mathcal{N}\mathsf{P} \simeq \mathcal{B}(P,P,\leqslant) = \mathcal{B}\mathcal{K}\mathsf{P}, \quad \mathcal{A}\mathsf{P} \simeq \mathcal{B}(P,P,\not\geqslant) = \mathcal{B}\mathcal{C}\mathcal{O}\mathcal{K}\mathsf{P}.$$

The following relationship between the two extremal join completions has been stated in [5].

Proposition 2.2. *For any poset* P, *the normal completion is the contraposition of the Aleksandrov completion, i.e.,* $\mathcal{N}\mathsf{P} \simeq \mathcal{C}\mathcal{O}\mathcal{A}\mathsf{P}$. *In particular, any complete lattice* L *is isomorphic to the contraposition* $\mathcal{C}\mathcal{O}\mathcal{A}\mathsf{L}$.

Example 2.3. For each ordinal number κ (considered as an ordered set), the completion $\mathcal{A}\kappa$ is isomorphic to $\kappa+1$. From 2.2, we get the isomorphism

$$\mathcal{C}\mathcal{O}(\kappa+1) \simeq \mathcal{N}\kappa \simeq \kappa \text{ for each successor ordinal } \kappa.$$

In particular, if L is a finite chain with $n+1$ elements and $n > 0$ then $\mathcal{C}\mathsf{L}$ and $\mathcal{C}\mathcal{O}\mathsf{L}$ are chains with n elements. On the other hand,

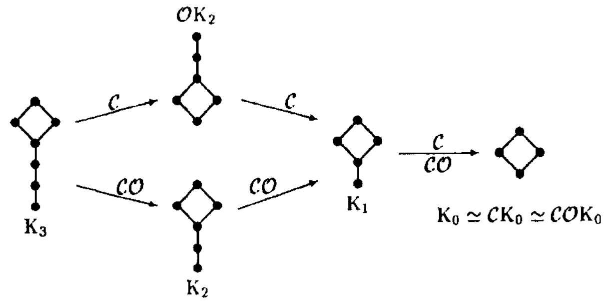
$$\mathcal{C}\mathcal{O}(\kappa+1) \simeq \mathcal{N}\kappa \simeq \kappa+1 \quad \text{if } \kappa \text{ is } 0 \quad \text{or a limit ordinal.}$$

Example 2.4. Let L be a complete lattice which is not small-based (like the real unit interval) and consider the lattice $\mathsf{K} = \mathcal{A}^n\mathsf{L}$. For every $k < n$, the k-th negation (resp. contraposition) of K is small-based, being isomorphic to $\mathcal{A}^{n-k}\mathsf{L}$ or to $\mathcal{A}^{n-k}\mathcal{O}\mathsf{L}$, while the n-th negation $\mathcal{C}^n\mathsf{K} \simeq \mathcal{O}^n\mathsf{L}$ is not small-based.

In the next section, we shall investigate *negation sequences* $(\mathcal{C}^n\mathsf{L})_{n\in\omega}$ of complete lattices L. Examples 2.5 and 2.6 below are symptomatic for the behavior of such sequences: up to isomorphism, the negation sequence in Example 2.5 becomes stationary, while the negation sequence in Example 2.6 ends with an oscillating pair.

Example 2.5. A kite loses its tail by iterated negation or contraposition.

For any ordinal κ, let K_κ denote the '*kite*' $\kappa \oplus 2^2$ obtained by putting the '*rhomb*' $\mathsf{K}_0 = 2^2$ above κ. Then the contraposition of K_κ is isomorphic to $\mathsf{K}_{\kappa-1}$ if κ is a successor ordinal, and it is isomorphic to K_κ itself if κ is 0 or a limit ordinal (see 4.1 and 2.3).

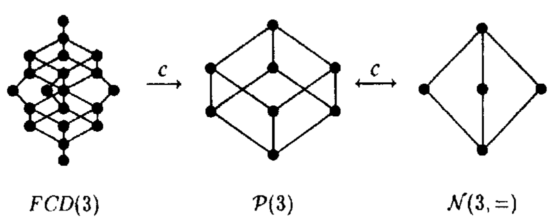


Here and in the following, symbol sequences of the form $\mathsf{X} \xrightarrow{\mathscr{H}} \mathsf{Y}$ mean $\mathscr{H}\mathsf{X} \simeq \mathsf{Y}$.

Example 2.6. Consider, for an arbitrary cardinal number κ, the free completely distributive lattice generated by κ elements, $FCD(\kappa)$, and the antichain $(\kappa, =)$. While (for $\kappa > 1$) the normal completion $\mathcal{N}(\kappa; =)$ adds a least and a greatest element, the Aleksandrov completion $\mathcal{A}(\kappa, =)$ is (isomorphic to) the power set $\mathcal{P}(\kappa)$. Each of these lattices is self-dual, and it is well known that $FCD(\kappa) \simeq \mathcal{AP}(\kappa) \simeq \mathcal{AA}(\kappa, =)$. By applying Proposition 2.2, we get the transformation rule

$$FCD(\kappa) \xrightarrow{\mathcal{CO}} \mathcal{P}(\kappa) \xleftrightarrow{\mathcal{CO}} \mathcal{N}(\kappa, =)$$

and, by self-duality, the corresponding rule with $\mathcal{C}$ instead of $\mathcal{CO}$. Especially, for $\kappa = 3$ we have the following diagram:



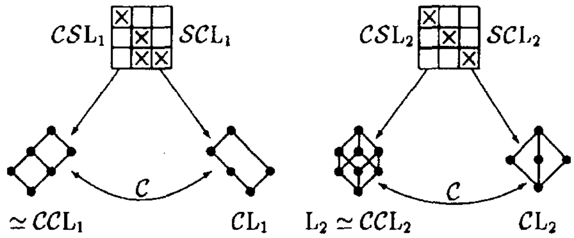
Notice that the non-distributive lattice $M_3 = \mathcal{N}(3, =)$ is a retract of the distributive lattice $FCD(3)$, but there is no lattice homomorphism from $FCD(3)$ onto M_3.

3. Double negation

From any reasonable negation operator one will expect that every object in the domain of the operator is at least comparable (in some obvious order) with its double negation, and one will be primarily interested in those objects which are isomorphic to their double negation. We are now going to investigate such situations in the realm of contexts and complete lattices.

A complete lattice L that is isomorphic to its double negation $\mathcal{C}\mathcal{C}\mathsf{L}$ will be referred to as a *flip-flop* (*lattice*). While trivially any context $\mathbb{K}$ is identical with its double complementation $\mathcal{C}\mathcal{C}\mathbb{K}$, it is not clear a priori which complete lattices L are flip-flops.

Example 3.1. Two pairs of flip-flops:



In order to show that every complete lattice L admits order embeddings (but in general no join- or meet-embeddings) of its double negation $\mathcal{C}\mathcal{C}\mathsf{L}$ in L, we have to analyze the so-called *arrow relations* of the *negative* context $\mathcal{C}\mathcal{S}\mathsf{L} = (\mathcal{J}\mathsf{L}, \mathcal{M}\mathsf{L}, \not\leqslant)$. They are defined as follows (cf. [5] or [8]): for any $j \in \mathcal{J}\mathsf{L}$ and $m \in \mathcal{M}\mathsf{L}$,

$$j \swarrow m \iff j \leqslant m \text{ and } \forall k \in \mathcal{J}\mathsf{L}: (\mathcal{M}\mathsf{L} \cap \uparrow k \subset \mathcal{M}\mathsf{L} \cap \uparrow j \Longrightarrow k \not\leqslant m),$$

$$j \nearrow m \iff j \leqslant m \text{ and } \forall n \in \mathcal{M}\mathsf{L}: (\mathcal{J}\mathsf{L} \cap \downarrow n \subset \mathcal{J}\mathsf{L} \cap \downarrow m \Longrightarrow j \not\leqslant n),$$

where $\subset$ denotes proper inclusion. These conditions may be simplified in case of small-based lattices (cf. [5]).

Lemma 3.2. *If j is a $\bigvee$-irreducible element and m is a $\bigwedge$-irreducible element of a small-based lattice L then*

$$j \swarrow m \iff [j,m] \cap \mathcal{J}\mathsf{L} = \{j\} \quad \textit{and} \quad j \nearrow m \iff [j,m] \cap \mathcal{M}\mathsf{L} = \{m\}.$$

Hence, $j \swarrow m$ means that j is a maximal $\bigvee$-irreducible element with $j \leqslant m$, and $j \nearrow m$ means that m is a minimal $\bigwedge$-irreducible element with $j \leqslant m$.

The following facts are borrowed from Hilfssatz 13 in [8]:

Lemma 3.3. *For any complete lattice L, an element of the negation $\mathcal{C}\mathsf{L}$ is $\bigvee$-irreducible iff it is of the form $\gamma_{\mathcal{C}\mathcal{S}\mathsf{L}}(j)$ for some $j \in \swarrow\mathcal{M}\mathsf{L}$, and dually. If L is finite then $\swarrow$ may be replaced with $\swarrow\!\!\!\nearrow$, where $j \swarrow\!\!\!\nearrow m$ means $j \swarrow m$ and $j \nearrow m$.*

These observations suggest to consider, for any complete lattice L, the subcontext $\mathcal{S}'\mathsf{L} = (\mathcal{J}'\mathsf{L}, \mathcal{M}'\mathsf{L}, \leqslant)$ of $\mathcal{S}\mathsf{L}$ with

$$\mathcal{J}'\mathsf{L} = \swarrow\mathcal{M}\mathsf{L} = \gamma_{\mathcal{C}\mathcal{S}\mathsf{L}}{}^{-1}[\mathcal{J}\mathcal{C}\mathsf{L}] \quad \text{and} \quad \mathcal{M}'\mathsf{L} = \mathcal{J}\mathsf{L}\nearrow = \mu_{\mathcal{C}\mathcal{S}\mathsf{L}}{}^{-1}[\mathcal{M}\mathcal{C}\mathsf{L}].$$

Observe that if 1_{L} is $\bigvee$-irreducible then $\gamma_{\mathcal{CS}\mathsf{L}}(1_{\mathsf{L}})$ is the bottom element of $\mathcal{C}\mathsf{L}$. Hence 1_{L} is never an element of $\mathcal{J}'\mathsf{L}$ and, dually, 0_{L} never belongs to $\mathcal{M}'\mathsf{L}$.

Proposition 3.4. *For any complete lattice* L, *a quasi-isomorphism* $\eta'_{\mathsf{L}} = (\gamma'_{\mathsf{L}}, \mu'_{\mathsf{L}})$ *from* $\mathcal{S}'\mathsf{L}$ *onto* $\mathcal{CSC}\mathsf{L}$ *is given by*

$$\gamma'_{\mathsf{L}}\colon\ \mathcal{J}'\mathsf{L} \longrightarrow \mathcal{JC}\mathsf{L}, \quad j \longmapsto \gamma_{\mathcal{CS}\mathsf{L}}(j),$$

$$\mu'_{\mathsf{L}}\colon\ \mathcal{M}'\mathsf{L} \longrightarrow \mathcal{MC}\mathsf{L}, \quad m \longmapsto \mu_{\mathcal{CS}\mathsf{L}}(m).$$

If L *is small-based then* η'_{L} *is an isomorphism.*

Proof. As the image of $\gamma_{\mathcal{CS}\mathsf{L}}$ is a join base of $\mathcal{C}\mathsf{L} = \mathcal{BCS}\mathsf{L}$, we know that $\mathcal{JC}\mathsf{L}$ is contained in $\gamma_{\mathcal{CS}\mathsf{L}}[\mathcal{J}\mathsf{L}]$. Hence, $\gamma_{\mathcal{CS}\mathsf{L}}[\mathcal{J}'\mathsf{L}]$ coincides with $\mathcal{JC}\mathsf{L}$. Together with the dual fact, this shows that η'_{L} is a surjective context morphism. For any $j \in \mathcal{J}'\mathsf{L}$ and $m \in \mathcal{M}'\mathsf{L}$, we have:

$$j \leqslant m \iff (j,m) \notin I_{\mathcal{CS}\mathsf{L}} \iff \gamma_{\mathcal{CS}\mathsf{L}}(j) \not\leqslant_{\mathcal{C}\mathsf{L}} \mu_{\mathcal{CS}\mathsf{L}}(m)$$

$$\iff \gamma_{\mathcal{CS}\mathsf{L}}(j)\, I_{\mathcal{CSC}\mathsf{L}}\, \mu_{\mathcal{CS}\mathsf{L}}(m).$$

If L is small-based then $\mathcal{S}\mathsf{L}$ and $\mathcal{CS}\mathsf{L}$ are purified, hence η'_{L} is bijective. □

Now we are ready for the announced embedding theorem:

Theorem 3.5. *For any complete lattice* L, *its double negation* $\mathcal{CC}\mathsf{L}$ *is isomorphic to the concept lattice* $\mathcal{BS}'\mathsf{L}$ *and order embeddable in* L, *hence a retract of* L.

Proof. Proposition 1.1 together with 3.4 yields an isomorphism $\gamma_{\mathsf{L}}'^{\rightarrow} = \mu_{\mathsf{L}}'^{\rightarrow}$ between $\mathcal{BS}'\mathsf{L}$ and $\mathcal{CC}\mathsf{L} = \mathcal{BCSC}\mathsf{L}$. By Corollary 1.2, we have order-embeddings $\bigvee_{\mathcal{S}'\mathsf{L},\mathsf{L}}$ and $\bigwedge_{\mathcal{S}'\mathsf{L},\mathsf{L}}$ from $\mathcal{BS}'\mathsf{L}$ into L. Composing them with the inverse of the established isomorphism between $\mathcal{BS}'\mathsf{L}$ and $\mathcal{CC}\mathsf{L}$, one obtains two explicit embeddings of $\mathcal{CC}\mathsf{L}$ in L, namely

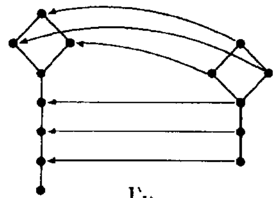
$$\Gamma_{\mathsf{L}}\colon \mathcal{CC}\mathsf{L} \longrightarrow \mathsf{L}, \quad (C,D) \longmapsto \bigvee \gamma_{\mathcal{CS}\mathsf{L}}{}^{-1}[C]$$

and

$$\mathrm{M}_{\mathsf{L}}\colon \mathcal{CC}\mathsf{L} \longrightarrow \mathsf{L}, \quad (C,D) \longmapsto \bigwedge \mu_{\mathcal{CS}\mathsf{L}}{}^{-1}[D]. \qquad \square$$

In general, the embeddings Γ_{L} and M_{L} do not agree, nor do they preserve (finite) joins, nor meets. In order to verify that claim, consider once more the 'kites' K_n of Example 2.5.

Example 3.6. The order embeddings Γ_{K_n} and $\mathrm{M}_{\mathsf{K}_n}$:



$$\mathrm{K}_4 \xleftarrow[\mathrm{M}_{\mathrm{K}_4}]{\Gamma_{\mathrm{K}_4}} \mathcal{C}\mathcal{C}\mathrm{K}_4 \simeq \mathrm{K}_2$$

A straightforward computation shows that the embeddings Γ_{K_n} and $\mathrm{M}_{\mathsf{K}_n}$ coincide for $n \geqslant 2$, but they never preserve all joins, nor all meets.

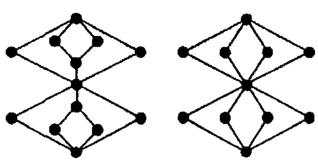
While $\mathcal{C}^n\mathsf{K}_0$ is isomorphic to the rhomb K_0, a different situation occurs with K_1:

$$\mathrm{K}_1 \xleftarrow{\Gamma_{\mathrm{K}_1}} \mathcal{C}\mathcal{C}\mathrm{K}_1 \simeq \mathrm{K}_0 \qquad \mathrm{K}_1 \xleftarrow{\mathrm{M}_{\mathrm{K}_1}} \mathcal{C}\mathcal{C}\mathrm{K}_1 \simeq \mathrm{K}_0$$

Here we have $\mathcal{C}\mathcal{C}\mathsf{K}_1 \simeq \mathcal{C}\mathsf{K}_1 \simeq \mathsf{K}_0$, and we obtain two distinct embeddings Γ_{K_1} and $\mathrm{M}_{\mathsf{K}_1}$ of $\mathcal{C}\mathcal{C}\mathsf{K}_1$ in K_1. The map Γ_{K_1} does not preserve all meets, while $\mathrm{M}_{\mathsf{K}_1}$ does not preserve the join of the empty set.

The next example shows that, in general, there need not exist any join- or meet-preserving embeddings of $\mathcal{C}\mathcal{C}\mathsf{L}$ in L.

Example 3.7 (*cf.* 3.13 *and* 4.1). For the lattice L below, the double negation $\mathcal{C}\mathcal{C}\mathsf{L}$ admits $(4!)^2 = 576$ equivalent order embeddings in L. None of them preserves joins or meets (consider the middle element!), while the corresponding retractions do preserve arbitrary joins and meets (but see 2.6).



L CCL

Call a complete lattice L *coreduced* if the negative context $\mathcal{C}\mathcal{S}\mathsf{L}$ is reduced. An important consequence of Proposition 3.4 is

Corollary 3.8. *The following statements on a complete lattice* L *are equivalent*:

(a) L *is coreduced.*
(b) η'_L *is an isomorphism between the small context* $\mathcal{S}\mathsf{L}$ *and the context* $\mathcal{CSC}\mathsf{L}$.
(c) $\mathcal{S}\mathsf{L}$ *is purified and agrees with* $\mathcal{S}'\mathsf{L}$.
(d) $\mathcal{J}\mathsf{L} = \swarrow\mathcal{M}\mathsf{L}$ *and* $\mathcal{J}\mathsf{L}\nearrow = \mathcal{M}\mathsf{L}$ *in the purified negative context* $\mathcal{CS}\mathsf{L}$.
Each of these conditions implies that $\mathcal{CS}\mathsf{L}$ *is isomorphic to* $\mathcal{SC}\mathsf{L}$.

The next theorem exhibits a large class of flip-flop lattices:

Theorem 3.9. *For a small-based lattice* L, *the following conditions are equivalent*:
(a) L *is coreduced.*
(b) L *is a flip-flop, and* $\mathcal{C}\mathsf{L}$ *is coreduced and small-based.*
(c) $\mathcal{SC}\mathsf{L} \simeq \mathcal{CS}\mathsf{L}$, *and* $\mathcal{C}\mathsf{L}$ *is small-based.*
(d) *For each* $j \in \mathcal{J}\mathsf{L}$, *there is an* $m \in \mathcal{M}\mathsf{L}$ *with* $[j,m] \cap \mathcal{J}\mathsf{L} = \{j\}$, *and dually.*

Proof. (a) $\Longrightarrow$ (c): Apply 3.8. The equation $\mathcal{JC}\mathsf{L} = \gamma_{\mathcal{CS}\mathsf{L}}[\mathcal{J}\mathsf{L}]$ and its dual imply that $\mathcal{C}\mathsf{L}$ is small-based.

(c) $\Longrightarrow$ (b): $\mathcal{CSC}\mathsf{L} \simeq \mathcal{S}\mathsf{L}$ is reduced (since L is small-based). Hence, $\mathcal{C}\mathsf{L}$ is coreduced, and further, $\mathcal{CC}\mathsf{L} = \mathcal{BCSC}\mathsf{L} \simeq \mathcal{BS}\mathsf{L} \simeq \mathsf{L}$.

(b) $\Longrightarrow$ (a): Apply (a) $\Longrightarrow$ (b) to $\mathcal{C}\mathsf{L}$ instead of L.

(a) $\Longleftrightarrow$ (d) follows from the corresponding equivalence in 3.8 by 3.2. $\square$

Although we shall see below that every finite flip-flop is coreduced, there exist infinite small-based flip-flop lattices which fail to be coreduced.

Example 3.10. Let κ be any limit ordinal (for example the ordered set ω of natural numbers). Then, by 2.3, the lattice $\mathsf{L} = \kappa+1$ is a small-based flip-flop. But L cannot be coreduced since its bottom element is $\bigwedge$-irreducible.

Examples of a coreduced lattice which is not a flip-flop and of a flip-flop which is not small-based will be given in Section 4.

Theorem 3.9 can be improved essentially for finite lattices (where the symbol $\sharp$ assigns to each set its cardinality).

Theorem 3.11. *The following statements on a finite lattice* L *are equivalent*:
(a) L *is coreduced.*
(b) L *is a flip-flop.*
(c) $\mathcal{SC}\mathsf{L} \simeq \mathcal{CS}\mathsf{L}$.
(d) $\mathcal{J}\mathsf{L} = \swarrow\mathcal{M}\mathsf{L}$ *and* $\mathcal{J}\mathsf{L}\nearrow = \mathcal{M}\mathsf{L}$ *in* $\mathcal{CS}\mathsf{L}$.
(e) $\sharp\mathcal{J}\mathsf{L} = \sharp\mathcal{JC}\mathsf{L}$ *and* $\sharp\mathcal{M}\mathsf{L} = \sharp\mathcal{MC}\mathsf{L}$.
(f) $\sharp\mathcal{J}\mathsf{L} + \sharp\mathcal{M}\mathsf{L} = \sharp\mathcal{JC}\mathsf{L} + \sharp\mathcal{MC}\mathsf{L}$.
(g) $\sharp\mathsf{L} = \sharp\mathcal{CC}\mathsf{L}$.

Proof. (a) $\Longleftrightarrow$ (c) is clear by the corresponding equivalence in 3.9 and finiteness of L. Using 3.3, one derives (a) $\Longleftrightarrow$ (d) follows from the corresponding equivalence in 3.8. As every finite lattice is small-based, the maps $\gamma_{\mathcal{CS}\mathsf{L}}$ and $\mu_{\mathcal{CS}\mathsf{L}}$ are injective, i.e., the context $\mathcal{CS}\mathsf{L}$ is purified. Now, since $\mathcal{JC}\mathsf{L}$ is a *finite* subset of $\gamma_{\mathcal{CS}\mathsf{L}}[\mathcal{J}\mathsf{L}]$ and dually for $\mathcal{M}$, condition (a) is equivalent to (e).

By Theorem 3.9, (b) follows from (a). As $\sharp\mathcal{J}\mathsf{L} \geqslant \sharp\mathcal{JC}\mathsf{L} \geqslant \sharp\mathcal{JCC}\mathsf{L}$ by 3.4, condition (b) implies (e), and (f) is just another formulation of (e). If (g) holds then every embedding of $\mathcal{CC}\mathsf{L}$ in L is an isomorphism. Hence 3.5 yields the implication (g)$\Longrightarrow$(b), and the converse is trivial. □

Condition (d) of the above theorem is a convenient criterion to work with: it states that the context table of the negative context $\mathcal{CS}\mathsf{L}$ contains at least one double arrow in each row and in each column. Sometimes, it may be easier to check Condition (d) of 3.9 directly.

As the negation operator cannot increase the number of $\bigvee$- or $\bigwedge$-irreducible elements, the sequence $(\sharp\mathcal{JC}^n\mathsf{L} + \sharp\mathcal{MC}^n\mathsf{L})_{n\in\omega}$ is monotone decreasing for every finite lattice L. Together with the equivalence (b) $\Longleftrightarrow$ (f) of Theorem 3.11, this fact yields the stationarity of the double negation sequence $(\mathcal{C}^{2n}\mathsf{L})_{n\in\omega}$ for finite lattices L. More precisely, we have the following:

Corollary 3.12. *If L is a finite lattice then for $n \geqslant \sharp\mathcal{J}\mathsf{L} + \sharp\mathcal{M}\mathsf{L}$, the n-th negation $\mathcal{C}^n\mathsf{L}$ is coreduced, hence a flip-flop. If k denotes the least number i such that $\mathcal{C}^i\mathsf{L}$ is a flip-flop, then*

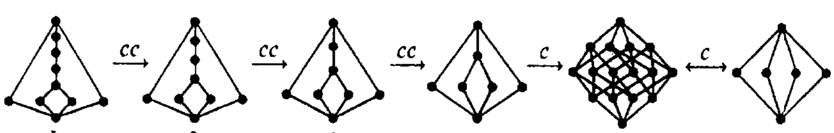
$$k + \sharp\mathcal{JC}^k\mathsf{L} + \sharp\mathcal{MC}^k\mathsf{L} \leqslant \sharp\mathcal{J}\mathsf{L} + \sharp\mathcal{M}\mathsf{L}. \qquad (\bigstar)$$

In fact, a straightforward induction, using (b) $\Longleftrightarrow$ (f) of 3.11 again, yields the inequality

$$i + \sharp\mathcal{JC}^i\mathsf{L} + \sharp\mathcal{MC}^i\mathsf{L} \leqslant \sharp\mathcal{J}\mathsf{L} + \sharp\mathcal{M}\mathsf{L}$$

for all $i \leqslant k$. The previous estimate is sharp:

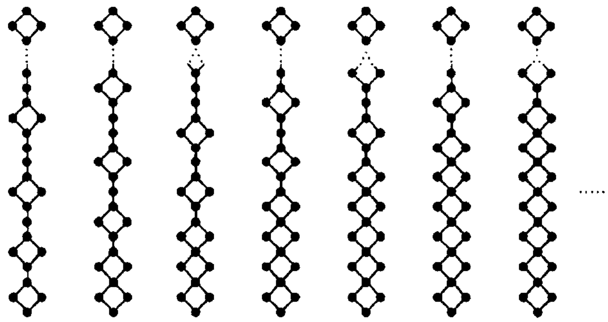
Example 3.13. For each odd integer k, there is an 'extended (dual) kite', obtained from a (dual) kite (see 2.5) by adding two 'wings', for which the inequality ($\bigstar$) of 3.12 becomes an equality. The case $k = 7$ is sketched below.



L $\quad$ $\mathcal{C}^2$L $\quad$ $\mathcal{C}^4$L $\quad$ $\mathcal{C}^6$L $\quad$ $\mathcal{C}^7\mathrm{L} \simeq \mathcal{C}^9\mathrm{L}$ $\quad$ $\mathcal{C}^8\mathrm{L} \simeq \mathcal{C}^{10}\mathrm{L}$

However, for an infinite small-based lattice L, neither of the sequences $(\mathcal{C}^n\mathsf{L})_{n\in\omega}$ and $((\mathcal{CO})^n\mathsf{L})_{n\in\omega}$ need contain any flip-flop lattice.

Example 3.14. Consider a lattice L with the diagram below. The iterated contrapositions are pairwise non-isomorphic, but mutually order embeddable (cf. [3]).



L $\quad \mathcal{CO}$L $\quad (\mathcal{CO})^2$L $\quad (\mathcal{CO})^3$L $\quad (\mathcal{CO})^4$L $\quad (\mathcal{CO})^5$L $\quad (\mathcal{CO})^6$L .....

4. Self-negative and self-contrapositive lattices

We call a complete lattice *self-negative* (respectively, *self-contrapositive*) if it is isomorphic to its own negation (respectively, contraposition). Though self-negative lattices are rather rare, it is possible to construct such lattices by vertical superposition of small-based coreduced lattices and their negations, as will be shown below (see, for example, Fig. 6 in Table 1).

Given two complete lattices L and L′, the *ordinal sum* $\mathsf{L} \oplus \mathsf{L}'$ is obtained by putting L′ above L, and the *vertical sum* $\mathsf{L} \mathbin{\widehat{\oplus}} \mathsf{L}'$ results from the latter by identifying the top element of L with the bottom element of L′ (cf. [3, 8, 10]). Call a complete lattice L *upper regular* if for each $j \in \mathcal{J}\mathsf{L}$, there exists an $m \in \mathcal{M}\mathsf{L}$ with $j \leqslant m$, and *lower regular* if the dual condition holds. If L is upper regular then the top element 1_{L} is $\bigvee$-reducible, and the converse holds provided L is small-based. By the equivalence (a)$\Longleftrightarrow$(d) in 3.8, every coreduced lattice and, in particular, every finite flip-flop (see 3.11) is upper and lower regular.

In [3], we have established the following rule for negations of vertical sums:

Proposition 4.1. *Let* L *and* L′ *be complete lattices. If* L *is upper regular or* L′ *is lower regular then*

$$\mathcal{C}(\mathsf{L} \mathbin{\widehat{\oplus}} \mathsf{L}') \simeq \mathcal{C}\mathsf{L}' \mathbin{\widehat{\oplus}} \mathcal{C}\mathsf{L}.$$

In all other cases,

$$\mathcal{C}(\mathsf{L} \mathbin{\widehat{\oplus}} \mathsf{L}') \simeq \mathcal{C}\mathsf{L}' \oplus \mathcal{C}\mathsf{L}.$$

From this rule, one easily derives

Theorem 4.2. *If* L *is a small-based coreduced lattice then* $\mathsf{L} \mathbin{\widehat{\oplus}} \mathcal{C}\mathsf{L}$ *and* $\mathcal{C}\mathsf{L} \mathbin{\widehat{\oplus}} \mathsf{L}$ *are self-negative.*

Proof. By 3.9, L is an upper regular flip-flop, and by 4.1, it follows that

$$\mathcal{C}(\mathsf{L}\,\widehat{\oplus}\,\mathcal{C}\mathsf{L}) \simeq \mathcal{CC}\mathsf{L}\,\widehat{\oplus}\,\mathcal{C}\mathsf{L} \simeq \mathsf{L}\,\widehat{\oplus}\,\mathcal{C}\mathsf{L}. \qquad \square$$

In order to apply 4.1 to vertical sums of flip-flops, we need two auxiliary lemmas.

Lemma 4.3. *Suppose* L *is a small-based lower regular complete lattice. Then the top element of the negation* $\mathcal{C}\mathsf{L}$ *is* $\bigvee$*-reducible. Hence, if* $\mathcal{C}\mathsf{L}$ *is small-based, too, then it is upper regular.*

Proof. Assume the top element of $\mathcal{C}\mathsf{L}$ is $\bigvee$-irreducible, hence of the form $\gamma_{\mathcal{CS}\mathsf{L}}(j)$ for some $j \in \mathcal{J}\mathsf{L}$. For each $k \in \mathcal{J}\mathsf{L}$, we have $\gamma_{\mathcal{CS}\mathsf{L}}(j) \geqslant \gamma_{\mathcal{CS}\mathsf{L}}(k)$; in other words, $\{m \in \mathcal{M}\mathsf{L} \mid j \not\leqslant m\} \subseteq \{m \in \mathcal{M}\mathsf{L} \mid k \not\leqslant m\}$, and consequently, $k \leqslant m$ implies $j \leqslant m$ for all $m \in \mathcal{M}\mathsf{L}$. By the hypothesis that L is small-based and lower regular, it would follow that j is the least element of L, which is impossible. $\square$

Lemma 4.4. *For each small-based complete lattice* L, *there is an (up to isomorphism unique) ordinal decomposition* $\mathsf{L} \simeq \kappa \oplus \mathsf{K}$ *into an ordinal* κ *and a lower regular lattice* K. *If* $\mathcal{C}\mathsf{L}$ *is small-based, too, and* L *is a flip-flop or self-contrapositive, respectively, then so is* K, *and* κ *is* 0 *or a limit ordinal.*

Proof. Choose the largest ordinal κ such that a decomposition $\mathsf{L} \simeq \kappa \oplus \mathsf{K}$ exists. Then K is a small-based lattice with a $\bigwedge$-reducible bottom element, and consequently, K is lower regular.

Now suppose L is a flip-flop. By 4.1, we have $\mathcal{C}\mathsf{L} \simeq \mathcal{C}(\kappa \oplus \mathsf{K}) \simeq \mathcal{C}\mathsf{K}\,\widehat{\oplus}\,\mathcal{C}(\kappa+1)$. If $\mathcal{C}\mathsf{L}$ is small-based then so is $\mathcal{C}\mathsf{K}$, and by 4.3, $\mathcal{C}\mathsf{K}$ is upper regular. Applying 4.1 once more, we get

$$\mathsf{L} \simeq (\kappa+1)\,\widehat{\oplus}\,\mathsf{K} \simeq \mathcal{CC}((\kappa+1)\,\widehat{\oplus}\,\mathsf{K}) \simeq \mathcal{C}(\mathcal{C}\mathsf{K}\,\widehat{\oplus}\,\mathcal{C}(\kappa+1)) \simeq \mathcal{CC}(\kappa+1)\,\widehat{\oplus}\,\mathcal{CC}\mathsf{K}.$$

By the dual of 4.3, the least element of $\mathcal{CC}\mathsf{K}$ is $\bigwedge$-reducible, while all non-maximal elements of ordinals are $\bigwedge$-irreducible. Hence, the above isomorphism entails that K is isomorphic to $\mathcal{CC}\mathsf{K}$, and $\kappa+1$ is isomorphic to $\mathcal{CC}(\kappa+1)$. By 2.3, this happens if and only if κ is 0 or a limit ordinal.

The case of small-based self-contrapositive lattices L (where $\mathcal{CO}\mathsf{L}$ and $\mathcal{C}\mathsf{L}$ are automatically small-based) is treated analogously. $\square$

Now we are in a position to describe the impact of vertical sums on (small-based) flip-flops and self-contrapositive lattices.

Theorem 4.5. *Let* L *and* L' *be complete lattices such that* L *and* $\mathcal{CO}\mathsf{L}$ *are upper regular or small-based, or* L' *and* $\mathcal{CO}\mathsf{L}'$ *are lower regular or small-based.*

If L *and* L' *are flip-flops or self-contrapositive, respectively, then so is their vertical sum* $\mathsf{L}\,\widehat{\oplus}\,\mathsf{L}'$. *In particular, finite flip-flops and finite self-contrapositive lattices are closed under vertical sums.*

Proof. The regular cases are easily settled with the help of 4.1.

Now assume, for example, that L and L' are flip-flops, and that L' and $\mathcal{CO}\mathsf{L}'$ are small-based but L' is not lower regular. Then 4.4 provides a decomposition $\mathsf{L}' \simeq \kappa \oplus \mathsf{K}$ into a limit ordinal κ and a small-based lower regular flip-flop K. The decomposition $\mathcal{C}\mathsf{L}' \simeq \mathcal{C}\mathsf{K} \widehat{\oplus} \mathcal{C}(\kappa+1)$ shows that not only $\mathcal{C}\mathsf{L}'$ but also $\mathcal{C}\mathsf{K}$ is small-based, and by 4.3, $\mathcal{C}\mathsf{K}$ is then upper regular. Again by 4.1 and 2.3, we obtain

$$\begin{aligned}\mathcal{CC}(\mathsf{L} \widehat{\oplus} \mathsf{L}') &\simeq \mathcal{CC}(\mathsf{L} \widehat{\ominus}(\kappa+1) \widehat{\oplus} \mathsf{K}) \simeq \mathcal{C}(\mathcal{C}\mathsf{K} \widehat{\ominus} \mathcal{C}(\kappa+1) \widehat{\oplus} m \widehat{\oplus} \mathcal{C}\mathsf{L})\\ &\simeq \mathcal{CC}\mathsf{L} \widehat{\ominus} n \widehat{\oplus} \mathcal{CC}(\kappa+1) \widehat{\oplus} \mathcal{CC}\mathsf{K} \simeq \mathsf{L} \widehat{\oplus} n \widehat{\ominus}(\kappa+1) \widehat{\ominus} \mathsf{K}\\ &\simeq \mathsf{L} \widehat{\oplus}(\kappa+1) \widehat{\oplus} \mathsf{K} \simeq \mathsf{L} \widehat{\oplus} \mathsf{L}',\end{aligned}$$

where m and n are natural numbers with $m \leqslant 2$ and $n \leqslant 3$.

The other cases are treated similarly. $\square$

Let us add a few examples demonstrating that things become more complicated when the hypothesis of small-basedness is dropped.

Example 4.6. The real unit interval $\mathsf{I} = [0, 1]$ is a self-dual upper and lower regular complete lattice but not small-based, and the negation $\mathcal{C}\mathsf{I}$ is a one-element lattice.

(1) The infinite kite $\mathsf{K}_\mathsf{I} = \mathsf{I} \widehat{\ominus} 2^2$ is coreduced but not a flip-flop: $\mathcal{C}\mathsf{L} \simeq \mathcal{CC}\mathsf{L} \simeq 2^2$.

(2) The ordinal sum $2\mathsf{I} = \mathsf{I} \ominus \mathsf{I}$ has a $\bigvee$-reducible top element and a $\bigwedge$-reducible bottom element but is neither upper nor lower regular. Moreover, $2\mathsf{I}$ admits no ordinal decomposition $\kappa \oplus \mathsf{K}$ such that K is lower regular (cf. 4.4).

(3) For any poset P, define the '*truncated Aleksandrov completion*' by $\mathcal{A}_0\mathsf{P} = \mathcal{A}\mathsf{P} \setminus \{\emptyset, P\}$. If P has a top element 1_P and a bottom element 0_P then $\mathcal{A}_0\mathsf{P}$ is a small-based complete lattice. Moreover, if 1_P is $\bigvee$-reducible and 0_P is $\bigwedge$-reducible then $\mathcal{A}_0\mathsf{P}$ is upper and lower regular, and clearly

$$\mathcal{A}\mathsf{P} \simeq 1 \oplus \mathcal{A}_0\mathsf{P} \oplus 1 \simeq 2 \widehat{\oplus} \mathcal{A}_0\mathsf{P} \widehat{\ominus} 2.$$

In that case, 2.2 combined with 4.1 yields

$$\mathcal{CO}\mathcal{A}_0\mathsf{P} \simeq \mathcal{N}\mathsf{P}.$$

In particular, $\mathsf{A} = \mathcal{A}_0(2\mathsf{I})$ is small-based, upper and lower regular, whereas its negation $\mathcal{C}\mathsf{A} \simeq \mathcal{O}(2\mathsf{I}) \simeq 2\mathsf{I}$ has none of these properties (cf. 2.2 and 4.3).

(4) The horizontal sum D of I and a three-element chain is obtained from the disjoint union by identifying the top elements and the bottom elements, respectively (cf. [3] and [8]). Then D is upper and lower regular, but not small-based, and its negation $\mathcal{C}\mathsf{D}$ is a two-element chain whose top element is $\bigvee$-irreducible and whose bottom element is $\bigwedge$-irreducible (cf. 4.3 again).

(5) The iterated truncated Aleksandrov completions $\mathsf{D}_n = \mathcal{A}_0{}^n\mathsf{D}$ are upper and lower regular, by (3) and (4). Consider the infinite vertical sum

$$\mathsf{S} = 2 \mathbin{\widehat{\oplus}} \mathsf{D} \mathbin{\widehat{\oplus}} \mathsf{D}_1 \mathbin{\widehat{\oplus}} \mathsf{D}_2 \ldots \oplus 1.$$

An infinite analogue of 4.1, derived in [3], yields

$$\mathcal{CO}\mathsf{S} \simeq \mathcal{CO}2 \mathbin{\widehat{\oplus}} \mathcal{CO}\mathsf{D} \mathbin{\widehat{\oplus}} \mathcal{CO}\mathcal{A}_0\mathsf{D} \mathbin{\widehat{\oplus}} \ldots \oplus 1 \simeq 1 \mathbin{\widehat{\oplus}} 2 \mathbin{\widehat{\oplus}} \mathsf{D} \mathbin{\widehat{\oplus}} \mathcal{A}_0\mathsf{D} \mathbin{\widehat{\oplus}} \ldots \oplus 1 \simeq \mathsf{S}.$$

Hence, S and its dual $\mathcal{O}\mathsf{S}$ are self-contrapositive. Furthermore, there is a unique ordinal decomposition $\mathsf{S} \simeq \kappa \oplus \mathsf{K}$ such that κ is an ordinal and K is lower regular, but κ is the finite successor ordinal 1, and $\mathsf{K} = \mathsf{D} \mathbin{\widehat{\oplus}} \mathsf{D}_1 \mathbin{\widehat{\oplus}} \mathsf{D}_2 \ldots \oplus 1$ is neither small-based nor self-contrapositive (cf. 4.4). Indeed, $\mathcal{C}\mathsf{K} \simeq \mathcal{O}\mathsf{S}$, and

$$\mathcal{CO}(\mathcal{O}\mathsf{S} \mathbin{\widehat{\oplus}} \mathsf{S}) \simeq \mathcal{CO}(\mathcal{O}\mathsf{K} \mathbin{\widehat{\oplus}} 3 \mathbin{\widehat{\oplus}} \mathsf{K}) \simeq \mathcal{C}\mathsf{K} \mathbin{\widehat{\oplus}} 2 \mathbin{\widehat{\oplus}} \mathcal{CO}\mathsf{K} \simeq \mathcal{O}\mathsf{S} \oplus \mathsf{S} \not\simeq \mathcal{O}\mathsf{S} \mathbin{\widehat{\oplus}} \mathsf{S}.$$

Thus we have found two self-contrapositive lattices S and $\mathcal{O}\mathsf{S}$ whose vertical sum $\mathsf{V} = \mathcal{O}\mathsf{S} \mathbin{\widehat{\oplus}} \mathsf{S}$ fails to be self-contrapositive (cf. 4.5). Moreover, the iterated negations $\mathcal{C}^n\mathsf{V} \simeq (\mathcal{CO})^n\mathsf{V} \simeq \mathcal{O}\mathsf{S} \mathbin{\widehat{\oplus}} (n+1) \mathbin{\widehat{\oplus}} \mathsf{S}$ are mutually embeddable but non-isomorphic (cf. 3.14).

(6) Now consider the horizontal sum D' of the interval I and a four-element chain, its negation $\mathcal{C}\mathsf{D}' = 3$, and the iterations $\mathsf{D}'_n = \mathcal{A}_0{}^n\mathsf{D}'$. For the infinite vertical sum

$$\mathsf{T} = 3 \mathbin{\widehat{\oplus}} \mathsf{D}'_1 \mathbin{\widehat{\oplus}} \mathsf{D}'_3 \mathbin{\widehat{\oplus}} \mathsf{D}'_5 \ldots \oplus 1$$

the aforementioned infinite analogue of 4.1 yields

$$\mathcal{CO}\mathsf{T} \simeq \mathcal{CO}3 \mathbin{\widehat{\oplus}} \mathcal{CO}\mathcal{A}_0\mathsf{D}' \mathbin{\widehat{\oplus}} \mathcal{CO}\mathcal{A}_0{}^3\mathsf{D}' \mathbin{\widehat{\oplus}} \ldots \oplus 1 \simeq 2 \mathbin{\widehat{\oplus}} \mathsf{D}' \mathbin{\widehat{\oplus}} \mathcal{A}_0{}^2\mathsf{D}' \mathbin{\widehat{\oplus}} \ldots \oplus 1.$$

Applying the above rule once more, we arrive at

$$\mathcal{C}^2\mathsf{T} \simeq (\mathcal{CO})^2\mathsf{T} \simeq \mathsf{T}.$$

Thus T is an upper but not lower regular small-based flip-flop, whereas its negation $\mathcal{C}\mathsf{T}$ is a lower but not upper regular flip-flop which is *not* small-based. In the unique ordinal decomposition $\mathsf{T} \simeq \kappa \oplus \mathsf{K}$ into an ordinal and a small-based lower regular lattice, κ is the finite successor ordinal 2, and K fails to be a flip-flop: $\mathcal{CC}\mathsf{K} \simeq 2 \mathbin{\widehat{\oplus}} \mathsf{K}$. This shows that in 4.3 and 4.4, it is essential to assume that *both* L and $\mathcal{C}\mathsf{L}$ are small-based. Furthermore, $\mathcal{C}\mathsf{T} \mathbin{\widehat{\oplus}} \mathsf{T}$ is not self-negative and not even a flip-flop. Indeed, induction shows that

$$\mathcal{C}^n(\mathcal{C}\mathsf{T} \mathbin{\widehat{\oplus}} \mathsf{T}) \simeq \mathcal{C}\mathsf{T} \mathbin{\widehat{\oplus}} (n+1) \mathbin{\widehat{\oplus}} \mathsf{T},$$

and these lattices are pairwise non-isomorphic.

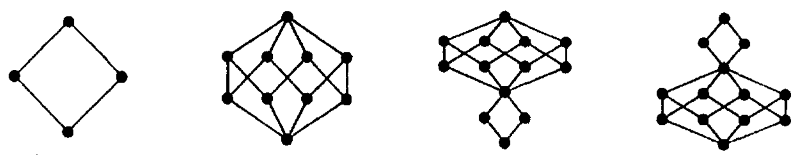
In all, we see that in 4.2, ‘coreduced’ cannot be replaced with ‘flip-flop’, and 4.5 becomes false when one of the small-basedness hypotheses is dropped.

To decide whether a given finite lattice L is self-negative requires to check the existence of an isomorphism between the small context $\mathcal{S}\mathsf{L}$ and its negation, which basically amounts to the classical problem of deciding whether a given graph is isomorphic to its complement. It is common sense to assume that there is no algorithm solving that graph-theoretical problem in polynomial time (cf. e.g. [9, p. 285]). This and other heuristic considerations suggest that there is no simple method to discern self-negativity of a given finite lattice. But, of course, for a small number of points, a list of such lattices can be produced with the aid of a computer. Using a PASCAL program, we have generated a complete list of pairwise non-isomorphic lattices which are normal completions of ordered sets with at most 8 points. Since for coreduced lattices L, we have $1_{\mathsf{L}} \notin \mathcal{J}\mathsf{L}$ and $0_{\mathsf{L}} \notin \mathcal{M}\mathsf{L}$, this list includes at least all coreduced lattices with less than 11 elements, but also some more. It turned out that among the 14 570 lattices generated this way,

2829 are (coreduced) flip-flops,
61 are self-negative, and
7 are self-contrapositive.

Incidentally, all of the seven generated self-contrapositive lattices are self-negative, too, but that is not a general phenomenon. Consider any two finite vertically indecomposable, self-dual and self-negative lattices which are not isomorphic. By 4.1, their vertical sum is self-contrapositive but not self-negative.

Example 4.7. A self-contrapositive but not self-negative finite lattice $\mathsf{L} = 2^2 \mathbin{\hat{\oplus}} \mathsf{E}$



$2^2 \simeq \mathcal{O}2^2 \simeq \mathcal{C}2^2 \simeq \mathcal{CO}2^2$ $\quad \mathsf{E} \simeq \mathcal{O}\mathsf{E} \simeq \mathcal{C}\mathsf{E} \simeq \mathcal{CO}\mathsf{E}$ $\quad 2^2 \mathbin{\hat{\oplus}} \mathsf{E} = \mathsf{L} \simeq \mathcal{CO}\mathsf{L}$ $\quad \mathsf{E} \mathbin{\hat{\oplus}} 2^2 \simeq \mathcal{O}\mathsf{L} \simeq \mathcal{C}\mathsf{L}$

The construction of self-negative and self-dual lattices like 2^2 and E may be generalized as follows. For any reduced context $\mathbb{K} \simeq \mathcal{O}\mathbb{K} \simeq \mathcal{C}\mathbb{K}$, the concept lattice $\mathsf{L} = \mathcal{B}\mathbb{K}$ is self-dual and self-negative:

$$\mathcal{C}\mathsf{L} \simeq \mathcal{BCSB}\mathbb{K} \simeq \mathcal{BC}\mathbb{K} \simeq \mathcal{B}\mathbb{K} = \mathsf{L}.$$

Example 4.8. Define, for each natural number n, a reduced context $\mathbb{K}_n = (J_n, M_n, I_n)$ by $J_n = M_n = 2n = \{0, 1, \dots, 2n-1\}$ and

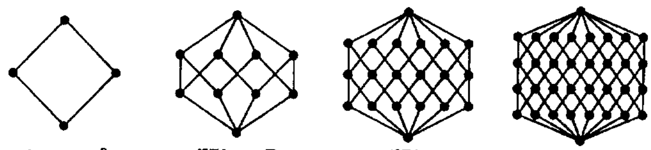
$$j\, I_n\, m \iff j + m = k \pmod{2n} \quad \text{for some } k \in \{0, 1, \dots, n-1\}.$$

Then

$$\mathbb{K}_n \simeq \mathcal{O}\mathbb{K}_n \simeq \mathcal{C}\mathbb{K}_n,$$

and consequently, the concept lattice $\mathcal{B}\mathbb{K}_n$ is self-dual and self-negative.

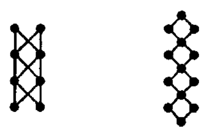
For $n \leqslant 4$, the lattices $\mathcal{B}\mathbb{K}_n$ are depicted below.



$\mathcal{B}\mathbb{K}_1 \simeq 2^2$ $\mathcal{B}\mathbb{K}_2 \simeq \mathrm{E}$ $\mathcal{B}\mathbb{K}_3$ $\mathcal{B}\mathbb{K}_4$

Surprisingly, there are no other finite distributive self-negative (respectively, self-contrapositive) lattices than *rhomb chains*, i.e., vertical sum of rhombs (Boolean lattices with four elements) or one-element lattices. Of course, '*diamond chains*' would be nicer, but lattice-theoreticians have reserved the word '*diamond*' for five-element modular but non-distributive lattices (see M_3 in 2.6). The first five lattices of Table 2 are rhomb chains. Generally, it is easy to see that rhomb chains are precisely the normal completions of so-called *doubled chains*, i.e., finite ordinal sums of two-element antichains.

Example 4.9. A doubled chain and its completion.



P $\mathcal{N}$P

Doubled chains have the exceptional property of possessing only one join completion. Indeed, one can prove the following fact (cf. [2]):

Proposition 4.10. *A finite ordered set* P *is a doubled chain if and only if* $\mathcal{N}\mathsf{P} \simeq \mathcal{A}\mathsf{P}$.

Proof. Recall that we have

$$\mathcal{N}\mathsf{P} = \{A^{\uparrow\downarrow} \mid A \subseteq P\} \subseteq \mathcal{A}\mathsf{P} = \{{\leqslant}A \mid A \subseteq P\}.$$

It is easy to check that a finite doubled chain P satisfies the equation $\mathcal{N}\mathsf{P} = \mathcal{A}\mathsf{P}$. Conversely, if P is any finite ordered set with $\mathcal{N}\mathsf{P} \simeq \mathcal{A}\mathsf{P}$ then the above inclusion forces $\mathcal{N}\mathsf{P}$ to coincide with $\mathcal{A}\mathsf{P}$. If P would possess only one maximal element m then the same would hold for the subposet $\mathsf{P}' = (P', \leqslant)$ where $P' = P \setminus \{m\}$ (since P' is a member of $\mathcal{A}\mathsf{P} = \mathcal{N}\mathsf{P}$), and as $\mathcal{A}\mathsf{P}'$ agrees with $\mathcal{N}\mathsf{P}'$, induction would lead to the conclusion that P is a finite chain, which is impossible since the empty set $\emptyset$ must be an element of $\mathcal{A}\mathsf{P}=\mathcal{N}\mathsf{P}$. Hence P has at least two maximal elements m, n. If $A \in \mathcal{A}\mathsf{P}=\mathcal{N}\mathsf{P}$

contains m but is distinct from the principal ideal $\downarrow m$, one may choose a $p \in A \setminus \downarrow m$ in order to obtain $\{m,p\}^{\uparrow} = \emptyset$, a fortiori $A^{\uparrow} = \emptyset$ and $A = A^{\uparrow\downarrow} = P$. This shows that there are no other maximal elements than m and n, and that the subposet $\mathsf{P}'' = (P'', \leqslant)$ with $P'' = P \setminus \{m,n\} = \{m,n\}^{\downarrow}$ satisfies $\mathcal{A}\mathsf{P}'' = \mathcal{A}\mathsf{P} \setminus \{\downarrow m, \downarrow n, P\}$. Now, $C \in \mathcal{A}\mathsf{P}''$ implies $C \in \mathcal{A}\mathsf{P} = \mathcal{N}\mathsf{P}$, i.e., $C = B^{\downarrow} = (B \setminus \{m,n\})^{\downarrow} \cap \{m,n\}^{\downarrow} = (B \cap P'')^{\downarrow} \cap P'' \in \mathcal{N}\mathsf{P}''$. Thus we get $\mathcal{N}\mathsf{P}'' = \mathcal{A}\mathsf{P}''$, and induction completes the proof. □

Notice that there are infinite chains P with $\mathcal{N}\mathsf{P} = \mathcal{A}\mathsf{P}$, for example the chain of integers. Now, applying the rule $\mathcal{COA}\mathsf{P} \simeq \mathcal{N}\mathsf{P}$ (cf. 2.2), we arrive at

Theorem 4.11. *The following statements on a finite lattice* L *are equivalent*:

(a) L *is a rhomb chain.*

(b) L *is self-negative and distributive.*

(c) L *is self-contrapositive and distributive.*

(d) L *and* $\mathcal{C}\mathsf{L}$ *are distributive flip-flops.*

Proof. By 4.2, (a) implies the other three statements.

(b) $\Longrightarrow$ (a): As L is finite and distributive, there exists an ordered set P with

$$\mathcal{A}\mathsf{P} \simeq \mathsf{L} \simeq \mathcal{C}\mathsf{L} \simeq \mathcal{CA}\mathsf{P} \simeq \mathcal{ON}\mathsf{P}.$$

In particular, $\sharp\mathcal{A}\mathsf{P} = \sharp\mathcal{N}\mathsf{P}$. As P is finite, this implies that $\mathcal{A}\mathsf{P} = \mathcal{N}\mathsf{P}$. By 4.10, P is a doubled chain. Thus $\mathsf{L} \simeq \mathcal{A}\mathsf{P}$ is a rhomb chain.

(c) $\Longrightarrow$ (a) follows by similar arguments.

(d) $\Longrightarrow$ (a): As L is a distributive flip-flop, $\mathsf{L} \widehat{\oplus} \mathcal{C}\mathsf{L}$ is distributive and self-negative (see 4.2). By (b) $\Longrightarrow$ (a), this implies that $\mathsf{L} \widehat{\oplus} \mathcal{C}\mathsf{L}$ is a rhomb chain, which is impossible unless L is a rhomb chain. □

For infinite complete lattices L, the statements (a) and (b) in 4.11 are independent. For example, the vertical sum of $\omega+1$ rhombs is distributive, but not self-negative. On the other hand, by 4.2, $(\omega+1) \widehat{\oplus} \mathcal{O}(\omega+1)$ is a self-negative and self-contrapositive chain (hence distributive) but certainly not a rhomb chain).

Our final result may be interpreted as a strong combinatorial argument for the claim that self-contrapositive lattices are much rarer than self-negative ones.

Proposition 4.12. *For every finite self-contrapositive lattice* L, *there exists a natural number* m *with* $\sharp\mathcal{J}\mathsf{L} = \sharp\mathcal{M}\mathsf{L} = 2m$, *and the small context of* L *has precisely* $2m^2$ *incident pairs.*

Proof. As L is self-contrapositive, $\mathcal{COS}\mathsf{L} = (\mathcal{M}\mathsf{L}, \mathcal{J}\mathsf{L}, \ngeqslant)$ is isomorphic to $\mathcal{S}\mathsf{L} = (\mathcal{J}\mathsf{L}, \mathcal{M}\mathsf{L}, \leqslant)$. Thus $\sharp\mathcal{J}\mathsf{L} = \sharp\mathcal{M}\mathsf{L}$. Now, if $n = \sharp\mathcal{J}\mathsf{L}$ then $k := \sharp I_{\mathcal{S}\mathsf{L}} = \sharp I_{\mathcal{COS}\mathsf{L}} = \sharp I_{\mathcal{CS}\mathsf{L}} = n^2 - \sharp I_{\mathcal{S}\mathsf{L}}$. Hence $n^2 = 2k$, and so $n = 2m, k = 2m^2$. □

Table 1
The class of all lattices that are isomorphic to normal completions of ordered sets with 8 or less elements is denoted by **L8**. Up to duality the diagrams represent those lattices in **L8** which are self-negative but not self-dual

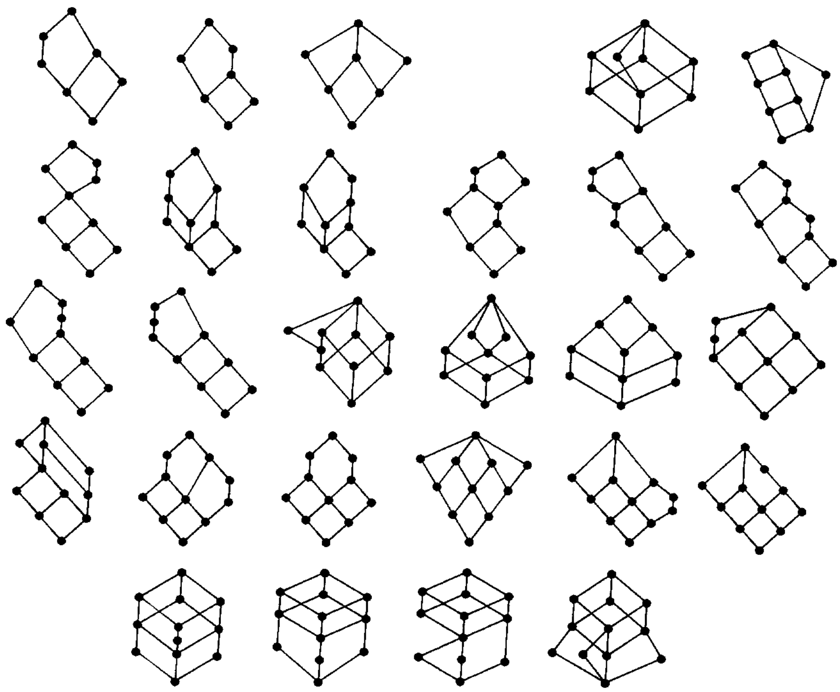
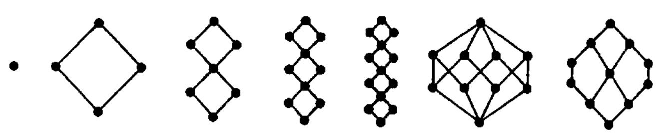


Table 2
The seven diagrams represent those lattices in **L8** which are self-negative and self-dual. At the same time, these are the diagrams of all self-contrapositive lattices in **L8**



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