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# On the local fractional derivative

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ABSTRACT

We present the necessary conditions for the existence of the Kolwankar–Gangal local fractional derivatives (KG-LFD) and introduce more general but weaker notions of LFDs by using limits of certain integral averages of the difference-quotient. By applying classical results due to Stein and Zygmund (1965) [16] we show that the KG-LFD is almost everywhere zero in any given intervals. We generalize some of our results to higher dimensional cases and use integral approximation formulas obtained to design numerical schemes for detecting fractional dimensional edges in signal processing.



## 1. Introduction

Fractional derivatives and fractional calculus have a long history and there are many applications [12–14]. However, these globally defined fractional derivatives do not generally reflect the local geometric behaviours for a given function. Attempts have been made recently [1,2,5–10] to define a local version of the fractional derivative. In this paper we focus on the right (left) local fractional derivatives defined by Kolwankar and Gangal [6–8]. In short we call this type of derivatives the KG-LFD. Since its introduction in 1996, the KG-LFD has been studied by several authors. However, there are still a number of basic issues to be addressed. For example, what does the class of locally fractionally differentiable functions look like? In this paper we first consider the relationship between the KG-LFD and that defined by the limit of the right (left) difference-quotient (DQ-LFD for short) [1,10], and we introduce two new notions of right (left) local fractional derivatives by the limit of families of (singular) integrals of the difference-quotient. We call them the right (left) SIDQ-LFD and the right (left) IDQ-LFD respectively (see Section 2 for details). We show that SIDQ-LFD is the weakest among all of the concepts of LFDs. Then we establish a structural theorem for the KG-LFG. Roughly speaking, it says that for a $C^\alpha$ function in $(a,b)$ with $0<\alpha<1$, if the KG-LFD exists almost everywhere (a.e. for short) in $(a,b)$, then the KG-$\alpha$-local fractional derivative equals zero a.e. in $(a,b)$. This not only confirms the observations made in previously constructed examples in [1,6–9] but also shows that the non-trivial KG-$\alpha$-LFD is a lower dimensional property for a $C^\alpha$ function when KG-$\alpha$-LFD exists. However this observation leads us to the constructions of numerical schemes for calculating modulus of LFD for $C^\alpha$ functions.

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The definition of KG-LFD is based on the Riemann–Liouville definition of the fractional integral for a real-valued function $f$ of a single variable. Following the notation of [13], the Riemann–Liouville fractional derivative of $f$ at $x > a$ is defined in [12,13] by

$$ {}_aD_x^\alpha f(x) = \frac{d^\alpha f(x)}{[d(x-a)]^\alpha} := \frac{1}{\Gamma(1-\alpha)} \frac{d^k}{dx^k} \int_a^x \frac{f(t)}{(x-t)^{\alpha-k+1}}\, dt, \quad k-1 < \alpha < k. \tag{1.1} $$

In this paper we only consider the case $k = 1$, that is, $0 < \alpha < 1$. The right (left) KG-LFD is defined as follows. For a function $f : (a, b) \to \mathbf{R}$, if the right (left) limit

$$ \mathcal{D}_\pm^\alpha f(y) := \lim_{x \to y_\pm} \frac{d^\alpha (f(x) - f(y))}{d(\pm(x-y))^\alpha} \tag{1.2} $$

exists and is finite, then $f$ has the right (left) LFD of order $\alpha$. Due to the symmetric nature of $\mathcal{D}_\pm^\alpha f(y)$, we mostly consider the right LFD in this paper.

Let us introduce some notation and preliminaries. Let $\Omega \subset \mathbf{R}^n$ be open. We denote by $C^\alpha(\Omega)$, $0 < \alpha < 1$ and $L^p(\Omega)$, $1 \leqslant p \leqslant +\infty$ the usual Hölder spaces and the Lebesgue spaces, respectively. For $x, y \in \mathbf{R}^n$, we denote by $\langle x, y\rangle$ its standard Euclidean inner product and $|x|$ the norm of $x \in \mathbf{R}^n$. An open ball in $\mathbf{R}^n$ with centre $x \in \mathbf{R}^n$ and radius $r > 0$ is denoted by $B(x,r)$. Let $S^{n-1} \subset \mathbf{R}^n$ be the unit sphere, we denote its area by $\omega_{n-1}$. For a measurable set $\Omega \subset \mathbf{R}^n$, we denote by $|\Omega|$ its Lebesgue measure. the integral average of a function over a ball $B \subset \mathbf{R}^n$ is defined by $\frac{1}{|B|}\int_B f\, dy$. We also denote by $D(0,r)$ the cube in $\mathbf{R}^n$ centred at 0 with side-length $2r$ whose sides are parallel to the coordinate axes.

The plan for the rest of the paper is as follows. In Section 2, we introduce two new notions of local fractional derivatives SIDQ-LFD and IDQ-LFD which are motivated from a necessary condition of KG-LFD. These LFDs are weaker than the KG-LFD. The relation between KG-LFD and yet another notion DQ-LFD defined by the limits of difference-quotient is also studied. We show that the existence of KG-LFD plus a local integrability condition implies the existence of the limits of the difference-quotient DQ-LFD. Then the relations among various LFDs are illustrated through a diagram (Theorem 1). In the later part of Section 2, we consider the example

$$ f(x) = \begin{cases} |x|^\alpha \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases} $$

We show that both SIDQ-LFD and IDQ-LFD exist while KG-LFD and DQ-LFD do not exist at 0. We conclude Section 2 by presenting a result for functions in the form $f(x) = x^\alpha g(x)$ where $g$ is a bounded function in $C^1(0,\delta)$. In Section 3, we apply a result due to Stein and Zygmund [16] for fractional differentiations to KG-LFD. We establish a structural theorem which says that if both the right and the left KG-LFDs exist a.e. in an interval, then they are both zero a.e. in that interval. Furthermore the IDQ-LFDs are also equal to zero a.e. in the same interval. We also make a partial extension of the one-dimensional result to higher dimensional cases. In Section 4, we illustrate some examples of numerical calculated LFDs based on our integral definition of local fractional derivatives IDQ-LFDs. The main reason for choosing IDQ-LFD is that it is the easiest to implement.

## 2. On various notions of local fractional derivatives

The following is a necessary condition for $\mathcal{D}_+^\alpha f(x)$ which forms the basis of our weaker notions of LFDs. It is also needed later in the proof of Theorem 2 in Section 3.

**Lemma 1.** *Let $f : (a, b) \to \mathbf{R}$ be continuous such that $\mathcal{D}_+^\alpha f(y)$ exists at some $y \in (0, 1)$, then*

$$ \lim_{h \to 0_+} \int_0^1 (1-t)^{-\alpha} \frac{f(ht+y) - f(y)}{h^\alpha}\, dt \tag{2.1} $$

*exists and*

$$ \mathcal{D}_+^\alpha f(y) = \frac{1}{\Gamma(1-\alpha)} \lim_{h \to 0_+} \int_0^1 (1-t)^{-\alpha} \frac{f(ht+y) - f(y)}{h^\alpha}\, dt. \tag{2.2} $$

**Proof of Lemma 1.** By definition, we have

$$ \mathcal{D}_+^\alpha f(y) = \lim_{x \to y_+} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_y^x \frac{f(t) - f(y)}{(x-t)^\alpha}\, dt. $$

Let

$$F_y(x) = \frac{1}{\Gamma(1-\alpha)} \int_y^x \frac{f(t) - f(y)}{(x-t)^\alpha}\, dt,$$

which can also be written as

$$F_y(x) = \frac{(x-y)^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{f(y + (x-y)s) - f(y)}{(1-s)^\alpha}\, ds$$

if we change the variable by $s = (t-y)/(x-y)$. We observe, from the definition of $\mathcal{D}_+^\alpha f(y)$ that $F_y'(x)$ must exist in a small interval $(y, y+\delta]$. Clearly $F(x)$ is also continuous in $[y, y+\delta]$ if we define $F_y(y) = 0$. Thus by the mean value theorem in calculus, we have, on one hand that for each fixed $h \in (0, \delta)$, there is some $\xi_h \in (0, h)$, such that

$$F_y'(y + \xi_h) = \frac{F_y(y+h) - F_y(y)}{h}.$$

On the other hand, we have, by the equivalent definition of $F_y(x)$ above, that

$$\begin{aligned}
\frac{F_y(y+h) - F_y(y)}{h} &= \frac{F_y(y+h)}{h} \\
&= \frac{h^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 \frac{f(y + (x-y)s) - f(y)}{h(1-s)^\alpha}\, ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-s)^{-\alpha} \frac{f(sh+y) - f(y)}{h^\alpha}\, ds.
\end{aligned}$$

Since by definition, the existence of $\mathcal{D}_+^\alpha f(y)$ is equivalent to $\lim_{x \to y_+} F_y'(x) = \mathcal{D}_+^\alpha f(y)$, we have $\lim_{h \to 0_+} F_y'(y + \xi_h) = \mathcal{D}_+^\alpha f(y)$ as $\xi_h \to 0$. Therefore

$$\lim_{h \to 0_+} \int_0^1 (1-s)^{-\alpha} \frac{f(sh+y) - f(y)}{h^\alpha}\, ds \quad \text{exists}$$

and

$$\mathcal{D}_+^\alpha f(y) = \frac{1}{\Gamma(1-\alpha)} \lim_{h \to 0_+} \int_0^1 (1-s)^{-\alpha} \frac{f(th+y) - f(y)}{h^\alpha}\, ds. \qquad \square$$

**Remark 1.** In the proof of Lemma 1, we have used some observations in [10]. However, Proposition 1 in [10] is essentially an attempt to calculate $F_y'(y)$ whose existence is not known. The use of upper and lower limit in [10] does not establish the equivalence

$$\text{existence of } (F_y)_+'(y) \quad \Longleftrightarrow \quad \text{existence of } \lim_{x \to y_+} \frac{f(x) - f(y)}{(x-y)^\alpha}.$$

In fact, in the proof of Lemma 1 we have also established that

$$(F_y)_+'(y) = \frac{1}{\Gamma(1-\alpha)} \lim_{h \to 0_+} \int_0^1 (1-t)^{-\alpha} \frac{f(th+y) - f(y)}{h^\alpha}\, dt.$$

The above expression of $(F_y)_+'(y)$ implies neither the existence of $\mathcal{D}_+^\alpha f(y)$ nor that of $\lim_{x \to y_+} \frac{f(x)-f(y)}{(x-y)^\alpha}$ (see Example 1 later in this section).

Based on our observations in Lemma 1, we define a weaker version of local fractional derivation than KG-LFD called *singular integral difference-quotient local fractional derivative* (SIDQ-LFD for short) as follows.

**Definition 1.** Suppose $f :\in C(a,b)$. We say that $f$ has a right singular integral difference-quotient local fractional derivative (right SIDQ-LFD) of order $\alpha$ $(0<\alpha<1)$, denoted by $\mathbf{D}_+^\alpha f(y)$ at $y \in (a,b)$ if the following limit exists:

$$\mathbf{D}_+^\alpha f(y) := \frac{1}{\Gamma(1-\alpha)} \lim_{h\to 0_+} \int_0^1 (1-t)^{-\alpha} \frac{f(th+y)-f(y)}{h^\alpha}\,dt. \tag{2.3}$$

Similarly, $f$ has a left SIDQ-LFD $\mathbf{D}_-^\alpha f(y)$ at $y \in (a,b)$ if the following limit exists:

$$\mathbf{D}_-^\alpha f(y) := -\frac{1}{\Gamma(1-\alpha)} \lim_{h\to 0_+} \int_0^1 (1-t)^{-\alpha} \frac{f(y-th)-f(y)}{h^\alpha}\,dt. \tag{2.4}$$

We only need a weaker version of Hölder continuity for the next result.

**Definition 2.** Let $f:(a,b)\to \mathbf{R}$ and $y \in (a,b)$. We say that $f$ is locally right (respectively left) $C^\alpha$ $(0<\alpha<1)$ at $y$ if there is a $\delta>0$ and a constant $C_y>0$ such that $|f(x)-f(y)| \leqslant C_y|x-y|^\alpha$ for $x \in (y, y+\delta)$ (respectively for $x \in (y-\delta, y)$).

We say that $f$ is locally $C^\alpha$ $(0<\alpha<1)$ at $y$ if $f$ is both locally left and right $C^\alpha$ at $y$.

A simple example of a local $C^\alpha$ function at 0 is $f(x)=|x|^\alpha \sin(1/x)$ $(x\neq 0)$ and $f(0)=0$ which will be examined in Example 1 later.

The following is a sufficient condition for the existence of the left (right) SIDQ-LFD $\mathbf{D}_\pm^\alpha f(y)$.

**Proposition 1.** *Suppose $f$ is locally right (respectively left) $C^\alpha$ at $y \in (a,b)$ such that*

$$\lim_{h\to 0_+} \int_0^1 \frac{f(y+th)-f(y)}{h^\alpha}\,dt \quad \left(\textit{respectively } \lim_{h\to 0_+} -\int_0^1 \frac{f(y)-f(y-th)}{h^\alpha}\,dt\right) \textit{ exists.} \tag{2.5}$$

*Then*

$$\mathbf{D}_+^\alpha f(y) = (1+\alpha)\Gamma(1-\alpha) \lim_{h\to 0_+} \int_0^1 \frac{f(y+th)-f(y)}{h^\alpha}\,dt,$$

$$\textit{respectively} \quad \mathbf{D}_-^\alpha f(y) = -(1+\alpha)\Gamma(1-\alpha) \lim_{h\to 0_+} \int_0^1 \frac{f(y)-f(y-th)}{h^\alpha}\,dt. \tag{2.6}$$

We define the right (respectively left) limit

$$D_+^\alpha(y) := (1+\alpha)\Gamma(1-\alpha) \lim_{h\to 0_+} \int_0^1 \frac{f(y+th)-f(y)}{h^\alpha}\,dt,$$

$$\text{respectively} \quad D_-^\alpha f(y) = -(1+\alpha)\Gamma(1-\alpha) \lim_{h\to 0_+} \int_0^1 \frac{f(y)-f(y-th)}{h^\alpha}\,dt$$

as the right (respectively left) *integral difference-quotient local fractional derivatives* (right (left) IDQ-LFD). Here DQ indicates that the local fractional derivative is still of difference-quotient in nature. However these LFDs are versions of integral averages of the difference-quotient, possibly with a singular weight. Proposition 1 implies that IDQ-LFD $\Rightarrow$ SIDQ-LFD. We do not know whether the converse is true or not.

**Proof of Proposition 1.** We only prove the proposition for $\mathbf{D}_+^\alpha f(y)$ and leave the other statement to interested readers. Let

$$\lim_{h\to 0_+} \int_0^1 \frac{f(y+th)-f(y)}{h^\alpha}\,dt = l$$

and let $C_y > 0$ be the local right Hölder constant for $f$ at $y$. We show that

$$\lim_{h\to 0_+}\int_0^1 (1-t)^{-\alpha}\frac{f(y+th)-f(y)}{h^\alpha}\,dt = (\alpha+1)\Gamma(1+\alpha)l.$$

Now we split the integral near 1 by a small $0<\delta<1$ and use integration by parts.

$$\int_0^1 (1-t)^{-\alpha}\frac{f(y+th)-f(y)}{h^\alpha}\,dt = \int_{1-\delta}^1 (1-t)^{-\alpha}\frac{f(y+th)-f(y)}{h^\alpha}\,dt + \int_0^{1-\delta} (1-t)^{-\alpha}\frac{f(y+th)-f(y)}{h^\alpha}\,dt$$
$$:= I_1 + I_2.$$

We have

$$|I_1| = \left|\int_{1-\delta}^1 (1-t)^{-\alpha}\frac{f(y+th)-f(y)}{h^\alpha}\,dt\right| \leqslant \int_{1-\delta}^1 (1-t)^{-\alpha}\frac{|f(y+th)-f(y|)}{h^\alpha}\,dt \leqslant \int_{1-\delta}^1 (1-t)^{-\alpha}C_y t^\alpha\,dt$$
$$\leqslant C_y\int_{1-\delta}^1 (1-t)^{-\alpha}\,dt \leqslant C_y\frac{\delta^{1-\alpha}}{1-\alpha}.$$

For $I_2$, we have

$$I_2 = \int_0^{1-\delta} (1-t)^{-\alpha}\frac{f(y+th)-f(y)}{h^\alpha}\,dt = \int_0^{1-\delta} (1-t)^{-\alpha}\frac{d}{dt}\left(\int_0^t \frac{f(y+sh)-f(y)}{h^\alpha}\,ds\right)dt$$
$$= \frac{1}{\delta^\alpha}\int_0^{1-\delta}\frac{f(y+sh)-f(y)}{h^\alpha}\,ds - \int_0^{1-\delta}\left[(1-t)^{-\alpha}\right]'\left(\int_0^t \frac{f(y+sh)-f(y)}{h^\alpha}\,ds\right)dt.$$

We also have, for $0<t\leqslant 1$,

$$\int_0^t \frac{f(y+sh)-f(y)}{h^\alpha}\,ds = t^{1+\alpha}\int_0^1 \frac{f(y+u(th))-f(y)}{(th)^\alpha}\,du \to t^{1+\alpha}l$$

uniformly as $h\to 0_+$ with respect to $t\in(0,1-\delta]$. Thus, as $h\to 0_+$,

$$I_2 \to \frac{l}{\delta^\alpha}(1-\delta)^{1+\alpha} - \int_0^{1-\delta}\left[(1-t)^{-\alpha}\right]' t^{1+\alpha} l\,dt = (1+\alpha)l\int_0^{1-\delta}(1-t)^{-\alpha}t^\alpha\,dt.$$

Since $\lim_{\delta\to 0_+}\int_0^{1-\delta}(1-t)^{-\alpha}t^\alpha\,dt = \Gamma(1-\alpha)\Gamma(1+\alpha)$, we see that

$$\lim_{h\to 0_+}\int_0^1 (1-t)^{-\alpha}\frac{f(y+th)-f(y)}{h^\alpha}\,dt = (1+\alpha)\Gamma(1-\alpha)\Gamma(1+\alpha)l,$$

so that

$$\mathbf{D}_+^\alpha f(y) = \frac{1}{\Gamma(1-\alpha)}\lim_{h\to 0_+}\int_0^1 (1-t)^{-\alpha}\frac{f(y+th)-f(y)}{h^\alpha}\,dt = \lim_{h\to 0_+}(1+\alpha)\Gamma(1+\alpha)l.$$

The proof is finished. □

As a consequence of Proposition 1, we have the following simple corollary which will be used later.

**Corollary 1.** *Suppose $f$ is locally right (respectively left) $C^\alpha$ at $y \in (a,b)$ such that*

$$\lim_{h\to 0_+} \int_0^1 \left| \frac{f(y+th)-f(y)}{h^\alpha} \right| dt = 0 \quad \left( \text{respectively } \lim_{h\to 0_+} \int_0^1 \left| \frac{f(y)-f(y-th)}{h^\alpha} \right| dt = 0 \right). \tag{2.7}$$

*Then*

$$\lim_{h\to 0_+} \int_0^1 (1-t)^{-\alpha} \left| \frac{f(y+th)-f(y)}{h^\alpha} \right| dt = 0 \quad \left( \text{respectively } \lim_{h\to 0_+} \int_0^1 (1-t)^{-\alpha} \left| \frac{f(y)-f(y-th)}{h^\alpha} \right| dt = 0 \right). \tag{2.8}$$

The following result, whose proof is easy, shows that if the limit of the difference-quotient (DQ-LFD) exists, then IDQ-LFD $\mathcal{D}_+^\alpha f(y)$ can be represented by that limit.

**Corollary 2.** *Suppose $f \in C(a,b)$ and for some $y \in (a,b)$ the limit*

$$\lim_{h\to 0_+} \frac{f(h+y)-f(y)}{h^\alpha} := d_+^\alpha f(y) \quad \text{exists}. \tag{2.9}$$

*Then*

$$D_+^\alpha f(y) = \Gamma(1+\alpha) d_+^\alpha f(y). \tag{2.10}$$

The proof is straightforward. We call the limit $\Gamma(1+\alpha)d_+^\alpha f(y)$ as the difference-quotient local fractional derivative (DQ-LFD for short). Clearly for $C^\alpha$ functions, DQ-LFD $\Rightarrow$ IDQ-LFD $\Rightarrow$ SIDQ-LFD.

Under stronger assumptions on ${}_yD_x^\alpha(f(x)-f(y))$ we have the following

**Proposition 2.** *Let $f:(a,b)\to \mathbf{R}$ be continuous such that $\mathcal{D}_+^\alpha f(y)$ exists and ${}_yD_x^\alpha(f(x)-f(y))$ belongs to $L^\infty(y, y+\delta)$ for some $\delta > 0$, then*

$$\mathcal{D}_+^\alpha f(y) = \Gamma(1+\alpha) \lim_{x\to y_+} \frac{f(x)-f(y)}{(x-y)^\alpha}. \tag{2.11}$$

*Consequently,*

$$f(x) = f(y) + \Gamma(1+\alpha)\mathcal{D}_+^\alpha f(y)(x-y)^\alpha + o(x-y)^\alpha, \tag{2.12}$$

*as $x \to y_+$.*

Proposition 2 shows that if we add the integrability condition for ${}_yD_x^\alpha(f(x)-f(y))$ on top of the existence of $\mathcal{D}_+^\alpha f(y)$, then we can recover the difference-quotient limit. If we call this integrability condition as INT for short, we have KG-LFD + INT $\Rightarrow$ DQ-LFD.

**Proof of Proposition 2.** Since ${}_yD_x^\alpha(f(x)-f(y))$ belongs to $L^\infty(y, y+\delta)$, the condition for [13], p. 71 (2.113) is satisfied and we have, for $x \in (y, y+\delta)$ that

$${}_yD_x^{-\alpha}\big[{}_yD_x^\alpha\big(f(x)-f(y)\big)\big] = f(x)-f(y) - \big[{}_yD_x^{\alpha-1}\big[f(x)-f(y)\big]\big]\big|_{x=y} \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)},$$

where

$${}_yD_x^{-\alpha}\big(g(x)\big) = \frac{1}{\Gamma(\alpha)} \int_y^x (x-t)^{\alpha-1} g(t)\, dt$$

and the evaluation here $|_{x=y}$ is understood as the limit $x \to y_+$. This is due to the facts that

$${}_yD_x^{\alpha-1}\big[f(x)-f(y)\big] = \frac{1}{\Gamma(1-\alpha)} \int_y^x \frac{f(t)-f(y)}{(x-t)^\alpha}\, dt \to 0$$

as $x \to y_+$ and that $f$ is continuous in $(a,b)$. Also, as we have assumed that ${}_yD_x^\alpha(f(x)-f(y))$ is bounded and measurable in $(y, y+\delta)$. Therefore

$$f(x)-f(y) = {}_yD_x^{-\alpha}\big[{}_yD_x^\alpha\big(f(x)-f(y)\big)\big], \quad x \in (y, y+\delta). \tag{2.13}$$

Note that as mentioned in [13], p. 70, we only need ${}_yD_x^\alpha(f(x)-f(y))$ to be integrable over $(y, y+\delta)$.

Now the proof follows from the fact that $\lim_{x\to y_+} D_x^\alpha(f(x)-f(y)) = \mathcal{D}_+^\alpha(f(y))$. We have

$$\begin{aligned}\frac{f(x)-f(y)}{(x-y)^\alpha} &= \frac{1}{(x-y)^\alpha}\,{}_yD_x^{-\alpha}\big[{}_yD_x^\alpha\big(f(x)-f(y)\big)\big]\\
&= \frac{1}{\Gamma(\alpha)(x-y)^\alpha}\int_y^x (x-t)^{\alpha-1}\big[{}_yD_t^\alpha\big(f(t)-f(y)\big)\big]\,dt\\
&= \frac{1}{\Gamma(\alpha)(x-y)^\alpha}\int_y^x (x-t)^{\alpha-1}\big[{}_yD_t^\alpha\big(f(t)-f(y)\big)-\mathcal{D}_+^\alpha\big(f(y)\big)\big]\,dt\\
&\quad+\frac{1}{\Gamma(\alpha)(x-y)^\alpha}\int_y^x (x-t)^{\alpha-1}\mathcal{D}_+^\alpha\big(f(y)\big)\,dt\\
&:= I_1+I_2.\end{aligned}$$

We have

$$I_2 = \frac{1}{\Gamma(\alpha)(x-y)^\alpha}\int_y^x (x-t)^{\alpha-1}\,{}_y\mathcal{D}_+^\alpha\big(f(y)\big)\,dt = \frac{\mathcal{D}_+^\alpha(f(y))}{\Gamma(\alpha)\alpha} = \frac{\mathcal{D}_+^\alpha(f(y))}{\Gamma(\alpha+1)}.$$

Thus we only need to show that $I_1 \to 0$ as $x \to y_+$ which is also easy to see. By definition we have, for any $\epsilon > 0$, there is some $0 < \eta < \delta$ such that

$$\big|{}_yD_t^\alpha\big(f(t)-f(y)\big)-\mathcal{D}_+^\alpha\big(f(y)\big)\big| \leqslant \epsilon$$

whenever $t \in (y, y+\eta]$. Therefore, for $x \in (y, y+\eta]$,

$$\begin{aligned}|I_1| &= \left|\frac{1}{\Gamma(\alpha)(x-y)^\alpha}\int_y^x (x-t)^{\alpha-1}\big[{}_yD_t^\alpha\big(f(t)-f(y)\big)-\mathcal{D}_+^\alpha\big(f(y)\big)\big]\,dt\right|\\
&\leqslant \frac{1}{\Gamma(\alpha)(x-y)^\alpha}\int_y^x (x-t)_y^{\alpha-1}\big|{}_yD_t^\alpha\big(f(t)-f(y)\big)-\mathcal{D}_+^\alpha\big(f(y)\big)\big|\,dt\\
&\leqslant \frac{1}{\Gamma(\alpha)(x-y)^\alpha}\int_y^x (x-t)^{\alpha-1}\epsilon\,dt\\
&= \frac{\epsilon}{\Gamma(1+\alpha)}.\end{aligned}$$

The proof is finished. □

**Remark 2.** The connections between the original definition of KG-LFD and the difference-quotient limits via Taylor's expansion were considered in [1,6,10]. Let us consider the right LFD only. In both [6] and [1] the following Taylor expansion was used while [1] applied it to show that KG-LFD equals DQ-LFD:

$$\begin{aligned}f(x)-f(y) &= \frac{1}{\Gamma(\alpha)}\int_0^{x-y}\frac{F(y,t,\alpha)}{(x-y-t)^{1-\alpha}}\,dt\\
&= \frac{1}{\Gamma(\alpha)}\left[F(y,t,\alpha)\int (x-y-t)^{\alpha-1}\,dt\right]_0^{x-y} + \frac{1}{\Gamma(\alpha)}\int_0^{x-y}\frac{dF(y,t,\alpha)}{dt}\frac{(x-y-t)^\alpha}{\alpha}\,dt,\end{aligned}\tag{2.14}$$

where (following the notation in [13] for the Riemann–Liouville fractional derivative)

$$F(y,t,\alpha) = {}_yD_t^\alpha\big(f(t)-f(y)\big).$$

We simply observe that for the first equality in (2.14) to be satisfied, we need ${}_yD_t^\alpha(f(t)-f(y))$ to be locally integrable. In order to make sense of the second equality, we need the Riemann–Liouville fractional derivative ${}_yD_t^\alpha(f(t)-f(y))$ to be

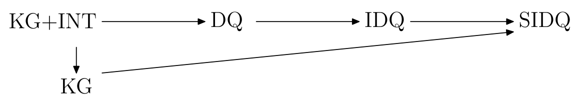


**Fig. 1.** Relations among various LFDs.

differentiable and the resulting derivative $d/dt({}_yD_t^\alpha(f(t)-f(y)))$ to be integrable. None of these were assumed in either [6] or [1]. Therefore further assumptions on the smoothness of $f$ are needed in order to make sense of (2.14).

To summarize what we have established above, we have, by dropping the affix LFD in the statements, that

**Theorem 1.** *The following implications of various LFDs hold.*

$$\text{KG}+\text{INT} \implies \text{DQ} \implies \text{IDQ} \implies \text{SIDQ}, \qquad \text{KG}+\text{INT} \implies \text{KG} \implies \text{SIDQ}.$$

*These relations can be viewed in Fig.* 1.

Let us examine the example mentioned earlier which shows that the right IDQ-LFD and SIDQ-LFD exist while the corresponding DQ-LFD and KG-LFD do not.

**Example 1.** For a fixed $0<\alpha<1$, let

$$f(x)=\begin{cases} |x|^\alpha \sin(\frac{1}{x}), & x\in\mathbf{R},\ x\neq 0,\\ 0, & x=0.\end{cases}$$

Clearly $f$ is locally $C^\alpha$ at 0.

We first show that the right SIDQ-LFD $\mathbf{D}_+^\alpha f(0)$ exists by showing that the right IDQ-LFD $D_+^\alpha f(0)$ exists and the value is zero. Then by Proposition 1, we may claim that $\mathbf{D}_+^\alpha f(0)=0$. We have

$$\int_0^1 \frac{f(th)-f(0)}{h^\alpha}\,dt=\int_0^1 \frac{(th)^\alpha \sin(1/(th))}{h^\alpha}\,dt=\int_0^1 t^\alpha \sin\bigl(1/(th)\bigr)\,dt=\int_1^{+\infty}\frac{\sin(\frac{s}{h})}{s^{2+\alpha}}\,ds.$$

Now we show that the last term above on the far right goes to zero as $h\to 0_+$. Given any $\epsilon>0$, we take some $M>1$ to be determined later and consider

$$\left|\int_M^{+\infty}\frac{\sin(\frac{s}{h})}{s^{2+\alpha}}\,ds\right|\leqslant \int_M^{+\infty}\frac{1}{s^{2+\alpha}}\,ds=\frac{1}{(\alpha+1)M^{\alpha+1}}.$$

Now we choose $M>1$ sufficiently large so that the last term above is less than $\epsilon$.

For this fixed $M>0$, we apply Riemann–Lebesgue Lemma in Fourier analysis to conclude that

$$\lim_{h\to 0_+}\int_1^M \frac{\sin(\frac{s}{h})}{s^{2+\alpha}}\,ds=0.$$

The conclusion then follows.

Next we show that neither the right KG-LFD $\mathcal{D}_+^\alpha f(0)$ nor the right DQ-LFD $d_+^\alpha f(0)$ exist. It is easy to see that the right DQ-LFD does not exist because for $x>0$,

$$\frac{f(x)-f(0)}{(x-0)^\alpha}=\sin\left(\frac{1}{x}\right)$$

which does not have a limit as $x\to 0_+$.

Now we show that the right KG-LFD $\mathcal{D}_+^\alpha f(0)$ does not exist. To avoid complicated calculations, we prove this by a contradiction argument using Proposition 2. Suppose $\mathcal{D}_+^\alpha f(0)$ exists, we only need to show that ${}_0D_x^\alpha(f(x)-f(0))$ belongs to $L^\infty(0,\delta)$ for some small $\delta>0$. Since by definition, $\mathcal{D}_+^\alpha f(0)=\lim_{x\to 0_+}{}_0D_x^\alpha(f(x)-f(0))$ which is assumed to exist, we see that ${}_0D_x^\alpha(f(x)-f(0))$ is bounded in a small interval $(0,\delta)$. We only need to show that ${}_0D_x^\alpha(f(x)-f(0))$ is continuous in $(0,\delta)$. In fact we only need the function to be measurable in $(0,\delta)$. We have

$$
\begin{aligned}
{}_0D_x^{\alpha}\big(f(x)-f(0)\big) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x \frac{t^{\alpha}\sin(1/t)}{(x-t)^{\alpha}}\,dt \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}x\int_0^1 \frac{s^{\alpha}\sin(1/(sx))}{(1-s)^{\alpha}}\,ds \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^1 \frac{s^{\alpha}\sin(1/(sx))}{(1-s)^{\alpha}}\,ds + \frac{x}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^1 \frac{s^{\alpha}\sin(1/(sx))}{(1-s)^{\alpha}}\,ds \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^1 \frac{s^{\alpha}\sin(1/(sx))}{(1-s)^{\alpha}}\,ds - \frac{1}{\Gamma(1-\alpha)}\frac{1}{x}\int_0^1 \frac{\cos(1/(sx))}{s^{1-\alpha}(1-s^{\alpha})}\,ds. \qquad (2.15)
\end{aligned}
$$

The derivative given by the last term above in (2.15) can be established for $x > 0$ by applying the Dominated Convergence Theorem [15]. Again by using the Dominated Convergence Theorem, we can see that both integrals in the last line of (2.15) are continuous for $x \in (0,\delta)$. We verify this claim at the end of the proof. Thus ${}_0D_x^{\alpha}(f(x) - f(0))$ belongs to $L^{\infty}(0,\delta)$. By Proposition 2, we claim that the DQ-LFD $d_+^{\alpha}(0)$ then exists. This clearly contradicts to our direct calculation earlier showing that $d_+^{\alpha} f(0)$ does not exist. The proof will be finished after we show that the two terms in the last line of (2.15) are continuous for $x \in (0,\delta)$. We prove that the first term is continuous in $(0,\delta)$. The proof for the second is similar. Let

$$
g(x) = \int_0^1 \frac{s^{\alpha}\sin(1/(sx))}{(1-s)^{\alpha}}\,ds, \quad x \in (0,\delta).
$$

We need to show that for each fixed $x_0 \in (0,\delta)$, $g(x)$ is continuous at $x_0$. This is a simple exercise in real analysis. We only need to show that for any sequence $x_j \in (0,\delta)$, $x_j \neq x_0$ and $x_j \to x_0$ as $j \to \infty$, we have $g(x_j) \to g(x_0)$ as $j \to \infty$. Now we prove this. For $j = 0,1,2,\ldots,$ let

$$
f_j(s) = \frac{s^{\alpha}\sin(1/(sx_j))}{(1-s)^{\alpha}}, \quad s \in (0,1).
$$

Clearly $f_j(s)$ is measurable in $(0,1)$ for each $j$. Also $f_j(s) \to f_0(s)$ for each fixed $s \in (0,1)$. Furthermore, for each $s \in (0,1)$,

$$
\big|f_j(s)\big| = \left|\frac{s^{\alpha}\sin(1/(sx_j))}{(1-s)^{\alpha}}\right| \leqslant \frac{1}{(1-s)^{\alpha}} := f(s).
$$

Since $0 < \alpha < 1$, we see that $f(s)$ is integrable in $(0,1)$. Thus by the Dominated Convergence Theorem, we have

$$
\lim_{j\to\infty}\int_0^1 f_j(s)\,ds = \int_0^1 f_0(s)\,ds, \quad \text{i.e.,}\ \lim_{j\to\infty} g(x_j) = g(x_0).
$$

**Remark 3.** Example 1 shows that KG-LFD and DQ-LFD are strictly stronger than IDQ-LFD and SIDQ-LFD. We also notice that the first term in the last line of (2.15) is actually the definition of the right SIDQ-LFD. We already know that this converges to zero. Indirectly we have shown that the second term in that line does not have a limit.

Example 1 motivates us to find criteria for functions with an isolated singularity in the form $f(x) = |x|^{\alpha} g(x)$ with $g \in C^1((-\delta,0) \cup (0,\delta))$ to have left and/or right KG-LFD. We have

**Proposition 3.** *For* $0 < \alpha < 1$ *and* $\delta > 0$, *let*

$$
f(x) = \begin{cases} x^{\alpha} g(x), & 0 < x < \delta, \\ 0, & x = 0, \end{cases}
$$

*where* $g \in C^1(0,\delta)$ *is bounded and for some* $C > 0$, $|g'(x)| \leqslant Cx^{-\beta}$ *for* $x \in (0,\delta)$ *with* $0 < \beta < 2 + \alpha$. *Let* $F(x) = x^{1+\alpha} g'(x)$ $(0 < x < \delta)$ *and define* $F(0) = 0$. *Then* $\mathcal{D}_+^{\alpha} f(0)$ *exists if and only if both* $\mathbf{D}_+^{\alpha} f(0)$ *and* $\mathbf{D}_+^{\alpha} F(0)$ *exist with* $\mathbf{D}_+^{\alpha} F(0) = 0$.

Note that the function $f(x) = |x|^{\alpha}\sin(1/x)$ considered in Example 1 with $g(x) = \sin(1/x)$ satisfies the assumptions that $g \in C^1(0,1)$, $g$ is bounded and $|g'(x)| \leqslant x^{-2}$, where $\beta = 2 < 2 + \alpha$.

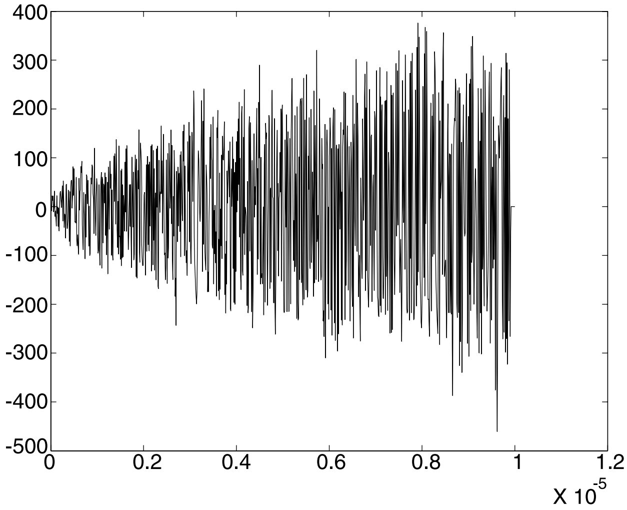


**Fig. 2.** The right 1/2-IDQ-LFD for $x^{1/2}\sin(1/x)$ near 0.

A numerical approximation of the 1/2-right-IDQ $D_\alpha^{1/2} f(x)$ for the above function $f$ with $\alpha = 1/2$ in a small interval $(0,\delta)$ is illustrated in Fig. 2. It suggests that $D_\alpha^{1/2} f(x)$ is in fact, approaching zero as $x \to 0_+$.

**Proof of Proposition 3.** By definition, we have

$$\mathcal{D}_+^\alpha f(0) = \lim_{x\to 0_+} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)-f(0)}{(x-t)^\alpha}\,dt.$$

By the change of variable $t = xs$ and by noticing that $f(0) = 0$, we have, for $0 < x < \delta$ that

$$\frac{d}{dx}\int_0^x \frac{f(t)-f(0)}{(x-t)^\alpha}\,dt = \frac{d}{dx}\int_0^x \frac{(xs)^\alpha g(xs)}{(x-xs)^\alpha}\,d(xs) = \frac{d}{dx}\left(x\int_0^1 \frac{s^\alpha g(xs)}{(1-s)^\alpha}\,ds\right) = \int_0^1 \frac{s^\alpha g(xs)}{(1-s)^\alpha}\,ds + x\frac{d}{dx}\left(\int_0^1 \frac{s^\alpha g(xs)}{(1-s)^\alpha}\,ds\right)$$
$$= I(x) + x\frac{d}{dx}I(x).$$

Now we show that $I(x)$ is differentiable and find the derivative of $I$ by using the Dominated Convergence Theorem. Fix $x \in (0,\delta)$ and let $\eta(x) > 0$ be such that $[x-\eta, x+\eta] \subset (0,\delta)$. Let $(x_j)$ be any sequence such that $x_j \to x$ as $j \to \infty$, $x_j \neq x$ and $x_j \in [x-\eta, x+\eta]$. We show that

$$\lim_{j\to\infty} \frac{I(x_j)-I(x)}{x_j - x} = \int_0^1 \frac{s^\alpha g'(xs)s}{(1-s)^\alpha}\,ds.$$

We have

$$\frac{I(x_j)-I(x)}{x_j-x} = \int_0^1 \frac{s^\alpha}{(1-s)^\alpha}\frac{g(x_js)-g(xs)}{x_j-x}\,ds.$$

Since $g \in C^1(0,\delta)$, we have, for a.e. $s \in [0,\delta)$

$$\lim_{j\to\infty} \frac{s^\alpha}{(1-s)^\alpha}\frac{g(x_js)-g(xs)}{x_j-x} = \frac{s^\alpha}{(1-s)^\alpha} g'(xs)s.$$

Also

$$\left|\frac{s^\alpha}{(1-s)^\alpha}\frac{g(x_js)-g(xs)}{x_j-x}\right| = \frac{s^\alpha}{(1-s)^\alpha}\left|\frac{\int_{xs}^{x_js} g'(\tau)\,d\tau}{x_j-x}\right| \leqslant \frac{s^\alpha}{(1-s)^\alpha}\left|\frac{\int_{xs}^{x_js} C\tau^{-\beta}\,d\tau}{x_j-x}\right|$$

$$= \frac{s^\alpha}{(1-s)^\alpha}\left|\frac{\int_x^{x_j} C(st)^{-\beta}\,d(st)}{x_j - x}\right| \leqslant \frac{s^\alpha}{(1-s)^\alpha} C s^{1-\beta}\big[x-\eta(x)\big]^{-\beta}$$
$$= C\big[x-\eta(x)\big]^{-\beta} s^{1+\alpha-\beta}(1-s)^{-\alpha} := H(x,s).$$

Since $1+\alpha-\beta > -1$, $H(x,\cdot) \in L^1(0,1)$. Thus by the Dominated Convergence Theorem, we have

$$\lim_{j\to\infty} \frac{I(x_j)-I(x)}{x_j - x} = \int_0^1 \frac{s^\alpha}{(1-s)^\alpha} g'(xs)s\,ds \quad \text{so that } \frac{d}{dx} I(x) = \int_0^1 \frac{s^{1+\alpha}}{(1-s)^\alpha} g'(xs)\,ds.$$

Now we have

$$\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x \frac{f(t)-f(0)}{(x-t)^\alpha}\,dt = \frac{1}{\Gamma(1-\alpha)}\int_0^1 \frac{s^\alpha g(xs)}{(1-s)^\alpha}\,ds + \frac{x}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\int_0^1 \frac{s^\alpha g(xs)}{(1-s)^\alpha}\,ds\right)$$
$$= \frac{1}{\Gamma(1-\alpha)}\int_0^1 \frac{(xs)^\alpha g(xs)}{x^\alpha(1-s)^\alpha}\,ds + \frac{1}{\Gamma(1-\alpha)}\int_0^1 \left(\frac{(xs)^{1+\alpha}}{x^\alpha(1-s)^\alpha}\right) g'(xs)\,ds. \tag{2.16}$$

If $\mathcal{D}_+^\alpha f(0)$ exists, the limit of the first term in (2.16) exists as $x \to 0_+$. By Proposition 1, $\mathbf{D}_+^\alpha f(0)$ exists and equals $\mathcal{D}_+^\alpha f(0)$. Thus the first term in the last line of (2.16) has a limit as $x \to 0_+$ which implies that the last term in the last line of (2.16) must go to zero as $x \to 0_+$. By definition we see that $\mathbf{D}_+^\alpha F(0)$ exists and equals zero.

Next we assume that both $\mathbf{D}_+^\alpha f(0)$ and $\mathbf{D}_+^\alpha F(0)$ exist with $\mathbf{D}_+^\alpha F(0) = 0$. Passing to the limit $x \to 0$ in (2.16) we see that $\mathcal{D}_+^\alpha f(0)$ exists and equals $\mathbf{D}_+^\alpha f(0)$. The proof is finished. □

As the final remark of this section, we have, by combining Example 1 and Proposition 3, we can show that for functions in the form

$$f(x) = |x|^\alpha \sin\big(1/x^\beta\big), \qquad f(0) = 0, \quad 0 < \alpha < 1,\ 0 < \beta < 1+\alpha$$

does not have either $\mathcal{D}_\pm^\alpha f(0)$ or $d_\pm^\alpha f(0)$ while both $D_\pm^\alpha f(0)$ and $\mathbf{D}_\pm^\alpha f(0)$ exist and both equal zero.

## 3. On the structure of KG-LFD

Now we turn to the issue concerning the implications of the existence of the KG-LFD a.e. in an interval. The following is our main structural theorem for the KG-LFD. Although so far we are not able to show that the existence of $\mathcal{D}_\pm^\alpha f(y)$ imply the existence of $D_\pm^\alpha f(y)$, the following theorem also implies that the existence of $\mathcal{D}_\pm^\alpha f$ a.e. implies the existence of $D_\pm^\alpha f(y)$ a.e. and both of them are in fact zero.

**Theorem 2.** *Suppose $f \in C^\alpha(a,b)$ for some $0 < \alpha < 1$ and $\mathcal{D}_\pm^\alpha f(y)$ exist for a.e. $y \in (a,b)$, then $\mathcal{D}_+^\alpha f(y) = \mathcal{D}_-^\alpha f(y) = 0$ for a.e. $y \in (a,b)$. Furthermore*

$$\lim_{h\to 0_+} \int_0^1 \left|\frac{f(y+th)-f(y)}{h^\alpha}\right| dt = 0, \qquad \lim_{h\to 0_+} -\int_0^1 \left|\frac{f(y-th)-f(y)}{h^\alpha}\right| dt = 0 \quad \text{a.e. } y \in (a,b).$$

*Consequently, $D_\pm^\alpha f(y) = 0$ for a.e. $y \in (a,b)$.*

Note that everywhere vanishing results in the form $\mathcal{D}_+^\alpha f(y) = \mathcal{D}_-^\alpha f(y) = 0$ were established in [2] where (in the case $0 < \alpha < 1$) the function $f$ is assume to belong to $C^r(a,b)$ for some $r > \alpha$.

Stein and Zygmund [16] considered the $\alpha$-fractional derivative in the sense of M. Riesz for functions defined on $\mathbf{R}$ and its variations including the Weyl fractional derivative. The results were generalized by Welland [17] to functions of several variables. We only describe the case when $0 < \alpha < 1$ here.

Let $\beta = 1-\alpha$. Given a measurable function $f : \mathbf{R} \to \mathbf{R}$, the $\beta$-th integral $f_\beta$ for $f$ is defined by [16]

$$f_\beta(x) = \int_{\mathbf{R}} \frac{f(y)}{|x-y|^{1-\beta}}\,dy = (f * K_{1-\beta})(x), \quad 0 < \beta < 1,$$

which is the convolution between $f$ and $K_\gamma(x) = |x|^{-\gamma}$. The $\alpha$-fractional derivative of $f$ at $x$, denoted by $f^{(\alpha)}(x)$, is defined by (see [16])

$$f^{(\alpha)}(x) = \frac{d}{dx} f_\beta(x), \quad \beta = 1 - \alpha.$$

In order to prove Theorem 1, we only need to state one of the results in [16].

To characterise the existence of such $\alpha$-fractional derivatives, the following conditions were given in [16] (Theorem 1):

(i) $f$ is said to satisfy $\Lambda_\alpha$ $(0 < \alpha < 1$ in our case) at $x$ if

$$R_x(t) = \mathcal{O}\big(|t|^\alpha\big) \quad \text{as } t \to 0, \tag{3.1}$$

where $R_x(t) = f(x+t) - f(x)$.

(ii) $f$ satisfies the condition $N_\alpha^2$ at $x$ if

$$\int_{-\delta}^{\delta} \frac{[R_x(t)]^2}{|t|^{1+2\alpha}}\, dt < +\infty \quad \text{for some } \delta > 0. \tag{3.2}$$

We apply the following result in [16] to establish Theorem 2.

**Proposition 4.** *Suppose $f \in L^1(\mathbf{R})$ and satisfies the condition $\Lambda_\alpha$ for each point $x$ of a set $E \subset \mathbf{R}$ of positive measure. Then $f^{(\alpha)}(x)$ exists almost everywhere in $E$ if and only if $f$ satisfies condition $N_\alpha^2$ almost everywhere in $E$.*

**Remark 4.** As mentioned in [16], the results in paper [16] remain valid and the proofs essentially unchanged if one replaces $f_\beta$ by Weyl's version of $\alpha$-fractional derivative

$$\frac{d}{dx} I_\beta(x), \quad \text{where } I_\beta(x) = \int_{-\infty}^{x} \frac{f(y)}{|x-y|^{1-\beta}}\, dy, \ 0 < \beta < 1.$$

It is easy to see and was observed in [9] that the KG-LFD is related to Weyl's fractional derivative as follows.

**Remark 5.** Let $f \in C^\alpha(a,b)$. For any fixed $y \in (a,b)$ we define

$$f_y^+(x) = \begin{cases} f(x) - f(y), & y < x < b, \\ 0, & x \leqslant y \text{ or } x \geqslant b, \end{cases} \qquad f_y^-(x) = \begin{cases} f(x) - f(y), & a < x < y, \\ 0, & x \geqslant y \text{ or } x \leqslant a. \end{cases} \tag{3.3}$$

Let

$$I_\beta^+ f_y^+(x) = \int_{-\infty}^{x} \frac{f_y^+(t)}{(x-t)^{1-\beta}}\, dt, \qquad I_\beta^- f_y^-(x) = \int_{x}^{\infty} \frac{f_y^-(t)}{(t-x)^{1-\beta}}\, dt, \quad \beta = 1 - \alpha. \tag{3.4}$$

Then the right and left KG-LFD are defined at $y \in (a,b)$ respectively by

$$\begin{aligned} \mathcal{D}_+^\alpha f(y) &= \frac{1}{\Gamma(1-\alpha)} \lim_{x \to y_+} \frac{d}{dx} I_\beta^+ f_y^+(x), \quad \text{and} \\ \mathcal{D}_-^\alpha f(y) &= \frac{1}{\Gamma(1-\alpha)} \lim_{x \to y_-} \frac{d}{dx} I_\beta^- f_y^-(x), \quad \beta = 1 - \alpha. \end{aligned} \tag{3.5}$$

**Proof of Theorem 2.** From Remark 5 we see that if $\mathcal{D}_\pm^\alpha f(y)$ exist at some $y \in (a,b)$, there is a neighbourhood $I_y = (y - \tau, y + \tau) \subset (a,b)$ with $\tau > 0$, such that

$$\frac{d}{dx} I_\beta^+ f_y^+(x) \text{ exists for } x \in (y, y+\tau), \qquad \frac{d}{dx} I_\beta^- f_y^-(x) \text{ exists for } x \in (y-\tau, y), \quad \beta = 1 - \alpha.$$

By our assumption that $f \in C^\alpha(a,b)$, Remark 5 and Proposition 4, we see that for a.e. $x \in (y, y+\tau)$, $N_\alpha^2$ holds for $f_y^+$ and for a.e. $x \in (y-\tau, y)$, $N_\alpha^2$ holds for $f_y^-$. Note that the function $R_x(t)$, applying to $f_+$ and $f_-$ at $x$, is independent of $f(y)$ for $x \in (y-\tau, y) \cup (y, y+\tau)$. Thus $R_x(t) = f(x+t) - f(x)$, hence for a.e. $x \in (y-\tau, y) \cup (y, y+\tau)$, $N_\alpha^2$ holds for $f$. Also by our assumptions we see that for a.e. $x \in (y-\tau, y+\tau)$, $\mathcal{D}_+^\alpha f(x)$ and $\mathcal{D}_-^\alpha f(x)$ both exist, and $N_\alpha^2$ holds for $f$.

Next we show that for such an $x \in (y-\tau, y+\tau)$, we have $\mathcal{D}_+^\alpha f(x) = 0$ and $\mathcal{D}_-^\alpha f(x) = 0$. Since $N_\alpha$ holds at $x$, there is some $\delta > 0$ sufficiently small such that $(x-\delta, x+\delta) \subset (y-\tau, y+\tau)$ and $\int_{-\delta}^{\delta} [R_x(t)]^2/|t|^{2\alpha+1}\, dt < +\infty$. By Vitali's equi-integrability theorem for Lebesgue integrals, we have

$$\lim_{h\to 0_+} \int_0^h \frac{[R_x(t)]^2}{|t|^{2\alpha+1}}\, dt = 0.$$

Since $\mathcal{D}_+^\alpha f(x)$ exists, we have, by Lemma 1 that

$$\mathcal{D}_+^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \lim_{h\to 0_+} \int_0^1 (1-t)^{-\alpha} \frac{f(ht+x) - f(x)}{h^\alpha}\, dt.$$

We only need to show that the limit on the right-hand side of the above is zero. In fact we can prove the following stronger statement which is needed later:

$$\lim_{h\to 0_+} \int_0^1 (1-t)^{-\alpha} \left| \frac{f(ht+x) - f(x)}{h^\alpha} \right| dt = 0. \tag{3.6}$$

Since $f \in C^\alpha(a,b)$, there is a constant $M > 0$ such that $|f(t) - f(s)| \leqslant M|t-s|^\alpha$ for all $t, s \in (a,b)$. Now for any $0 < \epsilon < 1$, we write, similar to the proof of Proposition 1 that

$$\begin{aligned}\int_0^1 (1-t)^{-\alpha} \left| \frac{f(ht+x) - f(x)}{h^\alpha} \right| dt &= \int_{1-\epsilon}^1 (1-t)^{-\alpha} \left| \frac{f(ht+x) - f(x)}{h^\alpha} \right| dt + \int_0^{1-\epsilon} (1-t)^{-\alpha} \left| \frac{f(ht+x) - f(x)}{h^\alpha} \right| dt \\ &:= I_1 + I_2,\end{aligned}$$

and we have

$$\begin{aligned} I_1 &= \int_{1-\epsilon}^1 (1-t)^{-\alpha} \left| \frac{f(ht+x) - f(x)}{h^\alpha} \right| dt \leqslant \int_{1-\epsilon}^1 (1-t)^{-\alpha} \left| \frac{f(ht+x) - f(x)}{h^\alpha} \right| dt \leqslant M \int_{1-\epsilon}^1 (1-t)^{-\alpha} t^\alpha\, dt \\ &\leqslant M \int_{1-\epsilon}^1 (1-t)^{-\alpha}\, dt = \frac{M}{1-\alpha} \epsilon^{1-\alpha}. \end{aligned}$$

We also have, by applying Cauchy–Schwarz inequality that

$$\begin{aligned} I_2 &= \int_0^{1-\epsilon} (1-t)^{-\alpha} \left| \frac{f(ht+x) - f(x)}{h^\alpha} \right| dt \leqslant \frac{1}{\epsilon^\alpha} \int_0^{1-\epsilon} \left| \frac{f(s+x) - f(x)}{h^\alpha} \right| ds \\ &\leqslant \frac{1}{\epsilon^\alpha h^{1+\alpha}} \int_0^h s^{\alpha+1/2} \left( \frac{|f(s+x) - f(x)|}{s^{\alpha+1/2}} \right) ds \\ &\leqslant \frac{1}{\epsilon^\alpha h^{1+\alpha}} \left( \int_0^h s^{2\alpha+1}\, ds \right)^{1/2} \left( \int_0^h \frac{|f(s+x) - f(x)|^2}{s^{2\alpha+1}}\, ds \right)^{1/2} \\ &= \frac{1}{\epsilon^\alpha \sqrt{2\alpha+2}} \left( \int_0^h \frac{|f(s+x) - f(x)|^2}{s^{2\alpha+1}}\, ds \right)^{1/2}. \end{aligned}$$

Since $N_\alpha^2$ holds for $f$ at $x$, we have

$$\lim_{h\to 0_+} \int_0^h \frac{|f(s+x) - f(x)|^2}{s^{2\alpha+1}}\, ds = 0,$$

hence there is some $\eta > 0$ such that

$$\frac{1}{\epsilon^\alpha\sqrt{2\alpha+2}}\left(\int_0^h \frac{|f(s+x)-f(x)|^2}{s^{2\alpha+1}}\,ds\right)^{1/2} \leqslant \epsilon^{1-\alpha}$$

whenever $0 < h < \eta$. Thus $I_1 + I_2 \leqslant (1+M)\epsilon^{1-\alpha}$ whenever $0 < h < \eta$. Thus (3.6) holds so that $\mathbf{D}^\alpha_+ f(x) = 0$ which implies that $\mathcal{D}^\alpha_+ f(x) = 0$. Similarly we see that $\mathcal{D}^\alpha_- f(x) = \mathbf{D}^\alpha_- f(x) = 0$.

We finish the first part of the proof by a simple covering argument. We may assume that both $\mathcal{D}^\alpha_+ f(y)$ and $\mathcal{D}^\alpha_- f(y)$ exists for $y \in (a,b)\setminus\mathcal{N}_0$ where $\mathcal{N}_0 \subset (a,b)$ is a subset of measure zero. As we have proved above, for each $y \in (a,b)\setminus\mathcal{N}_0$, there is an open interval $I_y \subset (a,b)$ centred at $y$ such that $\mathcal{D}^\alpha_+ f(x) = 0$ and $\mathcal{D}^\alpha_- f(x) = 0$ for a.e. $x \in I_y$. Clearly $\{I_y,\ y \in (a,b)\setminus\mathcal{N}_0\}$ is an open covering of $(a,b)\setminus\mathcal{N}_0$, that is, $(a,b)\setminus\mathcal{N}_0 \subset \bigcup_{y\in(a,b)\setminus\mathcal{N}_0} I_y$, there is a countable sub-covering $\{I_k,\ k = 1,2,\ldots\}$ such that $(a,b)\setminus\mathcal{N}_0 \subset \bigcup_{k=1}^\infty I_k$ and on each $I_k$, $\mathcal{D}^\alpha_+ f = 0$ and $\mathcal{D}^\alpha_- f = 0$ a.e. in $I_k$. Thus we conclude that $\mathcal{D}^\alpha_+ f = 0$ and $\mathcal{D}^\alpha_- f = 0$ a.e. in $(a,b)$. The proof of the first part is complete.

As for $0 \leqslant t < 1$ and $\alpha > 0$, we have $(1-t)^{-\alpha} \geqslant 1$. Thus by (3.6),

$$0 = \lim_{h\to 0_+}\int_0^1 (1-t)^{-\alpha}\left|\frac{f(ht+x)-f(x)}{h^\alpha}\right| dt \geqslant \overline{\lim}_{h\to 0_+}\int_0^1 \left|\frac{f(ht+x)-f(x)}{h^\alpha}\right| dt.$$

Therefore $\lim_{h\to 0_+}\int_0^1 |\frac{f(ht+x)-f(x)}{h^\alpha}|\,dt = 0$, hence $D^\alpha_+ f(x) = 0$. □

The following is a direct consequence of Theorem 1.

**Corollary 3.** *Suppose $f \in C^\alpha(a,b)$ and $\mathcal{D}^\alpha_+ f(y)$ (respectively $\mathcal{D}^\alpha_- f(y)$) exists at $y \in (a,b)$, then there is some $\delta > 0$ such that*

$$\mathbf{D}^\alpha_+ f(y) = D^\alpha_+ f(y) = 0 \quad \bigl(\textit{respectively } \mathbf{D}^\alpha_- f(y) = D^\alpha_- f(y) = 0\bigr),$$

*for a.e. $x \in (y, y+\delta)$ (respectively for a.e. $x \in (y-\delta, y)$).*

The conclusions of Corollary 3 are due to the fact that if $\mathcal{D}^\alpha_+ f(y)$ exists at $y$, then ${}_yD^\alpha_x f(x)$ exists in an interval $(y, y+\delta)$. Therefore $N^2_\alpha$ holds a.e. $(y, y+\delta)$ which implies, as in the proof of Theorem 2, that $\mathbf{D}^\alpha_+ f(y) = D^\alpha_+ f(y) = 0$ a.e. in $(y, y+\delta)$.

Numerically, the IDQ-LFDs $D^\alpha_+ f$ and $D^\alpha_- f$ are much easier to compute than the KG-LFDs $\mathcal{D}^\alpha_+ f$, $\mathcal{D}^\alpha_- f$ and the SIDQ-LFDs $\mathbf{D}^\alpha_+ f$ and $\mathbf{D}^\alpha_- f$. They are also more stable than the simple difference-quotient LFDs $d^\alpha_+ f$ and $d^\alpha_- f$ as the integral average can remove some of the high frequency noises. Therefore we believe that IDQ-LFDs are more suitable to be used as approximate LFDs which can also measure the modulus of $\alpha$-fractional derivatives. For example it seems reasonable to have the approximation

$$\begin{aligned}
\frac{1}{2}\bigl(\bigl|D^\alpha_+ f(y)\bigr| + \bigl|D^\alpha_- f(y)\bigr|\bigr) &\simeq \frac{(1+\alpha)\Gamma(1-\alpha)}{2}\left(\left|\int_0^1 \frac{f(y+th)-f(y)}{h^\alpha}\,dt\right| + \left|\int_0^1 \frac{f(y+th)-f(y)}{h^\alpha}\,dt\right|\right) \\
&= \frac{(1+\alpha)\Gamma(1+\alpha)}{2h^{1+\alpha}}\left(\left|\int_0^h \bigl[f(s+y)-f(y)\bigr]\,ds\right| + \left|-\int_0^h \bigl[f(s-y)-f(y)\bigr]\,ds\right|\right).
\end{aligned} \tag{3.7}$$

Next we give a partial generalization of our one-dimensional results to higher dimensional cases.

for a function $f : \Omega \mapsto \mathbf{R}$, the directional local fractional derivative at $y \in \Omega$ along a direction $v$ with $|v| = 1$ was defined in [8] as

$$\mathcal{D}^\alpha_v f(y) = \frac{d^\alpha}{dt^\alpha}\Phi(y,t)\bigg|_{t=0}, \tag{3.8}$$

where $t \mapsto \Phi(y+tv) = f(y+tv) - f(y)$. Similar to the one-dimensional case, we have, as in Lemma 1 that the existence of such a local fractional derivative implies that

$$\mathcal{D}^\alpha_v f(y) = \lim_{t\to 0_+}\int_0^1 (1-t)^{-\alpha}\frac{f(y+thv)-f(y)}{h^\alpha}\,dt.$$

As before, we only consider the case $0 < \alpha < 1$. Now we use a result due to Welland [17] which generalizes the result of Stein and Zygmund to $\mathbf{R}^n$.

Let $\beta = 1-\alpha$. Given a measurable function $f : \mathbf{R}^n \to \mathbf{R}$, the $\beta$-th integral $f_\beta$ for $f$ is now defined by

$$f_\beta(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\beta}}\, dy.$$

The function $f$ is said to have an $\alpha$-derivative at $x_0 \in \mathbf{R}^n$ if $f_\beta$ is differentiable at $x_0$. To characterise the existence of such $\alpha$-fractional derivatives, among other conditions, the following are stated in [17] which generalize the corresponding conditions in [16]:

(i) $f$ is said to satisfy $\Lambda_\alpha$ ($0<\alpha<1$ in our case) at $x_0$ if $|f(x_0+y)-f(x_0)| = \mathcal{O}(|y|^\alpha)$ as $y \to 0$;
(ii) $f$ is said to satisfy $N_\alpha^2$ ($0<\alpha<1$) at $x_0$ if for some $\rho>0$,

$$\int_{B(0,\rho)} \frac{|f(x_0+y)-f(x_0)|^2}{|y|^{2\alpha+n}}\, dy < +\infty.$$

Suppose $f$ satisfies condition $\Lambda_\alpha$ in a set $E \subset \mathbf{R}^n$ of positive measure, then $f$ has the $\alpha$-fractional derivative a.e. in $E$ if and only if $f$ satisfies $N_\alpha^2$ a.e. in $E$.

From the proof of Theorem 1, we have the following results in higher dimensional space. Let $S^{n-1} \subset \mathbf{R}^n$ be the unit sphere.

**Proposition 5.** *Suppose $\Omega \subset \mathbf{R}^n$ is an open set and $f \in C^\alpha(\Omega)$. If $f$ also satisfies $N_\alpha^2$ a.e. in $\Omega$, then for a.e. $y \in \Omega$ and a.e. $v \in S^{n-1}$,*

$$\lim_{h\to 0_+} \int_0^1 (1-t)^{-\alpha} \frac{f(y+thv)-f(y)}{h^\alpha}\, dt = 0. \tag{3.9}$$

**Proof.** Let $y \in \Omega$ be such that $N_\alpha^2$ holds at $y$. By Proposition 1, we only need to show that

$$\lim_{h\to 0_+} \int_{S^{n-1}} \int_0^h \left| \frac{f(y+\rho v)-f(y)}{h^{1+\alpha}} \right| d\rho\, dS = 0, \tag{3.10}$$

where $dS$ is the surface element of the sphere. By the fact that $y \in N_\alpha^2$ and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\int_{S^{n-1}} \int_0^h \left| \frac{f(y+\rho v)-f(y)}{h^{1+\alpha}} \right| d\rho\, dS &= \frac{1}{h^{1+\alpha}} \int_{S^{n-1}} \int_0^h \left| \frac{f(y+\rho v)-f(y)}{\rho^{n-1}} \right| \rho^{n-1} d\rho\, dS \\
&= \frac{1}{h^{1+\alpha}} \int_{B(0,h)} \frac{|f(y+x)-f(y)|}{|x|^{n-1}}\, dx \\
&= \frac{1}{h^{1+\alpha}} \int_{B(0,h)} \left( \frac{|f(y+x)-f(y)|}{|x|^{n/2+\alpha}} \right) \frac{1}{|x|^{n/2-1-\alpha}}\, dx \\
&\leqslant \frac{1}{h^{1+\alpha}} \left( \int_{B(0,h)} \frac{|f(y+x)-f(y)|^2}{|x|^{n+2\alpha}}\, dx \right)^{1/2} \left( \int_{B(0,h)} \frac{1}{|x|^{n-2-2\alpha}}\, dx \right)^{1/2} \\
&= \sqrt{\omega_{n-1}} \left( \int_{B(0,h)} \frac{|f(y+x)-f(y)|^2}{|x|^{n+2\alpha}}\, dx \right)^{1/2} \to 0,
\end{aligned}$$

as $h \to 0_+$ because the last term above goes to zero due to $N_\alpha^2$. The proof is finished. □

**Remark 6.** As Theorem 2 and Proposition 4 have shown, the existence of a non-trivial local fractional derivative is a lower dimensional feature of a function satisfying $\Lambda_\alpha$ and $N_\alpha^2$. Of course the integral defined in (3.10) captures the average modulus of directional fractional derivatives. However, the computation for such an integral could be complicated. Therefore we believe that in order to find the modulus of the $\alpha$-local fractional derivative numerically, the simple formula

$$\frac{1}{h^\alpha} \frac{1}{|D(0,r)|} \int_{D(0,h)} \left| f(y+x) - \frac{1}{|D(0,r)|} \int_{D(0,r)} f(y+z)\, dz \right| dx \tag{3.11}$$

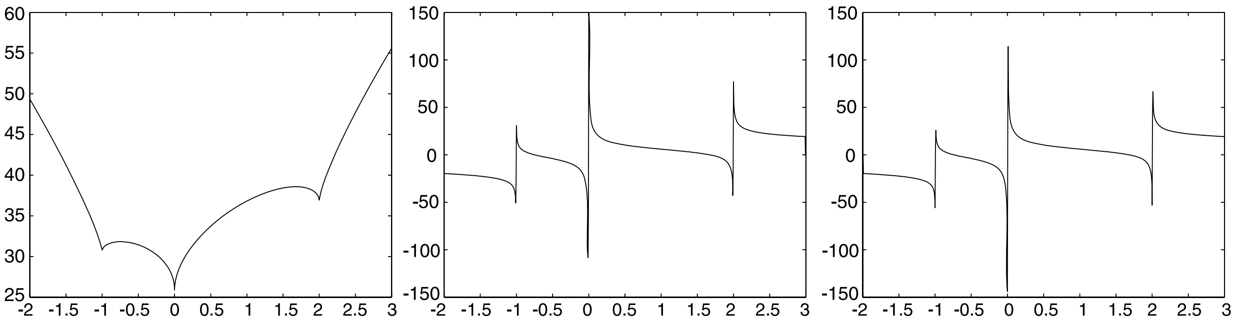


**Fig. 3.** $f(x) = 10(\sqrt{|x|} + |x-2|^{2/3} + |x+1|^{3/4})$ and right–left 1/2-IDQs.

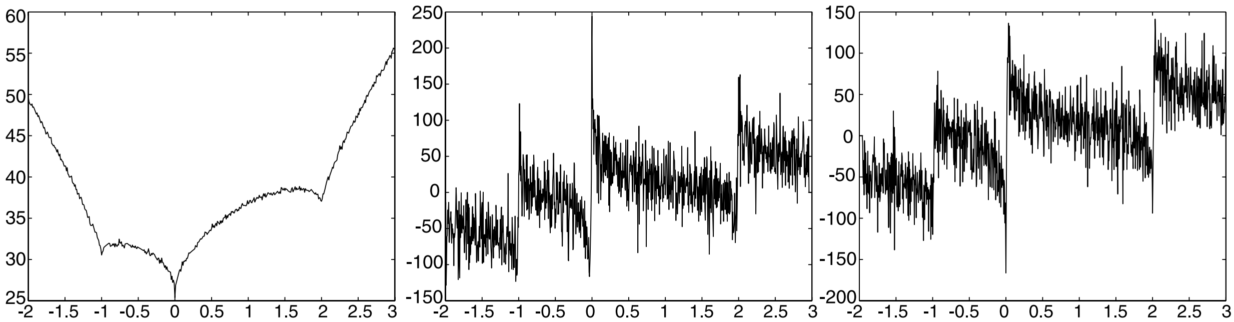


**Fig. 4.** $f(x)$ + small random noise and right–left 1/2-IDQs.

will serve our purpose well, partly due to the fact that if $y \in N_\alpha^2$, this quantity will go to zero. Note that by the approximate differentiation theorem due to Calderon and Zygmund [3,4], we have, when $\alpha = 1$ that

$$\lim_{h\to 0_+} \frac{1}{h} \frac{1}{|D(0,r)|} \int\limits_{D(0,h)} \left| f(y+x) - f(y) - \langle \nabla f(y), x\rangle \right| dx = 0$$

a.e. $y \in \Omega$ for any function in the Sobolev space $W^{1,1}(\Omega)$. Thus

$$\frac{1}{h} \frac{1}{|D(0,r)|} \int\limits_{D(0,h)} \left| f(y+x) - \frac{1}{|D(0,r)|} \int\limits_{D(0,r)} f(y+z)\,dz \right| dx$$

converges a.e. to a quantity proportional to $|\nabla f(y)|$.

## 4. Numerical approximations of IDQ-LFD

We conclude this paper by showing some test results of our numerical schemes for calculating the modulus of the integral difference-quotient LFDs based on formulas developed in this paper. Edge detections by using (global) fractional derivatives can be found in e.g. [11] and references therein.

In Figs. 3–5, we use the following approximations for a given scale $h > 0$ respectively,

$$\text{1/2-right IDQ-LFD:} \quad D_+^{1/2} f(y) \simeq \frac{1}{h^{3/2}} \int_0^h \big(f(y+t) - f(y)\big)\,dt, \quad h > 0,$$

$$\text{1/2-left IDQ-LFD:} \quad D_-^{1/2} f(y) \simeq -\frac{1}{h^{3/2}} \int_0^h \big(f(y-t) - f(y)\big)\,dt,$$

$$\text{modulus of 1/2-IDQ-LFD:} \quad \frac{1}{2}\big(\big|D_-^{1/2} f(y)\big| + \big|D_+^{1/2} f(y)\big|\big)$$

$$\simeq \frac{1}{2h^{3/2}} \left( \left| \int_0^h \big(f(y+t) - f(y)\big)\,dt \right| + \left| \int_0^h \big(f(y-t) - f(y)\big)\,dt \right| \right). \tag{4.1}$$

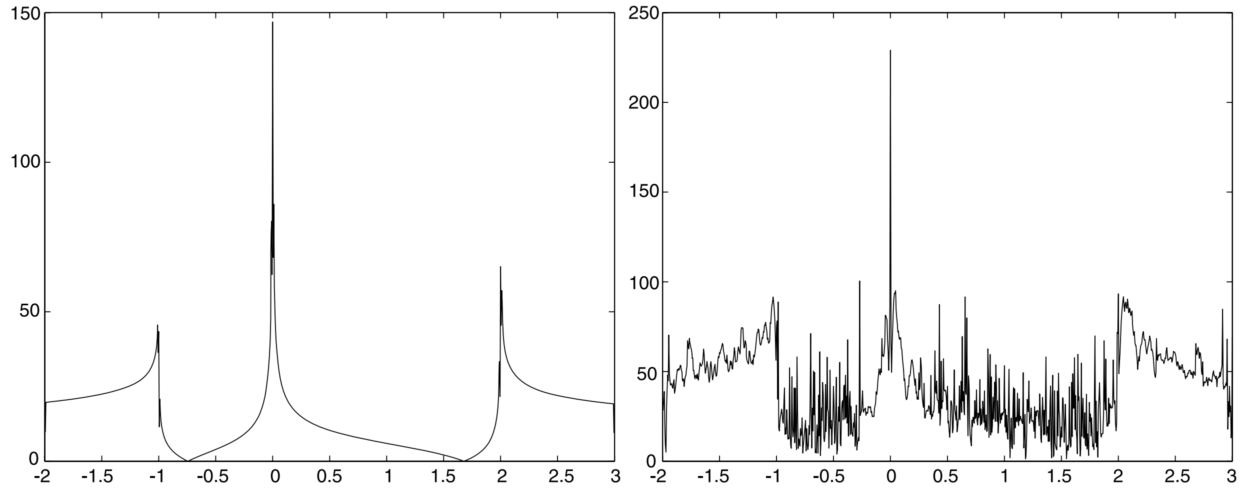


**Fig. 5.** The modulus $(|D_+^{1/2} f(y)| + |D_-^{1/2} f(y)|)/2$ for $f$ and $f(x) +$ small random noise.

We consider the function (see the left of Fig. 3)

$$f(x) = 10\sqrt{|x|} + 10|x-2|^{2/3} + 10|x+1|^{3/4}, \quad x \in [-2, 3].$$

Observe that $f$ has three non-smooth points at $-1$ with power $3/4$, at 2 with power $2/3$ and at 0 with power $1/2$. Fig. 4 is the half derivatives of $f(x) +$ a small random noise.

Fig. 5 gives the approximated modulus $(|D_+^{1/2} f(y)| + |D_-^{1/2} f(y)|)/2$ for the same function $f$ and the perturbed one. In all of these figures we see that IDQ-LFDs can pick up the strongest derivative at $x = 1/2$ even with an added small random noise.

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