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On the Lattice of Strong Radicals

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It is shown that the class of all strong radicals containing the prime radical is not a sublattice of the lattice of all radicals. This gives a negative answer to some questions of Sands and Puczylowski. © 1996 Academic Press, Inc.

1. INTRODUCTION

The fundamental definitions and properties of radicals can be found in [1, 5, 19, 20]. Note that in this paper the term ''radical'' means both ''radical class'' and ''radical property''.

Recall [6] that a radical α is *left (right) strong* if every left (right) α-ideal of any ring R belongs to $\alpha(R)$. A radical α is said to be *strong* if it is left and right strong.

It is well known that the class of all radicals forms a complete lattice with respect to set inclusion. In a number of papers [1–4, 7–17] different aspects of this lattice and some other lattices of radicals were investigated. The following questions of Sands and Puczylowski are still open.

(i) ([10, Question 2; 12, Question 1; 15; 16]). *Is the class of all strong radicals a sublattice of the lattice of all radicals*?

(ii) ([10, Question 3; 12, Question 1]). *Is the class of all left strong radicals containing the prime radical a sublattice of the lattice of all radicals*?

Our main result gives a negative answer to the above questions.

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THEOREM 1.1. *There exist strong radicals α and γ containing the prime radical such that the lower radical generated by $\alpha \cup \gamma$ is not strong.*

We shall make use of the following definitions and notations. For a set X, $|X|$ denotes its cardinality. If R is a ring then $I \triangleleft R$ ($I \triangleleft_l R$, $I < R$) will mean that I is an ideal (respectively, a left ideal, either a left or a right ideal but not both in general) of R. A subring A of a ring R is said to be accessible (one-sided accessible) if there exists a chain of subrings $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = R$ of R such that $A_i \triangleleft A_{i+1}$ ($A_i < A_{i+1}$) for $i = 0, 1, \ldots, n-1$. The prime radical is denoted by β.

We also consider the following question.

(iii) [10, Question 1] *If α is a right strong radical containing β and J is a right ideal of a α-semisimple ring R, does $\alpha(J) = \beta(J)$?*

The following theorem also gives a negative answer to Question (iii).

THEOREM 1.2. *There exist a strong radical α containing the prime radical β and an α-semisimple ring R with a nonzero right ideal J such that $\beta(J) = 0$ and $\alpha(J) \neq 0$.*

2. THE PROOFS OF THE MAIN THEOREMS

Given any nonempty class $\mathscr{M}$ of rings, we set $\mathscr{M}^1$ to be the class of all homomorphic images of rings from $\mathscr{M}$ and define $\mathscr{M}^\sigma$ to be the class of all rings R such that any nonzero homomorphic image R' of R contains a nonzero one-sided ideal in $\mathscr{M}^\tau$ for some ordinal $\tau < \sigma$. We set $LS(\mathscr{M}) = \cup_\sigma \mathscr{M}^\sigma$. According to [6, Theorem 2], $LS(\mathscr{M})$ is a strong radical and it is the smallest strong radical containing $\mathscr{M}$.

Let $\mathscr{Z}$ be the ring of integers and $R = M_2(\mathscr{Z})$ the (2×2)-matrix ring over $\mathscr{Z}$. Further, let $\{e_{ij} | 1 \leq i, j \leq 2\}$ be the set of matrix units of R. We set

$$I = 2R, \qquad J = e_{11}R + I \qquad \text{and} \qquad A = e_{12}\mathscr{Z} + I.$$

Clearly

$$I \triangleleft R, \qquad J \triangleleft_r R, \qquad \text{and} \qquad A \triangleleft J.$$

Let F be a two-element field. We put

$$\alpha = LS(\beta \cup \{A\}) \qquad \text{and} \qquad \gamma = LS(\beta \cup \{F\}).$$

By the above observation α and γ are strong radicals containing β. Define δ to be the lower radical generated by $\alpha \cup \gamma$. Clearly $A, F \in \delta$. Since $J/A \cong F$, $\delta(J) = J$. Therefore in order to prove Theorem 1.1 it is enough to show that $\delta(R) = 0$. We start with the following technical remarks which play an important role in the proof of our main results.

Remark 2.1. For any nonzero ideal K of A, $|A/K| < \infty$.

Proof. Since $(R, +)$ is torsion-free and $I = 2R$, we conclude that I has a nonzero intersection with any nonzero subring of R. Since $K \triangleleft A$ and $I \triangleleft A$, $L = K \cap I \triangleleft I$. But $I \triangleleft R$. By the Andrunakievich Lemma, $M = (RLR)^3 \subseteq L$. According to the above observation, M is a nonzero ideal of $R = M_2(Z)$. Therefore there exists a natural number n such that $M = nR$. Hence $|R/M| = n^4 < \infty$ and $|A/M| < \infty$. Since $M \subseteq L \subseteq K$, $|A/K| \leq |A/M| < \infty$.

Note that R is a free abelian group of rank 4. Any subring R' of R being a subgroup is again a free abelian group. We denote by $\mathrm{rk}(R')$ the rank of R'. Let $S = \mathscr{Z} \setminus \{0\}$. Clearly the localization $S^{-1}R$ of the ring R relative to the multiplicatively closed set S is the ring $M_2(Q)$ where Q is the rational number field. Note that $S^{-1}R'$ is a Q-subalgebra of $M_2(Q)$ and

$$\dim_Q(S^{-1}R') = \mathrm{rk}(R'). \tag{1}$$

Remark 2.2. Let $H \subseteq R$ be a subring and U a nonzero δ-radical one-sided ideal of H. Suppose that $\mathrm{rk}(H) = 4$. Then $\mathrm{rk}(U) = 4$ and $\beta(U) = 0$.

Proof. Without loss of generality we may assume that $U \triangleleft_r H$. Clearly $S^{-1}U \triangleleft_r S^{-1}H$. Since $\mathrm{rk}(H) = 4 = \dim_Q(S^{-1}H)$, $S^{-1}H = M_2(Q)$ (see (1)). Hence either $\dim_Q(S^{-1}U) = 2$ or $\dim_Q(S^{-1}U) = 4$. In the first case $S^{-1}U = eM_2(Q)$ for some idempotent e of rank 1. Clearly $eM_2(Q) = eM_2(Q)e + eM_2(Q)(1 - e) = eQ + eM_2(Q)(1 - e)$. Therefore $eM_2(Q)/\beta(eM_2(Q)) \cong Q$. Obviously $\beta(U) = \beta(S^{-1}U) \cap U$. Hence $U/\beta(U)$ is isomorphic to a subring of Q. In particular, the ring $T = U/\beta(U)$ is an infinite commutative domain. Hence any one-sided accessible subring of T is infinite. Since $U \in \delta$, $T \in \delta$. Therefore T contains a nonzero accessible subring from $\alpha \cup \gamma$ (see [18]). Suppose that T contains a nonzero accessible subring from γ. By [6, Lemma 3], T contains a nonzero one-sided accessible subring from $\beta \cup \{F\}$. Being a domain, T does not contain a β-radical subring. By the above result, it does not contain any finite one-sided accessible subrings. We get a contradiction. Now taking into account Remark 2.1 we conclude that T contains an isomorphic copy of A. But A is not commutative, a contradiction. Therefore $\dim_Q(S^{-1}U) = 4$ and $\mathrm{rk}(U) = 4$ (see (1)). Since

$$S^{-1}\beta(U) = \beta(S^{-1}U) = \beta\big(M_2(Q)\big) = 0,$$

$\beta(U) = 0$.

Remark 2.3. Suppose that $\delta(R) \neq 0$. Then there exists a chain of subrings

$$R = U_0 \supset U_1 \supset \cdots \supset U_n = V_0 \supset V_1 \supset \cdots \supset V_m$$

such that:

(1) $U_1 \triangleleft U_{i-1}$ for all $i = 1, 2, \dots, n$;
(2) $U_i \in \delta$ for all $i = 1, 2, \dots, n$;
(3) $V_j \in \alpha$ for all $j = 0, 1, \dots, m$;
(4) V_{j+1} is a onesided ideal of V_j for all $j = 1, 2, \dots, m-1$;
(5) $V_m \cong A$.

Proof. Taking $U_1 = \delta(R)$, we infer from [18, p. 418] that there exists a chain

$$R = U_0 \supset U_1 \supset \cdots \supset U_n$$

of nonzero subrings with properties (1) and (2) and with $U_n \in \alpha \cup \gamma$. Suppose now that $U_n \in \gamma$. We set $\mathscr{N} = \beta \cup \{F\}$. Then there exist an ordinal number σ_1 and a nonzero one-sided ideal V_1 of $V_0 = U_n$ such that $V_1 \in \mathscr{N}^{\sigma_1}$. By The definition of the class $\mathscr{N}^{\sigma_1}$ there exist an ordinal number $\sigma_2 < \sigma_1$ and a nonzero one-sided ideal V_2 of V_1 such that $V_2 \in \mathscr{N}^{\sigma_2}$. Since any strictly descending chain of ordinals has to be finite, we obtain a chain of subrings $U_n = V_0 \supset V_1 \supset \cdots \supset V_m$ with Property (4) and with V_m belonging to the homomorphic closure of $\beta \cup \{F\}$. Since $\gamma \subseteq \delta, V_j \in \delta$ for all $j = 0, 1, \dots, m$. It follows from Remark 2.2 that $\text{rk}(U_i) = 4 = \text{rk}(V_j)$ and $\beta(U_i) = 0 = \beta(V_j)$ for all i, j. Hence $V_m \cong F$ which is impossible since $|F| = 2$ and R is a free abelian group. Therefore $U_n \in \alpha$. Then as above, one may construct a chain of subrings $U_n = V_0 \supset V_1 \supset \cdots \supset V_m$ with properties (3), (4) and V_m belonging to the homomorphic closure of $\beta \cup \{A\}$. Taking into account Remark 2.1 and the equality $\beta(V_m) = 0$, we conclude that $V_m \cong A$.

Remarks 2.4. Let $\varphi\colon A \to R$ be a monomorphism of rings. Then there exists an invertible element $r \in R$ such that $r\varphi(A)r^{-1} = A$.

Proof. Note that the center $C(A)$ equals $2\mathscr{Z}$. Since $\text{rk}(\varphi(A)) = \text{rk}(A) = 4$, $S^{-1}\varphi(A) = M_2(Q)$. Hence $\varphi(C(A)) \subseteq C(M_2(Q))$ and $\varphi(C(A)) \subseteq C(R) = \mathscr{Z}$. Therefore $\varphi\colon C(A) \to \mathscr{Z}$. Since $\varphi(2)^2 = \varphi(4) = 2\varphi(2)$, $\varphi(2) = 2$.

We set

$$u_{11} = \varphi(2e_{11}),\ u_{12} = \varphi(e_{12}),\ u_{21} = \varphi(2e_{21}), \text{ and } u_{22} = \varphi(2e_{22}).$$

Note that $u_{11} + u_{22} = \varphi(2) = 2$. Consider a free abelian group M of rank 2. Clearly $\text{End}(M) = R$. We set $U = u_{11}M$ and $V = u_{22}M$. Since $u_{11}u_{22} = 0$ and $u_{11}^2 = 2u_{11}$, $U \cap V = 0$. Hence $\text{rk}(U) = 1 = \text{rk}(V)$.

Suppose that M/U is a torsion free group. Then M/U is a free abelian group. Hence $M = U \oplus H$ for some free rank 1 subgroup H of M. Since $\mathrm{rk}(U) = 1 = \mathrm{rk}(H)$, $U = \mathscr{Z}f$ and $H = \mathscr{Z}h$ for some $f \in U$ and $h \in H$. Recalling that $U = u_{11}M$, we infer that $f = u_{11}m$ for some $m \in M$ and $u_{11}f = u_{11}^2 m = 2u_{11}m = 2f$. Clearly $u_{11}h = tf$ for some $t \in \mathscr{Z}$. Since $f \in \mathscr{Z}u_{11}f + \mathscr{Z}u_{11}h$, we conclude that t must be odd, say, $t = 2k + 1$. We set $g = h - kf$ and $W = \mathscr{Z}g$. Then $u_{11}g = (2k+1)f - 2kf = f$ and therefore, in the basis $\{f, g\}$ we have $u_{11} = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since $u_{11} + u_{22} = 2$, $u_{22} = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 2 \end{smallmatrix}\right)$. Let $u_{12} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for some $a, b, c, d \in \mathscr{Z}$. Then

$$\begin{pmatrix} 4a & 4b \\ 4c & 4d \end{pmatrix} = 4u_{12} = u_{11}u_{12}u_{22} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}$$

and hence $u_{12} = \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)$. Analogously one can easily check that $u_{21} = \left(\begin{smallmatrix} 2c & c \\ -4c & -2c \end{smallmatrix}\right)$ for some $c \in \mathscr{Z}$. Since $u_{12}u_{21} = u_{11}$, we infer that $-2bc = 1$, a contradiction.

By the above result, there exist an element $f_1 \in M \setminus U$ and a number $s \neq 0$ such that $sf_1 \in U$. Since $u_{22}u_{11} = 0$, $u_{22}U = 0$ and $su_{22}f_1 = 0$. Hence $u_{22}f_1 = 0$. Now we have $2f_1 = (u_{11} + u_{22})f_1 = u_{11}f_1 \in U$. Therefore $2f_1 \in U$. Clearly $U' = \mathscr{Z}f_1 + \mathscr{Z}f$ is a subgroup of M of rank 1. Hence U' is an infinite cyclic group. Since $U' \supset U$ and $2U' \subseteq U$, we conclude that $U = 2U'$. Obviously $U' = \mathscr{Z}w_1$ for some $w_1 \in M$. Analogously one can find an element $w_2 \in M \setminus V$ such that $V = 2V'$ where $V' = \mathscr{Z}w_2$. Since $2U' = U$ and $2V' = V$, we conclude that $U' \cap V' = 0$. Then $M/(U \oplus V) \supseteq U'/U \oplus V'/V$ and $|M/(U \oplus V)| \geq 4$. On the other hand, we have that $U + V \supseteq (u_{11} + u_{22})M = 2M$ and $|M/(U \oplus V)| \leq |M/(2M)| = 4$. Therefore $|M/(U \oplus V)| = 4$, $M = U' \oplus V'$, and $2M = U \oplus V$. Define the endomorphisms $\{v_{ij} \mid 1 \leq i, j \leq 2\}$ of M by the rule $v_{ij}w_p = \delta_{jp}w_i$ (where δ_{jp} is Kronecker symbol) for all i, j, p. Clearly $u_{ii} = 2v_{ii}$ for $i = 1, 2$. As

$$4u_{12} = u_{11}u_{12}u_{22} = (2v_{11})u_{12}(2v_{22}) = 4v_{11}u_{12}U_{22},$$

we see that

$$u_{12} = v_{11}u_{12}v_{22} \in v_{11}Rv_{22} = v_{11}M_2(Z)v_{22} = Zv_{12}$$

and so $u_{12} = bv_{12}$ for some $b \in Z$. Analogously one can show that $u_{21} = cv_{21}$ for some $c \in Z$. As above one may show that $u_{12} = bv_{12}$ and $u_{21} = cv_{21}$ for some $b, c \in \mathscr{Z}$. Since $u_{12}u_{21} = u_{11}$, we have $bc = 2$. Only the following cases are possible:

Case 1. $(b, c) = (1, 2)$. Then we set $r = v_{11} + v_{22}$.

Case 2. $(b, c) = (-1, -2)$. We let $r = v_{11} - v_{22}$.

Case 3. $(b,c) = (2,1)$. We put $r = v_{12} + v_{21}$.

Case 4. $(b,c) = (-2,-1)$. Then we set $r = v_{12} - v_{21}$.

Since

$$\varphi(A) = \sum_{i,j=1}^{2} \mathscr{Z}u_{ij} = \mathscr{Z}2v_{11} + \mathscr{Z}bv_{12} + \mathscr{Z}cv_{21} + \mathscr{Z}2v_{22},$$

$r\varphi(A)r^{-1} = A$.

Proof of Theorem 1.1. Suppose that $\delta(R) \neq 0$. Then by Remark 2.3 there exists a chain

$$R = U_0 \supset U_1 \supset \cdots \supset U_n = V_0 \supset V_1 \supset \cdots \supset V_m$$

of subrings of R with Properties (1)–(5). Taking into account Remark 2.4, we can assume without loss of generality that $V_m = A$. Recall that $I = 2R \subset A$. Since $R/I \cong M_2(F)$ is a simple ring, we have $R = U_0 = U_1 = \cdots = U_n$. According to Property (3), $R = U_n = V_0 \in \alpha$. Hence $R/2R \in \alpha$ and $M_2(F) \in \alpha$. Note that $A^2 \subseteq 2R$ and $A^4 \subseteq 4R \subseteq 2A$. Therefore if B is a homomorphic image of A and $2B = 0$, then $B^4 = 0$. Hence $M_2(F)$ does not belong to the homomorphic closure of the class $\beta \cup \{A\}$. Now the inclusion $M_2(F) \in \alpha$ implies that there exists a proper one-sided ideal L of $M_2(F)$ belonging to α. Then $L/\beta(L) \cong F$. Hence $F \in \alpha$. F, being a field, has no proper nonzero one-sided accessible subrings. Therefore F is a homomorphic image of A which is impossible by the above result, a contradiction. Thus $\delta(R) = 0$ and the theorem is proved.

Proof of Theorem 1.2. By the above result $\delta(R) = 0$. Since $\alpha \subseteq \delta$, $\alpha(R) = 0$. Clearly $J = e_{11}R + I \lhd_r R$ and $A \lhd J$. Since $A \in \alpha$, $\alpha(J) \neq 0$. From the inclusion $J \supset I$, we infer that $\beta(J) = 0$ and the theorem is proved.

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