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GROUPS OF SMALL MORLEY RANK*

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1. Introduction

1.1. Recently attention has been devoted to problems in "applied categoricity theory", i.e. the attempt to classify first order axiom systems whose models admit a structure theory in one or another sense. In connection with the classification of $\aleph_1$-categorical theories the main model-theoretic tool has been:

Fact. (Morley [9], Baldwin [1]). *If the theory* T *is $\aleph_1$-categorical, then it is ω-stable of finite Morley rank.*

General categoricity and stability theory is developed in the text [11] and summarized at length in the introduction to [4]. (See also the expository article [12] and the more advanced text [13].)

The present article is concerned primarily with groups of Morley rank at most three. (As usual we say a structure is $\aleph_1$-categorical, stable, of Morley rank n, etc. iff its complete theory has the given property. In addition for us the phrase "Morley rank n" is synonymous with "Morley rank n and ω-stable".) The most interesting groups in this class are the groups PSL(2, K) over an algebraically closed field K. Of course any algebraic group over an algebraically closed field has finite Morley rank. I show as a weak converse:

Theorem. *If G is a nonsolvable group of Morley rank* 3 *with a definable subgroup of Morley rank* 2, *then the center Z of G is finite and*:

$$G/Z \simeq \mathrm{PSL}(2, K)$$

for some algebraically closed field K.

A nonsolvable group of Morely rank 3 which contains no definable subgroup of Morley rank 2 will be called a *bad* group – I expect none such exist [10]. In Section 5.1 it will be shown that bad groups satisfy some fairly stringent algebraic

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conditions, but the existence question is open, and seems very difficult. It will, however, be seen that a bad group is never locally finite.

1.2. The organization of this paper is as follows. Section 2 develops some general machinery for the analysis of groups of finite Morley rank, the main result being the Indecomposability Theorem of Section 2.2. Sections 3, 4, and 5 deal respectively with groups of Morley ranks 1, 2, and 3. Section 5.1 concerns good groups and Section 5.2 concerns bad groups. The results of Sections 4.1 and 4.2 (but not 4.3) are needed for 5.1.

1.3. It should be noted that Reineke pointed out several years ago that a group of Morley rank one and degree one is abelian. This is the basis for the inductive analysis of groups of small Morley rank.

1.4 (Added January 1978). Zil'ber has an interesting article closely connected with the work presented here [16], which was carried out independently. The connections and chronology are as follows.

Call a theory *weakly categorical* iff it is interpretable in an $\aleph_1$-categorical theory. Zil'ber proves a theorem equivalent to our Theorem 2.9 for the special case of weakly categorical groups [16, Theorem 3.2]. Our methods are the same, although Zil'ber uses types where I use definable sets and invokes the proof of Baldwin's Finiteness Theorem, as he does not have our Lemma 5 (cf. [16, Theorem 3.2]). He then observes that an argument of Reineke proves our Theorem 3.2 (via the corresponding special case of Theorem 3.1). His paper contains a number of significant results on weakly categorical groups and rings unrelated to the work described here. In particular he gives an elegant proof that a simple algebraic group over an algebraically closed field is $\aleph_1$-categorical [16, Corollary to Theorem 3.2]. I had observed (Fall 1976) that this result can be obtained easily, but at some length, from the known structure theory for such groups (using a good deal of [4] and the generators and relations of [14]). Zil'ber's proof is short and uses no structure theory.

Zil'ber's results were obtained in 1972, submitted for publication in April 1974, and published in 1977. Mine were obtained in August 1976 and were the subject of an informal seminar at the University of Hannover in June 1977.

1.5 (September 1978). Shelah has extended the Indecomposability Theorem (Theorem 2.9) to stable groups. It is likely that the results of this paper apply also to superstable groups of Shelah degree at most 3. This is discussed in [18].

2. Groups of finite Morley rank

2.1. *Ultra Morleyization*

In the inductive analysis of structures of finite Morley rank one deals frequently with the rank of substructures, quotients, and products. This creates certain formal complications. For example a definable subgroup H of a group G may have lower Morley rank as a group in its own right than as a definable subset of G (e.g. H may be abelian but have nontrivial G-conjugacy classes). The situation is even worse with respect to quotients of G by arbitrary equivalence relations. The simplest way to eliminate these problems is to introduce the *ultra Morleyization* of a structure (or else to ignore the issue).

Definition 1. Let M be a structure, θ an equivalence relation on M.

(1) M is *ultra Morley* iff for every definable relation ρ (whose definition may involve parameters from M) there is a relation symbol R such that for $a_1, \ldots, a_n \in M$: $\rho a_1 \cdots a_n$ iff $M \models Ra_1 \cdots a_n$.

(2) The *full quotient* $M//\theta$ is defined as the structure whose universe is the set of equivalence classes of θ and whose relations are exactly the relations induced on this set by the parameter definable relations on M whose truth value is constant on equivalence classes of θ.

Remark 2. Let M be a structure, θ a definable equivalence relation on M. Then:

(1) $M//\theta$ is ultra Morley.

(2) If M is ultra Morley, then every definable substructure of M is ultra Morley.

(3) Every structure has a natural ultra Morley expansion.

Throughout this article it is a tacit assumption that all structures are ultra Morley. The effect of this should perhaps be illustrated by an example. Let H be an elementary abelian p-group with p an odd prime and let $H = K \oplus L$ be decomposed as a direct sum in some arbitrary but fixed way. Let α be the automorphism of H fixing K pointwise and sending the elements of L to their inverses. Since α has order two we can form the semidirect product

$$G = H \rtimes \mathbb{Z}_2$$

in which the generator a of $\mathbb{Z}_2$ acts like α on H.

When H is considered as an abstract group the subgroups K, L are in general not definable. However, when H arises as a definable subgroup of G, then we naturally consider K and L definable (i.e. G-definable). One way of viewing this situation is to say that H as a subgroup of G comes equipped with the automorphism α which is "inherited" from G, and the use of the ultra Morleyization makes this manner of speaking rigorous.

2.2 *The degree of a group*

In this subsection we define connectedness for groups of finite Morley rank. If G has finite Morley rank n, then G has a degree d satisfying $1 \leq d < \infty$ (the *Morley* degree [11]). We prove that d is the index of the connected component of G in G. The rest of this paper is based on repeated applications of this fact. For the duration of this subsection we will work in the context of ω-stable groups G of Morley rank α (possibly infinite).

Definition 1. A group G is *connected* iff it has no definable proper subgroup of finite index.

This notion was introduced by Macintyre [3], following the lead of Kegel and Wehrfritz [6]. If G is an algebraic group (over an algebraically closed field) it is close to the classical notion. In one direction, if G is a connected algebraic group and H is a definable subgroup of finite index, then the elimination of quantifiers for algebraically closed fields means that H is constructible, hence closed [5, p. 54]. Since H has finite index, if H is unequal to G, then H disconnects G. Thus $H=G$ and G is connected in the above sense. The converse is true under a somewhat jesuitical interpretation. Suppose G is an algebraic group, connected in the above sense, where G is viewed as a subset of one (if G is affine) or more (in general) copies of K^n, K the algebraically closed base field. Here K^n carries all subvarieties as definable subsets, and G inherits the corresponding structure, if K^n is ultra Morleyized (nota bene). The connected component G° of G being closed, it is definable, and being of finite index in G, it equals G. Thus G is also connected as an algebraic group.

In our present context we deal with abstract groups, and connectedness is always understood in the above sense (and is therefore meaningful only for groups).

Definition 2. Let G be a group, H a connected definable subgroup of G having finite index in G. Then we call H the *identity component* of G, and write $H=G^\circ$.

The notation is justified by the following fact proved in [3]:

Fact 3. *If G is an ω-stable group, then G has a unique identity component, a characteristic subgroup of G.*

The uniqueness of G° is straightforward, as is the proof that it is characteristic in G. The existence comes from the descending chain condition on definable subgroups of ω-stable groups, which goes back to [7].

Before we can prove that the degree of an ω-stable group G equals the index of G° in G we need three technical lemmas: a straightforward result on invariance of rank and two weak results on the definability of rank.

Lemma 4. *Let $M<M^*$ be two structures. Let $A\subseteq M$, $B\subseteq M^*$ be definable sets (we do not assume A is M^*-definable), with $A\subseteq B$. Then* rank $A\leq$ rank B *(if A is not ω-stable this means B is not ω-stable).*

Proof. Suppose rank $A \geq \alpha$. We prove that rank $B \geq \alpha$ by induction on α, the only potentially nontrivial case being $\alpha = \beta + 1$, a successor. We may suppose that M, M^* are saturated sufficiently so that Morley rank equals Cantor-Bendixson rank. Then for $d \geq 1$ an integer, partition A into definable subsets $A_1, \ldots, A_d$ of rank $\geq \beta$. Define $B_i = A_i^* \cap B$, where A_i^* is the canonical extension of A_i to M^*. Then B contains the disjoint union of $B_1, \ldots, B_d$ and rank $B_i \geq \beta$. Since d is arbitrary, rank $B \geq \alpha$.

Our next result is a group-theoretic interpretation of Morley rank.

Lemma 5. *Let G be an ω-stable group of Morley rank α and degree d and let $X \subseteq G$ be a definable subset. Then the following are equivalent*:

(1) rank $X < \alpha$

(2) *for some $l \geq 1$ there are elements $g_{ki} \in G$ $(k \leq l,\ i \leq d+1)$ and definable subsets $X_k \subseteq X$ $(k \leq d)$ satisfying*

(a) $X = \bigcup_k X_k$ *and for* $k \leq l$, $i \neq j$:

(b) $g_{ki}X_k \cap g_{kj}X_k = \emptyset$.

Proof. Assume that $l \geq 1$, $g_{ki} \in G$, $X_k \subseteq X$ as in (2) are given. By (2(b)) G contains the disjoint union of the sets

$$\{g_{ki}X_k : i \leq d+1\}$$

for each k. Since $\deg G = d$ it follows that rank $X_k < \alpha$, hence by (2(a)) rank $X < \alpha$.

Now suppose conversely that rank $X < \alpha$. Assume that the language of G contains a name for each element of G and let T be the complete theory of G. Let a be a new constant symbol and extend T by the following axioms (the first axiom actually follows from the second group of axioms):

(1) $a \in X$,

(2) for each $(d+1)$-tuple $g_1, \ldots, g_{d+1}$ in G:

$$\bigvee_{i \neq j} (g_i^{-1} g_j a \in X).$$

We will now show that this axiom system is inconsistent. Suppose on the contrary that G^* is a model, and choose a maximal set $g_1, \ldots, g_r$ in G so that

$$g_i^{-1} g_j a \notin X \quad \text{for} \quad i \neq j, \quad i,j \leq r.$$

Then $r \leq d$ and for all g in G there is an $i \leq r$ with

$$g_i^{-1} g a \in X \quad \text{or} \quad g^{-1} g_i a \in X,$$

hence

$$g \in g_i X a^{-1} \quad \text{or} \quad g \in g_i a X^{-1}.$$

Thus

$$G \subseteq \bigcup_{i \leq r} (g_i X^* a^{-1} \cup g_i a (X^*)^{-1}),$$

a set of rank less than α in G^*, contradicting Lemma 4.

Since the above axiom system is inconsistent there are $(d+1)$-tuples $(g_{kl}, \ldots, g_{kd+1}) \in G^{d+1}$ $(k \leq l < \infty)$ such that G satisfies:

for all $a \in X$ there is a $k \leq l$ such that for
$i \neq j (i, j \leq d+1)$ $\quad g_{ki}^{-1} g_{kj} a \notin X$.

Then set:

$$X_k = \{a \in X\text{: for } i \neq j (i, j \leq d+1) g_{ki}^{-1} g_{kj} a \notin X\}.$$

Then for $i \neq j (i, j \leq d+1)$ $\quad g_{kj} X_k \cap g_{ki} X = \emptyset$, as desired.

Definition 6. Let $\phi(x; y_1, \ldots, y_r)$ be a formula defined in a group G of degree d. For $l \geq 1$ define a formula ϕ/l by

$$(\phi/l)(y_1, \ldots, y_r) = \text{“}\exists\, g_{11} \cdots g_{l,d+1} \text{ such that}$$

$$\{x \mid \phi(x; y_1, \ldots, y_r)\} =$$

$$\bigcup_{k \leq l} \{x \mid \phi(x; y_1, \ldots, y_r) \;\&\; \forall i \neq j \neg \phi(g_{ki}^{-1} g_k x; y_1, \ldots, y_r)\}\text{”}.$$

Remark 7. If $\phi(x; \bar{y})$ is defined in an ω-stable group G, then the following are equivalent for $\bar{g}$ in G:

(1) rank $\phi(x; \bar{g}) <$ rank G.

(2) $\bigvee_{l < \infty} (\phi/l)(\bar{g})$.

This is essentially a restatement of Lemma 5.

Lemma 8. *Let G be an ω-stable group of Morley rank α and degree d, and suppose the formula $\phi(x; y_1, \ldots, y_r)$ is defined in G. Define a relation R on G by*

$$R\bar{y} \equiv [\text{rank } \phi(x; \bar{y}) < \alpha].$$

Then R is first order definable over G.

Proof. Write $G = A_1 \cup \cdots \cup A_d$ with rank $A_i = \alpha$, degree $A_i = 1$. Define

$$R_i \bar{y} \equiv [\text{rank “}\phi(x; \bar{y}) \;\&\; x \in A_i\text{”} < \alpha].$$

Then $R\bar{y} \equiv \bigvee_i R_i \bar{y}$. It suffices to show that the relations R_i are G-definable. Fix i.

We will write $\psi(x; \bar{y}) =$ “$\phi(x; \bar{y}) \;\&\; x \in A_i$” and $-\psi(x; \bar{y}) =$ “$\neg \phi(x; \bar{y}) \;\&\; x \in A_i$”. Let T be the complete theory of G (with all elements of G named), let

$a_1, \ldots, a_r$ be new constant symbols, and extend T by the axioms:
(1) $\neg[(\psi/l)(\bar{a})], l<\infty$,
(2) $\neg[(-\psi/l)(\bar{a})], l<\infty$.

Since A_i has degree 1, Remark 7 shows that the resulting axiom system is inconsistent. It follows that for l sufficiently large, say for $l \geq l_0$, G satisfies

$$\forall \bar{y} \quad [(\psi/l)(\bar{y}) \vee (-\psi/l)(\bar{y})].$$

Thus $R_i\bar{y}$ iff $(\psi/l)(\bar{y})$ $(l > l_0)$.

Now we can compute the degree of an ω-stable group.

Theorem 9. *Let G be an ω-stable group of Morley degree d and identity component G°. Then $d=[G:G^\circ]$.*

Proof. It suffices to show that degree $G^\circ = 1$, so we may take G to be connected. Write G as the disjoint union of d definable sets A_i of degree 1 and rank α ($=$rank G). Define an action of G on $\{1, \ldots, d\}$ by the formula:

$$\text{rank}\,(gA_i \cap A_{gi}) = \alpha.$$

Since degree $A_i = 1$ this is well-defined and is an action. Applying Lemma 8 to the formulas

$$\phi_{ij}(x; g) = \text{“}x \in gA_i \cap A_j\text{”}$$

we see that this action is definable over G. In particular its kernel is a definable subroup of finite index in G, hence equal to G since G is connected. This means that

$$\text{rank}\,(gA_i \cap A_i) = \alpha$$

for $g \in G$, $i \leq d$, and hence also

$$\text{rank}\,(gA_i \cap A_j) < \alpha$$

when $i \neq j$. Similarly rank $(A_i g \cap A_j) < \alpha$ for $i \neq j$.

Now let T be the complete theory of G (with all elements of G named). Let a be a new constant symbol and extend T by the axioms:

$$ga \in A_1 \quad \text{for each} \quad g \in G.$$

This axiom system is consistent, for given $g_1, \ldots, g_k \in G$ we have

$$\text{rank}\, g_1^{-1}A_1 \cap \cdots \cap g_k^{-1}A_1 = \alpha$$

and choosing $a \in \bigcap_i g_i^{-1}A_1$ we have

$$g_1 a \in A_1 \;\&\; \cdots \;\&\; g_k a \in A_1.$$

Thus there is a model $G^* > G$ which contains an element a satisfying

$$Ga \subseteq A_1^*, \quad \text{so} \quad G \subseteq A_1^* a^{-1}.$$

If $d > 1$, then in particular $A_2 \subseteq A_1^* a^{-1} \cap A_2^*$, a definable subset of G^* of Morley rank less than α. This contradicts Lemma 4, proving that $d = 1$, as desired.

2.3. *Additivity of rank*

In this subsection we give an incomplete analysis of the behavior of Morley rank under passage to full quotients (Section 2.1, Definition 1(2)). The following result is adequate for our purposes. (A fuller discussion will appear in [13, Chapter V.7].)

Theorem 1. *Let θ be a definable equivalence relation on a structure M of rank n. Suppose the equivalence classes of θ all have equal Morley rank m and the full quotient $M//\theta$ has Morley rank k, with m, k finite. Suppose M is $\aleph_1$-saturated. Then*

(1) $n \geq m + k$,

(2) *If either of the following conditions is satisfied, then $n = m + k$:*

(a) $m \leq 1$,

(b) *M is a group, the equivalence classes of θ are the right cosets of a definable subgroup H of M, and M has a definable subgroup of every rank $\leq n$.*

Proof. (1). We proceed by induction on k, the case $k = 0$ being trivial. If $k = l + 1$ and $d \geq 1$ is any integer we may find disjoint subsets $B_1, \ldots, B_d$ of $M//\theta$ of rank l. Let $A_1, \ldots, A_d$ be the inverse images of $B_1, \ldots, B_d$ under θ. Then rank $A_i \geq m + l$ and $A_1, \ldots, A_d$ are disjoint. Since d is arbitrary rank $M \geq m + l + 1 = m + k$, as desired.

(2) Assume hypothesis (2(a)) or (2(b)). We prove $n \leq m + k$ by induction on k, the case $k = 0$ being trivial. We treat the case rank $M//\theta = k = l + 1$.

As a preliminary reduction we may assume $\deg M//\theta = 1$. Indeed in case (2(a)) write $M//\theta = A_1//\theta \cup \cdots \cup A_d//\theta$ with A_i closed under θ and rank $A_i//\theta = k$, degree $A_i//\theta = 1$. If rank $A_i = m + k$, then rank $M = m + k$. So we might as well assume that $M = A_i$, $\deg M//\theta = 1$. In case (2(b)) it suffices to replace M by M°. Suppose toward a contradiction that $n > m + k$. If $A \subseteq M$ is any definable subset closed under θ, then either

$$\text{rank } A//\theta < k \quad \text{and} \quad \text{rank } A = m + \text{rank } A//\theta < m + k \leq n - 1 \tag{1}$$

or

$$\text{rank } A//\theta = k \quad \text{and} \quad \text{rank } (M - A)//\theta < k, \quad \text{rank } (M - A) \leq n - 2, \quad \text{rank } A = n. \tag{2}$$

Fix $A_1, A_2, \ldots$ disjoint definable subsets of M of rank $n - 1$. Let A_i^θ be the closure of A_i under θ. Since rank $A_i^\theta \geq n - 1$, rank $A_i^\theta//\theta = k$, rank $A_1^\theta \cap \cdots \cap A_l^\theta//\theta = k$ for all l, and thus rank $A_1^\theta \cap \cdots \cap A_l^\theta = n$.

Suppose hypothesis (2(a)) applies. If $m=0$, then since M is $\aleph_1$-saturated, the degrees of the equivalence classes of θ are bounded by a fixed $d<\infty$. But $A_1^\theta \cap \cdots \cap A_{d+1}^\theta \neq \emptyset$, a contradiction. Suppose then that $m=1$. Look for an element $a \in M$ whose equivalence class a^θ satisfies:

$$a^\theta \cap A_i \quad \text{is infinite for each } i.$$

If this type is realized in M, then a^θ has rank 2, a contradiction. Hence this type is inconsistent, and for l large M satisfies:

$$\text{for all } a \text{ there is an } i<l \text{ so that } a^\theta \cap A_i \text{ has at most } l \text{ elements.}$$

Fix such an l.

Let $M_i=\{a \in M : a^\theta \cap A_i$ has at most l elements$\}$ for $i \leqslant l$. Then $M=\bigcup_{i\leqslant l} M_i$, so some M_i has rank n, so $M_i//\theta$ has rank k. It follows that $M_i \cap A_i^\theta//\theta$ has rank k. Let θ' be the restriction of θ to $M_i \cap A_i$. Then $M_i \cap A_i//\theta'$ has rank k. On the other hand the equivalence classes of θ' on $M_i \cap A_i$ have cardinality at most l, so by the case $m=0$ rank $M_i \cap A_i = k \leqslant n-2$. On the other hand $(A_i - M_i \cap A_i)^\theta//\theta$ has rank less than k, so rank$(A_i - M_i \cap A_i)^\theta$ is at most $n-2$, and $A_i = (A_i - M_i \cap A_i) \cup (M_i \cap A_i)$ has rank at most $n-2$, a contradiction. This disposes of the case $m=1$.

Suppose now that hypothesis (2(b)) applies. Then the sets A_i may be taken to be cosets of a subgroup K of rank $n-1$ in M, say $A_i = Km_i$.

The equivalence classes of θ on A_i have the form $(K \cap H)K_{m_i}$. Since K has rank $n-1$ and $A_i//\theta$ has rank k it follows that $K \cap H$ has rank $n-1-k \geqslant m$ (by induction on n). Hence $K \supseteq H^\circ$, so KH is a finite union of cosets of K, rank $KH = n-1$, contradicting eq. 2 above.

2.4 *Nilpotent and solvable groups*

We prove some simple results concerning nilpotent and solvable groups of finite Morley rank.

Theorem 1. *Let G be an infinite connected nilpotent group of finite Morley rank n. Then*

(1) *$Z(G)$ is infinite,*
(2) *G has nilpotency class at most n,*
(3) *If $H \subseteq G$ and $H \neq G$, then $[N(H):H]=\infty$.*

Proof. Note first that a connected group G cannot contain a finite normal noncentral subgroup H, since otherwise $C(H)$ is a definable proper subgroup of finite index in G. (This observation was used in [3].)

Let Z_1 be the preimage in G of $Z(G/Z(G))$ and suppose $Z=Z(G)$ is finite. If Z_1/Z contains an element x of finite order, then $H=\langle x, Z\rangle$ is noncentral finite normal in G, a contradiction. Thus Z_1/Z is torsion-free ω-stable, hence divisible. For $g \in G$ the commutation map $x \to [g,x]$ induces a group homomorphism from Z_1/Z to Z. Z being finite, this map is trivial, so $Z_1 \subseteq Z(G) = Z$, a contradiction. Both (2) and (3) now follow readily by induction on n. (For (3) cf. [5, p. 112].)

Theorem 2. *Let G be a connected solvable group of finite Morley rank n. Then G is solvable of class at most n and possesses a normal series of definable subgroups $1 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_c = G$ with infinite abelian quotients G_{i+1}/G_i.*

Proof. In a stable group the centralizer of any subset is definable [2]. In particular if $A \lhd G$ is abelian, then $H = C(A) \cap C^2(A)$ is a definable abelian subgroup normal in G, and $H \supseteq A$. If we merely show that G contains an infinite abelian normal subgroup A, then we may take A to be definable and prove the theorem by a simple induction.

Let $Z = Z(G)$. If Z is infinite we may take $A = Z$. Suppose therefore that Z is finite and let B be the preimage in G of a definable normal abelian subgroup of G/Z. Then B is infinite, nilpotent of class 2, and of finite Morley rank, hence B° has infinite center. Take $A = Z(B^\circ)$. Then A is the desired subgroup of G.

3. Groups of Morley rank 1

The only novelty in this section is the use of our Indecomposability Theorem (Section 2.2). The other arguments are repeated from Reineke's [14]. The following elementary group-theoretic fact is basic:

Lemma 0. *If all elements of the group G are of finite order and $G - 1$ is a single conjugacy class, then G has at most two elements.*

Proof. Fix g in $G - 1$ of prime order p. By the conjugacy condition G has exponent p. If $p = 2$, then by a well-known exercise G is abelian, and it follows that $G = \mathbb{Z}_2$.

Assuming p odd, a contradiction appears as follows. Choose x so that $g^x = g^{-1}$ (here and throughout g^x means $x^{-1}gx$). Then for all n:

$$g^{(x^n)} = g^{(-1)^n}.$$

Taking $n = p$ yields $g = g^{-1}$, so $p = 2$, a contradiction.

Theorem 1. *Let G be an infinite ω-stable group. Then G contains an infinite definable abelian subgroup.*

Proof. Suppose that G is a counterexample of minimal rank α and degree d. These minimality conditions imply that every proper definable subgroup of G is finite. In particular G is connected, so the degree d is 1 by the Indecomposability Theorem (Section 2.2).

We will show that Lemma 0 applies to the quotient $G/Z(G)$, so that $[G:Z(G)]\leq 2$ and therefore $Z(G)$ is infinite, definable, and abelian, as desired.

For any g in $G-Z(G)$ the centralizer $C(g)$ is a proper definable subgroup of G, hence finite. Since g is in $C(g)$ this implies that g has finite order. Furthermore since the conjugacy class g^G of g in G can be identified with the coset space $C(g)\backslash G$ and $C(g)$ is finite it follows that g^G has rank α. But the degree of G is 1, so there can be at most one such conjugacy class; thus the noncentral elements of G are conjugate.

Thus Lemma 0 applies to $G/Z(G)$, to complete the argument.

Theorem 2. *If G is a group of Morley rank 1, then G is abelian by finite.*

Proof. Let A be an infinite abelian definable subgroup of G. Then A has rank 1 and thus $[G:A]$ is finite.

Theorem 3. *Let T be an abelian group equipped with an automorphism α of order 2. If $\langle T, \alpha\rangle$ has rank 1, degree 1, then α is given by*

(1) $\alpha t = t$ *or*

(2) $\alpha t = t^{-1}$.

Proof. Define $\beta: T \to T$ by $\beta t = t\alpha t$. We consider two possibilities:

(1) $\ker\beta$ is finite. Then $\operatorname{Im}\beta$ is infinite, hence $\operatorname{Im}\beta = T$. But $\alpha t = t$ on $\operatorname{Im}\beta$, hence on T.

(2) $\ker\beta$ is infinite, hence $\ker\beta = T$. Then $\alpha t = t^{-1}$ on T.

4. Groups of Morley rank 2

We show that connected groups of Morley rank 2 are solvable, classify the nonnilpotent connected groups of Morley rank 2, and give an incomplete analysis of the connected nilpotent groups of Morley rank 2.

Our analysis is motivated in part by the theory of algebraic groups. In this we follow the strategy used in the classification of finite simple groups, as stated explicitly in the final chapter of [17]. We will devote Section 4.4 to a discussion of this aspect of the argument.

4.1 *Solvability*

Throughout this section assume that G is a connected nonsolvable group of Morley rank 2, and degree 1. We will eventually reach a contradiction.

Theorem 1. (1) *$Z(G)$ is finite.*

(2) *G contains a connected abelian definable subgroup of rank 1.*

Proof. (1) If $Z(G)$ is infinite, then $G/Z(G)$ is rank 1 connected, hence abelian, which proves that G is nilpotent, a contradiction.

(2) By Theorem 1 of Section 3 G contains an infinite abelian definable subgroup A. Clearly A has rank 1, and A° is the desired connected abelian definable rank 1 subgroup.

Notation 2. $H=G/Z(G)$ (another nonsolvable group of Morley rank 2, degree 1). $A\subseteq H$ is definable connected abelian of rank 1. $N=N(A)$.

Lemma 3. (1) $Z(H)=1$,

(2) $[H:N]=\infty$,

(3) $[N:A]<\infty$,

(4) $A^g\cap A^h=1$ *unless* $gh^{-1}\in N$.

Proof. (1) Let Z_1 be the preimage of $Z(H)$ in G. Since $Z_1\lhd G$ and Z_1 is nilpotent, clearly Z_1 is finite, hence central. Thus $Z(H)=1$.

(2) A is abelian and N/A has rank at most 1. Thus $N\neq H$, so $[H:N]=\infty$.

(3) By (2) rank $N\leq 1$ so $[N:A]<\infty$.

(4) If $b\in A^g\cap A^h$, then $A^g, A^h\subseteq C(b)$. If $C(b)=H$, then $b=1$. Otherwise $[H:C(b)]=\infty$, so rank $C(b)=1$ and A^g, A^h have finite index in $C(b)$. Since A^g, A^h are connected, $A^g=A^h$ and $gh^{-1}\in N$.

We can now make a satisfactory structural analysis of H.

Lemma 4. *If $w\in H-A$, then $H=AwA\cup A$. The element w may be chosen to be an involution. $A=N(A)$.*

Proof. We proceed in four steps.

Step 1: Fix w in $H-N$. Then $H=AwA\cup N$: We claim that AwA has rank 2. Then since H has rank 2 and degree 1 there can be only one such double coset, and the claim will follow. (Of course if we consider w in N, then the double coset AwA reduces to the simple coset $Aw=wA$ of rank 1.)

To see that AwA has rank 2 it suffices to verify that the expression of an element of AwA in the form a_1wa_2 is unique. This is a simple computation: if $a_1wa_2=a_3wa_4$, then

$$a_4a_2^{-1}=(a_3^{-1}a_1)^w\in A\cap A^w=1$$

and thus $a_2=a_4$, $a_1=a_3$, as claimed.

Step 2: H-N contains an involution: Start with an arbitrary element x of $H-N$. By Step 1 we can write

$$x^{-1}=a_1xa_2$$

with a_1, a_2 in A. Let $w=xa_1$. Then $w^2=a_1a_2^{-1}\in A$. Setting $a=w^2$ we have:

$$a=a^w\in A\cap A^w=1$$

and thus w is an involution. Fix such an involution w.

Step 3: Let $K=N\cap A^w$. Then $N=A\dot{\times}K$ (semi-direct product): Evidently $K\subseteq N$ normalizes A and $K\cap A=1$. It suffices therefore to show that $N=AK$.

For any n in N since $nw\notin N$ we may write:

$$nw=a_1wa_2$$

and hence $a_1^{-1}n=a_2^w\in A^w\cap N=K$. Thus $N=AK$.

Step 4: $K=1$ (hence $N=A$): If $a\in K^w$, then $a\in A$ and $a^w\in N$. For all $g\in H$ we claim that $a^g\in N$. This is clear if $g\in N$ while if $g=a_1wa_2$, then $a^g=(a^w)^{a_2}$ is also in N.

Let $B=\langle a^g:g\in H\rangle$. Then $B\lhd H$, $B\subseteq N$. If $a\neq 1$, then $[H:C(a)]=\infty$, so B is infinite. Since $[N:A]<\infty$ therefore $[B:A\cap B]<\infty$. Conjugating by $w, [B:A^w\cap B]<\infty$, so $[B:A\cap A^w\cap B]<\infty$, contradicting $A\cap A^w=1$.

Hence in fact $a=1$, so $K=1$, and $N=A$.

Note. Everything we have proved so far is true of the solvable group of rank 2 $H=F_+\dot{\times}F^\cdot$ (F an algebraically closed field), if we set $A=F^\cdot$ and $w=(1,-1)$. However, the next result is false for that group.

Lemma 5. $H=\bigcup_{g\in H}A^g$.

Proof. Let $X=\bigcup_H A^g$. Since $A^g\cap A^h=1$ when $gh^{-1}\notin A$ the Morley rank of X is 2. If $b\in H-X$ it follows that the conjugacy class of b has rank at most 1 (G has rank 2, degree 1), so $C(b)$ is infinite.

Let $B=C(b)^0$. Then Lemma 4 applies with B in place of A, so in particular $B=N(B)$, implying $B=C(b)$ and thus $b\in B$. Now let $Y=\bigcup_H B^g$. Then Y has rank 2, hence Y meets X in a large set. Therefore we may assume that $A\cap B\neq 1$. Since A, B are connected we get either $A\cap B$ is finite, in which case as usual the centralizer of a nontrivial element in the intersection is all of H, or else $A=B$, contradicting $b\in B-A$. Thus in either case we have a contradiction.

This proves that $H=X$, as desired.

Theorem 6. *A connected group G of Morley rank 2 is solvable.*

Proof. If G is a counterexample let $H=G/Z(G)$. Then for $a\in H-1$ $C(a)$ is a connected abelian group equal to its own normalizer. (Lemma 4 and remarks in the proof of Lemma 5). Let $w\in H$ be an involution outside $C(a)$ (Lemma 4). Then w is conjugate to an involution i of A (Lemma 5). Then $iw\neq wi$ and $(iw)^i=(iw)^{-1}$. Let $B=C(iw)$. Then $iw\in B\cap B^i$, so $B=B^i$, proving $i\in B$. Hence also $w\in B$, so $iw=wi$, a contradiction.

4.2. *Nonnilpotent groups*

We will give a classification of all nonnilpotent connected groups of Morley rank 2, with some supplementary information needed for Section 3.

Theorem 1. *Let G be a connected centerless nonnilpotent group of Morley rank 2. Then for some algebraically closed field F, G is isomorphic to the semidirect product*

$$F_+ \dot{\times} F^*$$

of the additive and multiplicative groups of F, F^* acting on F_+ by multiplication.

Proof. By Theorem 2 of Section 2.4 G has an infinite definable normal abelian subgroup U, which will turn out to be a copy of F_+ (U stands for "unipotent"). We may take U to be connected.

Next we seek an element b in $G - C(U)$ whose connected centralizer

$$T = C(b)^0.$$

is infinite (T stands for "torus"). If no such element exists, then $C(b)$ is finite for all b in $G - C(U)$ and then the proof of Theorem 1, Section 3 shows that the index of $C(U)$ in G is at most 2. But since G is connected this forces $G = C(U)$ and hence easily G is nilpotent, which is a contradiction.

Thus we may fix $T = C(b)^0$ infinite with b in $G - C(U)$, and clearly $T \neq U$. Then UT has rank 2 (since $U \cap T$ is finite), so $UT = G$. Furthermore $U \cap T = 1$ since G is centerless. Thus $G = U \dot{\times} T$.

Since we are about to prove that T is F^* acting on F_+ via multiplication, we will write U additively, T multiplicatively, and we define $t \cdot u = tut^{-1}$ for t in T, u in U (so that $u^t = t^{-1} \cdot u$). The identity element 1 of G is sometimes called 0 when it is considered as an element of U. (This notation can produce occasional peculiarities, but it is extremely efficient.)

Since G is connected centerless, for $a \in G - 1$ $[G : C(a)] = \infty$, so every nontrivial conjugacy class is infinite. Since U is connected of rank one it follows that $U - 1$ consists of a single conjugacy class in G. For $u \in U - 1$ it follows that every element of $U - 1$ has the form $t \cdot u$ ($t \in T$).

Fix $u \in U - 1$. We show now that $C(u) = U$. Suppose $t \in T \cap C(u)$. Then $t = t^{t'}$ centralizes $t' \cdot u$ for $t' \in T$, so $t \in T \cap C(U) = 1$. In particular the map

$$\hat{} : t \mapsto \hat{t} = t \cdot u$$

is a bijection from T to $U - 1$.

Adjoin to T a formal symbol 0, and extend the multiplication of T to $T \cup \{0\}$ by the rule $x \cdot 0 = 0 \cdot x = 0$. Define also $0 \cdot u = 0 \in U$. Let $F = T \cup \{0\}$, and define an addition + on F by

$$(x + y)\hat{} = \hat{x} + \hat{y}.$$

Then + is a commutative associative binary operation on F with identity 0 and inverses defined by

$$(-x)\hat{} = -\hat{x}.$$

To see that F is a field it suffices to check one distributive law: $z(x + y) = zx + zy$.

Indeed

$$(z(x+y))\hat{} = z \cdot (x+y)\hat{} = z \cdot (\hat{x}+\hat{y}) = \hat{zx} + \hat{zy} = (zx+zy)\hat{}.$$

Thus F is a field interpretable over G. Then F has finite Morley rank and is therefore algebraically closed [8]. Clearly $T = F^{\cdot}$ and $F_+ \simeq U$ via $\hat{}$. Note that u is identified with $1 \in F_+$, and u was fixed but arbitrary. It is also clear that under these identifications conjugation of U by T corresponds to multiplication.

Theorem 2. *Let G be connected nonnilpotent group of Morley rank 2, $Z = Z(G)$. Then G is a semidirect product:*

$$G = U \dot{\times} T$$

of connected abelian groups of rank 1, with T divisible, $Z \subseteq T$, Z finite, and for some algebraically closed field F:

$$U \simeq F_+, \qquad T/Z \simeq F^{\cdot},$$

and conjugation of U by T corresponds to multiplication in F.

Proof. Evidently Z is finite and G/Z is centerless, $G/Z \simeq F_+ \dot{\times} F^{\cdot}$ for some algebraically closed field F. Let U_1, T_1 be the inverse images of F_+, $F^{\cdot}$ in G, and let $U = U_1^{\circ}$, $T = T_1^{\circ}$. Then UT has rank 2, so $UT = G$. Since U, T are rank 1 connected, U and T are abelian.

Fix $u \in U - Z$. Define $\hat{t} = tut^{-1}$ for $t \in T$. U is connected (hence of degree 1), so $\hat{T}$ is cofinite in U. If $U \cap Z$ is nontrivial it follows that for some $t_1 \neq t_2$ in T there is an equation

$$\hat{t}_1 = z + \hat{t}_2 \quad \text{with} \quad z \in U \cap Z, \quad z \neq 1.$$

Then modulo Z $\hat{t}_1 = \hat{t}_2$, so $t_1 = t_2$ in $F^{\cdot}$, i.e. $t_1 t_2^{-1} \in Z \cap T$. Then $\hat{t}_1 = \hat{t}_2$, so $z = 1$, a contradiction.

Thus $U \cap Z = 1$, $U \simeq F_+$. In particular $U \cap T = 1$, $G = U \dot{\times} T$. Since $Z(F_+ \dot{\times} F) = 1$, $Z \subseteq T$. T being of rank 1 and connected, it follows that T is of prime exponent or divisible. Since $T/Z \simeq F^{\cdot}$, T is divisible. Since Z is finite it is contained in the torsion component of T. Let T_p, Z_p be the p-torsion components of T, Z. Then $T_p/Z_p = (F^{\cdot})_p$ is a Prufer p-group Z_{p^∞} (or trivial, if $p = \text{char } F$). Thus T_p is also a Prufer p-group Z_{p^∞} and Z_p is a finite subgroup. These last remarks pin down the structure of T, and hence of G, completely.

The rest of this section is devoted to a partial analysis of the subgroups and automorphisms of groups of the above types. Throughout the remainder of this section let G, Z, U, T, F be as in Theorem 2.

Theorem 3. *Let H be a subgroup of G. Suppose that the structure $\langle G; H\rangle$ has rank 2 and that U, T are connected in $\langle G; H\rangle$. Then H is definable in G. If H is infinite* *and unequal to G, then H has the following form:*

(1) $U \dot{\times} L$ *with* $L \subseteq T$ *finite or*

(2) T^u *with* $u \in U$.

Proof. Consider the projection maps $\pi_U : H \to U$, $\pi_T : H \to T$. If the image L of π_T is finite, then the kernel of π_T is infinite, hence equals U. Then $H = U \dot{\times} L$.

Suppose that π_T has infinite range. Then π_T is surjective. Let K be the kernel of π_T. Then K is finite. Define a map

$$f : T \to \text{subsets of } U$$

by $f(t) = \pi_U \pi_T^{-1}(t)$. Then:

$$f(t_1 t_2) \supseteq f(t_1) + t_1 \cdot f(t_2) \tag{$*$}$$

(indeed if $u_1 \in f(t_1)$, $u_2 \in f(t_2)$, then $u_1 t_1, u_2 t_2 \in H$ and an easy computation yields: $(u_1 + t_1 \cdot u_2) t_1 t_2 = u_1 t_1 u_2 t_2 \in H$, so $(u_1 + t_1 \cdot u_2) \in f(t_1 t_2)$, as claimed).

From $(*)$ we derive:

$$\operatorname{card} f(t_1 t_2) \geq \operatorname{card} f(t_1) \quad \text{for} \quad t_1, t_2 \in T.$$

Hence:

$$\operatorname{card} f(t_1) = \operatorname{card} f(t_2) \quad \text{for} \quad t_1, t_2 \in T.$$

In particular the inclusion in $(*)$ is an *equality* and (since $f(1) = K$):

$$\operatorname{card} f(t) = \operatorname{card} K$$

for $t \in T$. Take $t_1 = 1$, $t_2 = t$ in $(*)$. Then:

$$f(t) = K + f(t),$$

so $f(t)$ is a coset of K. Let $t_1 = t$, $t_2 = 1$ in $(*)$. Then:

$$f(t) = f(t) + t \cdot K.$$

It follows that $t \cdot K \subseteq K$ for $t \in T$. However, T contains elements of infinite order while K is finite, so $K \subseteq Z \cap U = 1$.

Thus f is a function from T to U. Since $t_1 t_2 = t_2 t_1$ in T, $(*)$ yields:

$$f(t_1) + t_1 f(t_2) = f(t_2) + t_2 \cdot f(t_1),$$

hence $(1 - t_2) \cdot f(t_1) = (1 - t_1) \cdot f(t_2)$. Thus for $t \neq 1$ the element $(1-t)^{-1} \cdot f(t) \in U$ is independent of t. Call it u. Thus finally:

$$f(t) = (1 - t) \cdot u$$

and $H = \{((1-t) \cdot u, t) : t \in T\} = T^u$ by explicit calculation.

Theorem 4. *Suppose that* $\alpha \in \operatorname{Aut} G$ *and that relative to the structure* $\langle G; \alpha \rangle$, *G has rank* 2 *and U, T are connected. Suppose that for some* $n > 0$, $\alpha^n = 1 \in \operatorname{Aut} G$. *Then* α *is an inner automorphism.*

Proof. From Theorem 3 it follows easily that $\alpha[U]=U$ and $\alpha[T]=T^u$ for some $u\in U$. Define $\beta\in\operatorname{Aut}G$ by $\beta(g)=\alpha(g)^{u^{-1}}$. It suffices to prove that β is inner. Here $\beta[U]=U$, $\beta[T]=T$.

Now for all k, $\alpha^k[T]=T^{u\alpha(u)\cdots\alpha^{k-1}(u)}$ and taking $k=n$ we see that $u\alpha(u)\cdots\alpha^{n-1}(u)\in N(T)\cap U=1$. Since $\beta^k(g)=\alpha^k(g)^{(u\alpha(u)\cdots\alpha^{k-1}(u))^{-1}}$ we see that $\beta^n=1$. Now $\beta[Z]=Z$, so β induces an automorphism γ of $G/Z=F_+\times F^{\cdot}$. We use additive notation for F_+ and multiplicative notation for the action of $F^{\cdot}$ on F_+, as usual. For $s, t\in F$, $u\in U-1$ we have:

$$\begin{aligned}\gamma(s+t)\cdot\gamma(u)&=\gamma((s+t)\cdot u)=\gamma(s\cdot u+t\cdot u)\\&=\gamma(s)\cdot\gamma(u)+\gamma(t)\cdot\gamma(u)=(\gamma(s)+\gamma(t))\cdot\gamma(u).\end{aligned}$$

Thus $\gamma(s+t)=\gamma(s)+\gamma(t)$ and γ is actually a field automorphism of F. Suppose m is the order of γ in $\operatorname{Aut}F$ ($m\leqslant n$). Define $\sigma:F\to F$ by

$$\sigma a=a+\gamma a+\cdots+\gamma^{m-1}a.$$

Then σ is a homomorphism of F_+ (the additive group of $F=T\cup\{0\}$).

If the range of σ is finite, then the kernel is infinite, hence is all of F_+, so $\sigma a\equiv 0$ and the maps $1, \gamma, \gamma^2,\dots,\gamma^{m-1}$ are linearly dependent but unequal, which is impossible. Thus the range of σ must be infinite, hence equal to F. But γ is the identity on range (σ), so γ is the identity on F.

Hence for $t\in T$ $\beta(t)=tz(t)$ where $z:T\to Z$ is a homomorphism. Here Z is finite and T is connected, so z is trivial and $\beta(t)=t$ on T. Fix $u\in U-1$, and let $\beta(u)=t_0\cdot u$. One sees easily that β coincides with the inner automorphism defined by t_0, as desired.

4.3. *Nilpotent groups*

We do not know whether there are nonabelian connected nilpotent groups of Morley rank 2.

Theorem 1. *Let G be a nonabelian connected nilpotent of Morley rank 2. Then G is of exponent p or p^2.* Let $Z_1=Z(G)$. *If G is of exponent p^2, then $G^p=Z_1^0$.*

Proof. Let $Z=Z_1^0$. Then Z, G/Z are connected abelian of rank 1. In particular $[G,G]\subseteq Z$. Let $A=G/Z_1$. Commutation induces a bilinear map:

$$A\times A\to Z.$$

A is either divisible or of prime exponent p. Suppose first that A is divisible. If $a\in A$ is a torsion element of order n, then

$$0=[na,A]=[a,nA]=[a,A]$$

so $a=0$. Thus A is torsion free. But $[a,a]\equiv 0$ so $a\in\ker(a)$ (viewing a as a linear map from A to Z), thus $\ker(a)\supseteq\mathbb{Z}a$ is infinite, $\ker(a)=A$, $A=(0)$, and G is abelian.

Hence in fact A has prime exponent p, and it follows that Z has exponent p as well. Also by considering the surjection $G/Z \to A$ and recalling that G/Z is connected we see that G/Z has exponent p.

In particular G is of exponent p or p^2.

Now it is easy to verify that the map $x \to x^p$ is a homomorphism from G to G, and its kernel K contains Z. Assume $K \neq G$. Then $[G:K] = \infty$, so G^p is infinite. But $G^p \subseteq Z$, so $G^p = Z$.

4.4. *Algebraic groups*

The only known examples of simple ω-stable groups are the simple matrix groups with coefficients in an algebraically closed field (and these are all $\aleph_1$-categorical, as mentioned in the introduction). The thrust of the present article is that in certain extremely special cases it is possible to prove that a simple ω-stable group is in fact a linear algebraic group.

Our results on groups of Morley rank 2 or 3 are based on the following idea. Consider a simple group G of Morley rank 2 or 3. Show that G resembles an algebraic group structurally, and obtain either a contradiction or a complete description of G. In fact even in the case of rank 2 our analysis is directed toward a proof that G resembles PSL $(2, F)$; since G cannot be PSL $(2, F)$ this eventually produces a contradiction. The contradiction is itself ad hoc, and it could indeed be presented in various forms, but the structural analysis preceding it is in its essential features canonical.

To explain this in more detail, we recall the *Bruhat decomposition* of a simple (more generally, reductive) algebraic group.

Let G be a simple algebraic matrix group. Let B be a *Borel subgroup*, i.e. a maximal connected solvable subgroup of G. It turns out that much of the structure of G can be elucidated in terms of the way G decomposes into double cosets modulo B. The double coset decomposition of G relative to B is called the *Bruhat decomposition* of G, and is written:

$$G = \bigcup_{w \in W} B\tilde{w}B.$$

Here W is the Weyl group of G, a finite group associated to G in a way that need not concern us here, and $\tilde{w}$ runs over representatives of the elements of W in G. In the case of PSL(n, F) the Weyl group W is the symmetric group S_n on n letters.

The important case for the present article is PSL $(2, F)$. In SL $(2, F)$ a natural choice for the Borel subgroup B is the group of upper triangular matrices:

$$\begin{vmatrix} a & b \\ 0 & a^{-1} \end{vmatrix}, \quad a \in F^{\cdot}; \qquad b \in F.$$

Then B is a semidirect product $U \rtimes T$ where T consists of the diagonal matrices and U consists of the upper triangular unipotent matrices:

$$\begin{vmatrix} 1 & b \\ 0 & 1 \end{vmatrix}.$$

(We pass from SL (2, F) to PSL (2, F) by factoring out the center of SL, which is also the center of B: $\pm I$, I = identity.)

The Bruhat decomposition in SL (2, F) then takes the form:

$$G = B \cup BwB$$

where w may be taken to be the matrix (essentially a permutation matrix):

$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}.$$

Here $w^2 = -I$, so that in PSL (2, F) we may write $w^2 = 1$. Thus in PSL (2, F) we have explicitly

$$G = \dot{\bigcup_{x \in S_2}} BxB$$

identifying S_2 with (I, w).

These specific facts are relevant to the calculations made in the following section (which correspond in part to matrix calculations in SL (2, F)). Of greater importance is the idea of a Bruhat decomposition as an initial goal for a group-theoretic analysis, together with additional information about Borel subgroups furnished by the general theory of algebraic groups. A number of general facts are available about Borel subgroups of algebraic groups which we are forced to piece together in an ad hoc manner in the cases which concern us, as we lack any general machinery for analyzing ω-stable groups. Chief among these are:

(1) the normalizer theorem: $N(B) = B$. In our analyses this is obtained simultaneously with the Bruhat decomposition.

(2) If B is nilpotent, then $B = G$. In our analysis of the rank 2 nonsolvable case we know from the start that B is abelian and $B \neq G$, but a contradiction is reached only after an involved argument. Similarly, the rank 3 nonsolvable groups that we have not succeeded in analyzing are exactly those in which Borel subgroups are of rank 1, hence abelian (see Section 5.2). Theorem 1 of Section 3 is our only general result concerning the size of Borel subgroups.

(As noted earlier, the importance of the Bruhat decomposition is suggested by the experience of finite group theorists as described in the final chapter of the book [17]. An elementary exposition of the structure theory of linear algebraic groups is in [5].)

5. Groups of Morley rank 3

5.1. *Good groups*

Definition 1. A group of Morley rank 3 is *good* if it contains a definable subgroup of rank 2, *bad* otherwise.

Theorem 1. *Let G be a good connected group of rank* 3. *Then either G is solvable or* $G/Z(G) \simeq \mathrm{PSL}(2, F)$ *for some algebraically closed field F. In the second case* $G \simeq \mathrm{PSL}(2, F)$ *or* $\mathrm{SL}(2, F)$.

The proof of this theorem consists of a detailed structural analysis of $G/Z(G)$ under the assumption of nonsolvability, much as in Section 4.1. We will soon get the Bruhat decomposition of $G/Z(G)$. In particular one should think of the group B introduced below as the group of upper triangular matrices, and

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Lemma 2. *Let G be a good connected nonsolvable group of Morley rank* 3. *Then* $Z(G)$ *is finite and* $G/Z(G)$ *contains a nonnilpotent definable subgroup B of rank* 2.

Proof. If G is nonsolvable, then the solvability of groups of rank 2 shows that any normal solvable subgroup of G is finite, hence central. It follows easily that $Z = Z(G)$ is finite and G/Z is centerless.

For simplicity suppose G is centerless and B is a definable connected subgroup of rank 2 of G (i.e. write G for G/Z). Let $N = N(B)$. Then $N \neq G$. Fix $x \in G - N$ and let $A = (B \cap B^x)^0$. Then rank $A \leq 1$. If A is trivial, then $B \cdot B^x$ has rank 4, which is impossible. Thus rank $A = 1$, and in particular A is abelian.

Suppose now that B is nilpotent. If B is abelian, then $C(B \cap B^x) \supseteq B, B^x$, hence $C(B \cap B^x) = G$, $B \cap B^x \subseteq Z = 1$, a contradiction. If B is nonabelian, then $A \cdot Z(B) \neq B$, so rank $A \cdot Z(B) = 1$ and $A \subseteq Z(B)$. Then $C(A) \supseteq B, B^x$, so $A \subseteq Z = 1$, a contradiction. Thus B is nonnilpotent.

Throughout the remainder of this subsection let G be a good connected nonsolvable centerless group of Morley rank 3 and let B be a connected definable subgroup of rank 2 in G, necessarily nonnilpotent by the above proof. We will call B a Borel subgroup of G. B is a semidirect product:

$$B \simeq U \rtimes T$$

with $U \simeq F_+$, $T/Z(B) \simeq F^\cdot$ for a suitable algebraically closed field F, called the *base field* of B. (Section 4.2, Theorem 2). Since $Z(G) = 1$, we may set $Z = Z(B)$ without danger of confusion.

We will prove that $G \simeq \mathrm{PSL}(2, F)$.

Lemma 3. (1) $G = UN(T)U$,
(2) $N(B) = B \cdot C(B)$,
(3) $N(B) \cap N(T) = T \cdot C(B)$.

Proof. (1) Fix $g \in G$. Since $\operatorname{rank} B \cdot B^g < 4$, $\operatorname{rank} B \cap B^g \geq 1$. If $g \in N(B)$, then part 2 of this lemma will show that $g \in B \cdot C(B) \subseteq UN(T)$.

Assume therefore that $\operatorname{rank} B \cap B^g = 1$. Then $B \cdot B^g$ has rank 3. Let $H = B \cap B^g$. According to Section 4.2, Theorem 3 if H does not contain U, then $H = T^u$ for some $u \in U$.

Suppose toward a contradiction that $U \subseteq H$, so that $U^{g^{-1}} \subseteq B$. By Section 4.2, Theorem 3 this yields $U^{g^{-1}} = U$, hence $B, B^g \subseteq N(U)$ proving $N(U) = G$, contradicting the nonsolvability of G.

Thus $H = T^u$ for some u, and $T^{ug^{-1}} \subseteq B$. Again by Theorem 3 of Section 4.2, $T^{ug^{-1}} = T^v$ for some v. Then $T^{vgu^{-1}} = T$, $vgu^{-1} \in N(T)$, $g \in UN(T)U$, as desired.

(2) If $a \in N(B)$, then by Theorem 3 of Section 4.2, $T^a = T^u$ for some $u \in U$. Then $au^{-1} \in N(B) \cap N(T)$. If we prove part 3, then $au^{-1} \in T \cdot C(B)$, so $a \in B \cdot C(B)$, as desired.

(3) Assume $a \in N(B) \cap N(T)$. Let $n = [N(B) : B]$. Then $a^n \in B$ and $T^{a^n} = T$, so $a^n \in B \cap N(T) = T$.

Define $g(t) = t(t^a) \cdots (t^{a^{n-1}})$ for $t \in T$, and fix $u \in U - 1$. If $u^a = u^t$ with $t \in T$, then

$$u^{a^n} = u^{g(t)}, \quad \text{so} \quad a^n = g(t). \tag{$*$}$$

If $\ker g = T$, then $a^n = 1$ in $\operatorname{Aut} B$, so a defines an inner automorphism of B by Theorem 4 of Section 4.2. Thus $a \in B \cdot C(B)$ in this case. In the remaining case $\ker g$ is finite, so eq. ($*$) has only finitely many solutions $t \in T$. Since U is connected there is a fixed $t \in T$ such that

$$u^a = u^t \quad \text{for } u \text{ in } U \text{ off a finite set.}$$

It follows easily that $u^a = u^t$ on U.

Replace a by at^{-1}. Then $a \in C(U)$, $T^a = T$, and $a^n \in C(T)$ as one sees easily. Thus $a^n \in C(B)$, so Theorem 4 of Section 4.2 applies to prove that $a \in B \cdot C(B)$.

Now we obtain the Bruhat decomposition.

Lemma 4. *$N(T) \not\subseteq N(B)$ and if $w \in N(T) - N(B)$, then*

$$G = B \cup BwB.$$

Also $B = N(B)$.

Proof. (1) $N(T) \not\subseteq N(B)$: If $g \in G - N(B)$ write $g = u_1 w u_2$ with $u_1, u_2 \in U$, $w \in N(T)$. Then $w \in N(T) - N(B)$.

(2) For $w \in N(T) - N(B)$, $G = N(B) \cup BwB$:

We will show that rank $BwB=3$. Since G is connected this will show that $G-N(B)$ can contain only one double coset of B, so $G=N(B)\cup BwB$.

Suppose that $b_1, b_2\in B$, $b_1wb_2=w$. Then $T=T^w=T^{b_1wb_2}$, so $T^{b_2^{-1}}=T^{b_1w}\subseteq B\cap B^w$. But $B\cap B^w$ has rank 1 and contains T, so $T=T^{b_2^{-1}}=T^{b_1w}$. It follows that $b_1, b_2\in N(T)\cap B=T$. In particular, if $b_2\in U$, then $b_2=b_1=1$. Thus BwU has rank 3, so BwB has rank 3.

(3) $N(B)=B$: By Lemma 3.2 it suffices to show that $C(B)\subseteq B$. Fix $c\in C(B)$ and write

$$cw=b_1wb_2 \quad \text{with} \quad b_1, b_2\in B.$$

Then $T^{b_2^{-1}}=T^{cwb_2^{-1}}=T^{b_1w}$ and it follows as in part 2 that $b_1, b_2\in T$. Then $cw=b_1wb_2=b_1b_2^{w^{-1}}w$, $c=b_1b_2^{w^{-1}}\in T$.

Lemma 5. *There is an element $w\in N(T)-T$ such that $w^2\in T$. Furthermore either*

(1) *$w\in C(T)$ and $w^2=1$ or*

(2) *$t^w=t^{-1}$ for $t\in T$, and $w^2\equiv\pm1$ in $F^{\cdot}$.*

Proof. Start with $x\in N(T)-T$ and write

$$x^{-1}=b_1xb_2 \quad \text{with} \quad b_1, b_2\in B.$$

Then $T^{b_2^{-1}}=T^{x^{-1}b_2^{-1}}=T^{b_1x}\subseteq B\cap B^x$, so $T=T^{b_2^{-1}}=T^{b_1x}$, proving $b_1, b_2\in T$. Let $w=xb_1$. Then

$$w^2=b_2^{-1}b_1\in T.$$

Then by Theorem 5 of Section 4.2 either $w\in C(T)$ or $t^w=t^{-1}$.

Let $t=w^2$. Assume first that $w\in C(T)$ and $b=t^{-1/2}\in T$. (T/Z is the multiplicative group of an algebraically closed field, so it is closed under formation of square roots; $t^{-1/2}$ denotes any element which represents the inverse square root of t modulo Z.) Then $(wb)^2=1$. On the other hand if $t^w=t^{-1}$ on T, then for $t=w^2$ we get $t=t^w=t^{-1}$, $t^2=1$. Hence $t\equiv\pm1 \pmod Z$.

For the remainder of this section w will be assumed chosen as specified in Lemma 5. In the remainder of this section we will produce generators and relations for G via a reasonably straightforward computation, and then recognize G as $\mathrm{PSL}(2,F)$, with F the base field for B.

Lemma 6. *$G=B\cup UTwU$ and the representation of an element $g\in G-B$ in the form u_1twu_2 is unique.*

Proof. It suffices to prove that $BwB=UTwU$, which is trivial, and the uniqueness statement, which was verified in the proof of Lemma 4.2.

Lemma 7. $w\notin C(T)$.

Proof. Assume w is in $C(T)$, and in particular $w^2=1$. Fix a in U with waw not in B, and write:

$$waw=u_1t_1wu_2$$

with $u_1=\hat{s}_1$, $u_2=\hat{s}_2$ (writing $\hat{s}$ for $s\cdot a$ and allowing $s=0$).

For s, t in T compute:

(1) $(w\hat{s}w)\hat{t}w=(s_1s+s_1s_2t_1s+s_1t_1^{-1}t)\hat{}.t_1^2w(s_2^2s+s_2t)\hat{}$.

(2) $w\hat{s}(w\hat{t}w)=(s_1s+s_1^2t)\hat{}t_1^2w(s_2t_1^{-1}s+s_1s_2t_1^{-1}t+s_2t)\hat{}$.

By Lemma 6 we may equate the coefficients from T modulo Z, getting two equations (modulo Z), of which the first is:

(3) $(s_1s_2t_1)s=(s_1^2-s_1t_1)t$.

Letting s, t vary we conclude: $s_1s_2=0$.

If e.g. $s_1=0$, then $u_1=0$ in U, i.e. $u_1=1$ in G and:

$$waw=t_1wu_2=wt_1u_2,$$

so $w=a^{-1}t_1u_2$ is an element of B, which is a contradiction.

Lemma 8. *For some $u\in U$, $u^w\in UwU$.*

Proof. Fix $a\in U-1$ and write $a^w=u_1t_1wu_2$. Letting $t=t_1^{-1/2}$ one gets

$$(a^t)^w=u_1^{t^{-1}}wu_2^{t^{-1}}$$

(this computation uses the fact that w acts on T by inversion). Now take $u=a^t$.

Proof of Theorem 1. We show that $G\simeq\mathrm{PSL}\ (2,F)$ with the above hypotheses and notation. In particular we fix u in U such that $u^w\in UwU$, say:

$$u^w=\hat{s}_1w\hat{s}_2$$

where $\hat{s}=s\cdot u$ for s in $T\cup\{0\}$. Easily $s_1, s_2\neq 0$. Let $\varepsilon=w^2$. In the proof of Lemma 5 we saw that $\varepsilon^2=1$.

For s, t in T a computation shows:

$$w\hat{s}w=(\varepsilon s_1s^{-1})\hat{}(\varepsilon s^{-2})w(s_2s^{-1})\hat{}, \tag{1}$$

$$(w\hat{s}w)\hat{t}w=(\varepsilon s_1s^{-1}+s_1s^{-1}(s_2+st)^{-1})\hat{}(s_2+st)^{-2}w(s_2s(s_2+st)^{-1})\hat{}, \tag{2}$$

$$w\hat{s}(w\hat{t}w)=(\varepsilon s_1t(\varepsilon s_1+st)^{-1})\hat{}(\varepsilon s_1+st)^{-2}w(\varepsilon s_2t^{-1}(\varepsilon s_1+st)^{-1}+s_2t^{-1})\hat{} \tag{3}$$

(we have cast these expressions into the normal form described in Lemma 6).

If we take s in the center Z of B in eq. (1) (recalling $Z\subseteq T$) we obtain:

$$wuw=(\varepsilon s_1)\hat{}(\varepsilon s^{-2})w\hat{s}_2.$$

Since this is independent of s, we conclude that $s^2=1$ for s in Z.

However, this yields: $s^w=s^{-1}=s$, so s commutes with w as well as all of B, proving that s is in the center of G, so that finally $s=1$.

Thus the center of B is trivial and B may be identified with $F_+\dot{\times}F^{\cdot}$. In particular a certain element of ambiguity once present in eqs. 2, 3 (concerning the meaning of $+$ in T) no longer exists. Also we can now say that $\varepsilon = \pm 1$.

Combining eqs. 2 and 3, the associative law in G, and Lemma 6 we obtain three equations, of which the first is:

$$\varepsilon s_1 s^{-1} + s_1 s^{-1}(s_2 + st)^{-1} = \varepsilon s_1 t(\varepsilon s_1 + st)^{-1}. \tag{4}$$

This may be reorganized to read: $s_1(\varepsilon s_2 + 1) + \varepsilon(s_1 + 1)st = 0$. Then varying s and t yields:

$$s_1 = -1, \tag{5}$$

$$s_2 = -\varepsilon. \tag{6}$$

It will suffice now to show that $\varepsilon = 1$, since the structural facts obtained (including Lemma 6) determine G and hold in PSL $(2, F)$ (see the discussion in Section 4.4). (The final statement in Theorem 1 can be extracted from the discussion of central extensions of Chevalley groups in Steinberg's notes [13, Section 7].)

To determine ε it suffices to compute u^ε. Start with the equations:

$$u^w = (-1)\hat{}\, w(-\varepsilon)\hat{}\,,$$
$$(u^{-1})^w = (u^w)^{-1} = \hat{\varepsilon} w^{-1}\hat{1} = \hat{\varepsilon}\,\varepsilon wu, \quad \text{hence } ((-1)\hat{}\,)^w = \hat{\varepsilon}\,\varepsilon wu,$$

and compute:

$$u^\varepsilon = (u^w)^w = w^{-1}(-1)\hat{}\, w(-\varepsilon)\hat{}\, w = ((-1)\hat{}\,)^w(-\varepsilon)\hat{}\, w$$
$$= \hat{\varepsilon}\,\varepsilon\, w(1-\varepsilon)\hat{}\, w = \hat{\varepsilon}((1-\varepsilon)\hat{}\,)^w = u^\varepsilon((1-\varepsilon)\hat{}\,)^w.$$

Thus $u^\varepsilon = u^\varepsilon((1-\varepsilon)\hat{}\,)^w$ and it follows that $\varepsilon = 1$. This completes the proof.

5.2. *Bad groups*

The bulk of our current knowledge concerning bad groups is contained in the following result.

Theorem 1. *Let G be connected nonsolvable centerless ω-stable and suppose that every proper definable subgroup has rank at most 1. For $a \in G - 1$ let $A = C(a)$. Then:*

(1) *$A = N(A)$ is connected abelian of rank 1,*
(2) *$G = \bigcup_G A^g$.*
(3) *Either G has prime exponent p or A is divisible.*
(4) *All finite subgroups of G are commutative of odd order.*

Proof. Let $X = \bigcup_G A^{\circ g}$, $X' = \bigcup_G C(A^\circ)^g$. Let $N = N(A^\circ)$. Then $N \neq G$, for otherwise G/A° would be solvable, and so G would be solvable. Thus $[N : A^\circ] < \infty$.

If $A^\circ \cap A^{\circ g} \neq 1$, then since G is centerless we find $A = A^{\circ g}$, $g \in N$. It follows easily that rank $X =$ rank G. If now $b \in G - X'$ and $B = C(b)^\circ$, $Y = \bigcup_G B^g$, then again rank $Y = \text{rank}\, G$, so $X \cap Y$ is nontrivial. We may assume without loss of generality that $A \cap B \neq 1$. It then follows that $A^\circ = B$, so $b \in C(A^\circ) \subseteq X'$, a contradiction. Thus $G = X'$.

Now we show $C(A^\circ) = A^\circ$, so that $G = X$. We know that rank $(G - X) < \text{rank}\, G$ and all centralizers have rank 1. By a slight extension of Theorem 2.3(2(a)) (with k infinite) for each conjugacy class K rank $G \leq \text{rank}\, K + 1$. It follows that $G - X$ is a finite union of conjugacy classes. We can conclude that $C(A^\circ) \cap G - X$ is finite, because the G-conjugacy classes in $C(A^\circ)$ are finite: if both b, $b^g \in C(A^\circ)$, then A°, $A^{\circ g} \subseteq C(b^g)$, so $A^{\circ g} = A^{\circ g}$, $g \in N$ – and N/A° is finite.

Thus if $C(A^\circ) \neq A^\circ$, then the infinite set $C(A^\circ) - A^\circ$ meets X. If $b = a^g \in (C(A^\circ) - A^\circ) \cap X$ with $a \in A^\circ$, then $A^\circ, A^{\circ g} \subseteq C(b)$, so $g \in N$, $b = a^g \in A^\circ$, a contradiction. Hence we have shown that $C(A^\circ) = A^\circ$, $G = X$.

Now A° is of prime exponent of divisible, so G is of prime exponent or A° is divisible. We will now show that $A^\circ = N(A^\circ)$, so that in particular $A^\circ = A$ and points (1)–(3) have been proved.

Suppose first that G has exponent p and $\alpha \in N(A^\circ)$ is viewed as an automorphism of A°. Define $1 - \alpha$ by:

$$a^{1-\alpha} = a(a^\alpha)^{-1}.$$

Since $(1-\alpha)^p = 0$, $\text{Ker}\,(1-\alpha)$ is infinite. Hence $\text{Ker}\,(1-\alpha) = A^\circ$, $\alpha = 1$ on A°, $\alpha \in C(A^\circ) = A^\circ$.

If on the other hand A° is divisible and $n = [N : A^\circ]$, then the subgroup $\langle x, A^\circ \rangle \subseteq C(x^n)$ for $x \in N$. Thus either $x \in A^\circ$ or $C(x^n) = G$, $x^n = 1$. If $x^n = 1$ let us assume that x has prime order p. Since $G = X$, A° has elements or order p, so $\text{Ker}\,(1-x)^p \neq 1$, and hence $\text{Ker}\,(1-x) \neq 1$. Thus $A^\circ \cap C(x) \neq 1$, hence $A^\circ = C(x)^\circ$. Then $x \in C(A^\circ) = A^\circ$, as desired.

Finally we treat the fourth point. If $i \in G$ is an involution then there is a conjugate involution j not commuting with i. Let $A = C(ij)$. Then $i \notin A$ but $ji \in A \cap A^i$, contradicting $A = N(A)$. Thus G has no involution, and every finite subgroup H of G has odd order and is therefore solvable. Let $A \lhd H$ be abelian. For $a \in A - 1$, since $A \subseteq C(a)$ it follows that H normalizes $C(a)$, so $H \subseteq N(C(a)) = C(a)^\circ$ is commutative.

Corrollary 2. *Let G be a connected nonsolvable locally finite ω-stable group. Then G contains a proper definable subgroup of rank at least 2. If in particular* rank $G = 3$, *then G is either* SL (2, K) *or* PSL (2, K) *for some locally finite algebraically closed field K.*

Proof. The proof is immediate in view of Theorem 1(4) above and Theorem 5.1. (Simple locally finite groups have involutions, see e.g. [6].)

Lemma 3. *With the above notation and hypotheses let $b \notin A$, $B = C(b)$. Then:*

(1) $bAb \cap A = \emptyset$,

(2) *ABA has rank* 3.

Proof. (1) If $ba_1b=a_2$, then $b^{-1}a_2b^{-1}=a_1$; since $a_1a_2=a_2a_1$ we get $b^{-1}a_2b^{-1}ba_1b=ba_1bb^{-1}a_2b^{-1}$, so $a_2a_1=b^2a_1a_2b^{-2}$, hence $a_2a_1=e$ or $b^2\in A\cap B=1$. But $b^2\neq 1$, so $a_2=a_1^{-1}$, $ba_1b=a_1^{-1}$, $(ba_1)^2=1$, so $ba_1=1$, a contradiction.

(2) Define $b_1\equiv b_2$ iff $b_2\in Ab_1A$. If the equivalence classes of $\equiv$ are finite, then ABA has rank 3, for given a fixed $b\in B$ and

$$a_1ba_2=a_3ba_4$$

we get $ba_2a_4^{-1}b^{-1}=a_1^{-1}a_2$. Suppose then that $\equiv$ has an infinite (hence cofinite) equivalence class. In particular for some $b\in B$ we have $b\equiv b^{-1}$. Then

$$a_1ba_2=b^{-1}, \qquad ba_2b=a_1^{-1},$$

contradicting part 1.

Corollary 4. *Assume* rank $G=3$. *Then given the above notation and hypotheses there are* $x_1,\ldots,x_k\in G$ *such that*

$$G=ABA\cup Ax_1A\cup\cdots\cup Ax_kA$$

and for $b\in B$, $AbA\cap B$ *is finite; also* $Ax_iA\neq Ax_i^{-1}A$ *for* $x_i\notin A$. *For any rank* 1 *subgroup* C, $C\cap ABA$ *is cofinite in* C.

Proof. For the final comment, if $C\cap Ax_iA$ is infinite, then taking inverses $C\cap Ax_i^{-1}A$ is infinite. But C has degree 1, a contradiction.

6. Conclusion

The following assertion seems reasonable:

Main Conjecture. *Every simple ω-stable group is an algebraic group over an algebraically closed field.*

There does not seem to be any method in sight for attacking this problem in full generality (even if we restrict it to groups of finite Morley rank, as is reasonable). A more manageable problem is obtained by restricting attention to the class of *linear* ω-stable groups, where by definition a linear group is a structure consisting of a group G of linear transformations of a finite-dimensional vector space V, where G is equipped with its structure as an abstract group together with its action on the underlying set of V.

A proof of the Main Conjecture for linear groups would be very interesting because it would show that as far as the structure theory of algebraic groups over algebraically closed fields is concerned, the notions and methods of algebraic geometry employed at present could be abandoned entirely, apart from the notion of dimension embodied in the concept of Morley rank.

A very different case of the Main Conjecture is obtained by specializing to the class of locally finite groups of finite Morley rank, dropping the linearity assumption. The sort of inductive analysis initiated here can perhaps be pushed through using the methods of the theory of finite simple groups in the manner of [6].

Presumably a solution to either of the above problems would be very elaborate technically.

The main conjecture includes the conjecture that a simple ω-stable group has finite Morley rank (a conjecture based more on ignorance than on conviction). Indeed, examples of ω-stable groups of infinite Morley rank seem hard to come by, so I will take this opportunity to mention two ways of constructing such groups (no proofs will be given).

(1) The Mal'cev correspondence. Associate to the ring R the group $G(R)$ of upper triangular unipotent $n\times n$ matrices (n at least 3). Then R and $G(R)$ are mutually interinterpretable, and the Morley rank of $G(R)$ is $n(n-1)/2$ times that of R. Examples of ω-stable rings of infinite Morley rank were noted in [4] - just take an extension of an algebraically closed field by an infinite set of mutually annihilating indeterminates with square $=0$.

(2) $\mathrm{GL}(n, K/F)$. Let $F\subseteq K$ be a pair of algebraically closed fields. Let $\mathrm{GL}(n, K/F)$ be the group of $n\times n$ matrices with coefficients in K and determinant in $F^{\cdot}$. Then $\mathrm{GL}(n, K/F)$ and the field K with distinguished subfield F are mutually interpretable, so $\mathrm{GL}(n, K/F)$ has infinite Morley rank.

This group arises naturally in the context of differentially closed fields K, where F is the subfield of constants. Perhaps more incisive examples can be defined in this context.

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References

[1] J. Baldwin, α_T is finite for $\aleph_1$-categorical T, Trans. Am. Math. Soc. 181 (1973) 37–51.
[2] J. Baldwin and J. Saxl, Logical stability in group theory, J. Austral. Math. Soc. 21 (1976) 267–276.
[3] W. Baur, G. Cherlin and A. Macintyre, Totally categorical groups, J. Algebra. 57 (1979) 407–440.
[4] G. Cherlin and J. Reineke, Categoricity and stability of commutative rings, Ann. Math. Logic 10 (1976) 367–399.
[5] J. Humphreys, Linear Algebraic Groups (Springer, New York, 1975).
[6] O. Kegel and B. Wehrfritz, Locally Finite Groups (North-Holland, Amsterdam, 1973).
[7] A. Macintyre, On ω_1-categorical theories of abelian groups, Fund. Math. 70 (1971) 253–270.
[8] A. Macintyre, On ω_1-categorical theories of fields, Fund. Math. 71 (1971) 1–25.
[9] M. Morley, Categoricity in power, Trans. Am. Math. Soc. 114 (1965) 514–538.

[10] W.V.O. Quine, On what there is, in: From a Logical Point of View (Harvard Univ. Press, Cambridge, MA, 1953).
[11] G. Sacks, Saturated Model Theory (Benjamin, Reading, MA, 1972).
[12] S. Shelah, The lazy model theoretician's guide to stability, Logique et Analyse 71–72 (1975) 242–308.
[13] S. Shelah, Classification Theory and the Number of Non-Isomorphic Models (North-Holland, Amsterdam, 1978).
[14] R. Steinberg, Chevalley groups, Mimeographed notes, Yale University.
[15] J. Reineke, Minimale Gruppen, Z. Math. Logik Grundlagen Math. 21 (1975) 357–379.
[16] B. Zil'ber, Groups and rings with categorical theories (in Russian), Fund. Math. 95 (1977) 173–188.
[17] D. Gorenstein, Finite Groups (Harper and Row, New York, 1968).
[18] G. Cherlin and S. Shelah, Superstable fields (in preparation).