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# Evolution of Inelastic Plane Curves

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**Abstract**—We derive the evolution equations for an inelastic plane curve, i.e., a curve whose length is preserved over all time. Necessary and sufficient conditions for an inelastic curve flow are expressed as a partial differential equation involving the curvature. Some examples of inelastic curve flows are considered for illustration. © 1999 Elsevier Science Ltd. All rights reserved.

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## INTRODUCTION

Recently, the study of the motion of inelastic plane curves has arisen in a number of diverse engineering applications. Chirikjian and Burdick [1] describe the motion of a planar hyper-redundant (or snake-like) robot as the flow of a plane curve, while Brockett [2] explicitly proposes the idea of an inelastic string machine as a robotic device. Another example from manufacturing is in mold design. In manufacturing a desired mold, it is often useful to have intermediate stages of the final design at successive levels of deformation while maintaining an isoperimetric equality throughout. This is also the case in sheet metal forming, where the goal is to bend the raw material into a desired shape without elongation or compression. The raw material may be a cylinder, for example, and the desired product may have a uniform cross-sectional shape specified by a simple closed curve. (This problem is essentially a simplified one-dimensional version of the more general developable surface design problem, see, e.g., [3].) One would like, if possible, a smoothly varying time-evolution from the original to final desired shape. Curve flows also arise in the context of many problems in computer vision, see, e.g., [4].

What many of the above problems share in common is the need to mathematically describe the time evolution of inelastic plane curves, i.e., plane curves whose lengths are preserved. In this article, we derive a set of partial differential equations that characterize the time evolution of such curves. It is well known that a plane curve can be uniquely specified up to rigid-body motion by its curvature; we derive necessary and sufficient conditions on the curvature function such that the curve flow is inelastic. Our main interest will be in simple closed plane curves, although many of our results carry over to more general plane curves.

Despite their growing importance in applications, inelastic plane curves have not received much attention in the literature. Closely related to the spirit of our approach is the work of Gage and

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Hamilton [5] and Grayson [6], who study the shrinking of closed plane curves to a circle via the heat equation. However, here the flows are not required to be inelastic, and the focus is rather on the conditions under which the curve will smoothly shrink to a point in the limit.

The article is organized as follows. We begin with some preliminaries on flows of plane curves, and derive necessary and sufficient conditions for a plane curve flow to be inelastic. Section 3 presents the main result, which are the partial differential equations for an inelastic curve flow expressed in terms of the curvature function. In Section 4, we consider some basic cases of inelastic curve flows, and discuss some properties of their solutions.

## FLOWS OF INELASTIC PLANE CURVES

Throughout this article, we assume that $F : S^1 \times [0,\tau) \to \Re^2$ is a family of smooth closed curves in the plane, $T$ its positively oriented unit tangential vector, $N$ the inward unit normal vector, and $\kappa$ the signed curvature. Let $S^1$ be parametrized by the local coordinate $u$ over the range $[0, L]$, where $L$ is the arclength of the initial curve. Then recall that the arclength of the curve is given as a function of $u$ by

$$s(u) = \int_0^u \left| \frac{\partial F}{\partial u} \right| du.$$

Defining $v = |\frac{\partial F}{\partial u}|$, the operator $\frac{\partial}{\partial s}$ is given by $\frac{\partial}{\partial s} = 1/v \frac{\partial}{\partial u}$, while the arclength parameter is $ds = v\,du$. See, e.g., [7] for a review of curve theory.

Clearly, any flow of $F$ can be expressed in the following form:

$$\frac{\partial F}{\partial t} = fT + gN,$$

where $f, g$ are, respectively, the tangential and normal speeds of the curve. Observe that since the curves we consider are closed, both $f$ and $g$ are smooth periodic scalar functions. Also recall from the Frenet equations that

$$\frac{\partial T}{\partial u} = v\kappa N, \qquad \frac{\partial N}{\partial u} = -v\kappa T.$$

Before deriving the necessary and sufficient conditions for an inelastic curve flow, we require the following lemma.

LEMMA 1. *Given $f, g, v$, and $\kappa$ as above,*

$$\frac{\partial v}{\partial t} = \frac{\partial f}{\partial u} - gv\kappa.$$

PROOF. Since $v^2 = \langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \rangle$ and noting that $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute, we have

$$\begin{aligned} 2v\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t} \left\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \right\rangle \\ &= 2\left\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial t}\right)\right\rangle \\ &= 2\left\langle \frac{\partial F}{\partial u}, \frac{\partial}{\partial u}(fT + gN)\right\rangle \\ &= 2\left\langle vT, \frac{\partial f}{\partial u}T + fv\kappa N + \frac{\partial g}{\partial u}N - gv\kappa T\right\rangle \\ &= 2v\left(\frac{\partial f}{\partial u} - gv\kappa\right), \end{aligned}$$

from which the lemma follows immediately.

Recall that the arclength is given up to a constant by $s(u,t) = \int_0^u v\,du$. For an inelastic flow, we then require that

$$\frac{\partial}{\partial t}s(u,t) = \int_0^u \frac{\partial v}{\partial t}\,du = 0,$$

for all $u \in [0, L]$. The necessary and sufficient conditions for inelastic flow are then given by the following theorem.

THEOREM 1. *Let*

$$\frac{\partial F}{\partial t} = fT + gN$$

*be a smooth flow of the curve $F$. The flow is inelastic if and only if $\frac{\partial f}{\partial s} = g\kappa$.*

PROOF. From Lemma 1,

$$\begin{aligned}\frac{\partial}{\partial t}s(u,t) &= \int_0^u \frac{\partial v}{\partial t}\,du \\ &= \int_0^u \left(\frac{\partial f}{\partial u} - gv\kappa\right)du \\ &= 0,\end{aligned}$$

for all $u \in [0, L]$. This implies that $\frac{\partial f}{\partial u} = gv\kappa$, or $1/v\frac{\partial f}{\partial u} = g\kappa$, or $\frac{\partial f}{\partial s} = g\kappa$ as claimed. The argument can be reversed to show sufficiency, completing the proof.

An interesting corollary of the above theorem is that in order to inelastically deform an arbitrary curve into a circle (which is one of our primary applications), the tangential component in the flow equations must be nonzero.

COROLLARY 1. *Any inelastic curve flow into a circle requires the tangential component $f$ to be nonzero.*

PROOF. Suppose the tangential component $f = 0$. Then $\frac{\partial f}{\partial s} = g\kappa = 0$. Since the curve is simply closed, clearly there exists some $u_0 \in [0, L]$ such that $\kappa(u_0, t) \neq 0$. By the continuity of $\kappa$, there exists an interval $I_0 \in [0, L]$ such that $u_0 \in I_0$ and the restriction of $\kappa$ to $I_0$ never vanishes. The restriction of $g$ to $I_0$ is therefore 0. This implies that over $I_0$, $\frac{\partial F}{\partial t} = 0$, or $F(s,t) = F(s)$ is constant with respect to time. The corollary now follows immediately.

## MAIN RESULT

We now restrict ourselves to arclength parametrized curves that undergo purely inelastic deformations. That is, $v = 1$, $\frac{\partial f}{\partial s} = g\kappa$, and the local coordinate $u$ corresponds to the curve arclength $s$. Before stating the main result, we require the following lemma.

LEMMA 2.

$$\begin{aligned}\frac{\partial T}{\partial t} &= \left(f\kappa + \frac{\partial g}{\partial s}\right)N, \\ \frac{\partial N}{\partial t} &= -\left(f\kappa + \frac{\partial g}{\partial s}\right)T.\end{aligned}$$

PROOF. Noting that $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ commute, we have $\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\frac{\partial F}{\partial s} = \frac{\partial}{\partial s}\frac{\partial F}{\partial t}$, or

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial}{\partial s}(fT + gN) \\ &= \frac{\partial f}{\partial s}T + f\kappa N + \frac{\partial g}{\partial s}N + g(-\kappa T) \\ &= \left(f\kappa N + \frac{\partial g}{\partial s}\right)N,\end{aligned}$$

where the last equality follows from Theorem 1. Also, since $\langle T, N\rangle = 0$,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\langle T, N\rangle \\ &= \left\langle \frac{\partial T}{\partial t}, N \right\rangle + \left\langle T, \frac{\partial N}{\partial t} \right\rangle \\ &= f\kappa + \frac{\partial g}{\partial s} + \left\langle T, \frac{\partial N}{\partial t} \right\rangle. \end{aligned}$$

Since $\langle N, N\rangle = 1$ implies $\langle N, \frac{\partial N}{\partial t}\rangle = 0$, $\frac{\partial N}{\partial t}$ and $T$ are parallel. Therefore,

$$\frac{\partial N}{\partial t} = -\left(f\kappa + \frac{\partial g}{\partial s}\right) T,$$

which completes the proof.

Recall by the fundamental theorem of the local theory of curves that if the curvature and torsion are given, then the curve is uniquely determined up to a rigid body motion. Since we only address plane curves, the torsion is zero. Hence, once the curvature is determined over the domain $[0, L] \times [0, T)$, the curve $F(s,t)$ can be uniquely determined from the initial conditions $F(s, 0) = F_0(s)$. The following theorem states the conditions on the curvature for the curve flow $F(s,t)$ to be inelastic.

THEOREM 2. *The curve flow* $\frac{\partial F}{\partial t} = fT + gN$ *is inelastic if and only if*

$$\frac{\partial \kappa}{\partial t} = g\kappa^2 + f\frac{\partial \kappa}{\partial s} + \frac{\partial^2 g}{\partial s^2},$$

*where* $\kappa$ *denotes the curvature of the curve.*

PROOF. From the previous lemma,

$$\frac{\partial}{\partial s}\frac{\partial T}{\partial t} = \left(\frac{\partial f}{\partial s}\kappa + f\frac{\partial \kappa}{\partial s} + \frac{\partial^2}{\partial s^2}\right) N - \left(f\kappa + \frac{\partial g}{\partial s}\right)\kappa T.$$

Similarly,

$$\frac{\partial}{\partial t}\frac{\partial T}{\partial s} = \frac{\partial}{\partial t}(\kappa N) = \frac{\partial \kappa}{\partial t} N - \kappa\left(f\kappa + \frac{\partial g}{\partial s}\right) T$$

and from the commutativity of $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$, it follows that

$$\frac{\partial \kappa}{\partial t} = \frac{\partial f}{\partial s}\kappa + f\frac{\partial \kappa}{\partial s} + \frac{\partial^2 g}{\partial s^2} = \frac{\partial}{\partial s}\left(f\kappa + \frac{\partial g}{\partial s}\right).$$

Invoking Theorem 1 now completes the proof.

## DISCUSSION

In this section, we consider some special cases of inelastic curve flows. First, let $\theta$ be the angle between the $x$-axis and the tangent vector $T$, so that $T = (\cos\theta, \sin\theta)$ and $N = (-\sin\theta, \cos\theta)$. By the chain rule and the Frenet equations,

$$\frac{\partial T}{\partial s} = \frac{\partial \theta}{\partial s}\cdot(-\sin\theta, \cos\theta) = \kappa N = \kappa\cdot(-\sin\theta, \cos\theta).$$

Therefore, $\kappa = \frac{\partial \theta}{\partial s}$. For a simple closed curve, we also have that $\int_0^L k\, ds = \theta(L) - \theta(0) = 2\pi$. Moreover,

$$F(s,t) = \int T\, ds = \left(\int \cos\theta(s)\, ds, \int \sin\theta(s)\, ds\right)$$

up to appropriate constants. We now consider an example.

EXAMPLE 1. Let the length $L = 2\pi$, the final time $\tau = 1$, and $\kappa_0(s) = \kappa(s,0)$ be the curvature function of the curve $F(s,0)$. Suppose the curvature varies linearly,

$$\kappa(s,t) = (1-t)\kappa_0(s) + \frac{2\pi}{L}t.$$

Then $\kappa(s,0) = \kappa_0(s)$, $\kappa(s,1) = (2\pi)/L$, and $\frac{\partial\kappa}{\partial t} = -\kappa_0(s) + (2\pi)/L$. Also, since $\kappa = \frac{\partial\theta}{\partial s}$, we have $\frac{\partial\theta}{\partial t} = \int \frac{\partial\kappa}{\partial t}\, ds$. Since $F$ can be expressed as $F = (\int \cos\theta(s)\, ds, \int \sin\theta(s)\, ds)$, it can be verified that

$$\frac{\partial F}{\partial t} = \left(\int \left[-\sin\theta \int \left(\frac{2\pi}{L} - \kappa_0\right) ds\right] ds, \quad \int \left[\cos\theta \int \left(\frac{2\pi}{L} - \kappa_0\right) ds\right] ds\right).$$

The tangential and normal components of the curve flow are then given by

$$f = \cos\theta \int \left[-\sin\theta \int \left(\frac{2\pi}{L} - \kappa_0\right) ds\right] ds + \sin\theta \int \left[\cos\theta \int \left(\frac{2\pi}{L} - \kappa_0\right) ds\right] ds,$$
$$g = -\sin\theta \int \left[-\sin\theta \int \left(\frac{2\pi}{L} - \kappa_0\right) ds\right] ds + \cos\theta \int \left[\cos\theta \int \left(\frac{2\pi}{L} - \kappa_0\right) ds\right] ds.$$

A straightforward calculation verifies that the conditions for Theorems 1 and 2 are satisfied, i.e., $\frac{\partial f}{\partial s} = g\kappa$ and $\frac{\partial\kappa}{\partial t} = \frac{\partial}{\partial s}\left(f\kappa + \frac{\partial g}{\partial s}\right)$, so that a linear variation in the curvature results in an inelastic curve flow.

EXAMPLE 2. Returning to the example of smooth deformation into a circle, consider the choice $g = \kappa - (2\pi)/L$. In this case, the normal component of the flow is proportional to $\kappa$, with $g = 0$ when $\kappa = (2\pi)/L$. Then

$$\frac{\partial f}{\partial s} = \kappa\left(\kappa - \frac{2\pi}{L}\right)$$

from Theorem 1, and

$$\frac{\partial\kappa}{\partial t} = \frac{\partial}{\partial s}\left(\kappa\int \kappa\left(\kappa - \frac{2\pi}{L}\right) ds + \frac{\partial g}{\partial s}\right)$$

from Theorem 2.

EXAMPLE 3. Suppose $f = (1/2)e^{\kappa^2}$. Then

$$\frac{\partial f}{\partial s} = \frac{1}{2}2\kappa e^{\kappa^2}\frac{\partial\kappa}{\partial s} = e^{\kappa^2}\frac{\partial\kappa}{\partial s}\kappa = g\kappa,$$

for an inelastic flow. Hence, $g = e^{\kappa^2}\frac{\partial\kappa}{\partial s}$, and by Theorem 2,

$$\frac{\partial\kappa}{\partial t} = \frac{\partial}{\partial s}\left(\frac{1}{2}\kappa e^{\kappa^2} + e^{\kappa^2}\frac{\partial^2\kappa}{\partial s^2} + 2\kappa e^{\kappa^2}\left(\frac{\partial\kappa}{\partial s}\right)^2\right).$$

REMARK. In this paper, we do not explicitly address the question of the existence of solutions to the partial differential equations in the previous and other examples.

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