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Fekete–Szegö problem for starlike and convex functions of complex order

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ABSTRACT

For nonzero complex b let $\mathcal{F}_n(b)$ denote the class of normalized univalent functions f satisfying $\operatorname{Re}\left[1 + (z(D^n f)'(z)/D^n f(z) - 1)/b\right] > 0$ in the unit disk $\mathcal{U}$, where $D^n f$ denotes the Ruscheweyh derivative of f. Sharp bounds for the Fekete–Szegö functional $\left|a_3 - \mu a_2^2\right|$ are obtained.

1. Introduction

Fekete and Szegö proved a noticeable result that the estimate

$$|a_3 - \lambda a_2^2| \le 1 + 2\exp\left(\frac{-2\lambda}{1-\lambda}\right)$$

holds for any normalized univalent function

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1.1}$$

in the open unit disk $\mathcal{U}$ and for $0 \le \lambda \le 1$. This inequality is sharp for each λ (see [1]). The coefficient functional

$$\Phi(f) = a_3 - \lambda a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\lambda}{2}[f''(0)]^2\right)$$

on normalized analytic functions f in the unit disk represents various geometric quantities, for example when $\lambda = 1$, $\Phi(f) = a_3 - a_2^2$, becomes $S_f(0)/6$ where S_f denotes the Schwarzian derivative $(f''/f')' - (f''/f')^2/2$. Note that, if we consider the nth root transform $[f(z^n)]^{1/n} = z + c_{n+1}z^{n+1} + c_{2n+1}z^{2n+1} + \cdots$ of f with the power series (1.1), then $c_{n+1} = a_2/n$ and $c_{2n+1} = a_3/n + (n-1)a_2^2/2n^2$, so that

$$a_3 - \lambda a_2^2 = n\left(c_{2n+1} - \mu c_{n+1}^2\right),$$

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where $\mu = \lambda n + (n-1)/2$. Moreover, $\Phi(f)$ behaves well with respect to the rotation, namely $\Phi(\mathrm{e}^{-\mathrm{i}\theta} f(\mathrm{e}^{\mathrm{i}\theta} z)) = \mathrm{e}^{2\mathrm{i}\theta}\Phi(f)$, $\theta \in \mathbb{R}$.

This is quite natural to discuss the behavior of $\Phi(f)$ for subclasses of normalized univalent functions in the unit disk. This is called Fekete-Szegö problem. Actually, many authors have considered this problem for typical classes of univalent functions (see, for instance [2–7,1,8–12]).

We denote by $\mathcal{S}$ the set of all functions normalized analytic and univalent in the unit disk $\mathcal{U}$ of the form (1.1). Also, for $0 \le \alpha < 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^c(\alpha)$ denote classes of starlike and convex univalent functions of order α, respectively, i.e.

$$\mathcal{S}^*(\alpha) = \left\{ f(z) \in \mathcal{S} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathcal{U} \right\} \tag{1.2}$$

and

$$\mathcal{S}^c(\alpha) = \left\{ f(z) \in \mathcal{S} : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in \mathcal{U} \right\}. \tag{1.3}$$

A notions of α-starlikeness and α-convexity were generalized onto a complex order α by Nasr and Aouf [13], Wiatrowski [14], Nasr and Aouf [15].

Observe that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{S}^c(0) = S^c$ represent standard starlike and convex univalent functions, respectively.

Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ be analytic functions in $\mathcal{U}$. The *Hadamard product (convolution)* of f and g, denoted by $f * g$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in \mathcal{U}.$$

Let $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The *Ruscheweyh derivative* of the n^{th} order of f, denoted by $D^n f(z)$, is defined by

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}, \quad n \in \mathbb{N}_0.$$

Ruscheweyh [16] determined that

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+1)(k-1)!} a_k z^k. \tag{1.4}$$

The Ruscheweyh derivative gave an impulse for various generalization of well known classes of functions. Exemplary, for α $(0 \le \alpha < 1)$ and $n \in \mathbb{N}_0$, Ahuja [17,18] defined the class of functions, denoted $\mathcal{R}_n(\alpha)$, which consists of univalent functions of the form (1.1) that satisfy the condition

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n f(z)} > \alpha, \quad z \in \mathcal{U}. \tag{1.5}$$

We note that $\mathcal{R}_0(\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{R}_1(\alpha) = \mathcal{S}^c(\alpha)$. The class $\mathcal{R}_n(0) = \mathcal{R}_n$ was studied by Singh and Singh [19].

With the aid of Ruscheweyh derivative Kumar et al. [20] introduced the class $\mathcal{F}_n(b)$ of function $f \in \mathcal{S}$ as follows:

Definition 1.1 (*[20]*). Let b be a nonzero complex number, and let f be an univalent function of the form (1.1), such that $D^n f(z) \neq 0$ for $z \in \mathcal{U} \setminus \{0\}$. We say that f belongs to $\mathcal{F}_n(b)$ if

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{z(D^n f(z))'}{D^n f(z)} - 1 \right) \right\} > 0, \quad z \in \mathcal{U}. \tag{1.6}$$

By giving specific values to n and b, we obtain the following important subclasses studied by various researchers in earlier works, for instance, $\mathcal{F}_0(b) = \mathcal{S}^*(1-b)$ (Nasr and Aouf [13]), $\mathcal{F}_1(b) = \mathcal{S}^c(1-b)$ (Wiatrowski [14], Nasr and Aouf [15]). Moreover, when $\alpha \in \langle 0, 1)$ $\mathcal{F}_n(1-\alpha) = \mathcal{R}_n(\alpha)$ (Singh and Singh [19], Darus and Akbarally [21]).

2. Main results

We denote by $\mathcal{P}$ a class of the analytic functions in $\mathcal{U}$ with $p(0 = 1)$ and $\operatorname{Re} p(z) > 0$. We shall require the following:

Lemma 2.1 (*[22], p. 166*). *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, then*

$$|c_n| \le 2, \quad \text{for } n \ge 1.$$

If $|c_1| = 2$ then $p(z) \equiv p_1(z) = (1 + \gamma_1 z)/(1 - \gamma_1 z)$ with $\gamma_1 = c_1/2$. Conversely, if $p(z) \equiv p_1(z)$ for some $|\gamma_1| = 1$, then $c_1 = 2\gamma_1$ and $|c_1| = 2$. Furthermore we have

$$\left| c_2 - \frac{c_1^2}{2} \right| \le 2 - \frac{|c_1|^2}{2}.$$

If $|c_1| < 2$ *and* $\left|c_2 - \frac{c_1^2}{2}\right| = 2 - \frac{|c_1|^2}{2}$, *then* $p(z) \equiv p_2(z)$, *where*

$$p_2(z) = \frac{1 + z\frac{\gamma_2 z + \gamma_1}{1+\bar{\gamma}_1\gamma_2 z}}{1 - z\frac{\gamma_2 z + \gamma_1}{1+\bar{\gamma}_1\gamma_2 z}},$$

and $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2 - c_1^2}{4-|c_1|^2}$. *Conversely if* $p(z) = p_2(z)$ *for some* $|\gamma_1| < 1$ *and* $|\gamma_2| = 1$, *then* $\gamma_1 = c_1/2$, $\gamma_2 = \frac{2c_2-c_1^2}{4-|c_1|^2}$ *and* $\left|c_2 - \frac{c_1^2}{2}\right| = 2 - \frac{|c_1|^2}{2}$.

Theorem 2.2. *Let* $n \geq 0$ *and let* b *be nonzero complex number. If* f *of the form* (1.1) *is in* $\mathcal{F}_n(b)$, *then*

$$|a_2| \leq \frac{2\,|b|}{(n+1)}, \tag{2.1}$$

$$|a_3| \leq \frac{2\,|b|}{(n+1)(n+2)} \max\{1, |1+2b|\}, \tag{2.2}$$

and

$$\left|a_3 - \frac{n+1}{n+2}a_2^2\right| \leq \frac{2|b|}{(n+1)(n+2)}.$$

Equality in (2.1) *holds if* $z(D^n f(z))'/(D^n f(z)) = 1 + b[p_1(z) - 1]$, *and in* (2.2) *if* $z(D^n f(z))'/(D^n f(z)) = 1 + b[p_2(z) - 1]$, *where* p_1, p_2 *are given in* Lemma 2.1.

Proof. Denote $F(z) = D^n f(z) = z + A_2 z^2 + A_3 z^3 + \cdots$, then

$$A_2 = (n+1)a_2, \qquad A_3 = \frac{(n+1)(n+2)}{2}a_3. \tag{2.3}$$

By the definition of the class $\mathcal{F}_n(b)$ there exists $p \in \mathcal{P}$ such, that $\frac{zF'(z)}{F(z)} = 1 - b + bp(z)$, so that

$$\frac{z(1 + 2A_2 z + 3A_3 z^2 + \cdots)}{z + A_2 z^2 + A_3 z^3 + \cdots} = 1 - b + b(1 + c_1 z + c_2 z^2 + \cdots),$$

which implies the equality

$$z + 2A_2 z^2 + 3A_3 z^3 + \cdots = z + (A_2 + bc_1)z^2 + (A_3 + bc_1 A_2 + bc_2)z^3 + (bc_1 A_3 + bc_2 A_2 + bc_3 + A_4)z^4 + \cdots.$$

Equating the coefficients of both sides we have

$$A_2 = bc_1, \qquad A_3 = \frac{b}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{(1+2b)}{4}bc_1^2, \tag{2.4}$$

so that, on account of (1.4)

$$a_2 = \frac{b}{n+1}c_1, \qquad a_3 = \frac{b}{(n+1)(n+2)}\left[c_2 + b\,c_1^2\right]. \tag{2.5}$$

Taking into account (2.5) and Lemma 2.1, we obtain

$$|a_2| = \left|\frac{b}{n+1}c_1\right| \leq \frac{2\,|b|}{n+1}, \tag{2.6}$$

and

$$\begin{aligned} |a_3| &= \left|\frac{b}{(n+1)(n+2)}\left[c_2 - \frac{c_1^2}{2} + \frac{1+2b}{2}c_1^2\right]\right| \\ &\leq \frac{|b|}{(n+1)(n+2)}\left[2 - \frac{|c_1|^2}{2} + \frac{|1+2b|}{2}\,|c_1|^2\right] \\ &= \frac{|b|}{(n+1)(n+2)}\left[2 + |c_1|^2\,\frac{|1+2b| - 1}{2}\right] \\ &\leq \frac{2|b|}{(n+1)(n+2)} \max\{1, [1 + |1+2b| - 1]\}. \end{aligned}$$

Thus

$$|a_3| \leq \frac{2\,|b|}{(n+1)(n+2)} \max\{1, |1+2b|\}.$$

Moreover

$$\begin{aligned}\left|a_3 - \frac{n+1}{n+2}a_2^2\right| &= \left|\frac{b}{(n+1)(n+2)}\left(c_2 + bc_1^2\right) - \frac{b^2c_1^2}{(n+1)^2}\frac{n+1}{n+2}\right| \\ &= \left|\frac{bc_2}{(n+1)(n+2)}\right| \\ &\leq \frac{2|b|}{(n+1)(n+2)},\end{aligned}$$

as asserted. □

Remark. In the above Theorem a special case of Fekete-Szegö problem e.g. for real $\mu = (n+1)/(n+2)$ occurred very naturally and simple estimate was obtained.

Now, we consider functional $\left|a_3 - \mu a_2^2\right|$ for complex μ.

Theorem 2.3. *Let b be a nonzero complex number and let $f \in \mathcal{F}_n(b)$. Then for $\mu \in \mathbb{C}$*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2|b|}{(n+1)(n+2)} \max\left\{1, \left|1 + 2b - 2\mu b\frac{n+2}{n+1}\right|\right\}.$$

For each μ there is a function in $\mathcal{F}_n(b)$ such that equality holds.

Proof. Applying (2.5) we have

$$\begin{aligned}a_3 - \mu a_2^2 &= \frac{b}{(n+1)(n+2)}\left[c_2 + bc_1^2\right] - \mu\frac{b^2c_1^2}{(n+1)^2} \\ &= \frac{b}{(n+1)(n+2)}\left[c_2 + bc_1^2 - \frac{\mu b(n+2)}{(n+1)}c_1^2\right] \\ &= \frac{b}{(n+1)(n+2)}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(1 + 2b - 2\mu b\frac{n+2}{n+1}\right)\right].\end{aligned}$$

Then, with the aid of Lemma 2.1, we obtain

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{|b|}{(n+1)(n+2)}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left|1 + 2b - 2\mu b\frac{n+2}{n+1}\right|\right] \\ &= \frac{|b|}{(n+1)(n+2)}\left[2 + \frac{|c_1|^2}{2}\left(\left|1 + 2b - 2\mu b\frac{n+2}{n+1}\right| - 1\right)\right] \\ &\leq \frac{2|b|}{(n+1)(n+2)} \max\left\{1, \left|1 + 2b - 2\mu b\frac{n+2}{n+1}\right|\right\}.\end{aligned}$$

An examination of the proof shows that equality is attained for the first case, when $c_1 = 0$, $c_2 = 2$, then the functions in $\mathcal{F}_n(b)$ is given by

$$\frac{z(D^n f(z))'}{D^n f(z)} = \frac{1 + (2b-1)z}{1-z}, \tag{2.7}$$

and, for the second case, when $c_1 = c_2 = 2$, so that

$$\frac{z(D^n f(z))'}{D^n f(z)} = \frac{1 + (2b-1)z^2}{1-z^2}, \tag{2.8}$$

respectively. □

We next consider the case, when μ and b are real. Then we have:

Theorem 2.4. *Let $b > 0$ and let $f \in \mathcal{F}_n(b)$. Then for $\mu \in \mathbb{R}$ we have*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{2b}{(n+1)(n+2)}\left[1+2b\left(1-\mu\dfrac{n+2}{n+1}\right)\right] & \text{if} \quad \mu \le \dfrac{n+1}{n+2}, \\ \dfrac{2b}{(n+1)(n+2)} & \text{if} \ \dfrac{n+1}{n+2} \le \mu \le \dfrac{(n+1)(1+2b)}{(n+2)b}, \\ \dfrac{2b}{(n+1)(n+2)}\left[2\mu b\dfrac{n+2}{n+1}-1-2b\right] & \text{if} \quad \mu \ge \dfrac{(n+1)(1+2b)}{b(n+2)}. \end{cases}$$

For each μ there is a function in $\mathcal{F}_n(b)$ such that equality holds.

Proof. First, let $\mu \le \frac{n+1}{n+2} \le \frac{(1+2b)(n+1)}{2b(n+2)}$. In this case (2.5) and Lemma 2.1 give

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &\le \frac{b}{(n+1)(n+2)}\left[2-\frac{|c_1|^2}{2}+\frac{|c_1^2|}{2}\left(1+2b-2\mu b\frac{n+2}{n+1}\right)\right] \\ &\le \frac{2b}{(n+1)(n+2)}\left[1+2b\left(1-\mu\frac{n+2}{n+1}\right)\right].\end{aligned}$$

Let, now $\frac{n+1}{n+2} \le \mu \le \frac{(1+2b)(n+1)}{2b(n+2)}$. Then, using the above calculations, we obtain

$$\left|a_3-\mu a_2^2\right| \le \frac{2b}{(n+1)(n+2)}.$$

Finally, if $\mu \ge \frac{(1+2b)(n+1)}{2b(n+2)}$, then

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &\le \frac{b}{(n+1)(n+2)}\left[2-\frac{|c_1|^2}{2}+\frac{|c_1^2|}{2}\left(2\mu b\frac{n+2}{n+1}-1-2b\right)\right] \\ &= \frac{b}{(n+1)(n+2)}\left[2+\frac{|c_1^2|}{2}\left(2\mu b\frac{n+2}{n+1}-2-2b\right)\right] \\ &\le \frac{2b}{(n+1)(n+2)}\left[2\mu b\frac{n+2}{n+1}-1-2b\right].\end{aligned}$$

Equality is attained for the second case on choosing $c_1=0,\ c_2=2$ in (2.7) and in (2.8) $c_1=2,\ c_2=2, c_1=2i,\ c_2=-2$ for the first and third case, respectively. Thus the proof is complete. □

Remark. (i) Setting $b=1-\alpha$ in the above results, we get the results from [21].

As an analogue to the complex nth starlikeness of a complex order we may introduce the notion of nth convexity of a complex order as follows:

Definition 2.1. Let b be a nonzero complex number, and let f be an univalent of the form (1.1). We say that f belongs to $\mathcal{S}_n^c(b)$ if

$$\operatorname{Re}\left\{1+\frac{1}{b}\frac{z(D^nf(z))''}{(D^nf(z))'}\right\}>0, \quad z\in\mathcal{U}. \tag{2.9}$$

Using the well known Alexander relation $f\in\mathcal{S}^c \iff zf'\in\mathcal{S}^*$ we easily obtain bounds of coefficients and a solution of the Fekete–Szegö problem in $\mathcal{S}_n^c(b)$.

Theorem 2.5. *Let b be a nonzero complex number and let $f\in\mathcal{S}_n^c(b)$. If f of the form (1.1) is in $\mathcal{S}_n^c(b)$, then*

$$|a_2| \le \frac{|b|}{(n+1)},$$
$$|a_3| \le \frac{2(1+2|b|)|b|}{3(n+1)(n+2)},$$

and

$$\left|a_3-\frac{4}{3}\frac{n+1}{n+2}a_2^2\right| \le \frac{2|b|}{3(n+1)(n+2)}.$$

Reasoning in the same line as in the proof of Theorem 2.3 we obtain

Theorem 2.6. *Let b be a nonzero complex number and let $f \in \mathcal{S}_n^c(b)$. Then, for $\mu \in \mathbb{C}$ holds*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2|b|}{3(n+1)(n+2)} \max\left\{1, \left|1 + 2b - \mu \frac{3b}{2} \frac{n+2}{n+1}\right|\right\}.$$

For each μ there is a function in $\mathcal{F}_n(b)$ such that equality holds.

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