**SOME SELF-BLOCKING BLOCK DESIGNS**

Dieter JUNGNICKEL  
*Mathematisches Institut, Justus-Liebig-Universität Giessen, Arndtstr. 2, D-6300 Giessen, F.R. Germany*

To Haim Hanani on the occasion of his 75th birthday.

Let $D$ be a block design which has a blocking set. We call $D$ self-blocking if the following two conditions hold: (i) The committees of $D$ (i.e. the blocking sets of minimum cardinality of $D$) form a block design, which we denote by $D^C$ and (ii) The committees of $D^C$ are precisely the blocks of $D$. (We also say that $D$ and $D^C$ are a pair of mutually blocking block designs, then.) We show that the classical projective planes $PG(2, q^2)$ are self-blocking; the same holds for $PG(2, 3)$ and $PG(2, 5)$ as well as for the classical affine planes $AG(2, q)$ with $q \geq 4$.

1. Introduction

Let $D$ be a finite incidence structure. A subset $S$ of the point set $P$ of $D$ is called a hitting set for $D$, if $S$ meets every block of $D$. If moreover $S$ does not contain any block of $D$, $S$ is called a blocking set for $D$. There are incidence structures not containing any blocking set; for instance, this holds for every Steiner triple system (see Drake [15]). We shall only consider structures $D$ admitting a blocking set in this paper. Then the blocking sets of smallest cardinality will be called the committees of $D$ (following Hirschfeld [16]).

Blocking sets arose in the theory of games, cf. Richardson [19], and have been studied extensively. The systematic investigation of blocking sets begins with Bruen's papers [6, 7] on blocking sets in projective planes. Later blocking sets in more general incidence structures were studied, in particular in affine planes (see Bruen and Silverman [10]), in general block designs (see de Resmini [14] and Drake [15]) and in $(r, \lambda)$-designs (see Jungnickel and Leclerc [18]).

In the present paper, we shall consider blocking sets in block designs and introduce a new type of question about these structures. Let $D$ be a block design admitting blocking sets. We denote by $D^C$ the incidence structure formed by all committees of $D$ (on the point set $P$ of $D$). Our first condition will be as follows:

1. The incidence structure $D^C$ formed by the committees of $D$ is a block design.

Note that this situation will arise quite often. (1) is certainly satisfied whenever $D$ admits a 2-transitive automorphism group. We shall call $D$ a self-blocking block design if it satisfies (1) and also the following condition (2).

2. $D^C$ admits blocking sets, and one has $(D^C)^C = D$; in other words, the committees of $D^C$ are precisely the blocks of $D$ (and vice versa).
In this case, we shall also say that $D$ and $D^C$ form a pair of mutually blocking block designs. The characterisation of all self-blocking block design seems to be a very hard problem, as we shall see. Our main result will be as follows.

**Main Theorem.** The Desarguesian projective planes $\text{PG}(2, q^2)$ and the Desarguesian affine planes $\text{AG}(2, q)$ (with $q \geq 4$) are self-blocking block designs (for every prime power $q$).

By a famous result of Bruen [7], the committees of $\text{PG}(2, q^2)$ are exactly the Baer subplanes. Thus the first half of our Main Theorem will be an immediate consequence of the following slightly stronger assertion: Every subset of $\text{PG}(2, q^2)$ which meets every Baer subplane has at least $q^2 + 1$ points; equality holds if and only if the subset is a line. We shall also use this result to study, more generally, the Baer subplanes of $\text{PG}(n, q^2)$. Finally, we shall also show that the designs $\text{PG}(2, 3)$ and $\text{PG}(2, 5)$ are self-blocking.

It should be mentioned that a related question is studied by Cameron and Mazzocca [12]. These authors prove that the smallest hitting sets of the incidence structure $D^b$ formed by all blocking sets of $D$ are the lines, whenever $D$ is a projective or affine plane containing blocking sets. Since most blocking sets do not contain a committee, this result is—though of a similar flavour—not related to our results. (Our Main Theorem is stronger, but it only applies for the planes $\text{PG}(2, q^2).$) In a sequel to [12], Cameron et al. [13] study those sets hitting every blocking set of $D$ which do not contain a line of $D$ (the so-called dual blocking sets of $D$).

We refer the reader to Beth et al. [1] for background from Design Theory and to Beutelspacher [3] and Hirschfeld [16] for background on blockings sets in projective planes and spaces.

2. Preliminaries

In this section we shall collect some well-known preliminary results on hitting sets and blocking sets of projective planes. The following simple lemma characterizes the smallest hitting sets:

**Lemma 2.1.** Let $D$ be a projective plane of order $n$, and let $S$ be a hitting set for $D$. Then $|S| > n + 1$; equality holds if and only if $S$ is a line of $D$.

We next state a fundamental result of Bruen [7] which gives a lower bound for the size of a blocking set in a projective plane of order $n$ and which implies a characterisation of the committees of $\text{PG}(2, q^2)$. Bruen's original proof was somewhat involved; a simpler proof was given by Bruen and Thas [11]. An even simpler version is a special case of a proof given in Jungnickel and Leclerc [18].
where Bruen’s result was generalized to $(r, \lambda)$-designs following a previous generalization to symmetric designs, due to de Resmini [14] and Drake [15]. A similar proof is also contained in Bruen and Silverman [10].

**Theorem 2.2 (Bruen).** Let $D$ be a projective plane of order $n$, and let $S$ be a blocking set for $D$. Then $|S| \geq n + \sqrt{n} + 1$; equality holds if and only if $S$ is a Baer subplane of $D$.

**Corollary 2.3 (Bruen).** The committees of the Desarguesian projective plane $PG(2, q^2)$ ($q$ a prime power) are precisely the Baer subplanes.

Writing $D = PG(2, q^2)$, we thus have that the blocks of $D^C$ are just the Baer subplanes of $D$. Since $D$ has a 2-transitive group, it is clear that $D^C$ is a design (and thus $D$ satisfies condition (1)). We compute the parameters of $D^C$:

**Proposition 2.4.** Let $D = PG(2, q^2)$, $q$ a prime power. Then the incidence structure $D^C$ (the blocks of which are the Baer subplanes of $D$) is a block design with parameters

\[ v = q^4 + q^2 + 1, \quad k = q^2 + q + 1, \quad b = q^3(q^2 + 1)(q^3 + 1), \]

\[ r = (q^2 + 1)q^3(q + 1) \quad \text{and} \quad \lambda = q^2(q + 1)^2. \]

**Proof.** The number $b$ of Baer subplanes of $PG(2, q^2)$ is well-known, see e.g. Hirschfeld [16, p. 88]. (Since each quadrangle of $D$ determines a unique Baer subplane this can be easily checked by the reader.) Then $r$ is determined from $vr = bk$, and $\lambda$ is obtained from $\lambda(v - 1) = r(k - 1)$. □

It is our aim to show that $D^C$ also satisfies condition (2), i.e. that the blocking sets of $D^C$ are the lines of $D$. We remark that the bounds of Drake [15] and of Jungnickel and Leclerc [18] yield only weak results here. The best result which can be obtained by standard inequalities seems to be the following: It is known that the minimum size of a blocking set $S$ satisfies $s \geq r/\lambda$ (see Jungnickel and Leclerc [18]), which here results in the bound $s \geq q^2 - q + 1$. Thus we require special arguments.

3. Sets meeting all Baer subplanes of $PG(2, q^2)$

In this section we shall prove that a hitting set $S$ for the design $D^C$ of Proposition 2.4 has at least $q^2 + 1$ points (with equality if and only if $S$ is a line of $PG(2, q^2)$). We will proceed by first proving the following result complementing Lemma 2.1:

**Proposition 3.1.** Let $D$ be a projective plane of order $n$, and let $S$ be a set of at
most \( n + 1 \) points of \( D \). Then one has one of the following alternatives:

There are three non-concurrent lines \( L, L', L'' \) which are disjoint from \( S \). 

(3) \( S \) contains \( n \) collinear points.

**Proof.** Assume that both (3) and (4) fail. Let \( G \) be a line that meets \( S \) in at least two points. Since (4) fails, there are two points \( x, x' \) in \( G \setminus S \). Then \( x \) and \( x' \) are on lines \( L \) and \( L' \) disjoint from \( S \), as \( |S| \leq n + 1 \). Since (3) fails, every line must contain a point of \( S \cup \{p\} \) where \( p = L \cap L' \). Considering the lines through \( x \) one sees that \( |S| = n + 1 \). Thus some line \( H \) through \( p \) meets \( S \) in two points. Choose a point \( q \) in \( H \setminus (S \cup \{p\}) \). Then \( q \) lines on a line \( L'' \) disjoint from \( S \cup \{p\} \), a contradiction to the assumption that (3) fails. \( \square \)

**Theorem 3.2.** Let \( S \) be a set of points of \( \text{PG}(2, q^2) \) which meets every Baer subplane. Then \( |S| \geq q^2 + 1 \), and equality holds if and only if \( S \) is a line.

**Proof.** We may assume that \( |S| \leq q^2 + 1 \); the assertion is that \( S \) is a line, then. Assume otherwise. By Proposition 3.1, there are two cases to be considered.

**Case 1.** There are three non-concurrent lines \( L, L', L'' \) which are disjoint from \( S \). Let \( p, q, r \) be the three points of intersection of these lines, and write \( T = L \cup L' \cup L'' \). Then each point not in \( T \) forms together with \( p, q, r \) a quadrangle and thus determines a unique Baer subplane of \( \text{PG}(2, q^2) \). Each such Baer subplane contains exactly \( (q - 1)^2 \) points not in \( T \); thus there are \( (q + 1)^2 \) Baer subplanes containing \( p, q, r \), and these subplanes split the points off \( T \) into \( (q + 1)^2 \) sets of \( (q - 1)^2 \) each. Since \( S \cap T = \emptyset \) and since \( |S| \leq q^2 + 1 \), \( S \) cannot meet all these Baer subplanes, a contradiction.

**Case 2.** \( S \) consists of \( n \) points of a line \( L \) and, possibly, of one further point \( p \) not on \( L \). Denote the unique point of \( L \) not in \( S \) by \( r \), and note that \( \operatorname{Aut} \text{PG}(2, q^2) \) is transitive on triples \( (L, p, r) \) with \( r \in L \) and \( p \notin L \), since it is transitive on triangles. Choose any Baer subplane \( B \), and let \( L' \) be a line meeting \( B \) only once, say in \( r' \). Moreover, choose a point \( p' \) not in \( B' \cup L' \). Mapping \( (L', p', r') \) onto \( (L, p, r) \), we obtain a Baer subplane disjoint from \( S \), a contradiction. \( \square \)

Theorem 3.2 shows that the smallest hitting sets for the design \( D^C \) defined in Proposition 2.4 are the lines of the original design \( D = \text{PG}(2, q^2) \). Since no line contains a Baer subplane, we see that these hitting sets are in fact the committees of \( D^C \); thus \( D^C \) satisfies condition (2) and we have proved the first half of our principal result:

**Theorem 3.3.** The Desarguesian projective plane \( \text{PG}(2, q^2) \) (\( q \) a prime power) is a self-blocking block design.
We shall consider some other designs in the following sections. But first we mention the following consequence of Theorem 3.3.

**Corollary 3.4.** Let $D = \text{PG}(2, q^2)$ and $D^C$ as in Proposition 2.4. Then $\text{Aut} D = \text{Aut} D^C$. In other words: Any bijection of the point set of $\text{PG}(2, q^2)$ which maps every Baer subplane onto a Baer subplane is a collineation of $\text{PG}(2, q^2)$, i.e. a member of $\text{PGL}(3, q^2)$.

Cameron and Mazzocca [12] have shown that any bijection of a projective plane of order $q \neq 2$ which preserves blockings sets is in fact a collineation. Corollary 3.4 strengthens this result for the planes $\text{PG}(2, q^2)$. As already mentioned, the main interest in the sequel [13] is in sets meeting each blocking set of a projective plane and not containing any line. This leads us to the following problem.

**Problem 3.5.** Let $S$ be a set of points of $\text{PG}(2, q^2)$ meeting every Baer subplane and not containing any line. What is the minimum size of $S$? (Note that such sets exist: The simplest example is the complement of a line.)

4. **Committees of $\text{PG}(n, q^2)$**

In this section we shall briefly consider the symmetric design $\text{PG}_{n-1}(n, q)$ with $n \geq 3$, the blocks of which are the hyperplanes of $\text{PG}(n, q)$. By the theorem of Bose and Burton [4], the committees of this design are the lines (if we use the standard definitions for arbitrary incidence structures given above). Thus we would have $D^C = \text{PG}_1(n, q)$ for $D = \text{PG}_{n-1}(n, q)$. Clearly $D^C$ is a design, and the hitting sets of minimal size of $D^C$ are the hyperplanes, i.e. the blocks of $D$ (again using the theorem of Bose and Burton [4]). However, $D$ is not self-blocking, since the hyperplanes are not blocking sets of $D^C$ (they contain lines).

Since the correspondence between lines and hyperplanes sketched above is somewhat trivial, Bruen [8] and Beutelspacher [2] have suggested to impose the stronger condition

\[ (*) \quad S \text{ meets every hyperplane, but } S \text{ contains no line} \]

to define blocking sets in $\text{PG}(n, q)$. To avoid confusion, we shall call such a set $S$ a **strong blocking set**. Using Corollary 2.3 as the starting point for an induction argument, one can prove the following result.

**Theorem 4.1** (Beutelspacher [2], Bruen [8]). Let $S$ be a strong blocking set of $\text{PG}_{n-1}(n, q)$. Then one has $|S| \geq q + \sqrt{q} + 1$; equality holds if and only if $S$ is a Baer subplane of some plane of $\text{PG}(n, q)$. 
Thus the strong committees of $\text{PG}_{n-1}(n, q^2)$ are the Baer subplanes of the planes of $\text{PG}(n, q^2)$. Clearly all these Baer subplanes form a block design; we will not bother determining its parameters. We shall now show that Theorem 3.2 may be used to obtain a lower bound on the cardinality of hitting sets for this design.

**Theorem 4.2.** Let $S$ be a subset of $\text{PG}(n, q)$, $q$ a square, which meets every Baer subplane. Then $|S| \geq q^{n-1} + \cdots + q + 1$.

**Proof.** We use induction on $n$; the case $n = 2$ is true by Theorem 3.2. Now assume that the assertion holds for $n - 1$, where $n \geq 3$. Let $H$ be any hyperplane of $\text{PG}(n, q)$, and put $S_H = S \cap H$. Clearly $S_H$ meets every Baer subplane of $\text{PG}(n, q)$ contained in $H$. Since $H$ is isomorphic to $\text{PG}(n - 1, q)$, we obtain $|S_H| \geq q^{n-2} + \cdots + q + 1$. Now count flags $(p, H)$ where $p$ is a point in $S$ and $H$ a hyperplane to obtain
\[
(q^n + \cdots + q + 1)(q^{n-2} + \cdots + q + 1) \leq |S| (q^{n-1} + \cdots + q + 1),
\]

hence
\[
|S| \geq q^{n-2} + \cdots + q + 1 + q^n(q^{n-2} + \cdots + q + 1)/(q^{n-1} + \cdots + q + 1)
\]
which gives the assertion. \( \square \)

We have not been able to characterize the case of equality in Theorem 4.2. Note that the hyperplanes do give examples, but there might be other ones. Of course, the hyperplanes are not blocking sets of the design formed by the Baer subplanes of $\text{PG}(n, q)$, $q$ a square, and thus $\text{PG}_{n-1}(n, q)$ is not self-blocking for $n \geq 3$, no matter whether one considers ordinary or strong blockings sets. We conclude this section with the following conjecture.

**Conjecture 4.3.** Let $S$ be a subset of $\text{PG}(n, q)$, $q$ a square, which meets every Baer subplane. Then $|S| = q^{n-1} + \cdots + q + 1$ if and only if $S$ is a hyperplane.

### 5. Committees of $\text{PG}(2, 3)$ and $\text{PG}(2, 5)$

In this section we shall show that $\text{PG}(2, 3)$ and $\text{PG}(2, 5)$ are self-blocking. First, let $D = \text{PG}(2, 3)$. It is known that the committees of $D$ are precisely the projective triangles, see Hirschfeld [16, Th. 13.4.41. This means the following (cf. Fig. 1). A committee consists of a triangle $p_1, p_2, p_3$ and of three collinear points $q_1, q_2, q_3$, where $q_i$ is on $p_ip_k$ ($i, j, k$ a permutation of 1, 2, 3). Note that the line $q_1q_2q_3$ contains a unique fourth point $q_4$ (which forms a quadrangle together with the $p_i$'s) and that the line joining the $q_i$'s is the unique line through $q_4$ not containing any $p_i$. So in fact the committees of $D$ are determined by the quadrangles with a distinguished point $q_4$. This shows that any triangle $p_1p_2p_3$ is contained in precisely four committees as the complement of a collinear triple. But since the
triangle \( p_iq_ip_k \) together with the fourth point on \( p_jp_kq_i \) determines the same committee as \( p_1p_2p_3 \) and \( q_4 \), each committee contains four triangles as the complement of a collinear triple. Thus the number of committees agrees with the number of triangles. Hence \( D^C \) is a block design with parameters

\[
v = 13, \quad b = 234, \quad k = 6, \quad r = 108 \quad \text{and} \quad \lambda = 45.
\]

Note that \( \text{PGL}(3, 3) \) acts transitively on committees.

We now claim that the blocking sets of \( D^C \) have size at least 4, and that equality occurs only for the lines of \( D \). (Clearly the lines of \( D \) are blocking sets for \( D^C \).) Thus let \( S \) be a blocking set of \( D^C \) and assume \( |S| \leq 4 \). We have to show that \( S \) is a line. This is accomplished by proving that any other configuration of at most 4 points will be disjoint from a suitable committee. Because of the transitivity properties of \( \text{PGL}(3, 3) \) it is clearly sufficient to consider a committee and to show that every type of configuration of at most 4 points is contained in its complement, excepting lines. This can be seen by elaborating Fig. 1 (see Fig. 2).

Let \( a = p_1q_4 \cap p_2p_3, \ b = p_1p_2 \cap aq_2, \ c = ab \cap p_1q_1, \ d = bq_1 \cap p_1a, \ e = bq_1 \cap p_1p_3, \ f = ae \cap p_1c. \) This gives most of \( \text{PG}(2, 3) \), and the complement of our committee contains both the quadrangle \( abeq_4 \) and the three collinear points \( bde \) together with the point \( a \) not on \( bde \). This proves the assertion. We collect our results:

**Theorem 5.1.** Let \( D = \text{PG}(2, 3) \). Then \( D^C \) is a design \( S_{45}(2, 6, 13) \), and the hitting sets of minimal size of \( D^C \) are the lines of \( D \). Thus \( D \) is self-blocking.
We now turn our attention to the case $D = \text{PG}(2, 5)$. Here the committees are determined by a conic $C$ together with two points $p$ and $q$ on $C$ as follows (cf. Hirschfeld [16, Th. 13.4.7]). Let $r$ be the point of intersection of the tangents at $C$ in $p$ and $q$, and let $L = pq$.

Then $S = (C \cup L \cup \{r\}) \setminus \{p, q\}$ is a committee. Note that $\text{PGL}(2, 5)$ is transitive on committees. Cf. Fig. 3. Clearly the committees form a design $D^C$; the determination of its parameters will be omitted. One can then use arguments similar to those for $\text{PG}(2, 3)$ to show that the smallest hitting sets of $D^C$ are the lines of $D$. The case analysis is, however, more involved. We omit all details and just state the following result.

**Theorem 5.2.** $\text{PG}(2, 5)$ is a self-blocking block design.

In the light of Theorems 5.1 and 5.2, the following problem is natural:

**Problem 5.3.** Is $\text{PG}(2, q)$ self-blocking for all prime powers $q$?

Since at present not even the committees of $\text{PG}(2, q)$ are known (unless $q$ is a square or very small), there seems to be no hope of solving this problem with the present methods. David A. Drake has shown that $\text{PG}(2, 7)$ is also self-blocking (private communication).

### 6. Committees of $\text{AG}(2, q)$

In this section we discuss the committees of the Desarguesian affine plane $\text{AG}(2, q)$, where $q \geq 4$. (It is well known that $\text{AG}(2, 2)$ and $\text{AG}(2, 3)$ do not contain any blocking sets.) We first recall the following fundamental result of Jamison [17].

**Theorem 6.1** (Jamison). Let $S$ be a hitting set of $\text{AG}(2, q)$. Then $|S| \geq 2q - 1$.

A somewhat simpler proof of 6.1 is given in Brouwer and Schrijver [5]. It should be noted that 6.1 does not hold for non-Desarguesian affine planes, see
Bruen and de Resmini [9]. For example, the Hughes plane of order 9 gives rise to an affine plane of order 9 containing a blocking set with 16 points only.

Unfortunately, the case of equality in Theorem 6.1 has not been characterised. In fact it seems that the committees of $AG(2, q)$ have not been discussed in the literature up to now (except for $q = 4$). As we shall see, the case $q = 4$ is exceptional. We thus start by exhibiting three classes of committees of $AG(2, q)$, where $q \geq 5$.

**Example 6.2.** Let $q \geq 5$ be a prime power, and let $D = AG(2, q)$. Choose a triangle $p, q, r$ and put $L = pq$, $L' = qr$. Let $s$ be the intersection point of the lines parallel to $L$ (resp. $L'$) passing through $r$ (resp. $p$), and let $t$ be any point $\neq p, r$ on $pr$. Then $S = (L \cup L' \cup \{s, t\}) \setminus \{p, r\}$ is a blocking set of cardinality $2q - 1$ and thus (by 6.1) a committee of $AG(2, q)$. Cf. Fig. 4.

**Example 6.3.** Let $q \geq 5$ be a prime power, and let $D = AG(2, q)$. Choose a $q$-arc $C$ meeting each line in the parallel class of some line $L$. ($C$ is a parabola obtained from a conic in $PG(2, q)$, where we take a tangent as line at infinity.) Let $p = C \cap L$, and choose a point $r \neq p$ on the tangent at $C$ through $p$. Then $S = (C \cup L \cup \{r\}) \setminus \{p\}$ is a blocking set of cardinality $2q - 1$ and thus a committee of $AG(2, q)$. Cf. Fig. 5.

**Example 6.4.** Let $q$ be any prime power $\geq 3$ and consider a Baer subplane $B$ of $PG(2, q^2)$. Choose a tangent line $L_\infty$ of $B$ and use this line in defining the affine
plane $AG(2, q^2)$. Denote the point of intersection of $B$ and $L_\infty$ by $p$ and write $B' = B \setminus \{p\}$. Then the subset $B'$ of $AG(2, q^2)$ meets every line of $AG(2, q^2)$ excepting the $q^2 - q - 1$ further tangents of $B$ through $p$. Select one point on each of these tangents (arbitrarily, but not using $q^2 - q - 1$ collinear points). Adjoining these points to $B'$ then results in a committee $S$ of $AG(2, q^2)$. Cf. Fig. 6.

**Problem 6.5.** Determine all committees of $AG(2, q)$, where $q \geq 5$.

Since we do not know whether there are any committees of $AG(2, q)$ different from those described in 6.2, 6.3 and 6.4, we cannot compute the parameters of the design $D^C$ formed by the committees of $D = AG(2, q)$. However, $D^C$ clearly is a design, since $\text{Aut } AG(2, q)$ is 2-transitive.

We now consider the case $q = 4$. Note that the constructions of 6.2, 6.3 and 6.4 do not necessarily result in blocking sets here but only in hitting sets: In 6.2, $S$ may contain the line $st$, in 6.3, the point $r$ may be on a line contained in $S$. We first exhibit a class of blocking sets of size 8 (which is a special case of blocking sets used by Cameron and Mazzocca [12]).

**Example 6.6.** Let $L$ and $L'$ be two parallel lines of $AG(2, 4)$, and choose points $p$ and $p'$ on $L$ and $L'$, respectively. Let $r, s$ be the remaining two points on the line $pp'$. Then $S = (L \cup L' \cup \{r, s\}) \setminus \{p, p'\}$ is a blocking set of size 8. Cf. Fig. 7.

There is some confusion in the literature regarding the size of the committees of $AG(2, 4)$. By Theorem 6.1, each hitting set has at least 7 points. Now Bruen and Thas [11] claim that it is easy to construct a blocking set of size 7 in $AG(2, 4)$ by using a Baer subplane of $PG(2, 4)$. On the other hand, Bruen and Silverman
prove the following result in [10]:

If \( S \) is a blocking set in an affine plane of square order \( n \),

\[
|S| \geq n + \sqrt{n} + 2.
\]

(Note that this result has been misquoted in [9] where the condition that \( n \) is a square was omitted.) We shall provide a proof at the end of this section. Note that (5) implies that any blocking set of \( \text{AG}(2, 4) \) has at least 8 points. We shall now give a proof of this fact and also determine the structure of these sets. More precisely, we show the following:

**Proposition 6.7.** All blocking sets of \( \text{AG}(2, 4) \) have 8 points and arise as described in Example 6.6.

**Proof.** Let \( S \) be a blocking set of \( \text{AG}(2, 4) \); as already noted, 6.1 implies \( |S| \geq 7 \). Assume that \( |S| = 7 \). Embed \( \text{AG}(2, 4) \) into the projective plane \( \text{PG}(2, 4) \) and add any point \( p \) on the line at infinity to \( S \). This results in a blocking set \( S' \) of size 8 of \( \text{PG}(2, 4) \). Now Theorem 3 of Bruen and Thas [11] yields two possible cases:

1. **Case 1.** \( S' \) is a Baer subplane \( B \) of \( \text{PG}(2, 4) \) together with a further point \( q \). Clearly \( q \) is one of the points of \( S \), since \( B \) has to meet the line at infinity (in \( p \)). Thus the point \( q \) has to be on the second line of \( \text{PG}(2, 4) \) which meets \( B \) exactly in \( p \). But this line is met by each of the four lines of \( B \) not containing \( p \), and so \( q \) is on one of these lines. Thus \( S \) contains a line of \( \text{AG}(2, 4) \) passing through \( q \), a contradiction.

2. **Case 2.** There is a triangle \( p, q, r \) and a point \( s \) on \( qr \), such that \( S' = (pq \cup pr \cup \{s\}) \setminus \{q, r\} \), see Fig. 8. Note that \( p \) is indeed on the line at infinity. Thus the lines \( pq \) and \( pr \) are parallel in \( \text{AG}(2, 4) \), and \( S \) does not meet one of the parallels of these two lines, a contradiction.

This shows that each blocking set of \( \text{AG}(2, 4) \) contains at least 8 points. Since the complement of a blocking set is also a blocking set, we see that all blocking sets of \( \text{AG}(2, 4) \) have size 8. Standard counting arguments show that \( b_3 = b_1 = 8 \) and \( b_2 = 4 \) where \( b_i \) is the number of \( i \)-secants of a blocking set \( S \) (i.e., of lines that meet \( S \) in exactly \( i \) points). Thus some parallel class of \( \text{AG}(2, 4) \) contains two 3-secants of \( S \). It now follows easily that \( S \) is of the type of Example 6.6.

\[\text{Fig. 8.}\]
Corollary 6.8. Let $D = AG(2, 4)$. Then the committees of $D$ form a resolvable design $D' \subseteq D$ with parameters

$$v = 16, \quad b = 120, \quad k = 8, \quad r = 60 \quad \text{and} \quad \lambda = 28.$$ 

Proof. Left to the reader. \(\square\)

We conclude this section by proving (5); our proof will be different from the one in [10]. Let $A$ be an affine plane of order $n$, where $n$ is a square, and let $S$ be a blocking set of $A$. Bruen and Thas [11] show that $|S| \leq n + \sqrt{n} + 1$. (It is in fact easy to deduce this from Theorem 2.2: Adding a point on the line at infinity to $S$ results in a blocking set of the projective extension $P$ of $A$.) Assume now $|S| = n + \sqrt{n} + 1$. Arguing as in the proof of 6.7, we get a blocking set $S'$ of $P$ with $|S'| = n + \sqrt{n} + 2$. We may assume $n > 4$; then only Case 1 above can occur (see [11, Th. 3]), and we obtain a contradiction as above. Thus we have:

Theorem 6.9 (Bruen and Silverman). Let $S$ be a blocking set in an affine plane of order $n$, where $n$ is a square. Then $|S| \geq n + \sqrt{n} + 2$.

7. Sets meeting all committees of $AG(2, q)$

In this section we prove our second principal result:

Theorem 7.1. Let $D = AG(2, q)$, $q \geq 4$, and let $S$ be a set of points of $D$ which meets every committee. Then $|S| \geq q$, and equality holds if and only if $S$ is a line of $D$.

Proof. We first assume $q \geq 5$. Assume that $S$ meets all committees of $D$, where $|S| \leq q$. We have to show that $S$ is a line, then. In fact we will prove that this assertion already follows from the assumption that $S$ meets all committees of the type described in Example 6.2. To this end, we consider $S$ as a subset of the projective extension $PG(2, q)$ of $D$. By Proposition 3.1, we see that either $S$ is a line of $AG(2, q)$ or that there are three non-concurrent lines of $PG(2, q)$ which are disjoint from $S$. We have to show that the second alternative is impossible. Assume otherwise; then there are two intersecting lines $L$ and $L'$ of $D$ which are disjoint from $S$. We can choose any one of the $q^2 - 2q + 1$ points outside of $L \cup L'$ as the point $s$ described in Example 6.2 by suitably selecting the points $p$ and $r$ on $L$ and $L'$, respectively. Thus there are at least $q^2 - 3q + 1$ choices of $s$ for which $s \notin S$. A computation shows that we may then select $s$ in such a way that there is a point $t$ on $pr$ which is not contained in $s$. But this means that $S$ misses the committee just constructed, a contradiction.

It remains to consider the case $q = 4$. The committees of $AG(2, 4)$ have been determined in Proposition 6.7 (see Fig. 7). Clearly the complement of the committee given in Fig. 7 contains all types of configurations of at most 4 points, excepting the lines. Using the transitivity properties of Aut $AG(2, 4)$ this will
yield the assertion (cf. the analysis for PG(2, 3)). The details are left to the reader. □

Corollary 7.2. The Desarguesian affine plane AG(2, q), q ≥ 4, is a self-blocking block design.

Acknowledgements

The final version of this paper was written while the author was a visiting professor at the University of Waterloo. He would like to thank this institution for its hospitality and NSERC for financial support under grant IS-0367. He is also grateful to Lynn Batten for providing him with a preprint copy of [10]. Last but not least, thanks are due to a referee for his detailed comments which resulted in several improvements of the presentation of this paper.

References