On $\varepsilon$-Equilibrium Point in a Noncooperative $n$-Person Game

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In this paper, we describe a noncooperative $n$-person game in strategic form (or normal form) and introduce $\varepsilon$-equilibrium point. We give mainly the characterization of such an $\varepsilon$-equilibrium point by applying Ekeland's theorem. © 1991 Academic Press, Inc.

1. INTRODUCTION

Much of the earliest work about game theory was introduced and was investigated by von Neumann and Morgenstern [9]. Both individual stability and collective stability have been studied in practical game problems. In view of individual stability in noncooperative $n$-person games, the concept of equilibrium point was introduced by Nash [8]. The concept is an extended one of saddle point in two-person zero-sum games. Such the equilibrium points have been investigated by many authors.

In [1–2], the Nash theorem for the existence of noncooperative equilibrium point is proved by means of the Ky Fan fixed point theorem and some selection theorems for fixed points [6–7]. But, in order to prove the Nash theorem, we need to assume the stronger condition such that a strategy set for each player is compact. So, we want to weaken the compactness condition. In this paper, we describe a noncooperative $n$-person game in strategic form (or normal form) and give a definition of $\varepsilon$-equilibrium point. Excluding the compactness condition of the strategy set for each player, we shall study the characterization of $\varepsilon$-equilibrium point in the $n$-person game. Then, Ekeland's theorem will play an important role [4–5].

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This paper is organized in the following way. In Section 2, we formulate a noncooperative \(n\)-person game and define \(\varepsilon\)-equilibrium point. Section 3 is the main part of this paper. Associated with the results of [1], we give the characterization of \(\varepsilon\)-equilibrium point in the game. Especially, using the results of Borwein [3], we replace the compactness condition by the closedness condition in Theorem 3.8.

2. FORMULATION OF A NONCOOPERATIVE \(n\)-PERSON GAME

We define a noncooperative \(n\)-person game by the following strategic form

\[(N, X, F), \tag{2.1}\]

where

(i) \(N = (1, 2, \ldots, n)\) is the set of \(n\) players.

(ii) \(X = \prod_{i=1}^{n} X^i \subseteq \mathbb{U} = \prod_{i=1}^{n} U^i\), for each \(i \in N\), \(X^i\) is the subset of a Banach space \(U^i\) and is called the strategy set of each player \(i\).

(iii) \(F = (f^1, f^2, \ldots, f^n) : X \rightarrow \mathbb{R}^n\) is a multiloss operator and, for each \(i \in N\), \(f^i : X^i \rightarrow \mathbb{R}\), denotes a loss function for player \(i\).

In this paper, denoting by \(i = N - i\) the coalition adverse to each player \(i\), the multistrategy set, \(X = \prod_{i=1}^{n} X^i\) is split as follows

\[X = X^i \times X^i \quad \text{and} \quad X^i = \prod_{j \neq i} X^j.\]

If \(\pi^i\) and \(\pi^j\) denote the projection from \(X\) into \(X^i\) and \(X^j\), we set \(x^i = \pi^i x\) and \(x^j = \pi^j x\) for a multistrategy \(x = (x^i, x^j) \in X\).

Now, we define, for each \(i \in N\),

\[\alpha^i = \inf_{x^i \in X^i} f^i(x)\]

and, throughout this paper, we assume that \(\alpha^i > - \infty\) for all \(i \in N\). In this case, the game is bounded below and \(\alpha - (\alpha^1, \alpha^2, \ldots, \alpha^n)\) is called shadow minimum of the game. Then, we have

\[F(X) \in \alpha + \mathbb{R}^n, \]

where

\[F(X) = \{ F(x) \in \mathbb{R}^n; \text{ for all } x = (x^1, x^2, \ldots, x^n) \in X \} \]
and
\[ R^n_+ = \{ x = (x^1, x^2, \ldots, x^n) \in R^n ; \; x^i \geq 0 \; \text{for all} \; i \in N \} . \]

If \( x = F(\bar{x}) \) belongs to \( F(X) \), the multistrategy \( \bar{x} \in X \) attains to the minimum of the loss function \( f^i \) for each player \( i \). In this case, \( \bar{x} \) is the best solution for each player. But, this situation is seldom the case and we have to investigate other solution concepts. So, we consider especially noncooperative equilibrium point.

**DEFINITION 2.1.** A multistrategy \( x = (x^1, x^2, \ldots, x^n) \in X \) is said to be \( \varepsilon \)-equilibrium point if, for some \( \varepsilon \geq 0 \) and all \( i \in N \),
\[ f^i(x) \leq \inf_{y \in X, x_{-i} = x_{-i}} f^i(y) + \varepsilon. \] (2.2)

**Remark 2.1.** If \( \varepsilon = 0 \) in Definition 2.1, the \( \varepsilon \)-equilibrium point \( x \) is called noncooperative equilibrium point or Nash equilibrium point and, given the complementary coalition's choice \( x_i \), player \( i \) responds by playing a strategy \( x_i \in X_i \) which minimizes \( f^i(\cdot, x_i) \) on \( X_i \), that is
\[ f^i(x', x_i) = \inf_{y \in X, x_{-i} = x_{-i}} f^i(y). \] (2.3)

**Remark 2.2.** If \( N = \{1, 2\} \) and \( f^1(x) + f^2(x) = 0 \) for all multistrategies \( x \in X = X_1 \times X_2 \), \( \varepsilon \)-equilibrium point is called \( \varepsilon \)-saddle point in the two-person zero-sum game.

3. **CHARACTERIZATION OF \( \varepsilon \)-EQUILIBRIUM POINT IN THE \( n \) PERSON GAME**

In order to show the characterization of \( \varepsilon \)-equilibrium point in the game, we introduce the function \( \varphi : X \times X \rightarrow \mathbb{R} \), defined by
\[ \varphi(x, y) = \sum_{i=1}^n [ f^i(x) - f^i(y, x_i) ]. \]

**THEOREM 3.1.** Suppose that, for some \( \varepsilon \geq 0 \) and a multistrategy \( \bar{x} \in X \), \( \varphi \) satisfies
\[ \sup_{y \in X} \varphi(\bar{x}, y) \leq \varepsilon. \] (3.1)

Then, \( \bar{x} \) is an \( \varepsilon \)-equilibrium point.

**Proof.** Let \( \bar{x} = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) \in X \) be the multistrategy satisfying (3.1).
We set \( y = (y^i, \bar{x}^i) \). Then, we have \( \pi^j y = \bar{x}^i \) for all \( j \neq i \) and \( \pi^i y = y^i \). It follows from (3.1) that
\[
\varphi(\bar{x}, y) = f^i(\bar{x}) - f^i(y) \leq \varepsilon.
\]
Therefore, we obtain
\[
f^i(\bar{x}) \leq \inf_{y \in X, \pi^j y = \bar{x}^i} f^i(y) + \varepsilon. \tag{3.2}
\]
Thus, (3.2) is true for each \( i \in N \), which proves the result.

Remark 3.1. We assume that, for each \( i \in N \), the strategy set \( X^i \) is a convex compact subset and the loss function \( f^i \) is continuous and \( f^i(\cdot, x^i) \) is convex for all \( x^i \in X^i \). Then, using the Ky Fan theorem, we can show that there exists a multistrategy \( \bar{x} \in X \) satisfying (3.1) with \( \varepsilon = 0 \). In this case, the multistrategy \( \bar{x} \) is a Nash equilibrium point.

Theorem 3.2. Suppose that \( \bar{x} \) is an \( \varepsilon \)-equilibrium point. For all multistrategies \( y \in X \),
\[
\varphi(\bar{x}, y) \leq n \varepsilon.
\]

Proof. From the definition of \( \varepsilon \)-equilibrium point, it follows that, for all \( i \in N \) and all \( y^i \in X^i \),
\[
f^i(\bar{x}) - f^i(y^i, \bar{x}^i) \leq \varepsilon.
\]
Adding the above inequalities, we obtain
\[
\varphi(\bar{x}, y) \leq n \varepsilon \quad \text{for all } y \in X.
\]
Thus, the proof is completed.

Now, we introduce the conjugate function defined by
\[
f^i*(p_i; x^i) = \sup_{y \in U, \pi^j y = x^i} [\langle p_i, y^i \rangle - f^i(y^i, x^i)] \quad \text{for all } p_i \in U^i,*
\]
where \( U^i,* \) denotes the dual space of the Banach space \( U^i \). Then, the conjugate function is lower semi-continuous convex function with respect to \( p_i \). See [1, p. 61, Sect. 2.4].

Definition 3.1. For some \( \varepsilon \geq 0 \), \( f^i*(\cdot; x^i): U^i,* \to R \), is called \( \varepsilon \)-subdifferentiable at \( \bar{p} \in U^i,* \) if there is \( x^{i,**} \in U^{i,**} \) such that
\[
f^i*(p; x^i) - f^i*(\bar{p}; x^i) \geq \langle x^{i,**}, p - \bar{p} \rangle - \varepsilon \quad \text{for all } p \in U^i,*
\]
and this \( x^i \ast \ast \) is called an \( \varepsilon \)-subgradient of \( f^i \ast (\cdot; x^i) \) at \( \tilde{p} \). The set of all subgradients is called \( \varepsilon \)-subdifferential of \( f^i \ast (\cdot; x^i) \) at \( \tilde{p} \in U^i \ast \) and is denoted by \( \partial_\varepsilon f^i \ast (\tilde{p}; x^i) \).

**Remark 3.2.** The \( \varepsilon \)-subdifferential \( \partial_\varepsilon f^i \ast (\tilde{p}; x^i) \) is a point-to-set mapping from \( U^i \ast \) into \( 2^{U^i \ast} \) and it may be empty if \( f^i \ast \) is not \( \varepsilon \)-subdifferentiable. If \( \varepsilon = 0 \) in the definition, \( \partial_0 f^i \ast (\tilde{p}; x^i) \) is said to be subdifferential of \( f^i \ast (\cdot; x^i) \) at \( \tilde{p} \).

**Theorem 3.3.** If \( \bar{x} = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) \) is \( \varepsilon \)-equilibrium point when \( X = U \), then for each \( i \in N \),

\[
\bar{x}^i \in \partial_\varepsilon f^i \ast (0; \bar{x}^i). \tag{3.3}
\]

**Proof.** Let

\[
f^i \ast (0; \bar{x}^i) = -\inf_{y \in U, \pi_y = \bar{x}^i} f^i(y, \bar{x}^i)
\]

(by the definition of the conjugate function)

\[
\leq -f^i(\bar{x}) + \varepsilon \quad (\bar{x} \text{ is } \varepsilon \text{-equilibrium point})
\]

\[
\leq \langle 0 - p_i, \bar{x}^i \rangle + f^i \ast (p_i; \bar{x}^i) + \varepsilon. \tag{3.4}
\]

Thus, it follows from (3.4) that

\[
\bar{x}^i \in \partial_\varepsilon f^i \ast (0; \bar{x}^i).
\]

This completes the proof.

**Theorem 3.4.** Suppose that, for a multistrategy \( \bar{x} \in X \) and all \( i \in N \), the loss function \( f^i(\cdot, \bar{x}^i) \) is lower semi-continuous convex function on \( U^i \) and

\[
\bar{x}^i \in \partial_\varepsilon f^i \ast (0; \bar{x}^i). \tag{3.5}
\]

Then, this \( \bar{x} \in X \) is an \( \varepsilon \)-equilibrium point.

**Proof.** From (3.5), it follows that, for all \( i \in N \) and all \( p_i \in U^i \ast \)

\[
f^i \ast (p_i; \bar{x}^i) - f^i \ast (0; \bar{x}^i) \geq \langle \bar{x}^i, p_i \rangle - \varepsilon,
\]

that is,

\[
\varepsilon - f^i \ast (0; \bar{x}^i) \geq \langle \bar{x}^i, p_i \rangle - f^i \ast (p_i; \bar{x}^i). \tag{3.6}
\]
Since for all $i \in N$, each loss function $f^i(\cdot, \bar{x}^i)$ is lower semi-continuous and convex, (3.6) shows that, for all $i \in N$,

$$\varepsilon + \inf_{y \in U^i} f^i(y^i, \bar{x}^i) \geq f^i(\bar{x}^i; \bar{x}^i)$$

($f^{\ast\ast}$ is the conjugate function of $f^\ast$)

$$= f^i(\bar{x}),$$

which completes the proof.

Now, in order to study the relations between $\varepsilon$-equilibrium points and differentiability of the loss function for each player $i$, we at first give the following definition.

**DEFINITION 3.2.** For any $x^i \in U^i$, the loss function $f^i(\cdot, x^i)$ is said to be Gâteaux-differentiable if at every point $x^i \in U^i$ there is a continuous linear functional $D_i f^i(x^i, x^i) \in U^{i*}$ such that, for any $y^i \in U^i$,

$$\lim_{t \to 0^+} \frac{1}{t} [f^i(x^i + ty^i, x^i) - f^i(x^i, x^i)] = \langle D_i f^i(x^i, x^i), y^i \rangle.$$

**THEOREM 3.5.** Let the strategy set $X^i$ for each player be closed subset in $U^i$. We assume a multistrategy $\bar{x} = (\bar{x}^1, \bar{x}^2, ..., \bar{x}^n) \in \text{int} X$ is $\varepsilon$-equilibrium point with $\varepsilon > 0$ and, for each $i \in N$, $f^i(\cdot, \bar{x}^i)$ is lower semi-continuous and Gâteaux-differentiable on $U^i$. Then, there exists $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, ..., \tilde{x}^n) \in X$ such that

$$\|\tilde{x} - \bar{x}\| \leq n \sqrt{\varepsilon}$$

(3.7)

and

$$\|D_i f^i(\tilde{x}^i, \bar{x}^i)\|_* \leq \sqrt{\varepsilon} \quad \text{for each } i \in N,$$

(3.8)

where $\text{int} X$ denotes the set of all interior points of $X$ and $\|\cdot\|_*$ denotes the norm in the dual space $U^{i*}$.

**Proof:** From the condition, the loss function $f^i(\cdot, \bar{x}^i)$ is lower semi-continuous on $U^i$. The game is bounded from below and $\bar{x}$ is $\varepsilon$-equilibrium point, that is,

$$f^i(\bar{x}) \leq \inf_{y \in X^i} f^i(y^i, \bar{x}^i) + \varepsilon.$$

Then, applying Ekeland's theorem (see Appendix) to $f^i(\cdot, \bar{x}^i)$, there exists a point $\tilde{x}^i \in X^i$ such that

$$\|\tilde{x}' - \bar{x}'\| \leq \sqrt{\varepsilon},$$

(3.9)
and
\[ f'(y', \bar{x}') \geq f'(\bar{x}', \bar{x}') - \sqrt{\varepsilon} \| y' - \bar{x}' \| \quad \text{for all } y' \in X'. \tag{3.10} \]

Take any \( u' \in U^i \) and set \( y' = \bar{x}' + tu' \) in (3.10), with sufficiently small \( t > 0 \). This holds
\[ \frac{1}{t} \left[ f'(\bar{x}' + tu', \bar{x}') - f'(\bar{x}', \bar{x}') \right] \geq -\sqrt{\varepsilon} \| u' \|. \]

Letting \( t \to 0 \) in the above inequality, we obtain
\[ \langle D_i f'(\bar{x}', \bar{x}'), u' \rangle \geq -\sqrt{\varepsilon} \| u' \|. \tag{3.11} \]

Taking the infimum of both sides of (3.11) over all \( u' \in U^i \) with \( \| u' \| = 1 \), we obtain
\[ -\| D_i f'(\bar{x}', \bar{x}') \|_* \geq -\sqrt{\varepsilon}. \]

Thus, the proof is completed.

Remark 3.3. By a similar argument to Theorem 3.5, it is well known that \( \| D_i f'(\bar{x}) \|_* = 0 \) for a Nash equilibrium point \( \bar{x} \in \text{int } X \).

We can show the following theorem similar to [4, Proposition 12] when we do not assume loss function of each player, \( f^i \) itself to be differentiable.

**Theorem 3.6.** Let a multistrategy \( \bar{x} \) be Nash equilibrium point when \( X \) is closed and \( \bar{x} \in \text{int } X \). Assume that, for every \( \varepsilon > 0 \), there exists a lower semi-continuous and Gâteaux-differentiable function \( f^i(\cdot, \bar{x}') \) such that
\[ f^i_\varepsilon(y', \bar{x}') \leq f^i(y', \bar{x}') \quad \text{for all } y' \in X^i, \tag{3.12} \]
\[ \inf_{y' \in X^i} f^i_\varepsilon(y', \bar{x}') \geq \inf_{y' \in X^i} f^i(y', \bar{x}') - \varepsilon, \tag{3.13} \]

and
\[ D_i f^i_\varepsilon(y', \bar{x}') \to \Phi^i(\bar{x}) \quad \text{as } \varepsilon \to 0 \text{ and } y' \to \bar{x}'. \tag{3.14} \]

Then, for each \( i \in N \),
\[ \Phi^i(\bar{x}) = 0 \quad \text{in } U^i \ast. \]

**Proof.** Since \( \bar{x} \) is Nash equilibrium point on \( X \), for each \( i \in N \), \( \bar{x}' \) minimizes \( f^i(\cdot, \bar{x}') \) on \( X^i \). So, it follows from (3.12) and (3.13) that
\[ f^i_\varepsilon(\bar{x}) \leq \inf_{y' \in X^i} f^i_\varepsilon(y, \bar{x}') + \varepsilon. \]
From Ekeland's theorem, it follows that there exists \( u^*_e \in X^i \) such that
\[
\| u^*_e - \tilde{x}^i \| \leq \sqrt{\varepsilon}
\]
and \( u^*_e \in X^i \) minimizes \( F_e(\cdot, \tilde{x}^i) = f_e^i(\cdot, \tilde{x}^i) + \sqrt{\varepsilon} \| u^*_e - \cdot \| \) on \( X^i \). Now, \( f_e^i(\cdot, \tilde{x}^i) \) is differentiable at \( u^*_e \) and the function \( y^i \to \| u^*_e - y^i \| \), although not differentiable at \( u^*_e \), has a directional derivative in every direction. It follows easily that \( u^*_e \) have to satisfy the condition
\[
0 \in D_i f^i_e(u^*_e, \tilde{x}^i) + \sqrt{\varepsilon} B^*
\]
with \( B^* \) the unit ball of \( U'^* \). Letting \( \varepsilon \to 0 \), \( u^*_e \) converges to \( \bar{x}^i \) and the left-hand side of (3.15) converges to \( \Phi^i(\tilde{x}) \) because of (3.14). This proves the result.

**Theorem 3.7.** Suppose that, for each \( i \in N \), \( f^i(\cdot, \tilde{x}^i) \) is convex on \( U^i \) and
\[
\| D_i f^i(\tilde{x}) \|_* \leq \varepsilon.
\]
Then, for each \( i \in N \),
\[
f^i(\bar{x}) \leq f^i(y^i, \tilde{x}^i) + \varepsilon \| y^i - \tilde{x}^i \| \quad \text{for all } y^i \in U^i.
\]

**Proof.** Since \( f^i(\cdot, \tilde{x}^i) \) is convex on \( U^i \) and Gâteaux-differentiable, we have for all \( y^i \in U^i \),
\[
f^i(y^i, \tilde{x}^i) \geq f^i(\tilde{x}) + \langle D_i f^i(\tilde{x}), y^i - \tilde{x}^i \rangle.
\]
It follows from (3.16) and (3.17) that
\[
f^i(y^i, x^i) \geq f^i(\tilde{x}) - \| D_i f^i(\tilde{x}) \|_* \| y^i - \tilde{x}^i \|
\]
\[
\geq f^i(\tilde{x}) - \varepsilon \| y^i - \tilde{x}^i \|,
\]
which completes the proof of the theorem.

**Theorem 3.8.** Suppose that, for each \( i \in N \), the strategy set \( X^i = U^i \) is reflexive, \( \bar{x} = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) \) is \( \varepsilon \)-equilibrium point with \( \varepsilon > 0 \), and the loss function \( f^i(\cdot, \bar{x}^i) \) is lower semi-continuous convex on \( V^i \). Then, there exists a point \( \bar{x}_e = (\bar{x}^1_e, \bar{x}^2_e, \ldots, \bar{x}^n_e) \in X \) and \( \bar{x}_e^* = (\bar{x}^1_e^*, \bar{x}^2_e^*, \ldots, \bar{x}^n_e^*) \in U^* = \prod_{i=1}^n U^i \) such that for each \( i \in N \),
\[
\| \bar{x}^i_e - \bar{x}^i \| \leq \sqrt{\varepsilon}
\]
\[
\| \bar{x}^i_e \|_* \leq \sqrt{\varepsilon}.
\]
Further, \( \bar{x}_e \) is a Nash equilibrium point of the game \((N, X, F)\), where
$F = (f^1, f^2, \ldots, f^n)$ is a multiloss operator and, when the players choose $x = (x^1, x^2, \ldots, x^n) \in X$, $f^i(x) = f^i(x^i, \tilde{x}^i) - \langle \tilde{x}^i, x^i \rangle$ denotes a loss for each player $i$.

**Proof.** Since $\tilde{x} \in X$ is an $\varepsilon$-equilibrium point, it follows from Theorem 3.3 that, for each $i \in N$,

$$\tilde{x}^i \in \partial \varepsilon f^i(0; \tilde{x}^i). \quad (3.20)$$

We set

$$g^i(p_i, \tilde{x}^i) = f^i*(p_i; \tilde{x}^i) - \langle p_i, \tilde{x}^i \rangle.$$

Then, $g^i(p_i, \tilde{x}^i)$ is lower semi-continuous convex function with respect to $p_i$ because the conjugate function $f^i*(\cdot; \tilde{x}^i)$ is lower semi-continuous and convex on $U^i*$. From (3.20), we easily obtain

$$g^i(0, \tilde{x}^i) \leq \inf_{p_i \in U^i*} g^i(p_i, \tilde{x}^i) + \varepsilon. \quad (3.21)$$

So, $g^i(p_i, \tilde{x}^i)$ is bounded from below on $U^i*$. Applying Ekeland's theorem to $g^i$, it follows that there exists a point $\tilde{x}^i* \in U^i*$ such that, for all $p_i \neq \tilde{x}^i*$,

$$g^i(p_i, \tilde{x}^i) > g^i(\tilde{x}^i*, \tilde{x}^i) - \sqrt{\varepsilon} \| p_i - \tilde{x}^i* \|_*. \quad (3.22)$$

and

$$g^i(\tilde{x}^i*, \tilde{x}^i) \leq g^i(0, \tilde{x}^i) - \sqrt{\varepsilon} \| \tilde{x}^i* \|_*. \quad (3.23)$$

From (3.22), $\tilde{x}^i*$ minimizes the function $G(\cdot, \tilde{x}^i) = g^i(\cdot, \tilde{x}^i) + \sqrt{\varepsilon} \| \cdot - \tilde{x}^i* \|_*$ on $U^i*$, so that $\tilde{x}^i*$ satisfies the following condition

$$0 \in \partial (g^i(\tilde{x}^i*, \tilde{x}^i) + \sqrt{\varepsilon} h^i(\tilde{x}^i*, \tilde{x}^i)),$$

where

$$h^i(p_i, \tilde{x}^i) = \| p_i - \tilde{x}^i* \|_*.$$

Using the subdifferential sum formula, (3.24) can be written as

$$0 \in \partial f^i*(\tilde{x}^i*; \tilde{x}^i) - \tilde{x}^i + \sqrt{\varepsilon} B^*$$

with $B^*$ the unit ball in $U^i*$. So, since $U^i$ is reflexive, there exists $\tilde{x}^i_c \in U^{i**} = U^i$ such that

$$\tilde{x}^i_c \in \partial f^i*(\tilde{x}^i*; \tilde{x}^i) \quad (3.25)$$
and
\[
\tilde{x}_e^i = \tilde{x}^i - \sqrt{\epsilon} \ p^*_i, \quad \|p^*_i\| \leq 1.
\] (3.26)

From (3.26), we obtain
\[
\|\tilde{x}_e^i - \tilde{x}^i\| \leq \sqrt{\epsilon} \|p^*_i\| \leq \sqrt{\epsilon},
\]
which shows that (3.18) in the theorem holds. Further, from (3.21) and (3.23), it follows easily that
\[
\|\tilde{x}_e^i\|^* \leq \sqrt{\epsilon}.
\]

Thus, (3.19) in the theorem holds.

In order to show that \( \tilde{x}_e \) is a Nash equilibrium point of the game \((N, X, F)\), from (3.25) we obtain
\[
f^i*(x_i; \tilde{x}^i) - f^i*(\tilde{x}_e^i; \tilde{x}^i) \geq \langle \tilde{x}_e^i, p_i - \tilde{x}_e^i \rangle \quad \text{for all } p_i \in U^i, \]
that is, for all \( p_i \in U^i, \)
\[
-f^i*(\tilde{x}_e^i; \tilde{x}^i) \geq \langle \tilde{x}_e^i, p_i \rangle - f^i*(p_i; \tilde{x}^i) - \langle \tilde{x}_e^i, \tilde{x}_e^i \rangle. \quad (3.27)
\]

Using the definition of the conjugate function to (3.27), it follows that
\[
-\sup_{y^i \in U^i} [\langle \tilde{x}_e^i, y^i \rangle - f^i(y^i, \tilde{x}^i)] \geq f^i**(\tilde{x}_e^i; \tilde{x}^i) - \langle \tilde{x}_e^i, \tilde{x}_e^i \rangle. \quad (3.28)
\]

Since \( f^i(\cdot, \tilde{x}^i) \) is lower semi-continuous and convex on \( U^i \), from (3.28), we obtain
\[
\inf_{y^i \in U^i} [f^i(y^i, \tilde{x}^i) - \langle \tilde{x}_e^i, y^i \rangle] \geq f^i(\tilde{x}_e^i, \tilde{x}^i) - \langle \tilde{x}_e^i, \tilde{x}_e^i \rangle.
\]
that is,
\[
\min_{y^i \in X, y^i = \tilde{x}_e^i} f^i(y) = f^i(\tilde{x}_e).
\]

Thus, the proof of the theorem is completed.

Remark 3.4. Since the loss function \( f^i(\cdot, \tilde{x}^i) \) for each player in Theorem 3.8 is convex on \( U^i \), \( f^i(\cdot, \tilde{x}^i) \) is Lipschitz on \( U^i \) except in pathological cases by the results of Roberts and Varberg [10]. So, in usual cases, we easily obtain
\[
|f^i(\tilde{x}) - f^i(\tilde{x}_e)| \leq |f^i(\tilde{x}_e, \tilde{x}^i) - f^i(\tilde{x}_e, \tilde{x}^i)| + |\langle \tilde{x}_e^i, \tilde{x}_e^i \rangle| \\
\leq \|\tilde{x}^i - \tilde{x}_e^i\| M + \|\tilde{x}_e^i\|^* \|\tilde{x}_e^i\|,
\]
where $M$ denotes a real number, but it may depend on $\bar{x}^i \in X^i$. Applying (3.18) and (3.19) to the above inequality, it follows that the loss $f^i(\bar{x})$ approximates $f^i(\bar{x})$ for sufficiently small $\varepsilon > 0$. Consequently, the theorem says that if the game $(N, X, F)$ has an $\varepsilon$-equilibrium point, there exists a perturbed game $(N, X, \tilde{F})$ which has a Nash equilibrium point and the points of both these games are close to each other.

**APPENDIX: EKELAND'S THEOREM [4-5]**

Let $(U, d)$ be a complete metric space, and $f: U \rightarrow R \cup \{+\infty\}$ a l.s.c. function, $\not= +\infty$, bounded from below. Let $\varepsilon > 0$ be given, and a point $u \in U$ such that

$$f(u) \leq \inf_U f + \varepsilon.$$

Then there exists some point $v \in U$ such that $f(v) \leq f(u)$, $d(u, v) \leq \sqrt{\varepsilon}$ and

$$f(w) \geq f(v) - \sqrt{\varepsilon} d(v, w) \quad \text{for all } w \in U.$$

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**REFERENCES**