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Exposed Points and Extremal Problems in H^1

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If $\phi \in L^\infty$, we denote by T_ϕ the functional defined on the Hardy space H^1 by

$$T_\phi(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\, \phi(e^{i\theta})\, d\theta/2\pi.$$

Let S_ϕ be the set of functions in H^1 which satisfy $T_\phi(f) = \|T_\phi\|$ and $\|f\|_1 \leqslant 1$. It is known that if ϕ is continuous, then S_ϕ is weak-* compact and not empty. For many noncontinuous ϕ each S_ϕ is weak-* compact and not empty. A complete description of S_ϕ if S_ϕ is weak-* compact and not empty is obtained. S_ϕ is not empty if and only if $S_\phi = S_\psi$ and $\psi = |f|/f$ for some nonzero f in H^1. It is shown that if $\phi = |f|/f$ and $f = pg$, where p is an analytic polynomial and g is a strong outer function, then S_ϕ is weak-* compact. As the consequence, if $f = p$, then S_ϕ is weak-* compact.

1. INTRODUCTION

Let U be the open unit disc in the complex plane and let ∂U be the boundary of U. If f is analytic in U and $\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta$ is bounded for $0 \leqslant r < 1$, then, $f(e^{i\theta})$ which we define to be $\lim_{r\to 1} f(re^{i\theta})$, exists almost everywhere on ∂U. If

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta = \int_{-\pi}^{\pi} \log^+ |f(e^{i\theta})|\, d\theta,$$

then f is said to be the class N_+. The set of all boundary functions in N_+ is denoted by N_+ again. For $0 < p \leqslant \infty$, the Hardy space H^p, is defined by $N_+ \cap L^p$ and $1 \leqslant p \leqslant \infty$, it coincides with the space of functions in L^p whose Fourier coefficients with negative indices vanish. If h in N_+ has the form

$$h(z) = \exp\left\{\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |h(e^{it})|\, dt/2\pi + ia\right\} \qquad (z \in U)$$

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for some real α, h is called an outer function. We call q in N_+ an inner function if $|q(e^{i\theta})| = 1$ a.e. on ∂U. Each nonzero f in H^1 has a unique factorization of the form $f = qh$, where q is an inner function and h is an ouher function.

DEFINITION. Let g be a nonzero function in H^1. The g is called a strong outer function (or strong outer) if it has the following properties: If f is a nonzero function in H^1 and $\arg f(e^{i\theta}) = \arg g(e^{i\theta})$ a.e. on ∂U, then $f = \gamma g$ for some positive constant $\gamma > 0$.

The author [5] determined strong outer functions in the set of all analytic polynomials. It is known (cf. [6]) that if g^{-1} belongs to H^1 or $\operatorname{Re} g(e^{i\theta}) \geqslant 0$ a.e. on ∂U, then g is strong outer. Of course a strong outer function is an outer function but the converse is not valid. In Section 1, we obtain a complete description of S_ϕ in case $\phi = \bar{z}^m |g|/g$, where $m \in Z_+$ and g is strong outer. Z_+ denotes the set of all nonnegative integers. In Section 2, as a consequence of the result in Section 1 we describe S_ϕ if S_ϕ is weak-* compact and not empty. In Section 3, we show that if $\phi = |f|/f$ and $f = pg$, where p is an analytic polynomial and g is strong outer, then S_ϕ is weak-* compact and not empty. In Section 4, we consider the exposed points of S and strong outer functions, where S denotes the unit sphere of H^1.

Let C denote the space of continuous functions on ∂U and set $A = H^\infty \cap C$. Then $H^1 = (C/zA)^*$. S is weak-* compact but in general S_ϕ is not weak-* compact. If $\phi \in C$, then S_ϕ is weak-* compact and not empty.

If $\phi \in L^\infty$, we denote by T_ϕ the functional defined on H^1 by

$$T_\phi(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\, \phi(e^{i\theta})\, d\theta/2\pi.$$

The norm of T_ϕ is $\|T_\phi\| = \sup\{|T_\phi(f)| : f \in S\}$ and let S_ϕ denote the set of all $f \in S$ for which $T_\phi(f) = \|T_\phi\|$. The functions in S_ϕ are called extremal functions. By the duality relation

$$\|T_\phi\| = \|\phi + zH^\infty\|,$$

where $\|\phi + zH^\infty\| = \inf\{\|\phi + h\|_\infty : h \in zH^\infty\}$. A function $\psi = \phi + k_0$ ($k_0 \in zH^\infty$) for which $\|T_\phi\| = \|\psi\|_\infty$ is called an extremal kernel. At least one extremal kernel must exist but it is not unique in general. For some ϕ, S_ϕ may be empty but if ϕ is continuous on ∂U, then S_ϕ is not empty. The author [4] showed that if $\|T_\phi\| > \lim \|T_{z^n\phi}\|$, then S_ϕ is not empty. If ϕ is continuous on ∂U, then $\lim \|T_{z^n\phi}\| = 0$.

2. $\phi = \bar{z}^m |g|/g$

In this section, we prove the main theorem which gives a description of S_ϕ for $\phi = \bar{z}^m |g|/g$, where $m \in Z_+$ and g is strong outer. If $g^{-1} \in H^1$, then it is easy to give the description of S_ϕ, using the following Lemma 1 which was proved by Helson and Sarason [2]. Suppose $S^1 = \{f \in H^1; \|f\|_1 = 1\}$. In particular if $g = 1$, then

$$S_{\bar{z}^m} = \left\{ \gamma \prod_{j=1}^{m} (z - a_j)(1 - \bar{a}_j z) \in S^1; \gamma > 0 \text{ and } 0 \leqslant |a_j| \leqslant 1 \right\}.$$

LEMMA 1. *If $F \in \bar{z}^n H^{1/2}$ is nonnegative a.e. on ∂U, then F is a trigonometric polynomial of degree at most n.*

LEMMA 2. *Let $\phi = \bar{q}|h|/h$, where q is an inner function and h is an outer function in H^1. Suppose $q = q_1 \cdots q_l$ and q_j $(1 \leqslant j \leqslant l)$ is a nonconstant inner function. Then, for any b_j with $|b_j| \leqslant 1$ $(1 \leqslant j \leqslant l)$ there exists $f \in S_\phi$ such that $f(b_1) = \cdots = f(b_l) = 0$.*

Proof. If $f_0 = \gamma_0 h \prod_{j=1}^{l} (q_j - q_j(0))(1 - \overline{q_j(0)}\, q_j) \in S^1$ for some positive constant γ_0, then $f_0 \in S_\phi$. For $\bar{q} \prod_{j=1}^{l} (q_j - q_j(0))(1 - \overline{q_j(0)}\, q_j) \geqslant 0$ a.e. on ∂U. Since $f_0/z^l \in H^1$, for any b_j with $|b_j| \leqslant 1$ $(1 \leqslant j \leqslant l)$

$$\gamma(f_0/z^l) \prod_{j=1}^{l} (z - b_j)(1 - \bar{b}_j z)$$

belongs to S_ϕ for some positive constant γ.

LEMMA 3. *Suppose $\phi = \bar{z}^m |g|/g$ $(m \in Z_+)$ and g is a strong outer function. If $f \in S_\phi$, then the inner part of f is a finite Blashke product which does not have more than m zeros in U.*

Proof. If there exists an $f \in S_\phi$ such that $f(a_1) = \cdots = f(a_{m+1}) = 0$ and $|a_j| < 1$ $(1 \leqslant j \leqslant m+1)$, then there exists $f_0 \in S_\phi$ with $f_0 = z^{m+1} k$ for some $k \in H^1$ Then $zkg^{-1} \geqslant 0$ a.e. on ∂U and so $zk = g$ because g is strong outer. This contradicts that g is an outer function.

THEOREM 1. *If $\phi = \bar{z}^m |g|/g$, where $m \in Z_+$ and g is strong outer, then*

$$S_\phi = (\{\gamma\} \times S_{\bar{z}^m} \times g) \cap S^1.$$

Proof. The proof is by induction on m. Set $\phi_m = \bar{z}^m |g|/g$. If $m = 0$, then $S_{\phi_0} = \{g/\|g\|_1\}$ clearly. Assume $S_{\phi_m} = (\{\gamma\} \times S_{\bar{z}^m} \times g) \cap S^1$ for $1 \leqslant m \leqslant n$. If $f \in S_{\phi_{n+1}}$, then

$$F = (f + z^{n+1} g)/2 \in S_{\phi_{n+1}}.$$

The F is not an outer function by a theorem of de Leeuw and Rudin [1, Theorem 1] because $\|F\|_1 = 1$ and F is not an extreme point of S. By Lemma 3, F has the form

$$F = \prod_{j=1}^{l} \frac{z - a_j}{1 - \bar{a}_j z} h,$$

where $1 \leqslant l \leqslant n+1$, $|a_j| < 1$ $(1 \leqslant j \leqslant l)$ and h is an outer function. Set

$$k = \frac{1}{\alpha} \prod_{j=1}^{l} \frac{h}{(1 - \bar{a}_j z)^2}$$

for some positive constant α and $\|k\|_1 = 1$. Then the k is an outer function in H^1 and $F = \alpha \prod_{j=1}^{l} (z - a_j)(1 - \bar{a}_j z)\, k$. Since $F\bar{z}^{n+1} g^{-1} \geqslant 0$ a.e. on ∂U and $F^{-1} z^l k \geqslant 0$ a.e. on ∂U, $z^l k \bar{z}^{n+1} g^{-1} \geqslant 0$ a.e. on ∂U and so the k belongs to $S_{\phi_{n-l+1}}$. By the hypothesis of induction,

$$k = \beta \prod_{j=1}^{n-l+1} (z - b_j)(1 - \bar{b}_j z)\, g,$$

where $\beta > 0$ and $|b_j| \leqslant 1$. Set $a_{l+j} = b_j$ for $1 \leqslant j \leqslant n - l + 1$, then

$$F = \alpha\beta \prod_{j=1}^{n+1} (z - a_j)(1 - \bar{a}_j z)\, g.$$

Set $f_0 = 2\alpha\beta \prod_{j=1}^{n+1} (z - a_j)(1 - \bar{a}_j z) - z^{n+1}$, then $f = 2F - z^{n+1} g = f_0 g$. Since $\bar{z}^{n+1} g^{-1} f \geqslant 0$ a.e. on ∂U and so $\bar{z}^{n+1} f_0 \geqslant 0$ a.e. on ∂U, $f_0 \in \{\gamma\} \times S_{\bar{z}^{n+1}}$ and hence f belongs to $(\{\gamma\} \times S_{\bar{z}^{n+1}} \times g) \cap S^1$.

3. S_ϕ is Weak-* Compact

The following lemma is known in the proof of [1, Theorem 10]. We shall give a simple proof in which we do not use a theorem of O. Frostman.

LEMMA 4. *If $\phi \notin zH^\infty$ and S_ϕ is weak-* compact and nonempty, then the inner part of f for every $f \in S_\phi$ is a finite Blashke product which does not have more than m zeros on U.*

Proof. If there were functions in S_ϕ with arbitrarily many zeros in U, then by Lemma 2, S_ϕ would contain functions f_n $(n = 1, 2, \ldots)$ with

$$f_n(1/2) = f_n(1/3) = \cdots = f_n(1/n) = 0.$$

This contradicts the weak-* compactness of S_ϕ. Lemma 2 shows that the inner part of every $f \in S_\phi$ is a finite Blashke product and hence the lemma follows.

THEOREM 2. *If $\phi \in L^\infty$ and $\phi \notin zH^\infty$, then the following are equivalent:*

(1) *S_ϕ is weak-* compact and not empty.*

(2) *The inner part of every f in S_ϕ is a finite Blashke product which does not have more than m zeros in U.*

(3) *The extremal kernel ψ of ϕ has the form: $\psi = \bar{z}^m |g|/g$, where g is strong outer and $m \in Z_+$.*

(4) *$S_\phi = (\{\gamma\} \times S_{\bar{z}^m} \times g) \cap S^1$, where g is strong outer and $m \in Z_+$.*

(5) *The dimension of the linear span of S_ϕ is finite and $2m+1$.*

Proof. (1) $\Rightarrow$ (2) is proved in Lemma 4. (2) $\Rightarrow$ (3) (cf. [1, Theorem 10]). $z^m g \in S_\phi$ for some $g \in H^1$ and $g(0) \neq 0$. Then the g is strong outer. For if $\arg g_1(e^{i\theta}) = \arg g(e^{i\theta})$ a.e. on ∂U for some $g_1 \in S^1$, then by [1, Theorem 1] $(g + g_1)/2$ is not an outer function. By Lemma 2, there exists $g_2 \in S^1$ such that $\arg g_2(e^{i\theta}) = \arg g(e^{i\theta})$ a.e. on ∂U and $g_2(0) = 0$. Then $z^m g_2 \in S_\phi$ and this contradicts (2). (3) $\Rightarrow$ (4) is proved in Theorem 1. (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (2), and (4) $\Rightarrow$ (1) are clear.

The author [4] proved that if $\|z^n\phi + zH^\infty\| < \|\phi + zH^\infty\|$ for some $n \in Z_+\setminus\{0\}$, then $S_\phi \neq \varnothing$ and S_ϕ is weak-* compact. We shall consider the case $\|z^n\phi + zH^\infty\| = \|\phi + zH^\infty\|$ for any $n \in Z_+\setminus\{0\}$.

LEMMA 5 [3, p. 231]. *Let $\phi \in L^\infty$ and $|\phi| = 1$ a.e. Then there is a nonzero $k \in H^\infty$ such that*

$$\|\phi - k\|_\infty \leqslant 1$$

if and only if there is a nonzero $h \in H^1$ with

$$\phi = h/|h|.$$

THEOREM 3. *Suppose $S_\phi \neq \varnothing$ and ϕ is an extremal kernel. If there exists a nonzero k_0 in zH^∞ such that $\|z^{n+1}\phi + k_0\|_\infty \leqslant \|\phi + zH^\infty\|$ for some $n \in Z_+$ then*

$$h \times S_\phi \subset S_{\bar{z}^n}$$

for some $m \in Z_+$ and some h in H^1. Hence S_ϕ is weak- compact.*

Proof. If $f \in S_\phi$, then $\|z^{n+1}|f|/f + k_0\| \leqslant 1$ and so

$$z^n \frac{|f|}{f} = \frac{h}{|h|}$$

for some nonzero $h \in H^1$ by Lemma 5. Hence $\bar{z}^n fh \geqslant 0$ a.e. on ∂U and $fh \in H^1$ by Lemma 1. Thus $\gamma \times h \times S \subset S_{\bar{z}^n}$.

Suppose $\phi = |f|/f$ for some nonzero $f \in H^1$. If there exists a nonzero $h \in H^1$ such that hf is an analytic polynomial, then S_ϕ is weak-* compact by Lemma 5 and Theorem 3 [5].

THEOREM 4. *Suppose $\phi = |f|/f$ for some nonzero f in H^1. If $f = pg$, where p is an analytic polynomial of degree n and g is strong outer, then the inner part of every F in S_ϕ is a finite Blashke product which does not have more than n zeros in U and so S_ϕ is weak-* compact.*

Proof. If the inner part of $F_1 \in S_\phi$ is a finite Blashke product which has $n+1$ zeros in U, then $z^{n+1}h \in S_\phi$ for some nonzero $h \in H^1$ by Lemma 2. If $p = \alpha \prod_{j=1}^{s} (z - a_j) \prod_{j=s+1}^{n} (1 - \bar{a}_j z)$, where $|a_j| \leqslant 1$ $(1 \leqslant j \leqslant s)$ and $|a_j| > 1$ $(s+1 \leqslant j \leqslant n)$, then

$$F_2 = \beta z \prod_{j=1}^{n} (z - a_j)(1 - \bar{a}_j z)\, h \in S_\phi.$$

Since $F_2/pg \geqslant 0$ a.e. on ∂U, $\beta\alpha^{-1} z \prod_{j=1}^{s} (1 - \bar{a}_j z) \prod_{j=s+1}^{n} (z - a_j)\, h/g \geqslant 0$ a.e. on ∂U. Thus

$$g = \gamma\beta\alpha^{-1} z \prod_{j=1}^{s} (1 - \bar{a}_j z) \prod_{j=s+1}^{n} (z - a_j)\, h$$

for some $\gamma > 0$ and this contradicts that g is strong outer.

4. EXPOSED POINTS OF S

An element g of S is called an exposed point of S if $g \in S_\phi$ for some $\phi \in L^\infty$ and $S_\phi = \{g\}$. It is clear that $g \in S$ is strong outer if and only if g is an exposed point in S. While de Leeuw and Rudin [1] showed that $h \in S$ is an outer function if and only if h is an extreme point in S. The following proposition is known [6]. We shall give another proof.

PROPOSITION 5. *Let g be in H^1.*

(1) *If $g^{-1} \in H^1$, then g is strong outer.*

(2) *If $\operatorname{Re} k(e^{i\theta})\, g(e^{i\theta}) \geqslant 0$ a.e. on ∂U for some $k \in H^\infty$ with $k^{-1} \in H^\infty$, then g is strong outer.*

Proof. (1) is clear by Lemma 1. (2) Let $h = kg/\|kg\|_1$ and $\phi = |h|/h$. If there exists $f \in S_\phi$ and $f \neq h$, then $(f+h)/2 \in S_\phi$ and $(f+h)/2$ is not an outer function by [1, Theorem 1]. While $\operatorname{Re}(f+h)/2$ is nonnegative a.e. on ∂U and so $(f+h)/2$ is an outer function. This contradiction implies that h is strong outer and so g is strong outer.

THEOREM 6. *If f is a nonzero function in H^1 and S_ϕ is weak-* compact for $\phi = |f|/f$, then the f has the form:*

$$f = g \prod_{j=1}^{m} (z - a_j)(1 - \bar{a}_j z),$$

where $|a_j| \leqslant 1$ and g is strong outer. Moreover, the factorization is unique.

Proof. From Theorem 2, the theorem follows immediately.

If $f = pg$ where p is an analytic polynomial and g is strong outer, by Theorem 4 S_ϕ is weak-* compact. Thus we can factorize f as in Theorem 6. The author [5] showed that we can factorize f in case $g^{-1} \in H^1$. In particular if $g = 1$ and so $f = p$, we can factorize f and as the consequence we can determine strong outer functions in all analytic polynomials.

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