Periodic solutions of second order superquadratic Hamiltonian systems with potential changing sign (I)

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Received 12 August 2004
Available online 10 August 2005

Abstract

We consider the periodic solutions of the second order Hamiltonian system

\[-\ddot{x} + \lambda x = h(t)V'(x)\]

with $V$ being positive and superquadratic at infinity, $h$ being continuous, 2π-periodic, sign changing and satisfying $\{t|h(t) > 0\} \cap \{t|h(t) < 0\} = \emptyset$. Some existence and multiplicity results of periodic solutions are given.

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1. Introduction

The existence of periodic solutions of the second order Hamiltonian system

\[-\ddot{x} + \lambda x = V'_x(t, x)\]  \hspace{0.5cm} (1.1)
where \( x \in \mathbb{R}^m \) and \( V \) is \( 2\pi \)-periodic in \( t \), has been extensively studied in the last two decades by the variational methods. The case that the potential \( V \) is positive at infinity, there are many results of periodic solutions for both subquadratic and superquadratic potentials, (see [9,19]) and the references therein. In particular, if \( V(t,x) = V(x) + f(t) \) with \( f \) being periodic and \( V \) satisfying the Ambresetti–Rabinowitz superquadratic condition: there are constants \( \theta > 2 \) and \( r > 0 \) such that
\[
V'(x) \cdot x \geq \theta V(x), \quad |x| \geq r,
\]
(V1)
it is proved in [2,18] that (1.1) has an unbounded sequence of periodic solutions.

We consider the case that the potential has the form \( V(t,x) = h(t)V(x) \) with \( h \) being sign changing by the Morse theory. This problem has been considered by several authors (see [1,4,8,10–14,16,17,21]). Roughly speaking, in these papers, the potential \( V \) is either positive homogeneous of degree \( p \) or asymptotic to \( |x|^p \) at infinity for some \( p > 2 \). We assume in this paper that \( V \) satisfies (V1) and \( h \) is a continuous, \( 2\pi \)-periodic function and satisfies the thick zero condition
\[
\{ t \in [0, 2\pi] | h(t) < 0 \} \cap \{ t \in [0, 2\pi] | h(t) > 0 \} = \emptyset,
\]
(h0)
and the sets \( S_+ = \{ t \in [0, 2\pi] | h(t) > 0 \} \), \( S_- = \{ t \in [0, 2\pi] | h(t) < 0 \} \), \( S_0 = S^1 \setminus (S_+ \cup S_-) \), are nonempty, consist of finite intervals. The case that \( h \) satisfies the thin zero condition
\[
h \in C^1, \quad h'(t) \neq 0 \text{ whenever } h(t) = 0
\]
(h1)
will be considered in a sequel paper to this one [15].

In case of \( h > 0 \), it is known that the Palais–Smale condition holds for the associated functional \( I \) if \( V \) satisfies (V1). Besides the (P.S) condition, in applying the Morse theory to (1.1), the fact that all critical groups of \( I \) at infinity are zero plays an important role. However, in the case \( h \) changes sign in \( t \), both the (P.S) condition and the above fact on the critical groups of \( I \) at infinity become nontrivial. Some additional conditions are needed in order to get these conclusions.

In considering the thick zero case, the set \( \sigma(S_0) \) of the eigenvalues of
\[
-\ddot{x} = \lambda x
\]
(1.2)
with Dirichlet boundary value on \( S_0 \) plays an important role. Let \( \lambda_* > 0 \) be the first eigenvalue in \( \sigma(S_0) \). Let \( \sigma(S^1) = \{ k^2, k \text{ is an integer} \} \) be the set of the eigenvalues of \( (1.2) \) with periodic boundary conditions \( x(0) = x(2\pi), \dot{x}(0) = \dot{x}(2\pi) \). It is shown in [21] that if \( \lambda < \lambda_* \) and \( \lambda \notin \sigma(S^1) \), \( V \) is superquadratic at 0 and asymptotic to \( |x|^p \) for some \( p > 2 \) at infinity, then (1.1) has a nonzero solution, and moreover, if \( V \) is even, then (1.1) has an unbounded sequence of periodic solutions.

We will prove the following result in this paper.
Theorem 1. Let $V_1$ and $V_2$ be $C^2$ functions satisfying the superquadratic condition (V1), and let $h$ be a continuous $2\pi$-periodic function satisfying the thick zero condition $(h_0)$. Set $h_- = \min\{0, h\}$ and $h_+ = \max\{0, h\}$. Suppose $\lambda \notin \sigma(S_0)$. Then

$$-\ddot{x} - \lambda x = h_-(t)V_1'(x) + h_+(t)V_2'(x)$$

has a nonzero $2\pi$-periodic solution if either $\lambda \notin \sigma(S^1)$, $V_1$ and $V_2$ satisfy

$$|V_1'(x)| = |V_2'(x)| = o(|x|) \quad \text{at } x = 0;$$

or there is a symmetric neighborhood $U$ of 0 in $\mathbb{R}^m$ such that

$$V_1(-x) = V_1(x), \quad V_2(-x) = V_2(x), \quad x \in U.$$  

(V3)

If $V_1$ and $V_2$ are even in $x$, then (1.3) has an unbounded sequence of $2\pi$-periodic solutions.

Comparing this theorem with the known results, the potential $V$ is much more general, and $V_1$ and $V_2$ may have different behaviors at infinity. Moreover, the assumption $\lambda < \lambda_*$ is removed.

Our proof of the theorem is variational, which means we are going to look for the critical points of the functional

$$I(x) = \frac{1}{2} \int_0^{2\pi} (|\dot{x}|^2 - \lambda |x|^2) \, dt - \int_0^{2\pi} h_-(t)V_1(x) \, dt - \int_0^{2\pi} h_+(t)V_2(x) \, dt$$

which is defined on

$$H^1(S^1) = \left\{ x : [0, 2\pi] \rightarrow \mathbb{R}^m, \ x(0) = x(2\pi), \ \int_0^{2\pi} (|\dot{x}|^2 + |x|^2) \, dt < \infty \right\}.$$  

The basic idea of the proof is same as in [7] where a semilinear elliptic BVP with superlinear and indefinite nonlinearities is discussed. Indeed, the results for (1.3) are quite similar to those for the semilinear elliptic BVP. We refer to [7] and the references therein for related results of the elliptic BVP. The key point is that with the assumptions of the theorem, the functional $I$ satisfies the (P.S) condition, the critical groups of $I$ at infinity are well defined and are zero as in the case of $h > 0$. Having these facts, the existence and multiplicity of periodic solutions of (1.3) follow from the Morse inequality easily. In case of $V_1$ and $V_2$ are even, this approach also yields an unbounded sequence of $2\pi$-periodic solutions as pointed out in [3]. This is different from the usual symmetric mountain pass theorem argument in [11,19], which needs the functional satisfying some additional geometric conditions at 0 and infinity, and has been used in [7,21].
2. The Palais–Smale condition

This section concerns with the (P.S) condition for the functional $I$. We begin with a simple decomposition lemma of the space $H^1(S^1)$. For simplicity, we assume that $S_+$ and $S_-$ have only one component. Then the condition $(h_0)$ implies that $S_0$ has two components $S_1^0$ and $S_2^0$, $S^1 = S_+ \cup S_1^0 \cup S_- \cup S_2^0$. For any interval $[a, b] \subset S^1$, we identify $H^1_0([a, b])$ with its image of the natural inclusion in $H^1(S^1)$.

Lemma 2. There is a direct sum decomposition

$$H^1(S^1) = X_1 \oplus X_2$$

with $X_1 = H^1_0(S_0^1 \cup S_- \cup S_2^0)$ and

$$X_2 = \left\{ x \in H^1_0(S_0^1 \cup S_+ \cup S_2^0) \mid \int_0^{2\pi} \dot{x} \cdot \dot{y} \, dt = 0, \forall y \in H^1_0(S_0^1 \cup S_2^0) \right\}.$$

Proof. The proof is quite simple. It is same as in [7]. Indeed, it is simpler here since the domain is an interval. First, we note that $H^1_0(S_-), H^1_0(S_+)$ and $H^1_0(S_0)$ are orthogonal to each other in $H^1(S^1)$. Let $E_1 = H^1_0(S_0) \oplus H^1_0(S_-) \oplus H^1_0(S_+)$, and let

$$E_2 = \left\{ x \in H^1(S^1) \mid \int_0^{2\pi} \dot{x} \cdot \dot{y} \, dt = 0, \forall y \in E_1 \right\}.$$

Then $H^1(S^1)$ is the direct sum of $E_1$ and $E_2$, $H^1(S^1) = E_1 \oplus E_2$. It is easy to see that

$$E_2 = \{ x \in H^1(S^1) \mid -\ddot{x} = 0 \text{ on } S_-, S_+, S_0 \},$$

which is the direct sum of $E_3$ and $E_4$, where

$$E_3 = \{ x \in H^1_0(S_0 \cup S_-) \mid -\ddot{x} = 0 \text{ on } S_-, S_0 \},$$

$$E_4 = \{ x \in H^1_0(S_0 \cup S_+) \mid -\ddot{x} = 0 \text{ on } S_+, S_0 \}.$$

Let $X_1 = E_3 \oplus H^1_0(S_0) \oplus H^1_0(S_-) = H^1_0(S_0 \cup S_-)$ and $X_2 = E_4 \oplus H^1_0(S_+)$, then

$$H^1(S^1) = X_1 \oplus X_2.$$

This completes the proof. □
We note that the dimension of $E_2$ is $4m$ and that of $E_3, E_4$ is $2m$. Hence all norms on these spaces are equivalent. We will use this fact later.

For $x \in H^1(S^1)$, set $x = y + z$ with $y \in X_1$ and $z \in X_2$. Then we have

\[
I(x) = \frac{1}{2} \int_0^{2\pi} (|\dot{y}|^2 + 2s \dot{y} \cdot \dot{z} + |\dot{z}|^2 - \lambda |y + z|^2) \, dt - \int_0^{2\pi} h_-(t)V_1(y + z) \, dt - \int_0^{2\pi} h_+(t)V_2(y + z) \, dt = \frac{1}{2} \int_0^{2\pi} (|\dot{y} + \dot{z}|^2 - \lambda |y + z|^2) \, dt - \int_0^{2\pi} h_-(t)V_1(y + z) \, dt - \int_0^{2\pi} h_+(t)V_2(y + z) \, dt = I(y, z). \tag{2.2}
\]

Now, we introduce a family of functionals $I_s$, $s \in [0, 1]$, as follows:

\[
I_s(x) = \frac{1}{2} \int_0^{2\pi} (|\dot{y}|^2 + 2s \dot{y} \cdot \dot{z} + |\dot{z}|^2 - \lambda (|y|^2 + 2s y \cdot z + |z|^2)) \, dt - \int_0^{2\pi} h_-(t)V_1(y) \, dt - \int_0^{2\pi} h_+(t)V_2(z) \, dt, \tag{2.3}
\]

which connects $I$ to

\[
I_0(x) = J_1(y) + J_2(z),
\]

and the later is of separable variables, where

\[
J_1(y) = \frac{1}{2} \int_0^{2\pi} (|\dot{y}|^2 - \lambda |y|^2) \, dt - \int_0^{2\pi} h_- V_1(y) \, dt,
\]

\[
J_2(z) = \frac{1}{2} \int_0^{2\pi} (|\dot{z}|^2 - \lambda |z|^2) \, dt - \int_0^{2\pi} h_+ V_2(z) \, dt.
\]

The following is the main result of this section.

**Proposition 3.** Let $V_1, V_2 \in C^1$ satisfy (V1) and let $h$ be a continuous, $2\pi$-periodic function satisfying $(h_0)$ and $\lambda \notin \sigma(S_0)$. Let $C$ be a constant and $s_n \in [0, 1], x_n \in H^1(S^1)$
such that
\[ I_{s_n}(x_n) \leq C, \quad \|I'_{s_n}(x_n)\| = o(\|x_n\|) \text{ as } n \to \infty. \tag{2.4} \]

Then \(\{x_n\}\) is bounded and contains a convergent subsequence.

**Proof.** As in the elliptic BVP problem, it suffices to show that \(\{x_n\}\) is bounded. In following, we denote \(C\) a constant independent of \(n\). Let
\[
\tilde{x}_n = \frac{x_n}{\|y_n\|_2 + \|z_n\|_2}, \quad \tilde{y}_n = \frac{y_n}{\|y_n\|_2 + \|z_n\|_2}, \quad \tilde{z}_n = \frac{z_n}{\|y_n\|_2 + \|z_n\|_2}.
\]
We assume that \(\|y_n\|_2 + \|z_n\|_2 \geq 1\). If \(\|y_n\|_2 + \|z_n\|_2 \leq 1\), the following arguments show that \(\{y_n\}\) and \(\{z_n\}\) are bounded. Therefore, \(\{x_n\}\) is bounded.

**Claim 1.** \(\{\tilde{y}_n\}\) and \(\{\tilde{z}_n\}\) are bounded.

First, we have
\[
\frac{\langle \varepsilon_n, \tilde{y}_n \rangle}{\|y_n\|_2 + \|z_n\|_2} = \frac{\langle I'_{s_n}(x_n), \tilde{y}_n \rangle}{\|y_n\|_2 + \|z_n\|_2}
\]
\[
= \int_0^{2\pi} (\tilde{y}_n^2 - \lambda \tilde{y}_n^2) dt - \frac{\int_0^{2\pi} h_- V'(y_n) \cdot \tilde{y}_n dt}{\|y_n\|_2 + \|z_n\|_2}
\]
\[
+ s_n \int_0^{2\pi} (\tilde{y}_n \cdot \tilde{z}_n - \lambda \tilde{y}_n \cdot \tilde{z}_n) dt
\]
\[
\geq \int_0^{2\pi} (\tilde{y}_n^2 + s_n \tilde{y}_n \cdot \tilde{z}_n) dt - C, \tag{2.5}
\]

since \(h_- \leq 0, \|\tilde{y}_n\|_2 \leq 1\) and \(\|\tilde{z}_n\|_2 \leq 1\). Now we estimate \(\int_0^{2\pi} \tilde{y}_n \cdot \tilde{z}_n dt\). Since \(\tilde{z}_n \in X_2 = H^1_0(S_+) \oplus E_4\), we have \(\tilde{z}_n = \tilde{z}_{1,n} + \tilde{z}_{2,n}\) with \(\tilde{z}_{1,n} \in H^1_0(S_+)\) and \(\tilde{z}_{2,n} \in E_4\).

Using \(\text{supp}(\tilde{y}_n) \subset S^1_+ \cup S_- \cup S^2_0\) we have
\[
\left| \int_0^{2\pi} \tilde{y}_n \cdot \tilde{z}_n dt \right| = \left| \int_0^{2\pi} \tilde{y}_n \cdot \tilde{z}_{2,n} dt \right|
\]
\[
\leq \|\tilde{y}_n\| \|\tilde{z}_{2,n}\|_2 + 1
\]
\[
\leq C\|\tilde{y}_n\| \|\tilde{z}_{2,n}\|_2 + 1
\]
\[
\leq C(1 + \|\tilde{y}_n\|) \tag{2.6}
\]

since \(E_4\) is finite dimensional and
\[
\|\tilde{z}_{2,n}\| \leq C\|\tilde{z}_{2,n}\|_2 \leq C\|\tilde{z}_n\|_2 \leq C.
\]
Substituting (2.6) to (2.5), we get

\[ \| \tilde{y}_n \|^2 \leq C + o(1) \| \tilde{y}_n \| \| \tilde{x}_n \| \leq C + o(1) \| \tilde{y}_n \| (\| \tilde{y}_n \| + \| \tilde{z}_n \|). \]  

(2.7)

This implies

\[ \| \tilde{y}_n \|^2 \leq C + o(1) \| \tilde{z}_n \|^2. \]  

(2.8)

Simple computation shows

\[
\frac{1}{(\| y_n \|_2 + \| z_n \|_2)^2} \left( I_{s_n}(x_n) - \frac{1}{\theta}(e_n, z_n) \right) \\
= \frac{1}{2} \int_0^{2\pi} (|\dot{\tilde{y}}_n|^2 - \lambda |\tilde{y}_n|^2) \, dt + \left( \frac{1}{2} - \frac{1}{\theta} \right) \int_0^{2\pi} (|\dot{\tilde{z}}_n|^2 - \lambda |\tilde{z}_n|^2) \, dt \\
+ \left( 1 - \frac{1}{\theta} \right) s_n \int_0^{2\pi} (\tilde{y}_n \cdot \dot{\tilde{z}}_n - \lambda \tilde{y}_n \cdot \tilde{z}_n) \, dt \\
- \int_0^{2\pi} h_-(t) V_1(y_n) \, dt + \int_0^{2\pi} h_+(t) \left( \frac{1}{2} V'_2(z_n) \cdot z_n - V_2(z_n) \right) \, dt \\
\geq \frac{1}{2} \left( \frac{1}{2} - \frac{1}{\theta} \right) \| \tilde{z}_n \|^2 - C (\| \tilde{y}_n \|^2 + 1) \\
\geq \epsilon_0 \| \tilde{z}_n \|^2 - C 
\] (2.9)

for some \( \epsilon_0 > 0 \) by (2.8). The LHS of (2.9) satisfies

\[
\frac{1}{(\| y_n \|_2 + \| z_n \|_2)^2} \left( I_{s_n}(x_n) - \frac{1}{\theta}(e_n, z_n) \right) \leq C + o(1) \| \tilde{z}_n \|^2
\]

by (2.4). Then from (2.9), we obtain \( \| \tilde{z}_n \| \leq C \) and \( \| \tilde{y}_n \| \leq C \) by (2.8).

Claim 2. \( \{ \| y_n \|_2 + \| z_n \|_2 \} \) is bounded, hence \( \{ \| y_n \| + \| z_n \| \} \) is bounded.

Assume \( \| y_n \|_2 + \| z_n \|_2 \to \infty \) as \( n \to \infty \), then by Claim 1, we have \( \| \tilde{z}_n \| \leq C \) and \( \| \tilde{y}_n \| \leq C \). We may assume

\[ \tilde{y}_n \to y_0, \quad \tilde{z}_n \to z_0, \quad \tilde{x}_n \to x_0 \text{ weakly in } H^1(S^1) \]

and

\[ \tilde{y}_n \to y_0, \quad \tilde{z}_n \to z_0, \quad \tilde{x}_n \to x_0 \text{ strongly in } L^2(S^1) \]

with \( x_0 = y_0 + z_0 \). Then \( x_0 \neq 0 \) as \( \| y_0 \|_2 + \| z_0 \|_2 = 1 \).
Let $\phi = \frac{y_n}{(\|y_n\|_2 + \|z_n\|_2)}$, we have

$$
(\varepsilon_n, \phi) = (I'_n(x_n), \phi)
= \int_0^{2\pi} (\hat{x}_n \cdot \hat{y}_n - \lambda \hat{x}_n \hat{y}_n) \, dt - \int_0^{2\pi} h_-(t) \frac{V'_1(y_n) \cdot y_n}{(\|y_n\|_2 + \|z_n\|_2)^2} \, dt
+ s_n \int_0^{2\pi} (\hat{y}_n \cdot \hat{z}_n - \lambda \hat{y}_n \cdot \hat{z}_n) \, dt.
$$

(2.10)

From this we get

$$
- \int_0^{2\pi} h_-(t) \frac{V'_1(y_n) \cdot y_n}{(\|y_n\|_2 + \|z_n\|_2)^2} \, dt \leq C.
$$

This and $V_1$ satisfies the superquadratic condition (V1) imply

$$
-(\|y_n\|_2 + \|z_n\|_2)^{\theta-2} \int_0^{2\pi} h_-|\tilde{y}_n|^\theta \, dt \leq C.
$$

(2.11)

Since for $n \to \infty$, $\tilde{y}_n \to y_0$ weakly in $X$ so

$$
\int_0^{2\pi} h_-|\tilde{y}_n|^\theta \, dt \to \int_0^{2\pi} h_-|y_0|^\theta \, dt.
$$

As $\theta > 2$ and $\|y_n\|_2 + \|z_n\|_2 \to \infty$, from (2.11) we conclude

$$
- \int_0^{2\pi} h_-|y_0|^\theta \, dt = 0.
$$

Therefore,

$$
y_0(t) = 0, \quad t \in S_-.
$$

Similarly, computing $(\varepsilon_n, \frac{z_n}{(\|y_n\|_2 + \|z_n\|_2)^2})$ we have

$$
z_0(t) = 0, \quad t \in S_+.
$$

Hence

$$
z_0 = 0$$
because \( \ddot{z} = 0 \) in \( S_0 \) and \( z = 0 \) on the boundary of \( S_0 \) by the definition of \( X_2 \). Consequently,

\[
x_0(t) = y_0(t) + z_0(t) = y_0(t) = 0 \quad \forall t \notin S_0.
\]

(2.12)

For any \( \phi \in H^1_0(S_0) \) we have

\[
\int_0^{2\pi} (\dot{x}_n \cdot \dot{\phi} - \lambda \dot{x}_n \phi) \, dx + s_n \int_0^{2\pi} (\dot{\phi} \cdot \ddot{z}_n - \dot{\lambda} \phi \cdot z_n) \, dt = (\tilde{\epsilon}_n, \phi),
\]

where \( \tilde{\epsilon}_n = \frac{\epsilon_n}{\|y_0\|_2 + \|z_0\|_2} \). Let \( n \to \infty \) we get

\[
\int (\dot{x}_0 \cdot \dot{\phi} - \lambda x_0 \phi) \, dt = 0
\]

provided by \( \tilde{z}_n \to 0 \) weakly. This shows

\[
\ddot{x}_0 + \lambda x_0 = 0.
\]

(2.13)

Since \( x_0 \neq 0 \), (2.12) and (2.13) imply \( \lambda \in \sigma(S_0) \). This contradicts with the assumption. Therefore \( \{x_n\} \) is bounded. Having this fact, now the proof that \( \{x_n\} \) contains a convergent subsequence is standard. □

The following is a simple consequence of Proposition 3, which is needed in the computation of critical groups of \( I \) at infinity by a deformation argument.

**Proposition 4.** There are constants \( A \) and \( \delta > 0 \) such that for \( s \in [0, 1] \),

\[
\|I_s'(x)\| \geq \delta \|x\| \quad \text{if} \quad I_s(x) \leq A.
\]

(2.14)

3. The critical groups and the existence of periodic solutions

In this section, we first give a result on the critical groups of the functional \( I \) at infinity by a deformation argument. Then we will apply the Morse theory to get the periodic solutions of

\[
-\ddot{x} - \lambda x = h_-(t)V_1'(x) + h_+(t)V_2'(x).
\]

(3.1)

From now on we assume \( V_1 \) and \( V_2 \) are \( C^2 \). Then the functional \( I \) defined by

\[
I(x) = \frac{1}{2} \int_0^{2\pi} (|\dot{x}|^2 - \lambda |x|^2) \, dt - \int_0^{2\pi} h_-(t)V_1(x) \, dt - \int_0^{2\pi} h_+(t)V_2(x) \, dt
\]
is $C^2$. We take a number $a < A$, where $A$ is the constant given by Proposition 4. Let

$$I_a = \{ x \in X | I(x) \leq a \}.$$  

The critical groups of $I$ at infinity are defined as

$$C_\ast(I, \infty) = H_\ast(X, I_a), \quad \ast = 0, 1, 2, \ldots.$$  

The coefficient group of the homology is $G$. They are well defined and independent of the choices of $a < A$. Set $m_\ast(\infty) = \text{rank} C_\ast(I, \infty)$ and

$$P(\infty, t) = \sum_{i \geq 0} m_i(\infty) t^i.$$  

For an isolated $2\pi$-periodic solution $x$ of (3.1), let

$$C_\ast(I, x) = H_\ast(U \cap I_c, (U \setminus \{ x \}) \cap I_c)$$

where $U$ is a neighborhood of $x$ such that $I$ has no critical point other than $x$ in $U$ and $c = I(x)$. Set $m_\ast(I, x) = \text{rank} C_\ast(I, x)$ and

$$P(x, t) = \sum_{i \geq 0} m_i(I, x) t^i.$$  

For any isolated critical point $x$, $P(x, t)$ is a finite sum. The Morse inequality is as follows: Let $\{ x_1, x_2, \ldots \}$ be the set of $2\pi$-periodic solutions of (3.1). We assume it is a finite set, hence each $x_i$ is isolated in particular. With this assumption, there is a polynomial $Q(t)$ with nonnegative integer as its coefficients such that

$$\sum P(x_i, t) = P(\infty, t) + (1 + t)Q(t). \quad (3.2)$$

**Proposition 5.** Let $h \in C^0$ satisfy $(h_0)$, $V_1$ and $V_2$ be $C^2$ functions satisfying the superquadratic condition $(V1)$ and $\lambda \notin \sigma(S_0)$. Then

$$C_\ast(I, \infty) = 0, \quad \ast = 0, 1, \ldots.$$  

We use the homotopy invariance of the critical groups to prove this result. Consider the family of functionals $I_s$ introduced in the last section. By Proposition 4, for $s \in [0, 1]$, the critical groups of $I_s$ at infinity $C_\ast(I_s, \infty)$ are well defined. We will show that for all $s \in [0, 1]$, $C_\ast(I_s, \infty)$ are isomorphic. The proof consists of several steps. We only give a sketch here since it is essentially the same as that in the superlinear elliptic BVP in [7].
**Step 1:** Consider the following differential equation on $X$:

\[
\frac{d}{ds}\sigma(s, x) = -\frac{\partial_x I_x(\sigma(s, x))}{\|I_x'(\sigma(s, x))\|^2}, \quad \sigma(0, x) = x.
\] (3.4)

The RHS of (3.4) is local Lipschitz since $I_x \in C^2$. Using Proposition 4 and $|\partial_x I_x(\sigma(s, x))| \leq C\|\sigma(s, x)\|^2$, we know that the RHS of (3.4) is at most linear growth if $I_x(\sigma(s, x)) \leq a$. Noting $\frac{dI_x(\sigma(s, x))}{ds} = 0$ from (3.4), hence the solution exists for $s \in [0, 1]$ if $x \in I_0(x) \leq a$ and the map

$$
x \to \sigma(1, x)
$$

is a homoeomorphism between $I_{0,a}$ and $I_a$. Hence

$$
C_*(I_0, \infty) = C_*(I, \infty).
$$ (3.5)

Noting $I_0(x) = J_1(y) + J_2(z)$.

**Step 2:** The functionals $J_1$ and $J_2$ satisfy (P.S) condition and the critical groups at infinity are well defined. Moreover, the following Künneth type formula holds, for a proof of this formula, (see [7])

$$
C_p(I_0, \infty) = \bigoplus_{i+j=p, i,j \geq 0} C_i(J_1, \infty) \otimes C_j(J_2, \infty).
$$ (3.6)

**Step 3:** All critical groups of $J_2$ at infinity are trivial,

$$
C_*(J_2, \infty) = 0, \quad * = 0, 1, \ldots.
$$ (3.7)

This follows from:

(i) For any $z \in X_2$ with $\|z\| = 1$,

$$
\lim_{t \to +\infty} J_2(tz) = -\infty.
$$

(ii) There is a constant $A$ such that

$$
\frac{d}{dt} J_2(tz) < 0 \quad \text{if} \quad J_2(tz) < A.
$$

By which, it can be proved that the level set $J_{2,a}$ is homotopy equivalent to the unit sphere in $X_2$ if $a \leq A$ as in [5,20]. Both (i) and (ii) are consequence of the superquadratic condition and the fact that if $z \in X_2$ and $\int_0^{2\pi} h_+(t)|z|^2 dt = 0$, then $z = 0$. 
Step 4: Determining the critical groups of $J_1$ at infinity. Using the decomposition $X_1 = H_0^1(S_0) \oplus H_0^1(S_-) \oplus E_3$, for $y \in X_1$, we have $y = y_1 + y_2$ with $y_1 \in H_0^1(S_0)$ and $y_2 \in H_0^1(S_-) \oplus E_3$. From

$$
\int_0^{2\pi} \dot{y}_1 \cdot \dot{y}_2 \, dt = 0
$$

and $\text{supp}(y_1) \subset S_0$ we have

$$
J_1(y) = J_1(y_1 + y_2)
= \int_0^{2\pi} \left( \frac{1}{2} |\dot{y}_1 + \dot{y}_2|^2 \, dt - \lambda |y_1 + y_2|^2 \right) \, dt - \int_0^{2\pi} h_-(t) V_1(y_1 + y_2) \, dt
= \frac{1}{2} \int_0^{2\pi} |\dot{y}_1|^2 \, dt - \frac{\lambda}{2} \int_0^{2\pi} |y_1|^2 \, dt - \int_0^{2\pi} y_1 \cdot y_2 \, dt
+ \frac{1}{2} \int_0^{2\pi} |\dot{y}_2|^2 \, dt - \frac{\lambda}{2} \int_0^{2\pi} |y_2|^2 \, dt - \int_0^{2\pi} h_-(t) V_1(y_2) \, dt
= J_3(y_1) + J_4(y_2) - \lambda \int_0^{2\pi} y_1 \cdot y_2 \, dt, \quad (3.8)
$$

where

$$
J_3(y_1) = \frac{1}{2} \int_0^{2\pi} |\dot{y}_1|^2 \, dt - \frac{\lambda}{2} \int_0^{2\pi} |y_1|^2 \, dt,
$$

$$
J_4(y_2) = \frac{1}{2} \int_0^{2\pi} |\dot{y}_2|^2 \, dt - \frac{\lambda}{2} \int_0^{2\pi} |y_2|^2 \, dt - \int_0^{2\pi} h_-(t) V_1(y_2) \, dt,
$$

they are defined on $H_0^1(S_0)$ and $H_0^1(S_-) \oplus E_3$, respectively. Similar to Step 2, $J_3$ and $J_4$ satisfy (P.S) condition, their critical groups at infinity are well defined and satisfy

$$
C_p(J_1, \infty) = \bigoplus_{i+j=p, i, j \geq 0} C_i(J_3, \infty) \otimes C_j(J_4, \infty). \quad (3.9)
$$

$J_3$ is a nondegenerate quadratic form, so

$$
C_*(J_3, \infty) = \delta_{*, j_0} G, \quad (3.10)
$$

where $j_0$ is the Morse index of $J_3$ on $H_0^1(S_0)$. As for $C_*(J_4, \infty)$ we have

$$
C_*(J_4, \infty) = \delta_{*, 0} G. \quad (3.11)
$$
This follows from the fact that $J_4$ is bounded from below since $\int_0^{2\pi} h_-(t)V_1(y_2) \, dt$ is superquadratic in $y_2$. It can be proved as follows. If there is a sequence $\{y_{2,n}\} \subset H^1_0(S_-) \oplus E_3$ such that $J_4(y_{2,n}) \leq -n$. Then

$$
\frac{1}{\|y_{2,n}\|^2_2} \left( \frac{1}{2} \int_0^{2\pi} |\dot{y}_{2,n}|^2 \, dt - \frac{\lambda}{2} \int_0^{2\pi} |y_{2,n}|^2 \, dt - \int_0^{2\pi} h_-(t)V_1(y_{2,n}) \, dt \right) \leq 0.
$$

Therefore, $\{\tilde{y}_{2,n}\}$ is bounded since $h_- \leq 0$, where $\tilde{y}_{2,n} = \frac{y_{2,n}}{\|y_{2,n}\|_2}$. We may assume $\tilde{y}_{2,n} \to \tilde{y}_2$ weakly as $n \to \infty$. As in the last section we have

$$
- \int_0^{2\pi} h_-|\tilde{y}_2|^2 \, dt = 0.
$$

This proves (3.12). Setting $t = -1$ in (3.13), we get $\sum_i P(x_i, -1) \neq 0$. This concludes that there must be a solution $x_i$ such that $P(x_i, -1) \neq 0$. Indeed, we can get more information on the critical point $x$. Let $Q(t) = q_0 + q_1 t + \cdots + q_k t^k + \cdots$ and let $N_0$ be the

\textbf{Proof of Theorem 1.} For the existence of one nonzero $2\pi$-periodic solution, we may assume that the number of $2\pi$-periodic solutions of (3.1) is finite. Otherwise, we have a sequence of solutions. In particular, all solutions are isolated. Let $\{x_1, \ldots, x_k\}$ be the set of solutions. Then by the Morse inequality (3.2) and (3.3) we have

$$
P(0, t) + \sum_i P(x_i, t) = P(\infty, t) + (1 + t)Q(t) = (1 + t)Q(t).
$$

Now we show

$$
P(0, -1) \equiv 1 \pmod{\mathbb{Z}_2}.
$$

With the condition (V2) and $\lambda \notin \sigma(S^1)$, 0 is a nondegenerate critical point of $I$, hence $C_*(I, 0) = \delta_{i_0} G$, and $P(0, -1) = (-1)^{i_0}$, where $i_0$ is the Morse index of 0. If (V3) holds, then the functional $I$ is even in a neighborhood of 0. $I'(x) = x - K(x)$ with $K : X \to X$ being a compact and odd map. 0 is an isolated zero of $x - K(x)$, so the fixed point index $\text{ind}(x - K(x), 0)$ is an odd number by the Borsuk-Ulam theorem. But according to a theorem in [5] we have

$$
P(0, -1) = \sum_i (-1)^{i} m_i(0) = \text{ind}(x - K(x), 0).
$$

This proves (3.14). Setting $t = -1$ in (3.13), we get $\sum_i P(x_i, -1) \neq 0$. This concludes that there must be a solution $x_i$ such that $P(x_i, -1) \neq 0$. Indeed, we can get more information on the critical point $x$. Let $Q(t) = q_0 + q_1 t + \cdots + q_k t^k + \cdots$ and let $N_0$ be the
degree of \( P(0, t) \). From (3.13), we conclude that there must be an integer \( 0 \leq i \leq N_0 + 1 \) and a critical point \( x_j \) such that
\[
m_i(x_j) \neq 0. \tag{3.15}
\]
Hence, the Morse index of \( x_j \) satisfies
\[
i(x_j) \leq N_0 + 1 \tag{3.16}
\]
by the Morse lemma and the shifting theorem for the critical groups for \( I \). For the details we refer to [5].

Indeed, if \( m_i(x_j) = 0 \) for \( 0 \leq i \leq N_0 \) and all \( x_j \), comparing the coefficients of the term \( t^{N_0} \) in the LHS and the RHS of (3.13), we have
\[
q_{N_0-1} + q_{N_0} \neq 0.
\]
If \( q_{N_0} = 0 \), from (3.13) we have
\[
P(0, t) = (1 + t) \left(q_0 + q_1 t + \cdots + q_{N_0-1} t^{N_0-1}\right).
\]
This contradicts with \( P(0, -1) \) is odd. Therefore,
\[
q_{N_0} \neq 0.
\]
Substituting this into (3.13), we obtain
\[
\sum_j m_{N_0+1}(x_j) \geq q_{N_0} > 0. \tag{3.17}
\]
This proves (3.15) for \( i = N_0 + 1 \) if (V2) or (V3) holds.

Now we assume that \( V_1 \) and \( V_2 \) are even in \( x \). Then, nonzero \( 2\pi \)-periodic solutions of (3.1) appear in pairs \( \{-x, x\} \). We will show that for any real number \( b \), there is a critical point \( x \) such that \( I(x) \geq b \). Hence, there is a sequence of critical points \( x_n \) such that \( I(x_n) \to +\infty \). This implies \( \|x_n\|_\infty \to +\infty \) as \( n \to \infty \). Since if \( \{x_n\} \) is bounded in \( L^\infty \), then it is bounded in \( H^1 \), hence \( I(x_n) \) is bounded.

Suppose for some \( b \), there is no critical point of \( I \) satisfying \( I(x) \geq b \). Then
\[
C_\ast(I, \infty) = H_\ast(X, I_A) = H_\ast(I_b, I_A) = 0, \quad \ast = 0, 1, 2, \ldots \tag{3.18}
\]
since \( I_b \) is a strong deformation retract of \( X \) by the deformation lemma. By the (P.S) condition, the set of critical points \( K = \{x | I'(x) = 0\} \) is compact. If 0 is an isolated
solution, then we may assume that the number of solutions is finite by the Marino-Prodi perturbation, for the details we refer to [11]. Let \( \{x_1, -x_1, x_2, -x_2, \ldots \} \) be the set of nonzero solutions. For each nonzero periodic solution \( x \), we have critical groups \( C_*(I, x) \), and \( C_*(I, x) = C_*(I, -x) \), \( * = 0, 1, \ldots \). Hence \( P(x, t) = P(-x, t) \). Consider the Morse inequality

\[
P(0, t) + \sum_i P(x_i, t) + \sum_i P(-x_i, t) = P(\infty, t) + (1 + t)Q(t) \tag{3.19}
\]

and set \( t = -1 \) in (3.19), we get that the LHS of (3.19) is odd due to (3.14), but the RHS is 0. This is impossible. In case of 0 is not an isolated solution, some additional arguments are needed. It can proceed as follows. Consider the following flow on \( X \):

\[
\frac{d}{ds} \eta(s, x) = -\xi(\eta(s, x)), \\
\eta(0, x) = x, \tag{3.20}
\]

where

\[
\xi(x) = \min\{\text{dist}(x, K), 1\} \frac{I'(x)}{\|I'(x)\|}.
\]

The flow \( \eta \) is well defined on \( X \times \mathbb{R} \). Set

\[
[K] = \left\{ x \in X | \lim_{s \to -\infty} \eta(s, x), \lim_{s \to +\infty} \eta(s, x) \in K \right\}.
\]

Then \( [K] \) is an isolated invariant set of the flow \( \eta \) and \( (I_b, I_A) \) is a Conley index pair of \( [K] \). We take a closed and symmetric neighborhood \( U \) of \( K \). Then by a result in [6], we have

\[
\deg(I'(x), U, 0) = \sum_i (-1)^i \text{rank}C_i(I_b, I_A) = 0. \tag{3.21}
\]

This is again impossible since \( I'(x) \) is an odd map and the LHS of (3.21) is odd by the Borsuk-Ulam theorem. \( \square \)

**Remarks.** (1) In case of (V2) or (V3) holds, the Morse index estimate (3.16) of the solution \( x \) is crucial in [15], in which we study the existence of periodic solutions of (3.1) in case that \( h \) satisfies \( (h_1) \) by an approximation argument. However, in order to get the Morse index estimate of the whole sequence of solutions \( x_n \) if \( V_1 \) and \( V_2 \) are even, some additional arguments are needed.
(2) The assumption that $h$ is continuous is not necessary. Theorem 1 holds if $h \in L^1(S^1)$ and satisfies

$$h(t) > 0 \text{ a.e. } t \in S_+, \quad h(t) < 0 \text{ a.e. } t \in S_-,$$

where $S_- \cap S_+ = \emptyset$, and $S^1 = S_- \cup S^1_0 \cup S_+ \cup S^2_0$, each $S_k$ is a union of finite intervals. Since in our proof, we need the functional $I$ is $C^2$, which only needs $V_1$ and $V_2$ are in $C^2$. In case of $h \in L^1(S^1)$, the solutions in Theorem 1 are weak solution, so (3.1) holds a.e.

4. A variant of Theorem 1

In Theorem 1, we assume that both $V_1$ and $V_2$ are superquadratic. From the proof, we see that the important point in our arguments is the (P.S) condition and all critical groups of $I$ at infinity are zero. In order to get these facts, it is not necessary to assume both $V_1$ and $V_2$ are superquadratic. In this section, we show that the same conclusion holds if $V_1$ is a $C^2$ function satisfying the asymptotically linear condition at infinity

$$V_1(x) = V \cdot x + o(|x|) \quad |x| \to \infty \quad (V4)$$

for a constant symmetric matrix $V$. The proof is same as that of Theorem 1. We assume that $S_+ = \{t \in S^1 | h(t) > 0\} \neq S^1$ and it is a union of finite intervals. Then $S_0 = S^1 \setminus S_+ \neq \emptyset$ is a finite union of intervals. Let $\sigma(S_0)$ be the set of the eigenvalues of

$$-\ddot{x} - h_-(t)V \cdot x = \lambda x \quad (4.1)$$

with Dirichlet boundary values on $S_0$.

**Theorem 6.** Let $V_1$ be a $C^2$ function satisfying (V4) and (V2), and let $V_2$ be a $C^2$ function satisfying (V1) and (V2). Then if $\lambda \notin \sigma(S_0) \cup \sigma(S^1)$,

$$-\ddot{x} - \lambda x = h_-(t)V'_1(x) + h_+(t)V'_2(x) \quad (4.2)$$

has a nonzero $2\pi$-periodic solution. Moreover, if $V_1$ and $V_2$ are even in $x$, then (4.2) has an unbounded sequence of $2\pi$-periodic solutions.

**Proof.** The proof is similar to that of Theorem 1, we give a sketch. Similar to Lemma 2, we have the following direct decomposition:

$$H^1(S^1) = H^1_0(S_0) \oplus E_1. \quad (4.3)$$
where

\[ E_1 = \left\{ x \in H^1(S^1) \mid \int_0^{2\pi} \dot{x} \cdot \dot{y} \, dt = 0, \ \forall y \in H^1_0(S_0) \right\}. \]

For \( x \in H^1(S^1) \), let \( x = y + z \), \( y \in H^1_0(S_0) \) and \( z \in E_1 \). Then

\[
I(x) = \frac{1}{2} \int_0^{2\pi} (|\dot{y}|^2 - \lambda |y|^2) \, dt - \int_0^{2\pi} h_-(t)V_1(y + z) \, dt \\
+ \frac{1}{2} \int_0^{2\pi} (|\dot{z}|^2 - \lambda |z|^2) \, dt - \int_0^{2\pi} h_+(t)V_2(z) \, dt \\
- \lambda \int_0^{2\pi} y \cdot z \, dt. \quad (4.4)
\]

**Step 1:** The functional \( I \) satisfies (P.S) condition. Let \( x_n = y_n + z_n \) be a (P.S) sequence of \( I \). It suffice to show that \( \{x_n\} \) is bounded. Let \( \tilde{y}_n = \frac{y_n}{\|y_n\|_2 + \|z_n\|_2} \), \( \tilde{z}_n = \frac{z_n}{\|y_n\|_2 + \|z_n\|_2} \).

Then

\[
\frac{1}{\|y_n\|_2 + \|z_n\|_2} \left( I(x_n) - \frac{1}{\theta} \langle e_n, z_n \rangle \right) = \frac{1}{2} \int_0^{2\pi} \left( |\dot{\tilde{y}}_n|^2 - \lambda |\tilde{y}_n|^2 \right) \, dt + \left( \frac{1}{2} - \frac{1}{\theta} \right) \int_0^{2\pi} \left( |\dot{\tilde{z}}_n|^2 - \lambda |\tilde{z}_n|^2 \right) \, dt \\
- \left( \frac{1}{\theta} \right) \lambda \int_0^{2\pi} \tilde{y}_n \cdot \tilde{z}_n \, dt \\
- \frac{1}{\|y_n\|_2 + \|z_n\|_2} \int_0^{2\pi} h_-(t) \left( V_1(y_n + z_n) \right. \\
\left. - \frac{1}{\theta} V_1'(y_n + z_n) \cdot z_n \right) \, dt \\
- \frac{1}{\|y_n\|_2 + \|z_n\|_2} \int_0^{2\pi} h_+(t) \left( V_2(z_n) \right. \\
\left. - \frac{1}{\theta} V_2'(z_n) \cdot z_n \right) \, dt. \quad (4.5)
\]

Using the condition (V4) and \( V_2 \) satisfying (V1), we can show that \( \int_0^{2\pi} |\dot{\tilde{y}}_n|^2 \, dt \) and \( \int_0^{2\pi} |\dot{\tilde{z}}_n|^2 \, dt \) are bounded. Thus, we may assume \( \tilde{z}_n \to z_0 \) and \( \tilde{y}_n \to y_0 \) weakly in \( H^1(S^1) \) as \( n \to \infty \). Then as in Proposition 3, if \( \{\|y_n\|_2 + \|z_n\|_2\} \) is unbounded, computing \( \frac{1}{\|y_n\|_2 + \|z_n\|_2} \langle e_n, z_n \rangle \), we have \( \int_0^{2\pi} h_+ |z_0|^2 \, dt = 0 \), hence \( z_0 = 0 \) by \( z_0 \in E_1 \).
and \( \tilde{z}_n \to 0 \) weakly. Combining this with (V4) we can show \( y_0 \) satisfies
\[
-\ddot{y} - \lambda y = h_-(t)V \cdot y.
\]
This contradicts with \( \lambda \notin \sigma(S_0) \). Hence \( \{ \| y_n \|_2 + \| z_n \|_2 \} \) and \( \{ \| z_n \| + \| y_n \| \} \) are bounded.

\textit{Step 2}: All critical groups of \( I \) at infinity are zero. This follows from the deformation
\[
I_s(x) = \frac{1}{2} \int_0^{2\pi} (|\dot{y}|^2 - \lambda|y|^2) \, dt - \frac{(1 - s)}{2} \int_0^{2\pi} h_-(t)(Vy, y) \, dt
+ \frac{1}{2} \int_0^{2\pi} (|\dot{z}|^2 - \lambda|z|^2) \, dt - \int_0^{2\pi} h_+(t)V_2(z) \, dt - \frac{(1 - s)}{2} \int_0^{2\pi} h_-(t)(Vz, z) \, dt
-s \lambda \int_0^{2\pi} y \cdot z \, dt - s \int_0^{2\pi} h_-(t)V_1(y + z) \, dt.
\]

We can prove as in Section 3 that the critical groups of \( I \) at infinity are well defined and same as those of
\[
\tilde{I}(x) = \frac{1}{2} \int_0^{2\pi} (|\dot{y}|^2 - \lambda|y|^2 - h_-(t)(Vy, y)) \, dt
+ \frac{1}{2} \int_0^{2\pi} (|\dot{z}|^2 - \lambda|z|^2 - h_-(t)(Vz, z)) \, dt - \int_0^{2\pi} h_+(t)V_2(z) \, dt
= \tilde{J}_1(y) + \tilde{J}_2(z).
\]
The functional \( \tilde{J}_1 \) is a nondegenerate quadratic form on \( H^1_0(S_0) \), and \( \tilde{J}_2 \) is superquadratic in \( z \in E_1 \) and satisfies \( C_*(\tilde{J}_2, \infty) = 0 \), \( * = 0, 1, 2, \ldots \). Then \( C_*(I, \infty) = C_*(\tilde{I}, \infty) = 0 \) follows from the Künneth formula (3.6). The remaining part of the proof is same as that of Theorem 1.

\textbf{References}

[13] M. Girardi, M. Matzeu, On periodic solutions of the system \( \ddot{x}(t) + b(t) (V_1(x(t)) + V_2(x(t))) = 0 \) where \( b(\cdot) \) changes sign and \( V_1, V_2 \) have different superquadratic growths, in: Proceedings Local and Variational Methods on Hamiltonian Systems, World Scientific, Singapore, 1995, pp. 65–76.