Homogeneity and $\sigma$-discrete sets

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Abstract

All countable dense homogeneous spaces and all densely homogeneous spaces satisfy the $T_1$-separation axiom. Such spaces may, in the presence of standard separation axioms, fail to satisfy stronger ones. Some results are given which pertain to the notion of homogeneity with respect to $\sigma$-discrete dense sets that are homeomorphic to one another.

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1. Introduction

This is a continuation of an ongoing study [1, 2] of densely homogeneous spaces. As before, if $X$ is a topological space, then $H(X)$ is the group of all autohomeomorphisms on $X$. A separable space $X$ is countable dense homogeneous (CDH) provided that if $A$ and $B$ are two countable dense subsets of $X$, then there is an $h \in H(X)$ such that $h(A) = B$. A space $X$ is densely homogeneous (DH) provided that (1) $X$ has a $\sigma$-discrete dense subset and (2) if $A$ and $B$ are two $\sigma$-discrete dense subsets of $X$, then there is an $h \in H(X)$ such that $h(A) = B$. Here, by a $\sigma$-discrete set we mean a subset of $X$ that is the union of countably many sets, each, with the relative topology, being a discrete space. This is a change from our former requirement that the terms in the countable union be closed in $X$, and we feel that this definition is preferable in that, under it, every countable set is $\sigma$-discrete.
The two are equivalent for Moore spaces. It is known [1] that connected spaces that are DH or CDH are homogeneous.

In Section 2 we discuss separation axioms in CDH spaces and in DH spaces.

In Section 3 we discuss \( \sigma \)-discrete dense subsets of metric spaces, using a theorem of Medvedev [3] to characterize those metric spaces all of whose \( \sigma \)-discrete dense subsets are homeomorphic. We also answer a question of E.K. van Douwen concerning a separable 2-manifold of Moore [4].

2. Separation axioms

We have previously discussed CDH spaces and DH spaces in the context of \( T_1 \)-spaces. Here we point out that this hypothesis is redundant. This theorem first appeared, in a different form, in [6].

**Theorem 2.1.** Suppose \( X \) is a topological space that is either DH or CDH. Then \( X \) is a \( T_1 \)-space.

**Proof.** If \( X \) is CDH, respectively, DH, let \( C \) denote the collection of all dense subsets of \( X \) that are countable, respectively, \( \sigma \)-discrete. Throughout, we adopt the notation that if \( x, y \in X \), then \( x \sim y \) means that \( x \in \text{CL}([y]) \), and \( x < y \) means that \( x \leq y \) and \( x \neq y \). Define an equivalence relation on \( X \) by \( x \sim y \) if and only if \( x \leq y \) and \( y \leq x \).

Suppose first that \( X \) is not \( T_0 \). Then there exist distinct points \( p, q \) such that \( p < q \) and \( q < p \). Let \( Q \) be the collection of all nontrivial equivalence classes under \( \sim \). Let \( K \in C \).

**Case 1:** \( K \cap (\bigcup Q) = \emptyset \). Let \( p \in \bigcup Q \), and let \( K' = K \cup \{p\} \). Then \( K' \in C \), but there is no \( h \in H(X) \) such that \( h(K) = K' \). This provides a contradiction.

**Case 2:** \( K \cap (\bigcup Q) \neq \emptyset \). For each \( A \in Q \), choose a representative \( p_A \) of \( A \), with \( p_A \in K \) if \( A \cap K \neq \emptyset \). Let \( K' = (K \setminus (\bigcup Q)) \cup \{p_A : A \in Q, A \cap K \neq \emptyset\} \). Then \( K' \in C \), and \( K' \) intersects each member of \( Q \) in at most one point. Choose one \( A \in Q \), and let \( p \) and \( q \) be distinct elements of \( A \), and let \( K'' = K' \cup \{p, q\} \). Then \( K'' \in C \), and there is no \( h \in H(X) \) such that \( h(K') = K'' \). Again, we have a contradiction.

Therefore, \( Q = \emptyset \), and \( X \) is a \( T_0 \)-space. It now follows that if \( x \leq y \) and \( y \leq x \), then \( x = y \).

Next, assume that \( X \) is not a \( T_1 \)-space. Let \( A = \{x \in X : [x] \) is not closed\}, and let \( B = \{y \in X : y < x \) for some \( x \in A\}\). Assume that there are points \( x, y \in X \) such that \( x < y \) and \( y \notin B \). In this case \( y \) is called a primitive of \( x \).

There exists a member \( K \) of \( C \) such that \( K \) contains some point and a primitive of that point. Let \( K_1 = \{x \in K \cap B : x \) and a primitive of \( x \) are in \( K\}\}; let \( K_2 = K \setminus K_1 \). Then \( K_2 \in C \), and \( K_2 \) contains no point and its primitive. There is no \( h \in H(X) \) such that \( h(K) = K_2 \), a contradiction.
Therefore, no point of $B$ has a primitive, and it follows that $A \subseteq B$.

Next, assume that there are points of $B$ that are not in $A$. There is a $K \in C$ that contains such a point. For each $x \in K \cap (B \setminus A)$, choose $p_x \in A$ with $x < p_x$. Let $K' = (K \setminus (B \setminus A)) \cup \{p_x \in A : x \in K \cap (B \setminus A)\}$. Then $K' \in C$, and no $h \in H(X)$ takes $K'$ onto $K$.

Therefore, we have $A = B$. It follows that if $y \in A$, then there exist $x, z \in A$ such that $x < y < z$. Let $K \in C$. Then $K \cap A \neq \emptyset$. Moreover, there exist points $y_i, i \in \mathbb{Z}$ (the set of all integers), such that each $y_i \in K \cap A$ and $y_i < y_j$ for $i < j$, $i, j \in \mathbb{Z}$. If this fails, then choose a $y_0 \in K \cap A$ and add such a double-ended sequence to $K$ to get a $K' \in C$ with no $h \in H(X)$ such that $h(K) = K'$.

Write $K \cap A = \bigcup_{i=1}^{\infty} K_i$, with each $K_i$ discrete. Let $L_1 = (K \cap A) \setminus \{k \in K : k < k_i \text{ for some } k_i \in K_i\}$. Then $L_1 = \bigcup_{i=1}^{\infty} K'_i$, where $K'_1 = K_1$ and $K'_i = K_i \cap L_1$. Let $L_2 = K'_1 \cup K'_2 \cup (\bigcup_{i=3}^{\infty} K_i) \setminus \{k \in K : k < k_2 \text{ for some } k_2 \in K_2\}$. Then $L_2 = \bigcup_{i=1}^{\infty} K''_i$, where $K''_1 = K'_1$, $K''_i = K'_i \cap L_2$. Continue this process, and consider the diagonal $L = K'_1 \cup K''_2 \cup K'''_3 \cup \cdots$. Then $(K \setminus A) \cup L$ is an element of $C$ that contains no double-ended sequence, a contradiction. The proof is now complete.  

Example 2.2. That not every CDH or DH space is Hausdorff is evinced by an uncountable space with the cofinite topology.

Example 2.3. That not every CDH $T_2$-space is regular is evinced by P. Minc’s example $(\mathbb{R}, \Omega')$, discussed in [1].

Example 2.4. That not every CDH $T_3$-space is normal is evinced by Moore’s manifold $\Sigma_n$, discussed in [2].

Example 2.5. (MA + CH) That not every CDH $T_4$-space is completely normal is evinced by $[0, 1]^{\omega_1}$, discussed in [5].

Example 2.6. That not every DH $T_5$-space is perfectly normal is evinced by the long line (without a first point). That space also shows that DH spaces need not be bihomogeneous.

Remark. We do not currently have examples to show that CDH $T_i$-spaces need not be completely regular, that CDH $T_i$-spaces need not be perfectly normal, or that DH $T_i$-spaces need not be $T_{i+1}$, $i = 2, 3, 4$.

3. $\alpha$-discrete dense subsets of metric spaces and Moore spaces

At the 1985 Independence Day Conference on Limits, held at City College, New York, on seeing Moore’s manifold $\Sigma_\theta$ presented as an example of a CDH space that is not DH, van Douwen asked whether $\Sigma_\theta$ is homogeneous with respect to $\alpha$-discrete dense sets that are homeomorphic to one another, and he suggested that such a weaker form of dense homogeneity might be worthy of study. Consideration
of this matter naturally leads to the question as to when all \( \sigma \)-discrete dense subsets of a space are homeomorphic to one another.

Consider the complex plane with the topology induced by the "hedgehog" metric. That is, if \( \text{Arg } z_1 \neq \text{Arg } z_2 \), then \( D(z_1, z_2) = |z_1| + |z_2| \), and if \( \text{Arg } z_1 = \text{Arg } z_2 \), then \( D(z_1, z_2) = |z_1 - z_2| \). This space, called a hedgehog of spininess \( c \), is densely homogeneous with respect to \( \sigma \)-discrete dense subsets that are homeomorphic to one another. It is connected, but it is not homogeneous. This seems to us to represent a flaw in this weaker form of DH. Note that the local density of the space is not constant.

If \( p \in X \), let \( \text{ld}(X, p) \), the local density of \( X \) at \( p \), be the smallest cardinal \( \alpha \) such that some open set in \( X \) containing \( p \) contains a dense subset of cardinality \( \alpha \).

**Theorem 3.1.** In order that all the \( \sigma \)-discrete dense subsets of a metric space \( X \) be homeomorphic it is necessary and sufficient that \( X \) be a discrete union of subspaces \( X_\alpha \) such that if \( p, q \in X_\alpha \), then \( \text{ld}(X, p) = \text{ld}(X, q) \).

**Proof.** The sufficiency follows from the following, which is a direct consequence of a theorem of Medvedev [3]. If \( a \) and \( \beta \) are infinite cardinals, \( a \leq \beta \), and if \( X \) and \( Y \) are two \( \sigma \)-discrete metric spaces of cardinality \( \beta \) which have bases whose members all have cardinality \( \alpha \), then \( X \) is homeomorphic to \( Y \).

To prove the necessity, assume that \( X \) is a metric space in which each two \( \sigma \)-discrete dense subsets are homeomorphic. Note that if \( D \) is a \( \sigma \)-discrete dense subset of \( X \) and \( p \in D \), then \( \text{ld}(D, p) = \text{ld}(X, p) \). Define an equivalence relation on \( X \) by \( p \sim q \) if and only if \( \text{ld}(X, p) = \text{ld}(X, q) \).

Consider the special case in which \( X \) is itself \( \sigma \)-discrete. Let \([p]\) be the equivalence class determined by \( p \). Assume some \([p]\) is not open, and let \( A \) be the set of all points of \([p]\) that are limit points of \( X \setminus [p] \). Then \( X = X \setminus A \) is \( \sigma \)-discrete and dense in \( X \), but \( X \) and \( Y \) are not homeomorphic. Therefore, each \([p]\) is open.

Finally, consider the general case. Let \( D \) be a \( \sigma \)-discrete dense subset of \( X \). Each two \( \sigma \)-discrete dense subsets of \( D \) are homeomorphic, so \( D \) is the union of a discrete (in \( D \)) collection of subspaces \( D_\alpha = \{ p \in D : \text{ld}(D, p) = \alpha \} \). Suppose some \([p]\) in \( X \) is not open in \( X \); let \( q \) be a point of \([p]\) that is a limit point of \( X \setminus [p] \). Now let \( D' = D \cup \{ q \} \). Again, \( D \) and \( D' \) are not homeomorphic. Therefore, the equivalence classes are all open, and the proof is complete. \( \square \)

**Corollary.** If \( X \) is a homogeneous metric space, then each two \( \sigma \)-discrete dense subsets of \( X \) are homeomorphic.

We conclude by answering van Douwen's question.

**Theorem 3.2.** \( \Sigma_B \) is not homogeneous with respect to \( \sigma \)-discrete dense subsets that are homeomorphic to one another.

**Proof.** We follow the description and terminology of the discussion of \( \Sigma_B \) given in [2].
Let \( \mathbb{R} \) be the real axis. For each \( r \in \mathbb{R} \), let \( I_r \) be the set of all new points in \( \Sigma_\beta \) attached at \( r \).

Let \( X \) and \( Y \) be two subsets of \( \bigcup_{r \in \mathbb{R}} I_r \) such that, for all \( r \), \( |X \cap I_r| = 1 \) and \( |Y \cap I_r| = 2 \). Suppose there is an \( h \in H(\Sigma_\beta) \) such that \( h(X) = Y \). For all \( r \in \mathbb{R} \), let \( J_r \subseteq I_r \) be an arc in \( \Sigma_\beta \) which connects the two points of \( Y \cap I_r \). Since \( h^{-1}(J_r) \subseteq \bigcup_{r \in \mathbb{R}} I_r \), it contains a point in the upper half-plane and hence contains an arc in the upper half-plane. This leads to a contradiction, since \( \{J_r: r \in \mathbb{R}\} \) is discrete in \( \Sigma_\beta \) and \( \{h^{-1}(J_r): r \in \mathbb{R}\} \) is not. Therefore, there is no such autohomeomorphism \( h \).

In \( \Sigma_\beta \), for each \( r \in \mathbb{R} \), let \( x_r \) be the vertical ray attached at \( r \); \( x_r \in \Sigma_\beta \). If \( M \subseteq \mathbb{R} \), let \( \tilde{M} = \{x_r: r \in M\} \). Let \( D \) be the set of all points in the upper half-plane with both coordinates rational. Let \( S \) be the open interval \((0, 1)\) in \( \mathbb{R} \); let \( A = D \cup \tilde{\mathbb{R}} \), \( B = D \cup \tilde{\mathbb{S}} \). Obviously, \( A \) and \( B \) are homeomorphic. Assume there is an \( h \in H(\Sigma_\beta) \) such that \( h(A) = B \). Consider the closed interval \( K = [-2, -1] \). Then \( \tilde{K} \) is a closed discrete set in \( \Sigma_\beta \), so \( h^{-1}(\tilde{K}) \) is closed and discrete. Thus there is an uncountable subset \( L \subseteq K \) such that \( h^{-1}(\tilde{L}) \subseteq \bigcup_{r \in \mathbb{R}} I_r \). Since \( h(A) = B \) and \( B \cap \tilde{K} = \emptyset \) it follows that \( h^{-1}(\tilde{L}) \cap A = \emptyset \). For \( r \in \mathbb{R} \) and \( h^{-1}(\tilde{L}) \cap I_r \neq \emptyset \), \( h^{-1}(\tilde{L}) \cap I_r \) contains only one element; call it \( u_r \). Let \( X = \bigcup\{\{x_r, u_r\}: r \in \mathbb{R}, h^{-1}(\tilde{L}) \cap I_r \neq \emptyset\} \). Let \( Y = \{x_r: 0 < r < 1\} \cup \{x_r: -2 \leq r \leq -1 \text{ and } r \in L\} \). Then, for uncountably many \( r \in \mathbb{R} \), \( |X \cap I_r| = 2 \) and \( |Y \cap I_r| = 1 \). As in the paragraph above, a contradiction arises.

References